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ON THE EXTENSION IN THE HARDY CLASSES AND IN THE NEVANLINNA CLASS

BY

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RÉSUMÉ. — En utilisant des méthodes de la théorie du potentiel on a établi des théorèmes d'extension pour les fonctions de quelques classes de Hardy et de la classe de Nevanlinna dans $\mathbb{C}^n$. Les ensembles exceptionnels sont polaires ou un peu plus grands ensembles n-petits selon que la fonction majorante sera harmonique ou séparément hyperharmonique.

ABSTRACT. — Using potential theoretic methods we give extension results for functions in various Hardy classes and in the Nevanlinna class in $\mathbb{C}^n$. Our exceptional sets are polar or slightly larger n-small sets depending whether the majorant is a harmonic or separately hyperharmonic function.

1. Introduction

1.1. Recently Järvi ([9]; Theorem 1, p. 597) gave the following result.

Let G be an open set in $\mathbb{C}^n$, $n \geq 1$. Let $E \subset G$ be closed in G and polar. Let $f: G \setminus E \to \mathbb{C}$ be a holomorphic function such that for some $p > 0$, $|f|^p$ has a harmonic majorant in $G \setminus E$. Then f has a unique holomorphic extension $f^*: G \to \mathbb{C}$ such that $|f^*|^p$ has a harmonic majorant in G.

In the case $n = 1$ Järvi's result is contained in Parreau's classical result ([13]; Théorème 20, p. 182). In the case $n \geq 1$ Järvi's result generalized Cima's and Graham's result ([3]; Theorem A, p. 241) which stated that analytic subvarieties are removable singularities for certain subdomains of

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$\mathbb{C}^n$. Note that in [15]; Theorem 3.2, p. 285 a similar result was given to Järvi's result, however, only in the case $p \geqslant 2$.

Järvi's proof was based on a lemma of Parreau ([9]; Lemma, pp. 596-597) (*see* also [7]; Lemma 1, p. 18) concerning quasibounded harmonic functions. In section 2 below we give a perhaps more elementary proof to the above result of Järvi. Our proof applies also to the case of n-harmonic, i. e. separately harmonic functions. In this case our exceptional sets are n-small. For the definition of these sets *see* [16]; Definition 2.2 and 2.2 below.

In [15]; Theorem 3.9, p. 287, it was observed that the following result is a direct consequence of [10]; Theorem 2, p. 279 (*see* also [11]; Theorem 4, p. 35 and [5]; Theorem 1.2 (*b*), p. 704) and of [1]; Corollary 2.10, p. 425.

Let G be an open set in $\mathbb{C}^n$, $n \geqslant 1$. Let $E \subset G$ be closed in G and polar. Let $f : G \setminus E \to \mathbb{C}$ be a holomorphic function such that $\log^+ |f|$ *has a pluriharmonic majorant in $G \setminus E$. Then f has a unique meromorphic extension to G.*

In the case $n=1$ this result is contained in the result of Parreau ([13]; Théorème 20, p. 182) (*see* also [1]; Corollary 2.10, p. 425). In the case $n \geqslant 2$ the above result generalized Cima's and Graham's result ([3]; Theorem C, p. 241) which stated that in this situation analytic subvarieties are removable singularities for certain subdomains of $\mathbb{C}^n$.

In section 3 below we show that in the above result it is sufficient to suppose that $\log^+ |f|$ *has a harmonic majorant in $G \setminus E$.* Our result gives thus a positive answer to a question posed by Cima and Graham ([3]; Remarks 7.4, p. 255).

The results for subharmonic functions are due to the first author, the results for n-hypoharmonic, i. e. separately hypoharmonic functions (except Remark 2.8) and for functions in the Nevanlinna class are due to the second author.

