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ON A RELATION BETWEEN TORSION NUMBERS AND ALEXANDER MATRIX OF A KNOT

BY

MADAN LAL MEHTA (*)

RÉSUMÉ. — Dans la théorie des nœuds, on considère deux sortes d'invariants; le polynôme d'Alexander $\Delta(x)$ déduit d'une matrice $A(x)$ dite d'Alexander, et les nombres de torsion d'ordre n pour $n=2, 3, 4, \ldots$, qui caractérisent l'espace complémentaire du nœud. Nous montrons que, pour n'importe quel nœud, les nombres de torsion d'ordre 2 sont les diviseurs élémentaires de la matrice $A(x)$ à $x=-1$. On en déduit une méthode rapide pour calculer à l'aide d'un ordinateur les nombres de torsion d'ordre n pour tous $n \geqslant 2$.

ABSTRACT. — In the theory of knots, one comes accross two sorts of invariants; the Alexander polynomial $\Delta(x)$ derived from the so-called Alexander matrix $A(x)$, and the torsion numbers of order n for $n=2, 3, 4, \ldots$ characterizing the space surrounding the knot. We show that for any knot the torsion numbers of order 2 are the elementary divisors of the matrix $A(x)$ for $x=-1$. A quick method is deduced to find torsion numbers of order n for all $n \geqslant 2$.

1. A knot is a simple closed curve in the ordinary three dimensional space. It can be represented as a diagram by its projection on a plane. The plane may always be chosen so that every multiple point of the projection is a double point. To indicate which segment of the curve lies over the other one, usually a small portion of the underlying curve is omitted near the double point (see *Fig.* 1 *a*). ALEXANDER [1] gave instead the following prescription. Orient the knot arbitrarily. Imagine an observer describing the projection in the positive sense (determined by the orientation of the knot) thereby passing twice through each crossing point. As the observer passes through a crossing point on the segment representing the upper branch he marks with dots the two corners on his right; as he passes through a crossing point on the lower segment, he makes no notation at all. The resulting

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system of dots will tell us which branch is upper and which is lower at every crossing point. (See *Fig.* 1 *b*, which represents the same knot as Figure 1 *a*, but in ALEXANDER'S notation.)

