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THE LAGRANGE COMPLEX

BY

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RESUME. — Nous définissons le complexe de co-chaines $(A, \delta)$, et nous prouvons le lemme de Poincaré pour l'opérateur $\delta$. L'opérateur $\delta$ est utilisé dans le calcul des variations en vue de déduire les équations d'Euler-Lagrange. Le lemme de Poincaré fournit alors le critère suivant lequel un système d'équations est un système d'Euler-Lagrange.

ABSTRACT. — A cochain complex $(A, \delta)$ is defined, and the $\delta$-Poincaré lemma is proved. The work is motivated by applications to the calculus of variations. The operator $\delta$ is used in the calculus of variations to construct the Euler-Lagrange equations, and the $\delta$-Poincaré lemma provides criteria for partial differential equations to be Euler-Lagrange equations.

The present paper generalizes results contained in earlier publications ([6], [8]) which were applicable to ordinary differential equations of the Euler-Poisson type.

1. Jets and tangent vectors

Let $M$ be a $C^\infty$-manifold. We denote by $T^{(k)}M$ the manifold $J^k_0(\mathbb{R}^p, M)$ of jets of order $k$ from $\mathbb{R}^p$ to $M$ with source 0 called by EHRESMANN [1] $p^k$-vitesse in $M$. Elements of $T^{(k)}M$ are equivalence classes of smooth mappings of $\mathbb{R}^p$ into $M$. Two mappings $\gamma$ and $\gamma'$ are equivalent if $D^n(f \circ \gamma)(0) = D^n(f \circ \gamma')(0)$ for each $C^\infty$-function $f$ on $M$ and each $n = (n_1, \ldots, n_p) \in \mathbb{N}^p$ such that $|n| = n_1 + \ldots + n_p \leq k$. The symbol $D^n g(0)$ is used to denote the partial derivative of a function $g$:

$$\mathbb{R}^p \to \mathbb{R} : (t_1, \ldots, t_p) \mapsto g(t_1, \ldots, t_p)$$

of orders $n_1, \ldots, n_p$ with respect to the arguments $t_1, \ldots, t_p$ respectively at $(t_1, \ldots, t_p) = (0, \ldots, 0)$. We denote by $j^k_0(\gamma)$ the jet of the mapping $\gamma$. For each $k \in \mathbb{N}$, there is the projection

$$\tau^{(k)} : T^{(k)}M \to M : j^k_0(\gamma) \mapsto \gamma(0)$$
and, if \( k' \leq k \), then there is the projection

\[ \rho_{(k')} : T^{(k)} M \rightarrow T^{(k')} M : j^k_0 (\gamma) \mapsto j^{k'}_0 (\gamma). \]

The manifold \( T^{(0)} M \) is identified with \( M \), and \( T^{(1)} M \) is the tangent bundle \( TM \) of \( M \). For each \( n \in \mathbb{N}^p \) such that \( |n| \leq k \) and each \( C^\infty \)-function \( f \) on \( M \) there is a \( C^\infty \)-function \( f^0 \) defined on \( T^{(k)} M \) by \( f^0 (j^k_0 (\gamma)) = D^n (f \circ \gamma) (0) \).

For each \( k \in \mathbb{N} \), we introduce an equivalence relation in the set of smooth mappings of \( \mathbb{R}^{p+1} \) into \( M \). Two mappings \( \chi \) and \( \chi' \) will be considered equivalent if \( D^{(r,n)} (f \circ \chi) (0) = D^{(r,n)} (f \circ \chi') (0) \) for each \( C^\infty \)-function \( f \) on \( M \), each \( n \in \mathbb{N}^p \) such that \( |n| \leq k \) and \( r = 0,1 \). The symbol \( D^{(r,n)} g (0) \) denotes the partial derivative of a function \( g : \mathbb{R}^n \rightarrow \mathbb{R}^n \) of orders \( r, n_1, \ldots, n_p \) with respect to the arguments \( s, t_1, \ldots, t_p \) respectively at \( (s, t_1, \ldots, t_p) = (0, 0, \ldots, 0) \). We denote the equivalence class of the mapping \( \chi \) by \( j^{(1,k)}_0 (\chi) \). The set of equivalence classes can be canonically identified with the tangent bundle \( TT^{(k)} M \) in such a way that

\[ \langle j^{(1,k)}_0 (\chi), df^0 \rangle = D^{(1,n)} (f \circ \chi) (0) \]

for each function \( f \) on \( M \) and each \( n \in \mathbb{N}^p \) such that \( |n| \leq k \) and also

\[ \tau_{T^{(k)} M} (j^{(1,k)}_0 (\chi)) = j^k_0 (\chi_0), \]

where \( \tau_{T^{(k)} M} : TT^{(k)} M \rightarrow T^{(k)} M \) is the tangent bundle projection, and \( \chi_0 \) is the mapping

\[ \chi_0 : \mathbb{R}^p \rightarrow M : (t_1, \ldots, t_p) \mapsto \chi_0 (0, t_1, \ldots, t_p). \]