1.2. We use mainly the same notation as in [8]. *See* also [16]. However, we recall the following.

If $a \in \mathbb{R}^k$, $k \geqslant 1$, and $r > 0$, we write

$$B^k(a, r) = \{ x \in \mathbb{R}^k \mid |x-a| < r \}, \qquad U = B^2(0, 1).$$

The complex space $\mathbb{C}^n$, $n \geqslant 1$, will be identified with the real space $\mathbb{R}^{2n}$. If $z_0 \in \mathbb{C}$ and $r > 0$, we write $S^1(z_0, r) = \partial B^2(z_0, r)$. If

$z=(z_1, \ldots, z_n)\in \mathbf{C}^n$, $n>1$, we set for each j, $1\leqslant j\leqslant n$,

$$Z_j=(z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n)\in \mathbf{C}^{n-1} \quad \text{and} \quad (z_j, Z_j)=z.$$

If $G \subset \mathbf{C}^n$ and $z_0=(z_j^0, Z_j^0)$ we write

$$G(z_j^0)=\{Z_j\in \mathbf{C}^{n-1} \mid (z_j^0, Z_j)\in G\},$$
$$G(Z_j^0)=\{z_j\in \mathbf{C} \mid (z_j, Z_j^0)\in G\}.$$

If $r=(r_1, \ldots, r_n)\in \mathbf{R}^n_+$, we write $R_1=(r_2, \ldots, r_n)$ and

$$D^n(z_0, r)=B^2(z_1^0, r_1)\times D^{n-1}(Z_1^0, R_1),$$

where

$$D^{n-1}(Z_1^0, R_1)=B^2(z_2^0, r_2)\times \ldots \times B^2(z_n^0, r_n).$$

If $G\subset \mathbf{C}^n$ is open and $f: G\to \mathbf{C}$ (resp. $[-\infty, \infty]$) we write for each $Z_1\in \mathbf{C}^{n-1}$ $f_{Z_1} : G(Z_1)\to \mathbf{C}$ (resp. $[-\infty, \infty]$),

$$f_{Z_1}(z_1)=f(z_1, Z_1).$$

For the Laplace of f (in the distribution sense) we write

$$\Delta f=\textstyle\sum_{j=1}^n \Delta_j f,$$

where

$$\Delta_j f=4\frac{\partial^2 f}{\partial z_j \partial \bar{z}_j}.$$

For the definition of n-hyperharmonic, i.e. separately hyperharmonic functions *see* [8]. A function $u : G\to [-\infty, \infty)$ is n-hypoharmonic, if $-u$ is n-hyperharmonic. Note that a function $u : G\to (-\infty, \infty]$ (resp. $[-\infty, \infty)$) is superharmonic (resp. subharmonic) if u is hyperharmonic and $\not\equiv \infty$ (resp. hypoharmonic and $\not\equiv -\infty$) on each component of G.

The k-dimensional Hausdorff measure is denoted by H_k (note the difference between the Hardy class H^p), the k-dimensional Lebesgue measure by m_k.

For the theory of holomorphic functions, Hardy classes and Nevanlinna class *see* [18] and [21]. For potential theory *see* [8] and [6].

2. On the extension in the Hardy classes

2.1. Let G be an open set in $\mathbf{R}^n$, $n\geqslant 2$. Let $p>0$. Set $h^p(G)=\{u : G\to \mathbf{R}_+ \mid u$ is subharmonic and u^p has a harmonic majorant in $G\}$.

If G is an open set in $\mathbb{C}^n$, $n \geqslant 1$, set

$h_n^p(G) = \{ u : G \to \mathbb{R}_+ \mid u$ is n-hypoharmonic and u^p has an n-hyperharmonic majorant in G which is $\not\equiv \infty$ on each component of $G\}$;

$H^p(G) = \{ f : G \to \mathbb{C} \mid f$ is holomorphic and $|f|^p$ has a harmonic majorant in $G\}$;

$H_n^p(G) = \{ f : G \to \mathbb{C} \mid f$ is holomorphic and $|f|^p$ has an n-hyperharmonic majorant in G which is $\not\equiv \infty$ on each component of $G\}$.

In Theorem 2.5 below we give extension results for the classes h^p and h_n^p. In the case of the class h^p the exceptional set is polar and the proof is based on the well-known result which states that polar sets are removable singularities for subharmonic functions which are locally bounded above (*see* [8]; Theorem 2, p. 25). In the case of the class h_n^p the exceptional set is n-small (*see* [16]; Definition 2.2 and 2.2 below) and the proof is based on a corresponding result according to which n-hypoharmonic functions which are locally bounded above can be extended across n-small sets (*see* [16]; Theorem 4.1). We recall here, however, the definition of n-small sets and give a property of these sets.

