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JAAKKO HYVÖNEN

JUHANI RIIHENTAUS

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## ON THE EXTENSION IN THE HARDY CLASSES AND IN THE NEVANLINNA CLASS

BY

JAAKKO HYVÖNEN and JUHANI RIIHENTAU (\*)

RÉSUMÉ. — En utilisant des méthodes de la théorie du potentiel on a établi des théorèmes d'extension pour les fonctions de quelques classes de Hardy et de la classe de Nevanlinna dans  $C^n$ . Les ensembles exceptionnels sont polaires ou un peu plus grands ensembles  $n$ -petits selon que la fonction majorante sera harmonique ou séparément hyperharmonique.

ABSTRACT. — Using potential theoretic methods we give extension results for functions in various Hardy classes and in the Nevanlinna class in  $C^n$ . Our exceptional sets are polar or slightly larger  $n$ -small sets depending whether the majorant is a harmonic or separately hyperharmonic function.

### 1. Introduction

1.1. Recently Järvi ([9]; Theorem 1, p. 597) gave the following result.

Let  $G$  be an open set in  $C^n$ ,  $n \geq 1$ . Let  $E \subset G$  be closed in  $G$  and polar. Let  $f: G \setminus E \rightarrow C$  be a holomorphic function such that for some  $p > 0$ ,  $|f|^p$  has a harmonic majorant in  $G \setminus E$ . Then  $f$  has a unique holomorphic extension  $f^*: G \rightarrow C$  such that  $|f^*|^p$  has a harmonic majorant in  $G$ .

In the case  $n=1$  Järvi's result is contained in Parreau's classical result ([13]; Théorème 20, p. 182). In the case  $n \geq 1$  Järvi's result generalized Cima's and Graham's result ([3]; Theorem A, p. 241) which stated that analytic subvarieties are removable singularities for certain subdomains of

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J. HYVÖNEN, department of mathematics, University of Joensuu SF-80100 Joensuu 10, Finland.

J. RIIHENTAU, department of mathematics, University of Joensuu SF-80100 Joensuu 10, Finland, and department of mathematics, University of Oulu, SF-90570 Oulu 57, Finland.

$\mathbb{C}^n$ . Note that in [15]; Theorem 3.2, p. 285 a similar result was given to Järvi's result, however, only in the case  $p \geq 2$ .

Järvi's proof was based on a lemma of Parreau ([9]; Lemma, pp. 596-597) (see also [7]; Lemma 1, p. 18) concerning quasibounded harmonic functions. In section 2 below we give a perhaps more elementary proof to the above result of Järvi. Our proof applies also to the case of  $n$ -harmonic, i. e. separately harmonic functions. In this case our exceptional sets are  $n$ -small. For the definition of these sets see [16]; Definition 2.2 and 2.2 below.

In [15]; Theorem 3.9, p. 287, it was observed that the following result is a direct consequence of [10]; Theorem 2, p. 279 (see also [11]; Theorem 4, p. 35 and [5]; Theorem 1.2 (b), p. 704) and of [1]; Corollary 2.10, p. 425.

*Let  $G$  be an open set in  $\mathbb{C}^n$ ,  $n \geq 1$ . Let  $E \subset G$  be closed in  $G$  and polar. Let  $f: G \setminus E \rightarrow \mathbb{C}$  be a holomorphic function such that  $\log^+ |f|$  has a pluriharmonic majorant in  $G \setminus E$ . Then  $f$  has a unique meromorphic extension to  $G$ .*

In the case  $n=1$  this result is contained in the result of Parreau ([13]; Théorème 20, p. 182) (see also [1]; Corollary 2.10, p. 425). In the case  $n \geq 2$  the above result generalized Cima's and Graham's result ([3]; Theorem C, p. 241) which stated that in this situation analytic subvarieties are removable singularities for certain subdomains of  $\mathbb{C}^n$ .

In section 3 below we show that in the above result it is sufficient to suppose that  $\log^+ |f|$  has a harmonic majorant in  $G \setminus E$ . Our result gives thus a positive answer to a question posed by Cima and Graham ([3]; Remarks 7.4, p. 255).