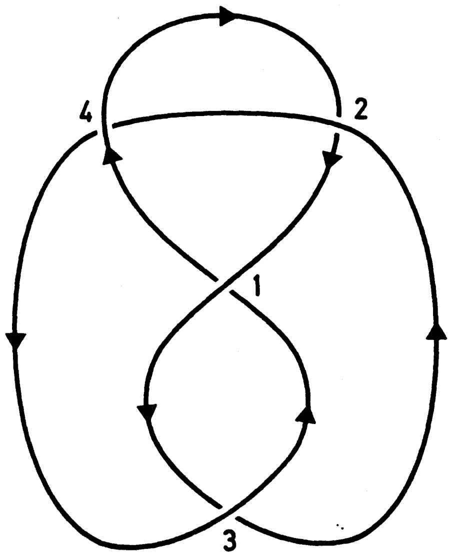

Fig. 1 *a*

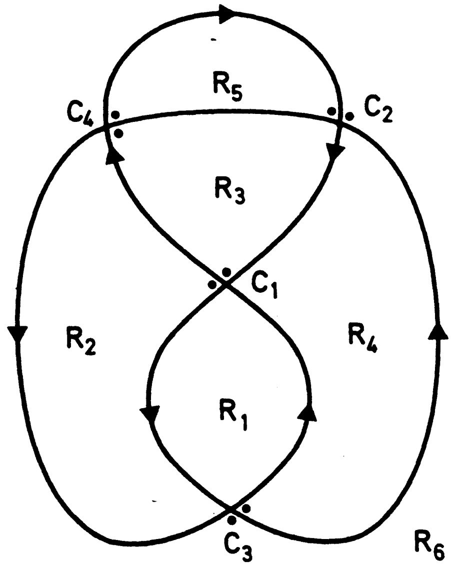

Fig. 1 *b*

The diagram of a knot divides the plane into regions. By studying how various dots are distributed along the corners of these regions, ALEXANDER and BRIGGS [2] and ALEXANDER [1] discovered quantities which remain invariant under continuous changes of the knot curve. For example, for each integer greater than one they derived a set of torsion numbers. Similarly an invariant polynomial $\Delta(x)$, the Alexander polynomial, is derived as the determinant of a matrix $A(x)$ whose elements are polynomials in a variable x.

We will show that the torsion numbers of order 2 are the elementary divisors of the matrix $A(-1)$. In particular, $\Delta(-1)$ is equal to the product of the torsion numbers of orders 2. We show that the matrix $A(x)$ contains all the information about torsion numbers of all orders. This information is not lost by elementary operations on $A(x)$. Thus $A(x)$ may be reduced and for this purpose a computer can be used. We thereby give a quick method to find torsion numbers of order n for all $n \geqslant 2$.

2. Let M be a (rectangular or square) matrix whose elements are integers. Its elementary divisors may be defined as follows [*see* for example [5], Chap. 6].

By elementary operations on M, we will mean :

(i) exchanging two rows or two columns;

(ii) changing the overall sign of a row or of a column;

(iii) adding an integer multiple of a row to any other row, or of a column to any other column.

These rules can be expressed in matrix notation as follows. The transformation $M \rightarrow PMQ$ is an elementary operation, if, and only if, the square matrices P and Q (of suitable orders, so that the product PMQ exists) have integer elements and det $P = \pm 1 = det\ Q$.

By a permutation of rows and columns one may bring any element of M in the leading position, i. e. in the first row and first column. So let us suppose that the element M_{11} has the smallest non-zero absolute value. If $M_{1j} \neq 0$, then we may add to the j-th column a suitable multiple of the first column so that the resulting M_{1j} has an absolute value smaller than M_{11}. If this smaller absolute value is non-zero, then we may by an exchange of columns bring this to the leading position and start the whole process again. Similar remarks apply for the elements in the first column. Also if M_{11} does not divide M_{ij}, $i \neq 1 \neq j$, while all the elements in the first row and the first column, except M_{11}, are zero, then we add the j-th column to the first column and add to the i-th row a suitable (perhaps negative) integral multiple of the first row to make the resulting M_{i1} have a smaller absolute value than M_{11}. We may then bring this smaller M_{i1} to leading position and start the whole process again. Hence, by elementary operations, we may transform M so that among its elements M_{11} has the smallest absolute value, $M_{i1} = M_{1j} = 0$, and M_{ij} is an integral multiple of M_{11} for every i and j different from 1.

The same process can then be repeated on the sub-matrix of M ignoring its first row and first column.

Thus, by elementary operations, we may transform M so that $M_{ij} = 0$ for $i \neq j$, and M_{pp} divides $M_{p+1,\,p+1}$ for every p. In matrix notation given any matrix M with integer elements we may always find square matrices P and Q also with integer elements and det $P = \pm 1 =$ det Q, such that all off-diagonal elements of PMQ are zero and the non-negative diagonal elements are successively multiples of the preceding ones. *Those diagonal elements which are different from one are called the elementary divisors of M.*

3. In a knot diagram let ν be the number of crossing points and therefore by Euler's formula $\nu+2$ the numbers of regions. We denote the crossing points as $c_1, c_2, \ldots, c_\nu$ and regions as $R_1, R_2, \ldots, R_{\nu+2}$. At every c_i there are 4 corners, two of which are dotted and two undotted. Every R_j has a certain number of corners belonging in general to different c_i's. (In special cases, two corners of R_j may belong to the same c_i, but such a diagram admits an obvious simplification. Also the two corners in question lie always opposite to each other, one of them being dotted and the other undotted. To include such cases only trivial modifications will be needed in what follows.) A crossing point c_i and a region R_j are said to be *incident* if, and only if, c_i lies on the boundary of R_j. If c_i and R_j are incident, then they are said to be *incident with* or *without a dot* according as the corner of c_i belonging to R_j is or is not a dotted corner of the diagram.