2.2. For each set $E \subset \mathbb{C}$ we define $\mathscr{C}^1(E) = \text{cap}^* E$, where cap* denotes the outer logarithmic capacity in $\mathbb{C}$. If $n \geqslant 2$, $1 \leqslant j \leqslant n$ and $\mathscr{C}^{n-1}$ is defined for subsets of $\mathbb{C}^{n-1}$, we define for $E \subset \mathbb{C}^n$

$$\mathscr{C}_j^n(E) = H_2\{ z_j \in \mathbb{C} \mid \mathscr{C}^{n-1}\{ Z_j \in \mathbb{C}^{n-1} \mid (z_j, Z_j) \in E \} > 0 \}.$$

Finally, set

$$\mathscr{C}^n(E) = \max_{1 \leqslant j \leqslant n} \mathscr{C}_j^n(E).$$

We say that $E \subset \mathbb{C}^n$ is *n-small*, if $\mathscr{C}^n(E) = 0$.

2.3. PROPOSITION. — *Let $E \subset \mathbb{C}^n$, $n \geqslant 2$. Then E is n-small, if for each k, $1 \leqslant k \leqslant n$, $H_{2n-2}(E_k) = 0$, where*

$$E_k = \{ Z_k \in \mathbb{C}^{n-1} \mid \text{cap}^* \{ z_k \in \mathbb{C} \mid (z_k, Z_k) \in E \} > 0 \}.$$

Conversely, if E is n-small and an $\mathscr{F}_\sigma$-set, then $H_{2n-2}(E_k) = 0$ for each k, $1 \leqslant k \leqslant n$.

Proof. — The first part of the Proposition is proved in [16]; Proposition 2.4. Note that there are (at least Lebesgue nonmeasurable) n-small sets E for which $H_{2n-2}(E_n) > 0$. *See* [16]; Remark 2.8. We give an induction proof for the second part. If $n = 2$ then the assertion clearly holds. Suppose then that $n \geqslant 3$ and take k, $1 \leqslant k \leqslant n$, arbitrarily. Since the outer logarithmic capacity and the Hausdorff outer measure are subadditive, we may suppose that E is compact. From [17]; Lemma 2.2.1, p. 87 it follows that E_k is an $\mathscr{F}_\sigma$-set and thus Lebesgue measurable.

Take $j \neq k$, $1 \leqslant j \leqslant n$, arbitrarily. Since E is n-small, there is $B_j \subset \mathbb{C}$ such that $H_2(B_j)=0$ and that for each $z_j \notin B_j$ the set

$$E(z_j)=\{ Z_j \in \mathbb{C}^{n-1} \,|\, (z_j, Z_j) \in E \}$$

is $(n-1)$-small. It follows from the induction hypothesis that $H_{2n-4}(E_k(z_j))=0$ for each $z_j \notin B_j$, where

$$E_k(z_j)=\{ Z_{kj} \in \mathbb{C}^{n-2} \,|\, \mathrm{cap}^*\{ z_k \in \mathbb{C} \,|\, (z_k, Z_{kj}) \in E(z_j)\} > 0\}.$$

If χ_{E_k} is the characteristic function of E_k, we get by Fubini's theorem

$$m_{2n-2}(E_k)=\int_{\mathbb{C}\setminus B_j}\left(\int_{\mathbb{C}^{n-2}\setminus E_k(z_j)} \chi_{E_k}(z_j, Z_{kj})\, dm_{2n-4}(Z_{kj})\right) dm_2(z_j)=0,$$

concluding the proof.

2.4. *Remark*. — From [19]; Lemma 6, p. 115 (*see* also [12]; Corollary 3.3) and Proposition 2.2 it follows that polar sets are n-small. Note that Lebesgue measurable n-small sets are of Lebesgue measure zero ([16]; Remark 2.3).

2.5. THEOREM. — *Let G be an open set in $\mathbb{R}^n$, $n \geqslant 2$ (resp. in $\mathbb{C}^n$, $n \geqslant 1$). Let $E \subset G$ be closed in G and polar (resp. n-small). Let $p > 1$. If $u \in h^p(G \setminus E)$ [resp. $h_n^p(G \setminus E)$], then u has a unique extension $u^* \in h^p(G)$ [resp. $h_n^p(G)$].*

Proof. — Let h be a harmonic majorant (resp. an n-hyperharmonic majorant which is $\not\equiv \infty$ on each component of $G \setminus E$) of u in $G \setminus E$. By [8]; Theorem 2, p. 25 (resp. [16]; Theorem 4.1) h has a unique superharmonic (resp. n-hyperharmonic which by [8]; Theorem, p. 31 is superharmonic) extension $h^* : G \to (-\infty, \infty]$. Thus the greatest harmonic minorant v of h^* in G exists by [8]; Corollary 1, p. 10.