The results for subharmonic functions are due to the first author, the results for  $n$ -hypoharmonic, i. e. separately hypoharmonic functions (except Remark 2.8) and for functions in the Nevanlinna class are due to the second author.

1.2. We use mainly the same notation as in [8]. See also [16]. However, we recall the following.

If  $a \in \mathbb{R}^k$ ,  $k \geq 1$ , and  $r > 0$ , we write

$$B^k(a, r) = \{x \in \mathbb{R}^k \mid |x - a| < r\}, \quad U = B^2(0, 1).$$

The complex space  $\mathbb{C}^n$ ,  $n \geq 1$ , will be identified with the real space  $\mathbb{R}^{2n}$ . If  $z_0 \in \mathbb{C}$  and  $r > 0$ , we write  $S^1(z_0, r) = \partial B^2(z_0, r)$ . If

$z = (z_1, \dots, z_n) \in \mathbb{C}^n$ ,  $n > 1$ , we set for each  $j$ ,  $1 \leq j \leq n$ ,

$$Z_j = (z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_n) \in \mathbb{C}^{n-1} \quad \text{and} \quad (z_j, Z_j) = z.$$

If  $G \subset \mathbb{C}^n$  and  $z_0 = (z_j^0, Z_j^0)$  we write

$$G(z_j^0) = \{Z_j \in \mathbb{C}^{n-1} \mid (z_j^0, Z_j) \in G\},$$

$$G(Z_j^0) = \{z_j \in \mathbb{C} \mid (z_j, Z_j^0) \in G\}.$$

If  $r = (r_1, \dots, r_n) \in \mathbb{R}_+^n$ , we write  $R_1 = (r_2, \dots, r_n)$  and

$$D^n(z_0, r) = B^2(z_1^0, r_1) \times D^{n-1}(Z_1^0, R_1),$$

where

$$D^{n-1}(Z_1^0, R_1) = B^2(z_2^0, r_2) \times \dots \times B^2(z_n^0, r_n).$$

If  $G \subset \mathbb{C}^n$  is open and  $f: G \rightarrow \mathbb{C}$  (resp.  $[-\infty, \infty]$ ) we write for each  $Z_1 \in \mathbb{C}^{n-1}$   $f_{Z_1}: G(Z_1) \rightarrow \mathbb{C}$  (resp.  $[-\infty, \infty]$ ),

$$f_{Z_1}(z_1) = f(z_1, Z_1).$$

For the Laplace of  $f$  (in the distribution sense) we write

$$\Delta f = \sum_{j=1}^n \Delta_j f,$$

where

$$\Delta_j f = 4 \frac{\partial^2 f}{\partial z_j \partial \bar{z}_j}.$$

For the definition of  $n$ -hyperharmonic, i.e. separately hyperharmonic functions see [8]. A function  $u: G \rightarrow [-\infty, \infty]$  is  $n$ -hypoharmonic, if  $-u$  is  $n$ -hyperharmonic. Note that a function  $u: G \rightarrow (-\infty, \infty]$  (resp.  $[-\infty, \infty)$ ) is superharmonic (resp. subharmonic) if  $u$  is hyperharmonic and  $\not\equiv \infty$  (resp. hypoharmonic and  $\not\equiv -\infty$ ) on each component of  $G$ .

The  $k$ -dimensional Hausdorff measure is denoted by  $H_k$  (note the difference between the Hardy class  $H^p$ ), the  $k$ -dimensional Lebesgue measure by  $m_k$ .

For the theory of holomorphic functions, Hardy classes and Nevanlinna class see [18] and [21]. For potential theory see [8] and [6].

## 2. On the extension in the Hardy classes

2.1. Let  $G$  be an open set in  $\mathbb{R}^n$ ,  $n \geq 2$ . Let  $p > 0$ . Set  $h^p(G) = \{u: G \rightarrow \mathbb{R}_+ \mid u \text{ is subharmonic and } u^p \text{ has a harmonic majorant in } G\}$ .