The tangent mapping \( T \rho_{(k')} : TT^{(k)} M \rightarrow TT^{(k')} M \) is given by

\[ T \rho_{(k')} (j^{(1,k)}_0 (\chi)) = j^{(1,k')}_0 (\chi). \]

For each \( k \in \mathbb{N} \) and each \( m \in \mathbb{N}^p \) there is the mapping

\[ F_m : TT^{(k)} M \rightarrow TT^{(k)} M : j^{(1,k)}_0 (\chi) \mapsto j^{(1,k)}_0 (\chi_m), \]

where \( \chi_m \) is the mapping

\[ \chi_m : \mathbb{R}^{p+1} \rightarrow M : (s, t_1, \ldots, t_p) \mapsto \chi (s t^m, t_1, \ldots, t_p), \]

and \( t^m = t^m_1 \ldots t^m_p \). Diagrams

\[
\begin{array}{ccc}
TT^{(k)} M & \xrightarrow{F_m} & TT^{(k)} M \\
\tau_{T^{(k)} M} & \Downarrow & \tau_{T^{(k)} M} \\
T^{(k)} M & \rightarrow & T^{(k)} M
\end{array}
\]

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and
\[
TT^{(k)} M \xrightarrow{T} TT^{(k)} M
\]
\[
\downarrow T_p(k') \downarrow T_p(k)
\]
\[
TT^{(k')} M \xrightarrow{T} TT^{(k')} M
\]
are commutative.

For each \( a = 1, \ldots, p \) and each \( k \in \mathbb{N} \), there is the mapping
\[
T^a : T^{(k + 1)} M \to TT^{(k)} M : j_0^{k+1}(\gamma) \mapsto j_0^{(1, k)}(\gamma^a),
\]
where \( \gamma^a \) is the mapping
\[
\gamma^a : \mathbb{R}^{p+1} \to M : (s, t_1, \ldots, t_p) \mapsto \gamma(t_1, \ldots, t_a + s, \ldots, t_p)
\]
(1).

Diagrams
\[
\begin{array}{ccc}
T^{(k+1)} M & \xrightarrow{T^a} & TT^{(k)} M \\
\downarrow T_p(k+1) & & \downarrow T_p^{(k)} M \\
T^{(k)} M & = & T^{(k)} M
\end{array}
\]
and
\[
\begin{array}{ccc}
T^{(k+1)} M & \xrightarrow{T^a} & TT^{(k)} M \\
\downarrow T_p(k+1) & & \downarrow T_p^{(k)} M \\
T^{(k'+1)} M & \xrightarrow{T^a} & TT^{(k')} M
\end{array}
\]
are commutative.

2. Forms and derivations

Let \( \Omega^q_{(k)} \) denote the \( \mathbb{R} \)-linear space of \( q \)-forms on \( T^{(k)} M \), and let \( \Omega_{(k)} \) be the nonnegative graded linear space \( \{ \Omega^q_{(k)} \} \). The exterior differential \( d \) is a collection \( \{ d^q \} \) of linear mappings
\[
d^q : \Omega^q_{(k)} \to \Omega^{q+1}_{(k)}
\]
and the exterior product \( \wedge \) is a collection \( \{ \wedge^{(q, q')} \} \) of operations \( \Omega^q_{(k)} \times \Omega^{q'}_{(k)} \to \Omega^{q+q'}_{(k)} \). For each \( k' \leq k \) and each \( q \), there is the cotangent mapping \( \rho^*_{(k')} : \Omega^q_{(k')} \to \Omega^q_{(k)} \) corresponding to the mapping \( \rho_{(k')} : T^{(k)} M \to T^{(k')} M \) and, if \( k'' \leq k' \leq k \), then
\[
\rho^*_{{(k'')}} \circ \rho^*_{(k')} = \rho^*_{(k')}
\]
(1) The mappings \( T^a \) are related to the holonomic lift \( \lambda \) defined by KUMPERA [3].