According to ALEXANDER [1], we write a $\nu\times(\nu+2)$ matrix A as follows. To every crossing point c_i corresponds a row of A and to every region R_j a column of A. We put $A_{ij}=0$ if c_i and R_j are not incident. Now suppose the four corners at c_i belong respectively to R_j, R_k, R_l and R_m, that we pass through these regions in the cyclic order just mentioned as we go round c_i in the positive sense, and that two dotted corners belong to R_j and R_k respectively. Then we put:

$$A_{ij}=x, \qquad A_{ik}=-x, \qquad A_{il}=1 \qquad \text{and} \qquad A_{im}=-1.$$

Thus each row of A contains four non-zero elements and the cyclic order with the two dotted corners first is essential.

For example, to the knot of Figure 1 corresponds the matrix

	R_1	R_2	R_3	R_4	R_5	R_6
c_1............	1	$-x$	x	-1	.	.
c_2............	.	.	1	-1	$-x$	x
c_3............	1	-1	.	$-x$	.	x
c_4............	.	-1	x	.	$-x$	1

The matrix A is the Alexander matrix and the determinant of any of its non-singular $\nu\times\nu$ sub-matrix is the Alexander polynomial (they all differ from each other at most by a factor $\pm x^m$) [1].

To define torsion numbers, ALEXANDER and BRIGGS [2] assign n variables $y_{i\alpha}$, $\alpha=1, 2, \ldots, n$ to each c_i. Thus each variable has two indices and there are in all νn variables $y_{i\alpha}$, $i=1, 2, \ldots, \nu$; $\alpha=1, 2, \ldots, n$. To allow α to take all integer values one makes the convention $y_{i\alpha}=y_{i(\alpha+n)}$. Corresponding to R_j they write the equations of homologies

$$\mathrm{R}_j\,;\quad \textstyle\sum_k y_{k\alpha}+\sum_{k'} y_{k'(\alpha+1)}=0, \qquad \alpha=1, 2, \ldots, n,$$

where the first sum runs over all c_k incident to R_j without a dot, and the second sum runs over all $c_{k'}$ incident to R_j with a dot.

For example, corresponding to the knot of Figure 1, the homology equations are

$$\begin{aligned}
R_1 &: \ y_{1\alpha}+y_{3\alpha}=0,\\
R_2 &: \ y_{1(\alpha+1)}+y_{3\alpha}+y_{4\alpha}=0,\\
R_3 &: \ y_{1(\alpha+1)}+y_{2\alpha}+y_{4(\alpha+1)}=0,\\
R_4 &: \ y_{1\alpha}+y_{2\alpha}+y_{3(\alpha+1)}=0,\\
R_5 &: \ y_{2(\alpha+1)}+y_{4(\alpha+1)}=0,\\
\mathrm{R}_6 &: \ y_{2(\alpha+1)}+y_{3(\alpha+1)}+y_{4\alpha}=0.
\end{aligned}$$

The coefficient matrix B of the homology equations is rectangular of order $\nu n\times(\nu n+2n)$ and its elements, either 0 or 1. Its elementary divisors are the torsion numbers [2] of order n.