If  $G$  is an open set in  $\mathbb{C}^n$ ,  $n \geq 1$ , set

$h_n^p(G) = \{u : G \rightarrow \mathbb{R}_+ \mid u \text{ is } n\text{-hypoharmonic and } u^p \text{ has an } n\text{-hyperharmonic majorant in } G \text{ which is } \neq \infty \text{ on each component of } G\}$ ;

$H^p(G) = \{f : G \rightarrow \mathbb{C} \mid f \text{ is holomorphic and } |f|^p \text{ has a harmonic majorant in } G\}$ ;

$H_n^p(G) = \{f : G \rightarrow \mathbb{C} \mid f \text{ is holomorphic and } |f|^p \text{ has an } n\text{-hyperharmonic majorant in } G \text{ which is } \neq \infty \text{ on each component of } G\}$ .

In Theorem 2.5 below we give extension results for the classes  $h^p$  and  $h_n^p$ . In the case of the class  $h^p$  the exceptional set is polar and the proof is based on the well-known result which states that polar sets are removable singularities for subharmonic functions which are locally bounded above (see [8]; Theorem 2, p. 25). In the case of the class  $h_n^p$  the exceptional set is  $n$ -small (see [16]; Definition 2.2 and 2.2 below) and the proof is based on a corresponding result according to which  $n$ -hypoharmonic functions which are locally bounded above can be extended across  $n$ -small sets (see [16]; Theorem 4.1). We recall here, however, the definition of  $n$ -small sets and give a property of these sets.

2.2. For each set  $E \subset \mathbb{C}$  we define  $\mathcal{C}^1(E) = \text{cap}^* E$ , where  $\text{cap}^*$  denotes the outer logarithmic capacity in  $\mathbb{C}$ . If  $n \geq 2$ ,  $1 \leq j \leq n$  and  $\mathcal{C}^{n-1}$  is defined for subsets of  $\mathbb{C}^{n-1}$ , we define for  $E \subset \mathbb{C}^n$

$$\mathcal{C}_j^n(E) = H_2 \{z_j \in \mathbb{C} \mid \mathcal{C}^{n-1} \{Z_j \in \mathbb{C}^{n-1} \mid (z_j, Z_j) \in E\} > 0\}.$$

Finally, set

$$\mathcal{C}^n(E) = \max_{1 \leq j \leq n} \mathcal{C}_j^n(E).$$

We say that  $E \subset \mathbb{C}^n$  is  $n$ -small, if  $\mathcal{C}^n(E) = 0$ .

2.3. PROPOSITION. — Let  $E \subset \mathbb{C}^n$ ,  $n \geq 2$ . Then  $E$  is  $n$ -small, if for each  $k$ ,  $1 \leq k \leq n$ ,  $H_{2, n-2}(E_k) = 0$ , where

$$E_k = \{Z_k \in \mathbb{C}^{n-1} \mid \text{cap}^* \{z_k \in \mathbb{C} \mid (z_k, Z_k) \in E\} > 0\}.$$

Conversely, if  $E$  is  $n$ -small and an  $\mathcal{F}_\sigma$ -set, then  $H_{2, n-2}(E_k) = 0$  for each  $k$ ,  $1 \leq k \leq n$ .

Proof. — The first part of the Proposition is proved in [16]; Proposition

2.4. Note that there are (at least Lebesgue nonmeasurable)  $n$ -small sets  $E$  for which  $H_{2, n-2}(E_n) > 0$ . See [16]; Remark 2.8. We give an induction proof for the second part. If  $n=2$  then the assertion clearly

holds. Suppose then that  $n \geq 3$  and take  $k$ ,  $1 \leq k \leq n$ , arbitrarily. Since the outer logarithmic capacity and the Hausdorff outer measure are subadditive, we may suppose that  $E$  is compact. From [17]; Lemma 2.2.1, p. 87 it follows that  $E_k$  is an  $\mathcal{F}_\sigma$ -set and thus Lebesgue measurable.