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Hence \((\Omega^q_{(k)}, \rho^*_p)_{(k)}\) is a directed system. Let \(\Omega^q\) denote the direct limit of this system, and let \(\Omega\) be the graded linear space \(\{\Omega^q\}\). The underlying set of \(\Omega^q\) is the quotient set of \(\bigcup_k \Omega^q_{(k)}\) by the equivalence relation according to which two forms \(\mu \in \Omega^q_{(k)}\) and \(v \in \Omega^q_{(k)}\) are equivalent if \(k' \leq k\) and \(\mu = \rho^*_{(k)}(k) v\), or \(k' \geq k\) and \(v = \rho^*_{(k)}(k') \mu\). The exterior differential \(d\) and the exterior product \(\wedge\) extend in a natural way to the direct limits giving the graded linear space \(\Omega\) the structure of both a cochain complex and a commutative graded algebra. We write \(\mu \in \Omega^q_{(k)}\) for an element \(\mu\) of \(\Omega^q\) if \(\mu\) has a representative in \(\Omega^q_{(k)}\). This notation could be justified by identifying \(\Omega^q_{(k)}\) with the image of the canonical injection \(\Omega^q_{(k)} \rightarrow \Omega^q\). A collection \(a = \{a^q\}\) of linear mappings \(a^q : \Omega^q \rightarrow \Omega^{q+r} : \mu \rightarrow a^q \mu\) is called a graded linear mapping of degree \(r\). We write \(a\) instead of \(a^q\) if this can be done without causing any confusion. The exterior differential \(d\) is a graded linear mapping of degree \(1\).

**Definition 2.1.** A graded linear mapping \(a = \{a^q\}\) of degree \(r\) is called a derivation of \(\Omega\) of degree \(r\) if

\[ a(\mu \wedge v) = a \mu \wedge v + (-1)^{qr} \mu \wedge a v, \quad \text{where } q = \deg \mu. \]

The exterior differential \(d\) is a derivation of \(\Omega\) of degree \(1\). If \(a\) and \(b\) are derivations of \(\Omega\) of degrees \(r\) and \(s\) respectively, then

\[ [a, b] = \{a^{q+s} b^q - (-1)^r s b^q + r a^q\} \]

is a derivation of \(\Omega\) of degree \(r+s\) called the commutator of \(a\) and \(b\).

It follows from the general theory of derivations [2] that derivations of \(\Omega\) are completely characterized by their action on \(\Omega^0\) and \(\Omega^1\). In fact, a derivation is completely determined by its action on equivalence classes of \(f\) and \(df^*\) for each function \(f\) on \(M\) and each \(n \in \mathbb{N}^p\). Following FRÖLICHER and NİJENHUIS [2], we call a derivation \(a\) a derivation of type \(i_q\) if it acts trivially on \(\Omega^0\). We call \(a\) a derivation of type \(d_q\) if \([a, d] = 0\).

For each \(m \in \mathbb{N}^p\), each \(k \in \mathbb{N}\) and each \(q > 0\) there is a linear mapping

\[ i_{F_m} : \Omega^q_{(k)} \rightarrow \Omega^q_{(k)} : \mu \mapsto i_{F_m} \mu, \]

defined by

\[
\langle w_1 \wedge \ldots \wedge w_q, i_{F_m} \mu \rangle \\
= \langle F_m(w_1) \wedge w_2 \wedge \ldots \wedge w_q, \mu \rangle \\
+ \langle w_1 \wedge F_m(w_2) \wedge \ldots \wedge w_q, \mu \rangle + \ldots + \langle w_1 \wedge w_2 \wedge \ldots \wedge F_m(w_q), \mu \rangle,
\]

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where $w_1, ..., w_q$ are vectors in $TT^{(k)} M$ such that $\tau_{T^{(k)} M} (w_1) = ... = \tau_{T^{(k)} M} (w_q)$ and $F_m : TT^{(k)} M \to TT^{(k)} M$ is the mapping defined in Section 1. Due to commutativity of diagrams

$$
\begin{array}{c}
\Omega^q_{(k')} \xrightarrow{i_{F_m}} \Omega^q_{(k')}
\end{array}
$$

$$
\begin{array}{c}
\rho_{(k')} (k) \downarrow \downarrow \rho_{(k')} (k)
\end{array}
$$

the mappings $i_{F_m}$ extend to a derivation $i_{F_m}$ of $\Omega$ of type $i_*$ and degree 0. If $\mu \in \Omega^q_{(k)}$, then $i_{F_m} \mu \in \Omega^q_{(k)}$ and $i_{F_m} \mu = 0$ if $\mu \in \Omega^q_{(k)}$ and $m \geq k$.

