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ABSOLUTE COHOMOLOGICAL PURITY

BY

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RÉSUMÉ.— Cette note démontre la conjecture de pureté cohomologique absolue de GROTHENDIECK pour la cohomologie étale l-adique à coefficients $\hat{\mathbb{Q}_l}$, et même pour la cohomologie étale à coefficients $\mathbb{Z}/l^v$ si l est grand. Les ingrédients principaux dans la preuve sont le théorème de localisation pour la K-théorie algébrique, et le théorème de comparaison entre la K-théorie algébrique et la K-théorie topologique.

ABSTRACT. – GROTHENDIECK's absolute cohomological purity conjecture is proved for $\hat{\mathbb{Q}_l}$-etale cohomology, and for etale cohomology with $\mathbb{Z}/l^v$ coefficients if l is large. The proof depends on Quillen's localization theorem for algebraic K-theory and my comparison of algebraic and topological K-theory.

In this note, I show how my theorem relating algebraic and topological K-theory yields an absolute cohomological purity theorem for topological K-theory as the analogue of Quillen's localization theorem for algebraic K-theory. Under various conditions the degeneration of the Atiyah-Hirzebruch spectral sequence allows one to deduce various purity results for etale cohomology. In particular, there are very general purity theorems for $\hat{\mathbb{Q}_l}$-cohomology in paragraph 3.

1. The standard statement of GROTHENDIECK'S purity conjecture is:

CONJECTURE 1.1 ([16], I 3.1.4). – Let X be a regular prescheme, $X' \subseteq X$ an open subscheme, and $i: Y' \to X'$ a closed immersion. Suppose that Y' is regular, and that $Y' \subseteq X'$ has codimension d at each point. Let l be a prime number invertible in $\mathcal{O}_{X'}$. Then the local cohomology sheaves are given by (1.1):

$$(1.1) \qquad \underline{H}^i_{Y'}(X'; \mathbb{Z}/l^v) = \begin{cases} 0, & i \neq 2d, \\ i_* \mathbb{Z}/l^v(-d), & i = 2d. \end{cases}$$

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Considering the usual relations between local and global cohomology as in [14], V.6, one easily sees that Conjecture 1.1 for a scheme X and all schemes X' etale over it is equivalent to Conjecture 1.2.

CONJECTURE 1.2. — Let X be a regular prescheme, X' etale over X, and $i: Y' \to X'$ a closed immersion. Suppose Y' is regular, and has codimension d in X' at every point. Suppose that l is invertible in X'. Then there is an isomorphism (1.2):

$$H^k_{Y'}(X'; \mathbb{Z}/l^\nu) \cong H^{k-2d}(Y'; \mathbb{Z}/l^\nu(-d)). \tag{1.2}$$

Substituting (1.2) into the usual long exact sequence for cohomology with supports yields a long exact sequence (1.3):

$$\begin{aligned} \ldots \to H^{k-1}(X'-Y'; \mathbb{Z}/l^\nu) &\xrightarrow{\partial} H^{k-2d}(Y'; \mathbb{Z}/l^\nu(-d)) \\ &\to H^k(X'; \mathbb{Z}/l^\nu) \to H^k(X'-Y'; \mathbb{Z}/l^\nu) \xrightarrow{\partial} \ldots . \end{aligned} \tag{1.3}$$

These conjectures are known to be true if X is smooth over a perfect field by [14], XVI 3.9, or if X is an excellent prescheme of equicharacteristic 0, by [14], XIX 3.2, or if X is noetherian of Krull dimension one by [16], I 5.1. A footnote to [16] announces that O. GABBER can prove the case where X is excellent of dimension 2. ARTIN's relative cohomological purity theorem of [14], XVI 3.7, gives (1.1) if $Y' \subseteq X'$ is a smooth pair over a base scheme S. There are a few semipurity theorems giving vanishing in (1.1) for some i, as in [15], Cycle 2.2.8. To complete the list of all hitherto known purity results, I add [6], § 6, and [5] for the Brauer group.