Take q, $1<q<p$, and $\varepsilon>0$ arbitrarily. Then the function $u_\varepsilon : G \setminus E \to [-\infty, \infty)$,

$$u_\varepsilon(z)=u(z)^q-\varepsilon\, h(z),$$

is subharmonic (resp. n-hypoharmonic). Since

$$u_\varepsilon(z) \leqslant u(z)^q - \varepsilon\, u(z)^p$$

for each $z \in G \setminus E$ and $q<p$, u_ε is bounded above in G. By [8]; Theorem 2, p. 25 (resp. [16]; Theorem 4.1) u_ε has a unique subharmonic (resp. n-hypoharmonic which by [8]; Theorem, p. 31 is subharmonic) extension $u_\varepsilon^* : G \to [-\infty, \infty)$.

For each $z \in G \setminus E$ we have

$$h^*(z) - u_\varepsilon^*(z) = h(z) - u(z)^q + \varepsilon h(z) \geqslant u(z)^p - u(z)^q + \varepsilon h(z) \geqslant -1.$$

Since E is of Lebesgue measure zero, it follows that

$$u_\varepsilon^*(z) \leqslant h^*(z) + 1$$

for each $z \in G$. Thus by [8]; Corollary 1, p. 10

$$u_\varepsilon^*(z) \leqslant v(z) + 1$$

for each $z \in G$. But then

$$u(z)^q - \varepsilon h(z) \leqslant v(z) + 1$$

for each $z \in G \setminus E$. Since $\varepsilon > 0$ was arbitrary, we get

$$\text{(A)} \qquad u(z)^q \leqslant v(z) + 1$$

for each $z \in G \setminus E$. Therefore u is locally bounded above in G and thus by [8]; Theorem 2, p. 25 (resp. [16]; Theorem 4.1) has a unique subharmonic (resp. n-hypoharmonic) extension $u^* : G \to [0, \infty)$. Since (A) holds for all $q<p$ and E is of Lebesgue measure zero,

$$u^*(z)^p \leqslant v(z) + 1$$

for all $z \in G$. Thus $u^* \in h^p(G)$. [Resp. it follows directly that $u^*(z)^p \leqslant h^*(z)$ for each $z \in G$. Thus $u^* \in h_n^p(G)$.]

2.6. Corollary ([9]; Theorem 1, p. 597 and [16]; Theorem 5.1). — *Let G be an open set in $\mathbb{C}^n$, $n \geqslant 1$. Let $E \subset G$ be closed in G and polar (resp. n-small). Let $p>0$. If $f \in H^p(G \setminus E)$ [resp. $H_n^p(G \setminus E)$], then f has a unique extension $f^* \in H^p(G)$ [resp. $H_n^p(G)$].*

Proof. — Choose $u = |f|^{p/2}$ and observe that $u \in h^2(G \setminus E)$ [resp. $h_n^2(G \setminus E)$]. By Theorem 2.5 u has a unique extension $u^* \in h^2(G)$ [resp. $h_n^2(G)$]. Using then [8]; Theorem 2, p. 25 (resp. [16]; Theorem 4.1) to the harmonic functions $\operatorname{Re} f$ and $\operatorname{Im} f$ locally bounded in G we see that f has a unique extension $f^* \in H^p(G)$ [resp. $H_n^p(G)$].

2.7. COROLLARY ([16]; Theorem 5.2). — *Let* $E \subset U^n$, $n \geqslant 1$, *be closed in* U^n *and n-small. Let* $f: U^n \setminus E \to \mathbb{C}$ *be a holomorphic function such that for some* $p>0$, $|f|^p$ *has an n-harmonic majorant in* $U^n \setminus E$. *Then f has a unique holomorphic extension* $f^*: U^n \to \mathbb{C}$ *such that* $|f^*|^p$ *has an n-harmonic majorant in* U^n.

