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ON THE BOREL CLASS
OF THE DERIVED SET OPERATOR, II

BY

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RESUME. — Soit $X$ un espace non-en-numerable topologique métrisable compact, $2^X$ l'espace topologique des compacts de $X$ avec la topologie de Hausdorff et soit $D$ la dérivation de Cantor. KURATOWSKI a démontré que $D$ est borélienne et précisément de la deuxième classe, et a posé le problème de trouver la classe précise des dérivés successifs $D^n$. Nous démontrons que si $n$ est fini, alors $D^n$ est précisément de la classe $2n$ et si $\lambda$ est un ordinal de seconde espèce et $n$ fini, alors $D^{\lambda+n}$ est précisément $\lambda+2n+1$.

ABSTRACT. — KURATOWSKI showed that the derived set operator $D$, acting on the space of closed subsets of the Cantor space $2^X$, is a Borel map of class exactly two and posed the problem of determining the precise classes of the higher order derivatives $D^n$. In part 1 of our work [Bull. Soc. Math. France, 110, 4, 1982, p. 357-380], we obtained upper and lower bounds for the Borel class of $D^n$ and in particular showed that for limit ordinals $\lambda$, $D^\lambda$ is exactly of class $\lambda+1$. The first author recently showed, using different methods (cf. [1]) that for finite $n$, $D^n$ is exactly of Borel class $2n$. We now complete the solution of KURATOWSKI's problem by showing that for any limit ordinal $\lambda$ and any finite $n$, the operator $D^{\lambda+n}$ is of Borel class exactly $\lambda+2n+1$.

In this paper, we determine the exact Borel classes of the iterated derived set operators $D^n$, acting on the space $\mathcal{M}$ of closed subsets of the Cantor space $2^X$ with the usual Vietoris topology. This completes the solution of the problem of KURATOWSKI [3] which was begun in part I of our work [2].


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The results in the present paper depend strongly on those of its predecessor. We begin with some basic definitions and results from [2].

The derived set operator \( D \) maps \( \mathcal{H} \) into \( \mathcal{H} \) and is defined by:

\[
D(F) = F' = \{ x : x \in \text{Cl}(F - \{ x \}) \}.
\]

The \( \alpha \)'th iterate \( D^\alpha \) of the derived set operator map be defined for all ordinals \( \alpha \) by letting \( D^0(F) = F \), \( D^{\alpha+1}(F) = D(D^\alpha(F)) \) for all \( \alpha \) and \( D^\lambda(F) = \bigcap \{ D^\alpha(F) : \alpha < \lambda \} \) for limit ordinals \( \lambda \). The set \( F \) is said to be scattered if \( D^{\alpha+1}(F) = \emptyset \) for some \( \alpha \); the derived set order \( o(F) \) of \( F \) is the least such ordinal \( \alpha \).

The countable subset \( S \) of \( 2^\mathbb{N} \) is defined to be \( \{ x : (\exists m) (\forall n > m), x(n) = 0 \} \). If \( 2^\mathbb{N} \) is identified with the family \( \mathcal{P}(\mathbb{N}) \) of subsets of \( \mathbb{N} \), then \( S \) corresponds to the family of finite sets. Let \( 0 = (0, 0, 0, \ldots) \). The stitching operator \( \Phi \) mapping \( \mathcal{H}^\mathbb{N} \) into \( \mathcal{H} \) is defined as follows:

\[
\Phi(F_0, F_1, F_2, \ldots) = \{ 0 \} \cup \{(0, 0, \ldots, 0, 1, x(0), x(1), \ldots) : x \in F_\alpha \}.
\]

Note that \( \Phi \) preserves both finite intersections and unions, that is:

\[
\Phi(F_0 \cup G_0, F_1 \cup G_1, \ldots) = \Phi(F_0, F_1, \ldots) \cup \Phi(G_0, G_1, \ldots)
\]

and similarly for intersections. This also implies that \( \Phi \) is monotone, that is, whenever \( F_i \subset H_i \) for all \( i \), then \( \Phi(F_1, F_2, \ldots) \subset \Phi(H_0, H_1, \ldots) \). The two fundamental results on the stitching operator, Lemmas 3.7 and 3.8 of [2] concern the derived set order of the stitched set and the continuity of the stitching map. We actually need an extension of the former lemma to infinite ordinals; the proof goes through without difficulty.