None of these results give absolute cohomological purity for X smooth over a field $k(t_1, \ldots, t_n)$ with char $k \neq 0$, nor for X regular and of finite type over $\mathbb{Z}$ with $\dim X > 2$.

Absolute cohomological purity theorems are useful and even necessary in constructing Gysin maps in etale cohomology, in setting up the cycle class and using cohomology to study intersection theory as in [15], Cycle, and in constructing a coniveau spectral sequence and filtration by codimension of support in global cohomology, as in [6], § 10.

2. I turn aside from etale cohomology to invoke the black magic of algebraic K-theory. Given a scheme X, QUILLEN constructs, from the exact category of algebraic vector bundles on X, a spectrum in the sense of algebraic topology, $K(X)$. Its homotopy groups are the QUILLEN higher

K-groups $K_*(X)$. $K_0(X)$ is the usual GROTHENDIECK group. The spectrum $K(X)$ may be smashed with a mod l^ν Moore spectrum to produce $K/l^\nu(X)$. Its homotopy groups $K/l^\nu_*(X)$ are related to $K_*(X)$ by the usual universal coefficient sequence. At present, algebraic K-theory is a great mystery, although it is clearly very closely related to aspects of algebraic geometry like intersection theory. One of the few theorems one knows is absolute purity, in the form of QUILLEN's localization theorem.

THEOREM 2.1 (QUILLEN). — *Let X be a regular noetherian separated scheme. Let Y be a regular closed subscheme of X. Define K-theory with supports in Y, $K_Y(X)$ to be homotopy fibre of $K(X)\to K(X-Y)$. Then there is a weak homotopy equivalence* (2.1):

$$K_Y(X)\simeq K(Y). \tag{2.1}$$

Thus (2.2) *is a homotopy fibre sequence, yielding a long exact sequence* (2.3) *on homotopy groups:*

$$K(Y)\to K(X)\to K(X-Y). \tag{2.2}$$

$$\ldots\to K_{n+1}(X-Y)\xrightarrow{\partial} K_n(Y)\to K_n(X)\to K_n(X-Y)\xrightarrow{\partial}\ldots. \tag{2.3}$$

Pf: [7], §7, 1 and 3.2 yield (2.3) and (2.2). The statement (2.1) is just a reinterpretation.

The statements of 2.1 remain true if K is replaced by K/l^ν, as smashing with a fixed spectrum preserves homotopy fibre sequences. Similarly, they are true for $K/l^\nu[\beta^{-1}]$, the localization of K/l^ν by inverting the action of the Bott element β. For $l>3$ and schemes over $\mathbb{Z}[e^{2\pi i/l^\nu}]=R$, β is the class in $K/l^\nu_2(R)$ which corresponds in the universal coefficient sequence to the l^ν torsion class $e^{2\pi i/l^\nu}$ in $K_1(R)=GL_1(R)$. $K/l^\nu(X)$ is a module spectrum over $K/l^\nu(R)$, so $K/l^\nu(X)[\beta^{-1}]$ *makes sense. For* $l=2$ or 3 and X not over R, the story is a bit more complicated. Consult [2] or [9] for details.

CONDITIONS 2.2. — Let X be a regular noetherian separated scheme. Suppose that either:

(*a*) X is of finite type over $\mathbb{Z}$, or over a local or global field, or over a separably closed field, or over a ring of integers in a local field or;

(*b*) X is the inverse limit scheme of an inverse system of schemes X_α with affine etale transition maps $X_\alpha\to X_\beta$, and that each X_α satisfies (*a*).

Schemes flat and quasi-finite over schemes that satisfy 2.2(*a*) or 2.2(*b*) satisfy the same condition. Let R be a local ring of, or a strict local henselization of, or a residue field of a scheme that satisfies (*a*). Then a regular separated scheme of finite type over R satisfies (*b*). If k is a local, global, or separably closed field, a regular separated scheme of finite type over a field L of finite transcendence degree over k satisfies (*b*). Aside from schemes associated to formal schemes, every regular separated noetherian scheme that arises in everyday life satisfies 2.2.