For $n=2$, the matrix B of order $2\nu\times(2\nu+4)$ may be written as follows. To every c_i corresponds two rows and to every R_j corresponds two columns. We partition B into $\nu\times(\nu+2)$ blocks of sizes 2×2 each, so that to every pair (c_i, R_j) corresponds a 2×2 matrix. If c_i and R_j are not incident, then this 2×2 matrix is zero. If c_i and R_j are incident without a dot then it is the 2×2 unit matrix. And if c_i and R_j are incident with a dot, then the corresponding 2×2 matrix is

$$\sigma=\begin{bmatrix} 0 & 1\\ 1 & 0\end{bmatrix}.$$

Let us add column $(2j-1)$ to column $2j$ for $j=1, 2, \ldots, \nu+2$, and then subtract row $2i$ from row $(2i-1)$ for $i=1, 2, \ldots, \nu$. This does not disturb the 2×2 zero and unit matrices but replaces the σ's by

$$\sigma\rightarrow\begin{bmatrix} -1 & 0\\ 1 & 1\end{bmatrix}.$$

Finally, collect all odd numbered rows and columns in the upper left corner, the even numbered rows and columns being in the right lower corner. These elementary operations transform B into B' having the following structure :

$$B' = \begin{bmatrix} B'_1 & 0 \\ B'_2 & B'_3 \end{bmatrix},$$

where B'_1, B'_2, B'_3 and 0 are matrices of order $\nu \times (\nu + 2)$. The (i, j) element of B'_1 is 0,1 or -1 according as c_i and R_j are not incident, incident without a dot or incident with a dot, respectively. The (i, j) element of B'_3 is 0 or 1 according as c_i and R_j ate not incident or incident (independently of the dot) respectively. The structure of B'_2 is also clear, but it will not be needed.

4. The proof that the elementary divisors of B are identical to those of $A(x)$ for $x = -1$, consists of two parts.

4.1. $A(-1)$ and B'_1 differ by a change of sign of certain rows and certain columns and hence have the same elementary divisors.

To see that, put a plus or a minus sign on each crossing point and each region as follows. We decide arbitrarily c_1 to be positive. Let c_1 be incident with R_j, R_k, R_l and R_m, that we pass through these regions in the cyclic order just mentioned as we go around c_i in the positive sense, and that the two dotted corners belong to R_j and R_k respectively. Then we mark R_j and R_l to be positive and R_k and R_m to be negative. Any region having a commun boundary with R_j or R_l receives a negative sign. Similarly any region having a common boundary with R_k or R_m receives a positive sign. We proceed step by step deciding the signs of regions surrounding those already marked in such a way that any two regions having a common boundary get opposite signs. This will never lead us to a contradiction for in making a complete tour of the knot projection one goes around each region a certain number of times and this number changes by one on crossing the boundary separating any two regions. Thus the parity of this number decides the sign of the region.

Now starting at c_1 we move along the curve passing through each crossing point twice, once along the upper branch and once along the lower one. When we reach c_i following the upper branch, the two dotted corners are on our right and belong to two regions having opposite signs. We choose the sign of c_i to be the same as that of the region containing the first dotted corner we encounter. When we pass through c_i following a lower branch, we do nothing. (The sign of c_i is either already decided or will be decided later, during the upper pass.)

If we change signs of the rows and the columns of $A(-1)$ corresponding to negatively marked crossing points and regions respectively, then what we obtain is a matrix identical to B'_1.

4.2. B'_3 has no elementary divisors. In other words B'_3 can be reduced by elementary operations to a matrix all of whose off-diagonal elements are zero and all diagonal elements unity.

The reasoning of section 4.1 above shows that if we make the same changes of signs (of rows corresponding to negative crossing points and of columns corresponding to negative regions) in $A(x)$ for $x=1$, we will get B'_3. Thus B'_3 and $A(1)$ have the same elementary divisors.

Reduce $A(1)$ by elementary operations as explained in section 2 to a matrix whose off-diagonal elements are all zero and the diagonal elements are integers, say, $b_1, b_2, \ldots, b_\nu$. But the Alexander polynomial $\Delta(x)$, apart from a factor $\pm x^m$, is equal to the determinant of the $\nu \times \nu$ non-singular sub-matrix of $A(x)$, and equals one at $x=1$, $\Delta(1)=1$, for any knot ([1], [3], [7], [8]). Thus the product of the integers $b_1, b_2, \ldots, b_\nu$ is unity and hence each one of them is equal to one. Consequently, $A(1)$ (or B'_3) has no elementary divisors.