Take  $j \neq k$ ,  $1 \leq j \leq n$ , arbitrarily. Since  $E$  is  $n$ -small, there is  $B_j \subset C$  such that  $H_2(B_j) = 0$  and that for each  $z_j \notin B_j$ , the set

$$E(z_j) = \{Z_j \in C^{n-1} \mid (z_j, Z_j) \in E\}$$

is  $(n-1)$ -small. It follows from the induction hypothesis that  $H_{2n-4}(E_k(z_j)) = 0$  for each  $z_j \notin B_j$ , where

$$E_k(z_j) = \{Z_{kj} \in C^{n-2} \mid \text{cap}^* \{z_k \in C \mid (z_k, Z_{kj}) \in E(z_j)\} > 0\}.$$

If  $\chi_{E_k}$  is the characteristic function of  $E_k$ , we get by Fubini's theorem

$$m_{2n-2}(E_k) = \int_{C \setminus B_j} \left( \int_{C^{n-2} \setminus E_k(z_j)} \chi_{E_k}(z_j, Z_{kj}) dm_{2n-4}(Z_{kj}) \right) dm_2(z_j) = 0,$$

concluding the proof.

**2.4. Remark.** — From [19]; Lemma 6, p. 115 (see also [12]; Corollary 3.3) and Proposition 2.2 it follows that polar sets are  $n$ -small. Note that Lebesgue measurable  $n$ -small sets are of Lebesgue measure zero ([16]; Remark 2.3).

**2.5. THEOREM.** — Let  $G$  be an open set in  $\mathbb{R}^n$ ,  $n \geq 2$  (resp. in  $\mathbb{C}^n$ ,  $n \geq 1$ ). Let  $E \subset G$  be closed in  $G$  and polar (resp.  $n$ -small). Let  $p > 1$ . If  $u \in h^p(G \setminus E)$  [resp.  $h_n^p(G \setminus E)$ ], then  $u$  has a unique extension  $u^* \in h^p(G)$  [resp.  $h_n^p(G)$ ].

*Proof.* — Let  $h$  be a harmonic majorant (resp. an  $n$ -hyperharmonic majorant which is  $\neq \infty$  on each component of  $G \setminus E$ ) of  $u$  in  $G \setminus E$ . By [8]; Theorem 2, p. 25 (resp. [16]; Theorem 4.1)  $h$  has a unique superharmonic (resp.  $n$ -hyperharmonic which by [8]; Theorem, p. 31 is superharmonic) extension  $h^* : G \rightarrow (-\infty, \infty]$ . Thus the greatest harmonic minorant  $v$  of  $h^*$  in  $G$  exists by [8]; Corollary 1, p. 10.

Take  $q$ ,  $1 < q < p$ , and  $\varepsilon > 0$  arbitrarily. Then the function  $u_\varepsilon : G \setminus E \rightarrow [-\infty, \infty)$ ,

$$u_\varepsilon(z) = u(z)^q - \varepsilon h(z),$$

is subharmonic (resp.  $n$ -hypoharmonic). Since

$$u_\varepsilon(z) \leq u(z)^q - \varepsilon u(z)^p$$

for each  $z \in G \setminus E$  and  $q < p$ ,  $u_q$  is bounded above in  $G$ . By [8]; Theorem 2, p. 25 (resp. [16]; Theorem 4.1)  $u_q$  has a unique subharmonic (resp.  $n$ -hypoharmonic which by [8]; Theorem, p. 31 is subharmonic) extension  $u_q^* : G \rightarrow [-\infty, \infty)$ .

For each  $z \in G \setminus E$  we have

$$h^*(z) - u_q^*(z) = h(z) - u(z)^q + \varepsilon h(z) \geq u(z)^p - u(z)^q + \varepsilon h(z) \geq -1.$$