Proof. — To see that $|f^*|^p$ has an n-harmonic majorant in U^n just proceed as in [16]; proof of Theorem 5.2 (*see* also [15]; p. 287).

2.8. *Remark.* — Note that in Corollary 2.7 it is not possible to replace the polydisc U^n by an arbitrary open set G.

For example, let

$$G=\{(z_1, z_2)\in B^4(0,1)\,|\,1/|1-z_1| + 1/|1-z_2| < \log(1/|z_1|)+\log(1/|z_2|)\},$$

where conventionally $\log \infty = \infty$. The function $f: G \to \mathbb{C}$,

$$f(z)=1/(1-z_1)+1/(1-z_2),$$

belongs to the class $H_2^1(G)$. Moreover, $|f|$ has a 2-harmonic majorant outside the 2-small set

$$E=\{z\in G\,|\,z_1=0 \text{ or } z_2=0\}.$$

Since $G \cap (\mathbb{C}\times\{0\}) = U \times \{0\}$ and the function

$$U \ni z \mapsto 1/(1-z)\in\mathbb{C}$$

does not belong to $H^1(U)$, it follows that $|f|$ has no 2-harmonic majorant in G.

2.9. COROLLARY. — *Let G be an open set in* $\mathbb{C}^n$, $n \geqslant 1$. *Let* $E \subset G$ *be closed in G and polar* (*resp. n-small*). *Let* $f: G\setminus E \to \mathbb{C}$ *be a holomorphic function such that for some* $p>1$, $(\log^+|f|)^p$ *has a harmonic majorant in* $G\setminus E$ (*resp. n-hyperharmonic majorant which is* $\not\equiv \infty$ *on each component of* $G\setminus E$). *Then f has a unique holomorphic extension* $f^*: G \to \mathbb{C}$.

Proof. — Observe that the subharmonic (resp. n-hypoharmonic) function $u: G\setminus E \to [-\infty, \infty)$,

$$u(z)=\log^+|f(z)|,$$

has by Theorem 2.5 a unique extension $u^* \in h^p(G)$ [resp. $h_n^p(G)$].

Therefore $|f|$ is locally bounded in G. Proceeding then as in the proof of Corollary 2.6 we see that f has a unique holomorphic extension $f^* : G \to \mathbb{C}$.

3. On the extension in the Nevanlinna class

3.1. Let G be an open set in $\mathbb{C}^n$, $n \geq 1$. Let f be meromorphic in G. It is easy to see that for each point $z_0 \in G$ there is a neighborhood U_{z_0} in G and an analytic subvariety E_{z_0} in U_{z_0} such that f is holomorphic in $U_{z_0} \setminus E_{z_0}$ and $\log^+ |f|$ has a pluriharmonic majorant in $U_{z_0} \setminus E_{z_0}$.

In Theorem 3.4 below we consider the converse situation. To be more precise, we show that if $E \subset G$ is closed in G and polar and if $f : G \setminus E \to \mathbb{C}$ is holomorphic such that $\log^+ |f|$ has a harmonic majorant in $G \setminus E$, then f has a unique meromorphic extension to G. For the proof of this result we give two definitions and one Lemma.

3.2. Let G be an open set in $\mathbb{C}^n$, $n \geq 1$. Let $E \subset G$ be closed in G. Let $\varphi : G \setminus E \to [-\infty, \infty)$ be subharmonic. By [8]; Theorem 1, p. 11 $\Delta\varphi$ is a measure in $G \setminus E$. We say that $\Delta\varphi$ has *locally finite mass near E*, if $\Delta\varphi(K \setminus E)$ is finite for each compact set $K \subset G$. Moreover, we say that φ has *locally a harmonic majorant near E*, if for each $z_0 \in E$ there is $R > 0$ such that $\bar{B}^{2n}(z_0, R) \subset G$ and a harmonic function $h : B^{2n}(z_0, R) \setminus E \to \mathbb{R}$ such that $\varphi(z) \leq h(z)$ for each $z \in B^{2n}(z_0, R) \setminus E$.