THEOREM 2.3. — *Let l^ν be a prime power. Let X be a scheme satisfying conditions 2.2. Assume l is invertible in the structure sheaf $\mathcal{O}_X$, and that $\mathcal{O}_X$ contains a square root of -1 if $l=2$. Then there is a strongly converging spectral sequence with differentials d_r of bidegree $(r, r-1)$:*

$$(2.4)\qquad E_2^{p,q}=\begin{Bmatrix} H^p_{et}(X;\, \mathbb{Z}/l^\nu(i)),\ q=2i \\ 0,\ q \text{ odd} \end{Bmatrix} \Rightarrow K/l^\nu_{q-p}(X)[\beta^{-1}].$$

The DWYER-FRIEDLANDER *map induces a weak homotopy equivalence:*

$$(2.5)\qquad \rho : K/l^\nu(X)[\beta^{-1}] \xrightarrow{\sim} K/l^{\nu\,\mathrm{Top}}(X).$$

Pf: This is a very deep theorem relating algebraic geometry as seen by algebraic K-theory to topological invariants. It is a special case of the slightly more general theorem of [9].

The right-hand side of (2.5) is the topological or etale K-theory of FRIEDLANDER and DWYER. One may consult ([3], [4], [8]), and [2] for this. For X of finite type over $\mathbb{C}$, it agrees with the usual topological K-theory of the space X with the analytic topology. Alternatively, one may take (2.5) as the definition of $K/l^{\nu\,\mathrm{Top}}(X)$, and show that this then has the usual properties of topological K-theory by appealing to the Atiyah-Hirzebruch type spectral sequence (2.4) and QUILLEN's theorems on $K(X)$.

Combining 2.3 with QUILLEN's localization theorem 2.1, one obtains:

THEOREM 2.4. — *Let X be a regular noetherian separated scheme satisfying conditions 2.2. Let X' be etale or even proetale over X, and let $i : Y' \to X'$ be a closed immersion with Y' regular. Let l be a prime number invertible in $\mathcal{O}_{X'}$, and let $\mathcal{O}_{X'}$ contain $\sqrt{-1}$ if $l=2$. Then there is a weak* *homotopy equivalence* (2.6), *a homotopy fibre sequence* (2.7), *and a long exact sequence* (2.8):

(2.6) $$K/l^{\nu}{}_{Y'}^{\mathrm{Top}}(X') \simeq K/l^{\nu\,\mathrm{Top}}(Y'),$$

(2.7) $$K/l^{\nu\,\mathrm{Top}}(Y') \to K/l^{\nu\,\mathrm{Top}}(X') \to K/l^{\nu\,\mathrm{Top}}(X'-Y'),$$

(2.8) $$\ldots \xrightarrow{\partial} K/l^{\nu\,\mathrm{Top}}_n(Y') \to K/l^{\nu\,\mathrm{Top}}_n(X') \to K/l^{\nu\,\mathrm{Top}}_n(X'-Y') \xrightarrow{\partial} \ldots.$$

Thus $K/l^{\nu\,\mathrm{Top}}$ satisfies absolute cohomological purity in that the analogue of Conjectures 1.1 and 1.2 hold for it under the mild hypotheses on X. This suggests that cohomological intersection and cycle theory should be expressed in terms of the generalized etale cohomology theory $K/l^{\nu\,\mathrm{Top}}$. This would be the topological analogue of the close connections between algebraic K-theory and the Chow ring.

Under various hypotheses, known cohomological purity theorems allow another topological construction of the Gysin sequences (2.7) and (2.8). The Riemann-Roch theorem of [11], 4.13, says that these Gysin sequences agree with the above.

Theorem 2.3 holds under more general hypotheses than 2.2, e.g. it holds under the hypotheses of [9], 2.45.

3. Soulé has shown in [8] that the generalized Atiyah-Hirzebruch spectral sequence (2.4) degenerates at E_2 modulo torsion of a bounded order depending only on the etale cohomological dimension of X. This generalizes the well-known collapse of the classical Atiyah-Hirzebruch spectral sequence for classical $K^{\mathrm{Top}}\otimes\mathbb{Q}$. The collapse allows one to reinterpret purity for $K/l^{\nu\,\mathrm{Top}}$ in terms of purity results for etale cohomology.