5. Alexander's recipe of writing $A(x)$ as given in section 3, though conceptually easy, is not convenient for routine programming. For computer purposes it is easier to adopt other equivalent constructions ([3], [6], [7], [8]). We indicate below one of these giving a $(\nu-1)\times(\nu-1)$ matrix $A_1(x)$ such that the $A(x)$ of section 3 above can be transformed by elementary operations to the form

$$\begin{bmatrix} 1 & O_{1(\nu-1)} & O_{12} \\ O_{(\nu-1)1} & A_1(x) & O_{(\nu-1)2} \end{bmatrix},$$

where O_{mn} represents the $m \times n$ zero matrix. For mathematical details, *see* for example, CROWELL and FOX [3].

Choose any point other than a double point of the knot projection as origin and as before describe the projection in the positive sense. As we pass through a double point along a lower branch we number it as 1, 2, 3, ... in sequence; as we pass through a double point along the upper branch, we do nothing, the crossing point is either already numbered or will be numbered later when we pass through it along the lower branch. Once all the crossing points (or underpasses) thus numbered, we assign the number j to the arc or segment of the curve lying between underpasses $j-1$ and j. The arc between the underpasses ν and 1 receives the number 1. Next we attach a sign to each crossing point, + or − according as the priority of passage from the right, say, is respected or not respected (*Fig.* 2).

Fig. 2

Thus to every crossing point correspond two numbers and a sign; the underpass number, the arc number of the overpass and the sign of the priority of passage. We construct a $\nu\times\nu$ matrix $A'(x)$ as follows. The rows of $A'(x)$ correspond the underpasses and the columns to the overpassing arcs. Let j be the arc number of the overpass and s the sign corresponding to the underpass No. i. The elements of $A'(x)$ are:

(i) if $j=i$ or $j=i+1$ (mod ν), then $A'_{ii}=1$, $A'_{i(i+1)}=-1$;
(ii) if $i\neq j\neq i+1$ (mod ν) and $s=+$, then $A'_{ii}=1$, $A'_{i(i+1)}=-x$, $A'_{ij}=x-1$;
(iii) if $i\neq j\neq i+1$ (mod ν) and $s=-$, then $A'_{ii}=-x$, $A'_{i(i+1)}=1$, $A'_{ij}=x-1$;
(iv) all other elements of A' are zero.

Any $(\nu-1)\times(\nu-1)$ sub-matrix of $A'(x)$ can be taken as $A_1(x)$. As an example, corresponding to the knot of Figure 1 *a*, we have the correspondances:

number of the underpass	1	2	3	4
overpassing arc No.	3	4	1	2
sign	+	−	+	−

and therefore

$$A'(x)=\begin{bmatrix} 1 & -x & x-1 & . \\ . & -x & 1 & x-1 \\ x-1 & . & 1 & -x \\ 1 & x-1 & . & -x \end{bmatrix}.$$

6. In what follows, a polynomial will mean a polynomial in x with integer coefficients. Let us reduce $A_1(x)$ by the following elementary transformations. (Compare with section 2):

(i) exchanging two rows or two columns;

(ii) multiplying a row or a column by $\pm x^{\alpha}$, where α is any integer, positive, negative or zero;

(iii) adding a polynomial multiple of a row to another row or of a column to another column;

(iv) if the only non-zero element in a row and a column is found at their intersection and if this non-zero element is 1, then we delete that row and that column.

It is clear from the above rules that if a row contains only one non-zero element and if this non-zero element is a monomial, $\pm x^{\alpha}$, then we can delete that row and the column containing the monomial in question.

