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## CONTINUITY POINTS IN $\{x\} \times Y$

BY

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RÉSUMÉ. — Le résultat principal de cet article est un peu plus fort que le théorème suivant : soit  $X$  un espace à base dénombrable en tout point soit  $Y$  un espace de Baire et soit  $Z$  un espace métrique. Si une fonction  $f: X \times Y \rightarrow Z$  est séparément continue, l'ensemble des points de continuité de  $f$  est un dense  $G_\delta$  dans  $\{x\} \times Y$ , pour chaque  $x \in X$ .

ABSTRACT. — The main result of this paper is somewhat stronger than the following: let  $X$  be a first countable space, let  $Y$  be a Baire one and let  $Z$  be a metric space. If a function  $f: X \times Y \rightarrow Z$  is separately continuous, then the set of points of continuity of  $f$  is a dense  $G_\delta$  subset in  $\{x\} \times Y$ , for all  $x \in X$ .

There are many papers which deal with the classical problem of determination of points of continuity of a separately continuous function, for some references, see [1].

The general problem is: find conditions on topological spaces  $X$ ,  $Y$  and  $Z$  so that each separately continuous function  $f: X \times Y \rightarrow Z$  (i. e., function continuous in each variable while the other is fixed) is jointly continuous at points of a "substantial" (in some topological sense) subset of  $X \times Y$ , cf. [1], p. 515.

We will answer this problem showing how the set of points of continuity looks like in the sets of form  $\{x\} \times Y$ , for each  $x$ , while  $X$  is assumed to be first countable,  $Y$  is Baire,  $Z$  is metric and  $f$  is somewhat weaker than separately continuous. As a useful tool we make use of quasi-continuous functions. Namely:

A function  $f: X \rightarrow Y$  is called *quasi-continuous* if for every point  $x \in X$  and every neighborhoods  $U$  of  $x$  and  $V$  of  $f(x)$ , there exists an open, non-empty set  $G$ ,  $G \subset U$ , such that  $f(G) \subset V$ .

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Recall that S. Marcus proved that there exists a quasi-continuous function which is not Lebesgue measurable. Of course, every continuous function is quasi-continuous.

A function  $f: X \times Y \rightarrow Z$  ( $X, Y, Z$ , arbitrary topological spaces) is said to be *quasi-continuous with respect to the variable  $x$* , if for every point  $(p, q)$  of  $X \times Y$  and for every neighborhood  $N$  of  $f(p, q)$  and for every neighborhood  $U \times V$  of  $(p, q)$  there exists a neighborhood  $U'$  of  $p$ , with  $U' \subset U$  and a non-empty open set  $V' \subset V$  such that for all  $(x, y) \in U' \times V'$  we have  $f(x, y) \in N$ . Analogously, one may define functions which are quasi-continuous with respect to the variable  $y$ . If  $f$  is quasi-continuous with respect to the variable  $x$  and quasi-continuous with respect to the variable  $y$ , then we say that  $f$  is *symmetrically quasi-continuous*.

One can easily construct symmetrically quasi-continuous functions which are not separately continuous. From [2], Theorem 1 it follows:

LEMMA. — *Let  $X$  be first countable,  $Y$  be Baire and  $Z$  be metric. If  $f: X \times Y \rightarrow Z$  is a function such that all its  $x$ -sections  $f_x$  are quasi-continuous and all its  $y$ -sections  $f_y$  are continuous, then  $f$  is quasi-continuous with respect to  $x$ .*

Now, under the same assumptions as in Lemma, let us fix an arbitrary element  $x$  from  $X$ . Consider the function  $y \rightarrow \omega(x, y)$ . Observe, that the open set  $\{y \mid \omega(x, y) < 1/n\}$  is dense in  $Y$ ! Hence, standard arguments let us state the following:

THEOREM. — *Let  $X$  be first countable,  $Y$  be Baire and  $Z$  be metric. If a function  $f: X \times Y \rightarrow Z$  has all its  $x$ -sections  $f_x$  quasi-continuous and all its  $y$ -sections  $f_y$  continuous, then for all  $x \in X$ , the set of points of continuity of  $f$  is a dense,  $G_\delta$  subset in  $\{x\} \times Y$ .*

COROLLARY. — *Let  $X$  and  $Y$  be first countable, Baire spaces and  $Z$  be metric. If a function  $f: X \times Y \rightarrow Z$  is separately continuous, then the set of points of continuity of  $f$  is dense,  $G_\delta$  in the sets of form  $X \times \{y\}$  and  $\{x\} \times Y$ , for all  $x \in X$  and all  $y \in Y$ .*

The following Question remains open:

Question. — For which "nice" topological (neither metric nor satisfying any sort of countability conditions, see [1], p. 515<sub>20</sub>) spaces  $X$  and  $Y$ , our Lemma holds?

Good answers to this Question will let to extend our Theorem.

## REFERENCES

- [1] NAMIOKA (I.). — *Separate continuity and joint continuity*, *Pacific J. Math.*, vol. 51, 1974, p. 515-531.
  - [2] PIOTROWSKI (Z.). — *Quasi-continuity and product spaces*, *Proc. Conf. Geom. Topology*, Warsaw, 1978.
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