Since  $E$  is of Lebesgue measure zero, it follows that

$$u_q^*(z) \leq h^*(z) + 1$$

for each  $z \in G$ . Thus by [8]; Corollary 1, p. 10

$$u_q^*(z) \leq v(z) + 1$$

for each  $z \in G$ . But then

$$u(z)^q - \varepsilon h(z) \leq v(z) + 1$$

for each  $z \in G \setminus E$ . Since  $\varepsilon > 0$  was arbitrary, we get

$$(A) \quad u(z)^q \leq v(z) + 1$$

for each  $z \in G \setminus E$ . Therefore  $u$  is locally bounded above in  $G$  and thus by [8]; Theorem 2, p. 25 (resp. [16]; Theorem 4.1) has a unique subharmonic (resp.  $n$ -hypoharmonic) extension  $u^* : G \rightarrow [0, \infty)$ . Since (A) holds for all  $q < p$  and  $E$  is of Lebesgue measure zero,

$$u^*(z)^p \leq v(z) + 1$$

for all  $z \in G$ . Thus  $u^* \in h^p(G)$ . [Resp. it follows directly that  $u^*(z)^p \leq h^*(z)$  for each  $z \in G$ . Thus  $u^* \in h_n^p(G)$ .]

2.6. COROLLARY ([9]; Theorem 1, p. 597 and [16]; Theorem 5.1). — Let  $G$  be an open set in  $\mathbb{C}^n$ ,  $n \geq 1$ . Let  $E \subset G$  be closed in  $G$  and polar (resp.  $n$ -small). Let  $p > 0$ . If  $f \in H^p(G \setminus E)$  [resp.  $H_n^p(G \setminus E)$ ], then  $f$  has a unique extension  $f^* \in H^p(G)$  [resp.  $H_n^p(G)$ ].

*Proof.* — Choose  $u = |f|^{p/2}$  and observe that  $u \in h^2(G \setminus E)$  [resp.  $h_n^2(G \setminus E)$ ]. By Theorem 2.5  $u$  has a unique extension  $u^* \in h^2(G)$  [resp.  $h_n^2(G)$ ]. Using then [8]; Theorem 2, p. 25 (resp. [16]; Theorem 4.1) to the harmonic functions  $\operatorname{Re} f$  and  $\operatorname{Im} f$  locally bounded in  $G$  we see that  $f$  has a unique extension  $f^* \in H^p(G)$  [resp.  $H_n^p(G)$ ].

2.7. COROLLARY ([16]; Theorem 5.2). — Let  $E \subset U^n$ ,  $n \geq 1$ , be closed in  $U^n$  and  $n$ -small. Let  $f: U^n \setminus E \rightarrow \mathbb{C}$  be a holomorphic function such that for some  $p > 0$ ,  $|f|^p$  has an  $n$ -harmonic majorant in  $U^n \setminus E$ . Then  $f$  has a unique holomorphic extension  $f^*: U^n \rightarrow \mathbb{C}$  such that  $|f^*|^p$  has an  $n$ -harmonic majorant in  $U^n$ .

*Proof.* — To see that  $|f^*|^p$  has an  $n$ -harmonic majorant in  $U^n$  just proceed as in [16]; proof of Theorem 5.2 (see also [15]; p. 287).

2.8. Remark. — Note that in Corollary 2.7 it is not possible to replace the polydisc  $U^n$  by an arbitrary open set  $G$ .

For example, let

$$G = \{(z_1, z_2) \in B^4(0, 1) \mid 1/|1 - z_1| + 1/|1 - z_2| < \log(1/|z_1|) + \log(1/|z_2|)\},$$

where conventionally  $\log \infty = \infty$ . The function  $f: G \rightarrow \mathbb{C}$ ,

$$f(z) = 1/(1 - z_1) + 1/(1 - z_2),$$

belongs to the class  $H_2^1(G)$ . Moreover,  $|f|$  has a 2-harmonic majorant outside the 2-small set

$$E = \{z \in G \mid z_1 = 0 \text{ or } z_2 = 0\}.$$

Since  $G \cap (\mathbb{C} \times \{0\}) = U \times \{0\}$  and the function

$$U \ni z \mapsto 1/(1 - z) \in \mathbb{C}$$

does not belong to  $H^1(U)$ , it follows that  $|f|$  has no 2-harmonic majorant in  $G$ .

2.9. COROLLARY. — Let  $G$  be an open set in  $\mathbb{C}^n$ ,  $n \geq 1$ . Let  $E \subset G$  be closed in  $G$  and polar (resp.  $n$ -small). Let  $f: G \setminus E \rightarrow \mathbb{C}$  be a holomorphic function such that for some  $p > 1$ ,  $(\log^+ |f|)^p$  has a harmonic majorant in  $G \setminus E$  (resp.  $n$ -hyperharmonic majorant which is  $\neq \infty$  on each component of  $G \setminus E$ ). Then  $f$  has a unique holomorphic extension  $f^*: G \rightarrow \mathbb{C}$ .