Note that we do not allow multiplying a row by a constant $c \neq 1$. If we did that, then $A_1(x)$, as any other matrix, can be reduced to the diagonal form and the diagonal elements are polynomial multiples of the preceding ones ([5], chap. 6); elementary polynomial divisors (with coefficients not necessarily integers). As multiplying a row by a constant is not allowed, $A_1(x)$ can not in general be reduced to the diagonal form.

By a permutation of the rows and columns, if necessary, we can bring any monomial element of $A_1(x)$ in the first row and first column. If any other element in the first row is non-zero, we can make it zero by adding to the column containing it a suitable polynomial multiple of the first column. We can then delete the first row and the first column. Thus as long as a matrix has an element $\pm x^{\alpha}$ somewhere, we can reduce its size.

In matrix notation, we look for square matrices $P(x)$ and $Q(x)$ with polynomial elements such that their determinants are monomials, det $P(x) = \pm x^{\alpha}$, det $Q(x) = \pm x^{\beta}$, α, β integers, and $P(x) A_1(x) Q(x)$ has the form

$$P(x)A_1(x)Q(x) = \begin{bmatrix} I & 0 \\ 0 & A_2(x) \end{bmatrix} \equiv I \dotplus A_2(x), \tag{6.1}$$

where I is a unit matrix and no element of $A_2(x)$ is a monomial in x.

Now if we replace 1 and x in (6.1) by the $n\times n$ square matrices

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \hdashline 0 & 0 & \dots & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \hdashline 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

respectively (these matrices have integer elements and determinants ± 1), then we see from the reasoning of section 4 that the elementary divisors of B are identical to those of $A_2(x)$.

In general, $A_2(x)$ is a much smaller matrix than $A_1(x)$. And the calculation of torsion numbers is thereby greatly simplified. We illustrate this by two examples.

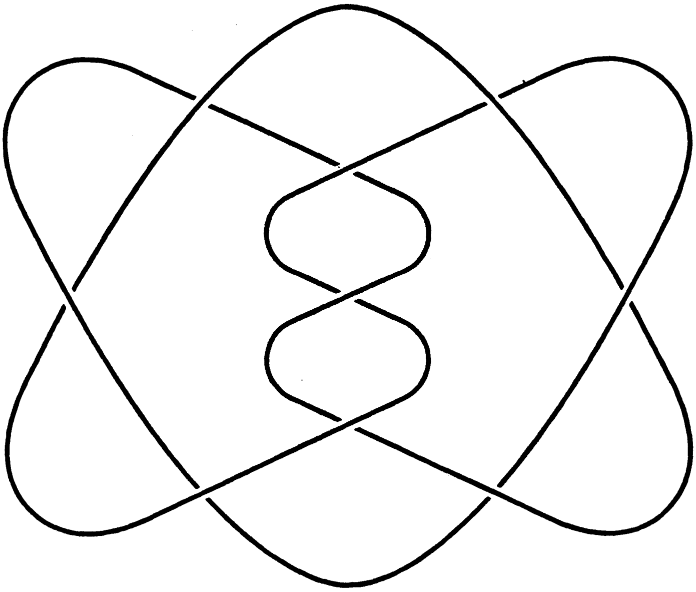

Fig. 3

7. Our first example is the knot $9*46$ of Alexander and Briggs table, depicted on Figure 3. One writes down $A_1(x)$ according to the prescription of section 5 and it is straightforward to reduce it. One finds (*cf.* CROWELL and FOX [3], p. 129) $A_2(x)$ to be a 2×2 diagonal matrix

$$A_2(x)=[(1-2x)\dot{+}(2-x)].$$

And the $n \times n$ matrix

$$1-2x \equiv \left[\begin{array}{cccccc} 1 & -2 & 0 & \dots & 0 & 0 \\ 0 & 1 & -2 & \dots & 0 & 0 \\ \hdashline 0 & 0 & 0 & \dots & 1 & -2 \\ -2 & 0 & 0 & \dots & 0 & 1 \end{array}\right]$$

has one elementary divisor 2^n-1. The elementary divisor of $2-x$ is the same. Therefore, the torsion numbers of order n for the knot $9*46$ (*Fig.* 3) are 2^n-1, 2^n-1.