3.3. LEMMA (*cf.* [2]; p. 283). — *Let G be an open set in* $\mathbb{C}^n$, $n \geq 1$. *Let* $\varphi : G \setminus E \to [-\infty, \infty)$ *be subharmonic. If* $\Delta\varphi$ *has locally finite mass near E, then* φ *has locally a harmonic majorant near E.*

Proof. — Take $z_0 \in E$ arbitrarily. Choose R and R_1 such that $0 < R < R_1$ and $\bar{B}^{2n}(z_0, R_1) \subset G$. Set $\mu = (1/c_{2n}) \Delta\varphi \,|\, (B^{2n}(z_0, R) \setminus E)$, where c_{2n} is the Poisson constant (*see* [8]; *p.* 4). Proceeding as Cegrell in [2]; proof of Proposition, (ii) $\Rightarrow$ (i), p. 283 one gets the desired harmonic majorant as follows. Define ψ,

$$\psi(z) = \varphi(z) + G_\mu(z)$$

where G_μ is the Green potential of μ in $B^{2n}(z_0, R_1)$. By [8]; p. 4 one sees that $\Delta\psi = 0$ in $B^{2n}(z_0, R) \setminus E$ in the distribution sense. Using then Weyl's lemma ([8]; Corollary, p. 3) one finds a harmonic function $h : B^{2n}(z_0, R) \setminus E \to \mathbb{R}$ such that $h = \psi$ Lebesgue almost everywhere in $B^{2n}(z_0, R) \setminus E$. Since G_μ is positive, h gives a harmonic majorant to φ in $B^{2n}(z_0, R) \setminus E$.

3.4. THEOREM. — *Let G be an open set in $\mathbb{C}^n$, $n \geq 1$. Let $E \subset G$ be closed in G and polar. Let $f: G \setminus E \to \mathbb{C}$ be a holomorphic function such that* $\log^+ |f|$ *has a harmonic majorant u in $G \setminus E$. Then f has a unique meromorphic extension f^* to G.*

Proof. — Since E is polar, int $E = \varnothing$. Therefore it is sufficient to show that each point $z_0 \in E$ has a neighborhood U_{z_0} in G such that $f \,|\, U_{z_0} \setminus E$ has a meromorphic extension to U_{z_0}.

Since E is polar, $H_{2n-1}(E) = 0$ by [6]; Theorem 5.13, p. 225. Thus we find by [4] (*see* also [20]; Lemma 2, p. 114) a complex line P through the point $z_0 = (z_1^0, Z_1^0)$ such that $H_1(E \cap P) = 0$. By [10]; Proposition 2, p. 266 (*see* also [11]; Theorem 2, p. 35) and by [6]; p. 55 we may rotate the coordinate system and thus assume that $P = \mathbb{C} \times \{Z_1^0\}$.

Using the facts that $H_1(E \cap (\mathbb{C} \times \{Z_1^0\})) = 0$ and E is closed in G, we find

$$r_1, r_1' \in \mathbf{R}_+, \quad 0 < r_1' < r_1 \qquad \text{and} \qquad R_1 = (r_2, \ldots, r_n) \in \mathbf{R}_+^{n-1}$$

such that

$$\bar{B}^2(z_1^0, r_1) \times \bar{D}^{n-1}(Z_1^0, R_1) \subset G$$

and

$$(\bar{B}^2(z_1^0, r_1) \setminus B^2(z_1^0, r_1')) \times \bar{D}^{n-1}(Z_1^0, R_1) \subset G \setminus E.$$

Therefore

$$f \,|\, (B^2(z_1^0, r_1) \setminus \bar{B}^2(z_1^0, r_1')) \times D^{n-1}(Z_1^0, R_1)$$

is holomorphic.

Now we argue as in [2]; proofs of Proposition and Theorem, pp. 283-285. Since E is polar, we see using [8]; Theorem 2, p. 25 that the subharmonic functions $\log^+ |f| - u$ and $-u$ in $G \setminus E$ have subharmonic extensions $\varphi_1 : G \to [-\infty, \infty)$ and $\varphi_2 : G \to [-\infty, \infty)$, respectively. But then

$$\log^+ |f(z)| = \varphi_1(z) - \varphi_2(z)$$

for each $z \in G \setminus E$. Since $\Delta\varphi_1$ and $\Delta\varphi_2$ are measures in G, $\Delta \log^+ |f|$ *has locally finite mass near* E.