Soulé's paper [8] is written in terms of a preliminary version of the Dwyer-Friedlander topological K-theory. Because of technical problems with this version of the theory, Soulé makes some unnecessary assumptions in [8]. For example, he assumes $l\neq 2$. To avoid these unnecessary hypotheses, one may define $K/l^{\nu\,\mathrm{Top}}(X)$ by (2.5). Then (2.4) provides the Atiyah-Hirzebruch type spectral sequence for $K/l^{\nu\,\mathrm{Top}}(X)$. There are Adams operations on $K/l^{\nu\,\mathrm{Top}}(X)$ induced by the Adams operations ψ^k on $K/l^{\nu}(X)$. As these operations are natural with respect to etale maps in X, they operate on the etale local-to-global spectral sequence (2.4). On the etale sheaf $\tilde{\pi}_{2i} K/l^{\nu}(\ )[\beta^{-1}] \cong \mathbb{Z}/l^{\nu}(i) \cong \mu_{l^{\nu}}^{\otimes i}$, ψ^k acts by multiplication by k^i. These observations are the essential ingredients of the proof of the degeneration modulo torsion of the Atiyah-

Hirzebruch spectral sequence. A complete sketch of the proof is given below, but the inexperienced reader may wish to consult [8] for more details.

DEFINITION 3.1. — For X a scheme, the hereditary etale cohomological dimension (resp., away from a set of primes J) is the least integer N such that for all primes l (resp., all l not in J), all schemes X' etale over X, all l-torsion sheaves $\mathcal{F}$ on X', and all $n \geqslant N+1$, one has $H^n_{et}(X'; \mathcal{F})=0$.

LEMMA 3.2. — *Let X be a scheme of finite type over a field L, and suppose X has Krull dimension n. Suppose L has etale cohomological dimension k (resp., away from a set of primes J). Then X has hereditary etale cohomological dimension (resp., away from J) at most $2n+k$.*

Let X be of finite type over $\mathbb{Z}[\sqrt{-1}]$, or of finite type over $\mathbb{Z}$. If X has Krull dimension n, it has hereditary etale cohomological dimension (resp., away from 2) at most $2n+1$.

If X is proetale over a noetherian separated scheme of hereditary cohomological dimension N (resp., away from J) then X has hereditary cohomological dimension at most N (resp. away from J).

Pf: The usual estimates of [14], X 4.3, 5.2, 6.2, give the first two results. The last statement follows from the definition and the fact that etale cohomology of an inverse limit of schemes is the direct limit of the etale cohomologies, i.e., [14], VII, § 5.

3.3. I recall some constants from [8], 3.3.1, that give bounds for the degeneration of the Atiyah-Hirzebruch spectral sequence below. Proofs of these assertions may be found in [1]. Let j be a positive integer, and let w_j be the greatest common divisor of $k^\infty(k^j-1)$ as k runs over the positive integers. Then $w_j=2$ for j odd and is the denominator of $B_j/2j$ for j even, where B_j is the jth Bernoulli number. The l-adic valuation is given by (3.1):

$$v_2(w_j)=v_2(j)+2,$$

$$v_l(w_j)=\begin{cases} 0, & l-1 \text{ does not divide } j, \quad l \text{ odd}, \\ v_l(j)+1, & l-1 \text{ divides } j, \quad l \text{ odd}. \end{cases} \tag{3.1}$$

Let $M(k)$ be the product of the w_j for $2j<k$. An odd prime l divides $M(k)$ if and only if $l<(k/2)+1$.