**Lemma 1.** — For any sequence \( (F_o, F_1, \ldots) \) of sets from \( \mathcal{H} \cap \mathcal{P}(S) \) and any ordinal \( \alpha \):

\[
D^\alpha(\Phi(F_0, F_1, \ldots)) = \begin{cases} 
\Phi(D^\alpha(F_0), D^\alpha(F_1), \ldots), & \text{if } (\forall \beta < \alpha) \{ n : D^\beta(F_n) \neq \emptyset \} \text{ is infinite,} \\
\Phi(D^\alpha(F_0), D^\alpha(F_1), \ldots) - \{ 0 \}, & \text{otherwise.}
\end{cases}
\]

**Lemma 2.** — Let \( (H_0, H_1, \ldots) \) be a sequence of continuous functions mapping a topological space \( X \) into the space \( \mathcal{H} \) of closed subsets of \( 2^\mathbb{N} \)
such that each $H_n(x) \subseteq S$. Then the function $H$, defined by $H(x) = \Phi(H_0(x), H_1(x), \ldots)$ is also continuous. □

Calculation of the exact Borel classes of the iterated derived set operators begins with Theorem 1.3 of [2].

**Theorem 3.** — For any finite $n$ and any limit ordinal $\lambda$:

(a) $D^n$ is of Borel class $2n$;
(b) $D^{\lambda+n}$ is of Borel class $\lambda + 2n + 1$. □

Proofs that the Borel classes cited in Theorem 3 are exact proceed as follows. First we note that $\{\emptyset\}$ is both a closed and an open subset of $\mathcal{H}$. Thus if $D^n$ were of class $2n - 1$, then $T_n = (D^n)^{-1}(\{\emptyset\})$ would have to be a Borel subset of $\mathcal{H}$ of both additive and multiplicative class $2n - 1$; similarly, if $D^{\lambda+n}$ were of class $\lambda + 2n$, then $T^{\lambda+n} = (D^{\lambda+n})^{-1}(\{\emptyset\})$ would be of both additive and multiplicative class $\lambda + 2n$. To show that $T_n$ is not of multiplicative class $2n - 1$, we prove that $T_n$ is actually universal for Borel sets of additive class $2n - 1$; a similar result is given for $T^{\lambda+n}$. Both results will be proved by induction on $n$. We need two more propositions from [2]; the first is Proposition 4.1:

**Theorem 4.** — For any $F$, subset $B$ of $N^N$, there is a continuous function $H$ mapping $N^N$ into $\mathcal{H} \cap \mathcal{P}(S) \cap T_2$ such that, for all $x, x \in B$ if and only if $H(x) \in T_1$. □

We actually need the following improvement of Theorem 6.2 of [2].

**Theorem 5.** — For any countable limit ordinal $\lambda$ and any Borel subset $B$ of $N^N$ of additive class $\lambda$, there is a continuous function $H$ mapping $N^N$ into $\mathcal{H} \cap \mathcal{P}(S) \cap T_{\lambda+1}$ such that, for all $x, x \in B$ if and only if $H(x) \in T_\lambda$.

**Proof.** — Let $B$ be a Borel subset of $N^N$ of additive class $\lambda$. By Theorem 6.2 of [2], there is a continuous function $G$ from $N^N$ into $\mathcal{H} \cap \mathcal{P}(S) \cap T_{\lambda+2}$ such that, for all $x, x \in B$ if and only if $G(x) \in T_\lambda$; furthermore, $G(x)$ is also normal, as defined in 5.1 of [2]. Now let $C = C_\lambda$ be some canonical normal set with $o(C) = \lambda$ (see 5.10 of [2]). Define the function $H$ by:

$$H(x) = G(x) \cap C_\lambda.$$  

Recall from Lemma 5.2 of [2] that, for two normal sets $F$ and $G$:

$$o(F \cap G) = \min(o(F), o(G)).$$

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It follows that:

\[ o(H(x)) = \min(o(G(x)), \lambda). \]

This implies that \( H \) maps into \( T_{k+1} \) and that, for any \( x, x \in B \) if and only if \( H(x) \in T_k \). Recall from Lemma 5.12 of [2] that the intersection map is continuous for normal sets. Of course the constant map \( F(x) = C_k \) is continuous. It follows that \( H \) is also continuous. \( \square \)

It should be pointed out that the proof of Theorem 6.2 in [2] required the introduction of a more complex stitching operator acting on the family of normal sets.