*Proof.* — Observe that the subharmonic (resp.  $n$ -hypoharmonic) function  $u: G \setminus E \rightarrow [-\infty, \infty)$ ,

$$u(z) = \log^+ |f(z)|,$$

has by Theorem 2.5 a unique extension  $u^* \in h^p(G)$  [resp.  $h_n^p(G)$ ].



Therefore  $|f|$  is locally bounded in  $G$ . Proceeding then as in the proof of Corollary 2.6 we see that  $f$  has a unique holomorphic extension  $f^* : G \rightarrow \mathbb{C}$ .

### 3. On the extension in the Nevanlinna class

3.1. Let  $G$  be an open set in  $\mathbb{C}^n$ ,  $n \geq 1$ . Let  $f$  be meromorphic in  $G$ . It is easy to see that for each point  $z_0 \in G$  there is a neighborhood  $U_{z_0}$  in  $G$  and an analytic subvariety  $E_{z_0}$  in  $U_{z_0}$  such that  $f$  is holomorphic in  $U_{z_0} \setminus E_{z_0}$  and  $\log^+ |f|$  has a pluriharmonic majorant in  $U_{z_0} \setminus E_{z_0}$ .

In Theorem 3.4 below we consider the converse situation. To be more precise, we show that if  $E \subset G$  is closed in  $G$  and polar and if  $f : G \setminus E \rightarrow \mathbb{C}$  is holomorphic such that  $\log^+ |f|$  has a harmonic majorant in  $G \setminus E$ , then  $f$  has a unique meromorphic extension to  $G$ . For the proof of this result we give two definitions and one Lemma.

3.2. Let  $G$  be an open set in  $\mathbb{C}^n$ ,  $n \geq 1$ . Let  $E \subset G$  be closed in  $G$ . Let  $\varphi : G \setminus E \rightarrow [-\infty, \infty)$  be subharmonic. By [8]; Theorem 1, p. 11  $\Delta\varphi$  is a measure in  $G \setminus E$ . We say that  $\Delta\varphi$  has *locally finite mass near  $E$* , if  $\Delta\varphi(K \setminus E)$  is finite for each compact set  $K \subset G$ . Moreover, we say that  $\varphi$  has *locally a harmonic majorant near  $E$* , if for each  $z_0 \in E$  there is  $R > 0$  such that  $\bar{B}^{2n}(z_0, R) \subset G$  and a harmonic function  $h : B^{2n}(z_0, R) \setminus E \rightarrow \mathbb{R}$  such that  $\varphi(z) \leq h(z)$  for each  $z \in B^{2n}(z_0, R) \setminus E$ .

3.3. LEMMA (cf. [2]; p. 283). — Let  $G$  be an open set in  $\mathbb{C}^n$ ,  $n \geq 1$ . Let  $\varphi : G \setminus E \rightarrow [-\infty, \infty)$  be subharmonic. If  $\Delta\varphi$  has locally finite mass near  $E$ , then  $\varphi$  has locally a harmonic majorant near  $E$ .

*Proof.* — Take  $z_0 \in E$  arbitrarily. Choose  $R$  and  $R_1$  such that  $0 < R < R_1$  and  $\bar{B}^{2n}(z_0, R_1) \subset G$ . Set  $\mu = (1/c_{2n}) \Delta\varphi|_{(B^{2n}(z_0, R) \setminus E)}$ , where  $c_{2n}$  is the Poisson constant (see [8]; p. 4). Proceeding as Cegrell in [2]; proof of Proposition, (ii)  $\Rightarrow$  (i), p. 283 one gets the desired harmonic majorant as follows. Define  $\psi$ ,

$$\psi(z) = \varphi(z) + G_\mu(z)$$

where  $G_\mu$  is the Green potential of  $\mu$  in  $B^{2n}(z_0, R_1)$ . By [8]; p. 4 one sees that  $\Delta\psi = 0$  in  $B^{2n}(z_0, R) \setminus E$  in the distribution sense. Using then Weyl's lemma ([8]; Corollary, p. 3) one finds a harmonic function  $h : B^{2n}(z_0, R) \setminus E \rightarrow \mathbb{R}$  such that  $h = \psi$  Lebesgue almost everywhere in  $B^{2n}(z_0, R) \setminus E$ . Since  $G_\mu$  is positive,  $h$  gives a harmonic majorant to  $\varphi$  in  $B^{2n}(z_0, R) \setminus E$ .