For each $\alpha = 1, \ldots, p$, each $k \in \mathbb{N}$, and each $q \in \mathbb{N}$, there is a linear mapping

$$
\begin{array}{c}
i_{T^\alpha} : \Omega^q_{(k)} \to \Omega^q_{(k+1)} : \mu \mapsto i_{T^\alpha} \mu,
\end{array}
$$

defined by

$$
\langle w_1 \wedge \ldots \wedge w_q, i_{T^\alpha} \mu \rangle = \langle x \wedge u_1 \wedge \ldots \wedge u_q, \mu \rangle,
$$

where

$$
x = T^\alpha (v), \quad v = \tau_{T^{(k+1)} M} (w_1) = \ldots = \tau_{T^{(k+1)} M} (w_q),
$$

$$
u_1 = T \rho_{(k+1), (k)} (w_1), \ldots, u_q = T \rho_{(k+1), (k)} (w_q),
$$

and $T^\alpha : T^{(k+1)} M \to TT^{(k)} M$ is the mapping defined in Section 1. Due to commutativity of diagrams

$$
\begin{array}{c}
\Omega^{q+1}_{(k')} \xrightarrow{i_{T^\alpha}} \Omega^q_{(k'+1)}
\end{array}
$$

$$
\begin{array}{c}
\rho_{(k')} (k) \downarrow \downarrow \rho_{(k') + 1} (k+1)
\end{array}
$$

the mappings $i_{T^\alpha}$ extend to a derivation $i_{T^\alpha}$ of $\Omega$ of type $i_*$ and degree $-1$.

A derivation $d_{\mu^a}$ of $\Omega$ of type $d_*$ and degree 0 is defined by $d_{\mu^a} = [i_{T^\alpha}, d]$. If $\mu \in \Omega^q_{(k)}$, then $i_{T^\alpha} \mu \in \Omega^q_{(k+1)}$, and $d_{\mu} \mu \in \Omega^{q+1}_{(k+1)}$.

For each $\alpha = 1, \ldots, p$ let $e^\alpha$ denote the element $(e^\alpha_1, \ldots, e^\alpha_p)$ of $\mathbb{N}^p$ defined by $e^\alpha_\beta = 1$ if $\alpha = \beta$, and $e^\alpha_\beta = 0$ if $\alpha \neq \beta$. Let $\succeq$ denote the partial ordering relation in $\mathbb{N}^p$ defined by $(n_1, \ldots, n_p) \succeq (n'_1, \ldots, n'_p)$ if

$$
n_1 \geq n'_1, \ldots, n_{p-1} \geq n'_{p-1} \quad \text{and} \quad n_p \geq n'_p.
$$

For each $m \in \mathbb{N}^p$, let $m!$ denote $m_1! \ldots m_p!$. 

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PROPOSITION 2.1. — If $m \geq e^2$ then

$$[i_{F_m}, d_{T^r}] = \frac{m!}{(m-e^2)!} i_{F_{m-e^2}}, \quad \text{and} \quad [i_{F_m}, d_{T_2}] = 0$$

in all cases other than $m \geq e^2$.

Proof. — The commutator $[i_{F_m}, d_{T^r}]$ is a derivation and it is of type $i_{x}$ since it acts trivially on $\Omega$. It can be easily shown for each $n \in \mathbb{N}^p$ and each function $f$ on $M$ that $i_{F_m} df_n = (n!(n-m)!) df_{n-m}$ if $n \geq m$, and $i_{F_m} df_n = 0$ in all other cases. Also $d_{T^r} f_n = f_{n+e^2}$. It follows that

$$[i_{F_m}, d_{T^r}] df_n = \frac{m!}{(m-e^2)!} i_{F_{m-e^2}} df_n \quad \text{if} \quad m \geq e^2,$$

and $[i_{F_m}, d_{T^r}] df_n = 0$ in all cases other than $m \geq e^2$. This completes the proof since a derivation of type $i_{x}$ is completely determined by its action on equivalence classes of $df_n$ for each $f$ and each $n \in \mathbb{N}^p$.

PROPOSITION 2.2. — For each $\alpha, \beta = 1, \ldots, p$, $[d_{T^\alpha}, d_{T^\beta}] = 0$.