As a second example, we take the knot depicted on Figure 1 (number $4*1$ of Alexander and Briggs' table). Here, $A_1(x)$ reduces completely, $A_2(x)=\Delta(x)=1-3x+x^2$. The corresponding $n \times n$ matrix

$$\left[\begin{array}{ccccccc} 1 & -3 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -3 & 1 & \dots & 0 & 0 \\ \hdashline 1 & 0 & 0 & 0 & \dots & 1 & -3 \\ -3 & 1 & 0 & 0 & \dots & 0 & 1 \end{array}\right]$$

can by elementary operations be brought to the form

$$I_{n-2} \dotplus \begin{bmatrix} a_n+1 & b_n-3 \\ c_n & d_n+1 \end{bmatrix}$$

and for our purposes the $(n-2)\times(n-2)$ unit matrix can be ignored. The integers a_n, b_n, c_n, d_n are to be determined recursively from the relations

$$a_{n+1}=3a_n+b_n, \qquad b_{n+1}=-a_n,$$
$$c_{n+1}=3c_n+d_n, \qquad d_{n+1}=-c_n,$$

with the initial values

$$a_2=1, \qquad b_2=0, \qquad c_2=-3, \qquad d_2=1.$$

The table below lists their values for a few n:

$n=$	2	3	4	5	6	7	8	9
a.	1	3	8	21	55	144	377	987
b.	0	−1	−3	−8	−21	−55	−144	−377
c..	−3	−8	−21	−55	−144	−377	−987	−2584
d.	1	3	8	21	55	144	377	987

The corresponding torsion numbers are

5; 4,4; 3,15; 11,11; 8,40; 29,29; 21,105; 76,76; ...

8. For the reduction of $A_1(x)$, one can use a computer. For this purpose on the knot diagram we number consecutively the underpasses and mark the overpassing arc numbers and the signs of the crossings, as explained in paragraph 5 above. As the underpasses are numbered consecutively, we give only the successive overpassing arc numbers with signs to the computer on a card. From this information the computer constructs the matrix $A_1(x)$ according to the rules indicated and performs the elementary operations to reduce it as much as it can. The final result is printed.

Thus among the 250 prime knots upto 10 crossings, listed in D. ROLFSEN [7] :

(*a*) $A_1(x)$ reduces to a 2×2 matrix for the knots

$$9*35,\quad 9*38,\quad 9*47,\quad 9*48,\quad 9*49,$$
$$10*61,\quad 10*65,\quad 10*75,\quad 10*101,\quad 10*103,$$
$$10*115,\quad 10*140,\quad 10*142,\quad 10*157 \quad\text{and}\quad 10*160;$$

(*b*) it reduces to a 2×2 diagonal matrix for the knots

$$8*18,\quad 9*37,\quad 9*40,\quad 9*46,$$
$$10*74,\quad 10*99,\quad 10*122,\quad 10*123 \quad\text{and}\quad 10*155;$$

(*c*) it reduces to a 1×1 matrix, i. e. to the polynomial $\Delta(x)$, for all the remaining 226 knots.

Thus for most of the knots (226 out of 250) the Alexander polynomial $\Delta(x)$ contains all the information about torsion numbers of any order. For 9 knots listed in (*b*) above, torsion numbers of order n separate into two sets. Only for 15 knots listed in (*a*) above, the $A_1(x)$ does not seem to reduce further and therefore torsion numbers are a little more difficult to calculate (*see* Appendix).