3.4. THEOREM. — Let  $G$  be an open set in  $\mathbb{C}^n$ ,  $n \geq 1$ . Let  $E \subset G$  be closed in  $G$  and polar. Let  $f: G \setminus E \rightarrow \mathbb{C}$  be a holomorphic function such that  $\log^+ |f|$  has a harmonic majorant  $u$  in  $G \setminus E$ . Then  $f$  has a unique meromorphic extension  $f^*$  to  $G$ .

*Proof.* — Since  $E$  is polar,  $\text{int } E = \emptyset$ . Therefore it is sufficient to show that each point  $z_0 \in E$  has a neighborhood  $U_{z_0}$  in  $G$  such that  $f|_{U_{z_0} \setminus E}$  has a meromorphic extension to  $U_{z_0}$ .

Since  $E$  is polar,  $H_{2n-1}(E) = 0$  by [6]; Theorem 5.13, p. 225. Thus we find by [4] (see also [20]; Lemma 2, p. 114) a complex line  $P$  through the point  $z_0 = (z_1^0, Z_1^0)$  such that  $H_1(E \cap P) = 0$ . By [10]; Proposition 2, p. 266 (see also [11]; Theorem 2, p. 35) and by [6]; p. 55 we may rotate the coordinate system and thus assume that  $P = \mathbb{C} \times \{Z_1^0\}$ .

Using the facts that  $H_1(E \cap (\mathbb{C} \times \{Z_1^0\})) = 0$  and  $E$  is closed in  $G$ , we find

$$r_1, r'_1 \in \mathbb{R}_+, \quad 0 < r'_1 < r_1 \quad \text{and} \quad R_1 = (r_2, \dots, r_n) \in \mathbb{R}_+^{n-1}$$

such that

$$\bar{B}^2(z_1^0, r_1) \times \bar{D}^{n-1}(Z_1^0, R_1) \subset G$$

and

$$(\bar{B}^2(z_1^0, r_1) \setminus B^2(z_1^0, r'_1)) \times \bar{D}^{n-1}(Z_1^0, R_1) \subset G \setminus E.$$

Therefore

$$f|_{(B^2(z_1^0, r_1) \setminus \bar{B}^2(z_1^0, r'_1)) \times D^{n-1}(Z_1^0, R_1)}$$

is holomorphic.

Now we argue as in [2]; proofs of Proposition and Theorem, pp. 283-285. Since  $E$  is polar, we see using [8]; Theorem 2, p. 25 that the subharmonic functions  $\log^+ |f| - u$  and  $-u$  in  $G \setminus E$  have subharmonic extensions  $\varphi_1: G \rightarrow [-\infty, \infty)$  and  $\varphi_2: G \rightarrow [-\infty, \infty)$ , respectively. But then

$$\log^+ |f(z)| = \varphi_1(z) - \varphi_2(z)$$

for each  $z \in G \setminus E$ . Since  $\Delta\varphi_1$  and  $\Delta\varphi_2$  are measures in  $G$ ,  $\Delta \log^+ |f|$  has locally finite mass near  $E$ .