LEMMA 3.4. — *Let $X' = Spec(A')$ be an affine regular noetherian scheme satisfying the other conditions of 2.2. Let $Y' = Spec(A'/I')$ be a regular closed subscheme of codimension d at each point. Suppose there is a regular sequence $(t_1, t_2, \ldots, t_d)$ in A' that generates I'.*

Let $i_*[\mathcal{O}_{Y'}]$ be the image in $\pi_0 K/l^\nu{}_{Y'}^{\mathrm{Top}}(X')$ of the canonical class $[\mathcal{O}_{Y'}]=1$ in $\pi_0 K/l^{\nu \mathrm{Top}}(Y')$ under the Gysin equivalence (2.6). For k prime to l, the Adams operation ψ^k on $K/l^\nu{}_{Y'}^{\mathrm{Top}}(X')$ sends $i_*[\mathcal{O}_{Y'}]$ to $k^d i_*[\mathcal{O}_{Y'}]$.

Pf: Let Y'_i be the divisor $\mathrm{Spec}(A'/(t_i))$. Then $i_*[\mathcal{O}_{Y'_i}]$ is the class $t_i : \mathcal{O}_{X'} \to \mathcal{O}_{X'}$ in $\pi_0 K_{Y'}(X')$, i.e., it is the image under the boundary map in the localization sequence (2.3) of the unit t_i in $K_1(A'[1/t_i])$. One has $\psi^k(t_i) = t_i^k$ as a unit, so $\psi^k(t_i) = kt_i$ additively in $K_1(A'[1/t_i])$. Thus $\psi^k i_*[\mathcal{O}_{Y'_i}] = k i_*[\mathcal{O}_{Y'_i}]$ in $\pi_0 K_{Y'_i}(X')$. As $\psi^k(\beta x) = k\beta\psi^k(x)$, ψ^k induces a stable cohomology operation on $K/l^\nu_{Y'}(X')[\beta^{-1}] = K/l^\nu{}_{Y'}^{\mathrm{Top}}(X)$, which is the usual Adams operation on topological K-theory. For $d=1$ and $Y'=Y'_i$, the desired formula is induced. For other d, the result follows as ψ^k respects the pairings of topological K-groups with supports, and as $i_*[\mathcal{O}_{Y'}]$ in $\pi_0 K/l^\nu{}_{Y'}^{\mathrm{Top}}(X')$ is the product of the $i_*[\mathcal{O}_{Y'_i}]$ in $\pi_0 K/l^\nu{}_{Y'_i}^{\mathrm{Top}}(X')$ for $i=1, 2, \ldots, d$. Note that $Y' = Y'_1 \cap \ldots \cap Y'_d$.

THEOREM 3.5. — *Let X be a regular noetherian separated scheme satisfying the conditions 2.2. Let l be a prime, and let X have hereditary etale cohomological dimension N (at least at l). Let X' be etale over X, with l invertible in $\mathcal{O}_{X'}$. Let $i' : Y' \to X'$ be a closed immersion, with Y' regular and of codimension d at each point.*

Then there are maps with kernels and cokernels annihilated by multiplication by the integer $M(N)^2$ of 3.3:

$$(3.2) \qquad i_* \mathbb{Z}/l^\nu \leftarrow M(N)\, i_* \mathbb{Z}/l^\nu \to \underline{H}^{2d}_{Y'}(X'; \mathbb{Z}/l^\nu(d)).$$

For $j \neq 2d$, the sheaf $\underline{H}^j_{Y'}(X'; \mathbb{Z}/l^\nu)$ is torsion and is annihilated by $M(N)$.

Pf: The question is local in the etale topology. Thus one may assume X' contains primitive l^νth roots of unity. By [13], IV, 19.1.1 and 16.9, one may also assume that $Y' \to X'$ is as in Lemma 3.4.

Consider the sheafification of the Atiyah-Hirzebruch spectral sequence (2.4) with supports:

$$(3.3) \qquad E_2^{p,q} = \begin{Bmatrix} \underline{H}^p_{Y'}(X'; \mathbb{Z}/l^\nu(i)), & q=2i \\ 0, & q \text{ odd} \end{Bmatrix} \Rightarrow \underline{\pi}_{q-p} K/l^\nu{}_{Y'}^{\mathrm{Top}}(X').$$

As in [8] it is natural with respect to the Adams operations ψ^k for k prime to l. The operation ψ^k on $E_2^{p,2i}$ acts by multiplication by k^i. Also $E_2^{p,q}=0$ unless p is between 1 and $N+1$ inclusive.