After this work was completed the author came to know that the torsion numbers of any order as well as well as the Alexander polynomial can also be derived ([4], p. 150-158) from a $2h \times 2h$ matrix where h is the genus of the knot, and from this one can deduce in particular that torsion numbers of order 2 are the elementary divisors of $A(-1)$. However, the $2h \times 2h$ matrix in question requires for its construction manipulating a Seifert surface bounding the knot, hardly amenable to computer programming.

I am thankful to J. DES CLOIZEAUX for separating B'_1 from B'_3 and for pointing out that $\Delta(1)$ is related to B'_3. I am also indebted to A. GERVOIS and G. MAHOUX for many improvements in the presentation, and to V. POENARU and F. LAUDENBACH for insisting that such things must be known in the existing vast literature.

APPENDIX

Reduced form of the Alexander matrix for the prime knots upto 10 crossings, as listed in D. ROLFSEN [7]. For the non-diagonal 2×2 matrices, we adopt the notation (a, b, c, d) to indicate the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where the elements a, b, c and d are polynomials in x. When b and c are both zero so that the 2×2 matrix is a diagonal one, we write simply $a \dot{+} d$. When $A(x)$ reduces to the 1×1 matrix $\Delta(x)$, the Alexander polynomial, it can be looked up in Rolfsen's book and we do not reproduce them here.

Knot number	Reduced form of $A(x)$
9 * 35	$(x-2, 3x-3, 1-2x, x-2)$
9 * 38	$(5-9x+5x^2, 1+x, 0, 1-x+x^2)$
9 * 47	$(1-3x+4x^2-3x^3+x^4, x-2x^2, 2x-x^2, 1-x+x^2)$
9 * 48	$(2+x^2, 1-x+x^2, -1+8x, x+x^2)$
9 * 49	$(1-x^2, -1+3x-x^2, 1-3x+x^2, 2-2x^2)$
10 * 61	$(1-x+x^2, 0, 2, 2-3x+x^2-3x^3+2x^4)$
10 * 65	$(1-x+x^2, 0, 2, 2-5x+7x^2-5x^3+2x^4)$
10 * 75	$(1-4x+3x^2-x^3, 0, 1-2x, 1-3x+4x^2-x^3)$
10 * 101	$(-1+x-x^3, 2-3x+3x^2, 2-5x+3x^2, 3-2x)$
10 * 103	$(2-4x+3x^2-x^3, 2-2x+x^2, 2x-2x^2+x^3, -1+4x-3x^2+2x^3)$
10 * 115	$(1-8x+17x^2-8x^3+x^4, 2x^2, 2x, 1-x+x^2)$
10 * 140	$(1-x+x^2, 2, 0,, 1-x+x^2)$
10 * 142	$(2-x-x^2-x^3+2x^4, 2, 0, 1-x+x^2)$
10 * 157	$(1-3x+4x^2-4x^3+2x^4, 2-3x+2x^2, 1-3x+2x^2-2x^3+x^5, 1+x^3)$
10 * 160	$(1-3x+2x^2-3x^3+x^4, x-2x^2, 2x-x^2, 1-x+x^2)$
8 * 18	$(1-x+x^2)\dot{+}(1-4x+5x^2-4x^3+x^4)$
9 * 37	$(1-2x)\dot{+}(2-7x+5x^2-x^3)$
9 * 40	$(1-3x+x^2)\dot{+}(1-4x+5x^2-4x^3+x^4)$
9 * 46	$(1-2x)\dot{+}(2-x)$
10 * 74	$(1-2x)\dot{+}(4-8x+7x^2-2x^3)$
10 * 99	$(1-2x+3x^2-2x^3+x^4)\dot{+}(1-2x+3x^2-2x^3+x^4)$
10 * 122	$(2-3x+2x^2)\dot{+}(1-4x+5x^2-4x^3+x^4)$
10 * 123	$(1-3x+3x^2-3x^3+x^4)\dot{+}(1-3x+3x^2-3x^3+x^4)$
10 * 155	$(1-2x+x^2-x^3)\dot{+}(1-x+2x^2-x^3)$

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