L. Pignktiewicz has pointed out that in Proposition 5.8 of [2] \( \theta(\hat{F}) \) is actually normal if and only if \( \gamma = \lim_{n \to \infty} (o(F_n) + 1) \); this does not affect the proof of Theorem 6.2.

The induction step in the proofs that \( T_n \) and \( T_{k+1} \) are universal depends on Lemmas 1 and 2 and the following well-known result (a version of which can be found in LUSIN'S classic book [5]).

**Lemma 6.** Let \( X \) be a topological space with a countable basis of clopen sets (such as \( 2^\mathbb{N} \) and \( N^\mathbb{N} \)). Then for any countable ordinal \( \alpha \) and any Borel subset \( B \) of \( X \) of additive class \( \alpha \), \( B \) can be written as the disjoint countable union of Borel sets \( B_n \) each of multiplicative class \( \alpha \).

**Theorem 7.** (a) For any natural number \( k \) and any Borel subset \( B \) of \( N^\mathbb{N} \) of additive class \( 2k-1 \), there is a continuous function \( H \) mapping \( N^\mathbb{N} \) into \( \mathcal{H} \cap \mathcal{P}(S) \cap T_{k+1} \) such that, for all \( x, x \in B \) if and only if \( H(x) \in T_k \). (b) For any countable limit ordinal \( \lambda \), any natural number \( k \) and any Borel subset \( B \) of \( N^\mathbb{N} \) of additive class \( \lambda + 2k \), there is a continuous function \( H \) mapping \( N^\mathbb{N} \) into:

\[ \mathcal{H} \cap \mathcal{P}(S) \cap T_{\lambda+k+1} \]

such that, for all \( x, x \in B \) if and only if \( H(x) \in T_{\lambda+k} \).

**Proof.** The proofs of parts (a) and (b) proceed from, respectively, Theorems 4 and 5 in a similar manner. We will give the proof of (b), which is of course by induction on \( k \). Theorem 5 covers the case \( k = 0 \). Suppose therefore that the result is true for \( k \) and let \( B \) be a Borel subset of \( N^\mathbb{N} \) of additive class \( \lambda + 2k + 2 \). Since \( N^\mathbb{N} \setminus B \) is of multiplicative class \( \lambda + 2k + 2 \), there is a decreasing sequence \( \{ C_n : n \in N \} \) of sets of additive class \( \lambda + 2k + 1 \) such that \( N^\mathbb{N} \setminus B = \cap C_n \). Now by Lemma 6, there exists for each \( n \) a disjoint sequence \( \{ C_{n,m} : m \in N \} \) of sets of
multiplicative class \( \lambda + 2k \) such that \( C_n^* = \bigcup_m C_{n_m}^* \). It is now easy to see that, for all \( x \):

(i) \( x \in B \iff \{(n, m) : x \in C_{n_m}^* \} \) is finite.

Let \((n_0, m_1), (n_1, m_2), \ldots \) be some one-to-one enumeration of \( N \times N \) and let \( A_i = N^N \setminus C_{n_m}^* \). By the induction hypothesis, there exists a sequence \( \{H_i : i \in N\} \) of continuous functions from \( N^N \) into:

\[
\mathcal{H} \cap \mathcal{P}(S) \cap T_{\lambda+k+1}
\]
such that, for all \( x \):

(ii) \( x \in A_i \iff H_i(x) \in T_{\lambda+k} \).

The desired reduction \( H \) of \( B \) to \( T_{\lambda+k} \) is now defined by:

(iii) \( H(x) = \Phi(H_0(x), H_1(x), \ldots) \).