Set  $r = (r_1, R_1)$  and take an increasing sequence of test functions

$$\chi_j \in D_+(D^n(z_0, r) \setminus E), \quad j = 1, 2, \dots,$$

such that  $\chi_j(z) \rightarrow 1$  as  $j \rightarrow \infty$  for each  $z \in D^n(z_0, r) \setminus E$ . Since  $\Delta \log^+ |f|$  has locally finite mass near  $E$ , there is  $M \in \mathbb{R}_+$  such that

$$\int \log^+ |f(z)| \Delta \chi_j(z) dm_{2n}(z) \leq M$$

for each  $j=1, 2, \dots$ . Since  $\log^+ |f|$  is  $n$ -hypoharmonic, we see by [8]; Proposition 1, p. 33 that

$$\int \log^+ |f(z)| \Delta_1 \chi_j(z) dm_{2n}(z) \leq \int \log^+ |f(z)| \Delta \chi_j(z) dm_{2n}(z) \leq M$$

for each  $j=1, 2, \dots$ . From Fubini's theorem it follows that

$$(B) \quad \int \left( \int \log^+ |f_{z_1}(z_1)| \Delta \chi_{jz_1}(z_1) dm_2(z_1) \right) dm_{2n-2}(Z_1) \leq M$$

for each  $j=1, 2, \dots$ . Since the functions  $\log^+ |f_{z_1}|$ ,  $Z_1 \in D^{n-1}(Z_1^0, R_1)$ , are subharmonic, we see that the sequence

$$\int \log^+ |f_{z_1}(z_1)| \Delta \chi_{jz_1}(z_1) dm_2(z_1), \quad j=1, 2, \dots,$$

is increasing for each  $Z_1 \in D^{n-1}(Z_1^0, R_1)$ . Using then Monotone convergence in (B), we find a set  $B_1 \subset D^{n-1}(Z_1^0, R_1)$  such that  $H_{2n-2}(B_1) = 0$  and that

$$(C) \quad \lim_{j \rightarrow \infty} \int \log^+ |f_{z_1}(z_1)| \Delta \chi_{jz_1}(z_1) dm_2(z_1) < \infty$$

for each  $Z_1 \in B_1$ .

And now we continue with our previous techniques (see [14]; proof of Theorem 3.1, pp. 147-148). Since  $E$  is polar, there is by [19]; Lemma 6, p. 115 (see also [12]; Corollary 3.3) a set  $B_2 \subset D^{n-1}(Z_1^0, R_1)$  such that  $H_{2n-2}(B_2) = 0$  and that for each  $Z_1 \notin B_2$  the set  $E(Z_1)$  is polar in  $\mathbb{C}$ .

Set  $B = B_1 \cup B_2$ . It follows from (C) that for each  $Z_1 \in D^{n-1}(Z_1^0, R_1) \setminus B$   $\Delta \log^+ |f_{z_1}|$  has locally finite mass near  $E(Z_1) \cap B^2(z_1^0, r_1)$ . From Lemma 3.2 it follows that for each

$$Z_1 \in D^{n-1}(Z_1^0, R_1) \setminus B, \quad \log^+ |f_{z_1}| \mid B^2(z_1^0, r_1) \setminus E(Z_1)$$

has locally a harmonic majorant near  $E(Z_1) \cap B^2(z_1^0, r_1)$ . Since then  $E(Z_1)$  is polar in  $\mathbb{C}$ , it follows from [13]; Théorème 20, p. 182 (see also [1]; Corollary 2.10, p. 425) that  $f_{Z_1}$  has a unique meromorphic extension  $f_{Z_1}^*$  to  $D^n(z_0, r)(Z_1) = B^2(z_1^0, r_1)$ . Since  $H_{2n-2}(B) = 0$ , it follows from Levi's extension theorem ([5]; Theorem 2.1 (b), p. 710) (see also [14]; Lemma 2.4, p. 147) that  $f|_{D^n(z_0, r) \setminus E}$  has a unique meromorphic extension to  $D^n(z_0, r)$ .

3.5. *Remark.* — Using the fact that the Hardy classes are contained in the Nevanlinna class, Theorem 3.4 together with Cima's and Graham's rather difficult argument ([3]; pp. 251-252) we get another proof to the first part of Corollary 2.6 above.

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*Added in proof.* — In the meantime D. Singman has proved extension results for Hardy classes in his article *Removable singularities for  $n$ -harmonic functions and Hardy classes in polydiscs*, *Proc. Amer. Math. Soc.*, Vol. 90, 1984, pp. 299-302.