As in [8] 3.3.2, one gets $M(N)d_r=0$ for $r\geqslant 2$. This is because the different eigenvalues of ψ^k on $E_r^{p,2i}$ and $E_r^{p+2j+1,2i+2j}$ must be congruent modulo the order of the differential $d_r=d_{2j+1}$. This forces $w_j d_{2j+1}=0$. As $d_r=0$ for r even or for $r>N$, this yields $M(N)d_r=0$. Thus $M(N)E_\infty^{p,q}$ is canonically a subobject of $E_2^{p,q}$, containing $M(N)^2 E_2^{p,q}$.

By the Gysin equivalence (2.6) and the sheafification of (2.4) for Y', the abutment of (3.3) is $i_* \mathbb{Z}/l^\nu$ for $q-p=0$, and 0 for $q-p=1$. Lemma 3.4 shows that ψ^k acts on $i_*\mathbb{Z}/l^\nu$ by multiplication by k^d. As ψ^k acts on $E_\infty^{2j,2j}$ by k^j, one must have $w_{|j-d|}E_\infty^{2j,2j}=0$ for $j\neq d$.

The results claimed follow from these statements in the obvious way. This completes the proof.

3.6. One may give an alternative proof of 3.5 by using Chern classes to degenerate the spectral sequence. This method allows comparison of (3.2) to the usual cycle class map. By induction on d as in 3.4, one sees that the cycle map of [16], Cycle 2.2.2, is the cup product of d Chern classes $c_{1,2}$ with supports in the Y_i'. Thus the cycle map is $(d-1)!\,c_{d,2d}$ with supports in Y'. This map is a fixed rational multiple of the map (3.2). Thus the cycle map has kernel and cokernel annihilated by a divisor of $(d-1)!\,M(N)$.

COROLLARY 3.7. — *Let X be a regular separated noetherian scheme satisfying the other conditions of* 2.2. *Let l be a prime number with* $l\geqslant(N/2)+1$, *where N is the hereditary etale cohomological dimension of X. Then the purity conjecture* (1.1) *is true for X and l.*

Pf: By 3.3, the integer $M(N)$ is a unit in $\mathbb{Z}/l^\nu$. Thus 3.5 gives the result.

3.8. If X is of finite type over $\mathbb{Z}$ and has Krull dimension n, 3.7 applies if $l\geqslant n+2$. For X of Krull dimension n and of finite type over a field of etale cohomological dimension k, 3.7 applies if $l\geqslant n+1+(k/2)$.

COROLLARY 3.9. — *Let X be a regular separated noetherian scheme satisfying the other conditions of* 2.2. *Let X' be etale over X, with the prime number l invertible in $\mathcal{O}_{X'}$. Let $i' : Y'\to X'$ be a closed immersion,* *everywhere of codimension d, and with Y′ regular. Then in the category of constructible* $\hat{\mathbb{Q}_l}$ *-sheaves of* [16], VI, 1.4.2, *one has purity isomorphisms for* $\hat{\mathbb{Q}_l}$ *-cohomology:*

(3.4) $$i'_* \hat{\mathbb{Q}_l}(-d) \cong \underline{H}^{2d}_{Y'}(X'; \hat{\mathbb{Q}_l}),$$

(3.5) $$\underline{H}^{p}_{Y'}(X'; \hat{\mathbb{Q}_l}) = 0 \qquad \textit{for} \quad p \neq 2d.$$

Pf: The assertion is just that the analogous $\mathbb{Z}/l^{\nu}$ statements hold modulo torsion of exponent bounded independently of the value of ν. This is true by 3.5.

3.10. For a global purity result like 1.2 for a more naive $\hat{\mathbb{Q}_l}$ -etale cohomology, one may consult [10], 1.10. The principle of proof is the same as above.

While all these results do not in full generality establish absolute cohomological purity, they do institute a sweeping reform.

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