\( H \) is continuous by Lemma 2. We must now calculate the possible derived set order of \( H(x) \). First of all, from the induction hypothesis

\[
D^{\lambda+k+1}(H_i(x)) = \emptyset;
\]
it follows from Lemma 1 that \( D^{\lambda+k+2}(H(x)) = \Phi(\emptyset, \emptyset, \ldots) \setminus \{\emptyset\} = \emptyset \). Thus \( H(x) \in T_{\lambda+k+2} \) for any \( x \). Next suppose that \( x \in B \). Then by (i) and the definition of the \( A_i \), \( \{i : x \notin A_i\} \) is finite. It follows from (ii) that:

\[
\{i : D^{\lambda+k}(H_i(x)) \neq \emptyset\}.
\]
is finite. Then by Lemma 1, \( D^{\lambda+k+1}(H(x)) = \emptyset \) as desired. Finally, suppose that \( x \notin B \). Then again using (i) and (ii), it follows that:

\[
\{i : D^{\lambda+k}(H_i(x)) \neq \emptyset\}
\]
is infinite. Applying Lemma 1 and the fact that each \( D^{\lambda+k+1}(H_i(x)) = \emptyset \), we obtain:

\[
D^{\lambda+k+1}(H(x)) = \Phi(\emptyset, \emptyset, \ldots) = \{\emptyset\},
\]
so that \( H(x) \notin T_{\lambda+k+1} \). \( \square \)

**Theorem 8.** — (a) For any natural number \( k \), \( T_\lambda \) is a Borel subset of \( \mathcal{H} \) of additive class \( 2k-1 \) but not of multiplicative class \( 2k-1 \). (b) For any countable limit ordinal \( \lambda \) and any finite \( k \), \( T_{\lambda+k} \) is a Borel subset of \( \mathcal{H} \) of additive class \( \lambda+2k \) but not of multiplicative class \( \lambda+2k \).

**Proof.** — The positive direction is proved by induction, as follows. \( T_1 = \{F : F \text{ is finite}\} \) is an \( F_\sigma \) set by Lemma 1.1 of [2]. For any limit ordinal \( \lambda \), \( T_\lambda = \bigcup_{\alpha < \lambda} T_\alpha \) and will therefore be of additive class \( \lambda \).
if the result is assumed for $\alpha<\lambda$. Finally, $T_{\alpha+1} = D^{-1}(T_{\alpha})$; since $D$ is a mapping of Borel class 2, the result can always be extended from $\alpha$ to $\alpha+1$. The other direction has similar proofs for parts (a) and (b); we give the proof of (b). Let $B$ be an arbitrary subset of $N^N$ which is of additive class $\lambda+2k$ but not of multiplicative class $\lambda+2k$ (see [4], p. 425). By Theorem 7, there is a continuous function $H$ such that $B = H^{-1}(T_{\lambda+k})$. Now if $T_{\lambda+k}$ were of multiplicative class $\lambda+2k$, it would follow that $B$ must also be of multiplicative class $\lambda+2k$, contradicting our choice of $B$.

We can now give the complete solution of the problem of Kuratowski.

**Theorem 9.** — (a) For any natural number $k$, the iterated derived set operator $D^k$ is of Borel class exactly $2k$. (b) For any countable limit ordinal $\lambda$ and any natural number $k$, $D^{\lambda+k}$ is of Borel class exactly $\lambda+2k+1$.

**Proof.** — One direction is given by Theorem 3. The other direction has similar proofs for parts (a) and (b); we give the proof of (b). Recall that $\{\emptyset\}$ is a closed subset of $\mathcal{H}$. Thus if $D^{\lambda+k}$ were of Borel class $\lambda+2k$, then:

$$T_{\lambda+k} = (D^{\lambda+k})^{-1}(\{\emptyset\})$$

would have to be a Borel set of multiplicative class $\lambda+2k$, which would contradict Theorem 8.

The finite cases of Theorems 7, 8 and 9 were previously obtained by the first author in [1] using different methods.

**REFERENCES**