Set $r = (r_1, R_1)$ and take an increasing sequence of test functions

$$\chi_j \in D_+(D^n(z_0, r) \setminus E), \qquad j = 1, 2, \ldots,$$

such that $\chi_j(z) \to 1$ as $j \to \infty$ for each $z \in D^n(z_0, r) \setminus E$. Since $\Delta \log^+ |f|$ *has locally finite mass near* E, there is $M \in \mathbf{R}_+$ such that

$$\int \log^+ |f(z)| \Delta \chi_j(z)\, dm_{2n}(z) \leq M$$

for each $j = 1, 2, \ldots$ Since $\log^+ |f|$ is n-hypoharmonic, we see by [8]; Proposition 1, p. 33 that

$$\int \log^+ |f(z)| \Delta_1 \chi_j(z)\, dm_{2n}(z) \leq \int \log^+ |f(z)| \Delta \chi_j(z)\, dm_{2n}(z) \leq M$$

for each $j = 1, 2, \ldots$ From Fubini's theorem it follows that

$$\text{(B)} \qquad \int \left(\int \log^+ | f_{Z_1}(z_1) | \Delta \chi_{j Z_1}(z_1)\, dm_2(z_1) \right) dm_{2n-2}(Z_1) \leq M$$

for each $j = 1, 2, \ldots$ Since the functions $\log^+ | f_{Z_1} |$, $Z_1 \in D^{n-1}(Z_1^0, R_1)$, are subharmonic, we see that the sequence

$$\int \log^+ | f_{Z_1}(z_1) | \Delta \chi_{j Z_1}(z_1)\, dm_2(z_1), \qquad j = 1, 2, \ldots,$$

is increasing for each $Z_1 \in D^{n-1}(Z_1^0, R_1)$. Using then Monotone convergence in (B), we find a set $B_1 \subset D^{n-1}(Z_1^0, R_1)$ such that $H_{2n-2}(B_1) = 0$ and that

$$\text{(C)} \qquad \lim_{j \to \infty} \int \log^+ | f_{Z_1}(z_1) | \Delta \chi_{j Z_1}(z_1)\, dm_2(z_1) < \infty$$

for each $Z_1 \in B_1$.

And now we continue with our previous techniques (*see* [14]; proof of Theorem 3.1, pp. 147-148). Since E is polar, there is by [19]; Lemma 6, p. 115 (*see* also [12]; Corollary 3.3) a set $B_2 \subset D^{n-1}(Z_1^0, R_1)$ such that $H_{2n-2}(B_2) = 0$ and that for each $Z_1 \notin B_2$ the set $E(Z_1)$ is polar in $\mathbf{C}$.

Set $B = B_1 \cup B_2$. It follows from (C) that for each $Z_1 \in D^{n-1}(Z_1^0, R_1) \setminus B$ $\Delta \log^+ | f_{Z_1} |$ has locally finite mass near $E(Z_1) \cap B^2(z_1^0, r_1)$. From Lemma 3.2 it follows that for each

$$Z_1 \in D^{n-1}(Z_1^0, R_1) \setminus B, \qquad \log^+ | f_{Z_1} | \,|\, B^2(z_1^0, r_1) \setminus E(Z_1)$$

has locally a harmonic majorant near $E(Z_1)\cap B^2(z_1^0, r_1)$. Since then $E(Z_1)$ is polar in $\mathbb{C}$, it follows from [13]; Théorème 20, p. 182 (*see* also [1]; Corollary 2.10, p. 425) that f_{Z_1} has a unique meromorphic extension $f_{Z_1}^*$ to $D^n(z_0, r)(Z_1)=B^2(z_1^0, r_1)$. Since $H_{2n-2}(B)=0$, it follows from Levi's extension theorem ([5]; Theorem 2.1 (*b*), p. 710) (*see* also [14]; Lemma 2.4, p. 147) that $f \mid D^n(z_0, r)\setminus E$ has a unique meromorphic extension to $D^n(z_0, r)$.

3.5. *Remark.* – Using the fact that the Hardy classes are contained in the Nevanlinna class, Theorem 3.4 together with Cima's and Graham's rather difficult argument ([3]; pp. 251-252) we get another proof to the first part of Corollary 2.6 above.

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Added in proof. — In the meantime D. Singman has proved extension results for Hardy classes in his article *Removable singularities for n-harmonic functions and Hardy classes in polydiscs*, Proc. Amer. Math. Soc., Vol. 90, 1984, pp. 299-302.