Proof. — Obvious.

3 The Lagrange complex $(\Lambda, \delta)$ (2)

Let $\tau = \{ \tau^\alpha \}$ be the graded linear mapping of $\Omega$ into $\Omega$ of degree 0 defined by $\tau^0 = 1$ and

$$\tau^\varphi \mu = -\sum_{m} (-1)^{|m|} (m!)^{-1} d_T^m i_{F_m} \mu,$$

where $q > 0$, $\mu \in \Omega^P_{(q)}$ and $d_T^m = (d_T^1)^{m_1} \ldots (d_T^p)^{m_p}$. The sum in the above definition contains all nonzero terms $(-1)^{|m|} (m!)^{-1} d_T^m i_{F_m} \mu$ since $i_{F_m} \mu = 0$ unless $|m| \leq k$. We write

$$\tau^\alpha = \sum_{m} (-1)^{|m|} (m!)^{-1} d_T^m i_{F_m},$$

without explicitly restricting the summation range which is understood to be wide enough to include in the sum all nonzero terms when $\tau^\alpha$ is applied to an element of $\Omega^\varphi$.

PROPOSITION 3.1. — If $q > 0$, then $\tau^\alpha d_{T^\alpha} = 0$ for each $\alpha = 1, \ldots, p$.

(2) For definitions of algebraic topology terms used in this and the following sections, see reference [5].

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Proof:

\[ \tau^q d_T = \frac{1}{q} \sum_m (-1)^m m!^{-1} d_T^m i_{f_m} d_T = \]

\[ = \frac{1}{q} \sum_m (-1)^m m!^{-1} (d_T^m + e^x i_{f_m} + d_T^m [i_{f_m}, d_T]) = \]

\[ = \frac{1}{q} \sum_m (-1)^m m!^{-1} d_T^m + e^x i_{f_m} \]

\[ + \frac{1}{q} \sum_{m \geq e^x} (-1)^m (m-e^x)!^{-1} d_T^m i_{f_{m-e^x}} = 0. \]

It follows from proposition 3.1, that \( \tau\tau = \tau \) and \( \tau d\tau = d\tau \).

**Proposition 3.2.** The graded linear mapping \( \tau d = \{ \tau^{q+1} d^q \} \) is a differential of degree 1.

**Proof.** \( \tau d\tau = d\tau = 0 \) and degree \( (\tau d) = \) degree \( \tau + \) degree \( d = 1 \).

We introduce the graded linear space \( \Lambda = \{ \Lambda^q \} \), where \( \Lambda^q = \text{im } \tau^q \).

The differential \( \tau d \) can be restricted to \( \Lambda \) due to \( \tau d\tau = \tau d \).

The restriction of \( \tau d \) to \( \Lambda \) is a differential of degree 1 denoted by \( \delta \).

**Definition 3.1.** The differential \( \delta = \{ \delta^q \} \) is called the Lagrange differential, and the cochain complex \( \{ \Lambda^q, \delta^q \} \) is called the Lagrange complex.

**Theorem 3.1 (\( \delta \)-Poincaré lemma).** If the manifold \( M \) is contractible then the Lagrange complex \( \{ \Lambda^q, \delta^q \} \) is acyclic for \( q > 0 \).

Let \( R \) denote the subspace of \( \Lambda^0 = \Omega^0 \) consisting of equivalence classes of constant functions and let \( \gamma : G \rightarrow \Lambda^0 \) be the canonical injection of the subspace \( G = R \oplus (d_{T_1}(\Omega^0) + \ldots + d_{T_p}(\Omega^0)) \).

**Theorem 3.2.** The mapping \( \gamma : G \rightarrow \Lambda^0 \) is an augmentation of the Lagrange complex and the sequence

\[ 0 \rightarrow G \xrightarrow{\gamma} \Lambda^0 \xrightarrow{\delta^0} \Lambda^1 \xrightarrow{\delta^1} \ldots \xrightarrow{\delta^{q-1}} \Lambda^q \xrightarrow{\delta^q} \ldots \]

is a resolution of \( G \).

We give proofs of the two theorems in the following section after having constructed a resolution of the graded linear space \( \Lambda' = \{ \Lambda^q \}_{q \geq 0} \).

**4. A resolution of \( \Lambda' \)**

Let \( K \) be the simplicial complex with vertices \( 1, \ldots, p \), and let \( \Lambda_r(K) \) denote the free abelian group generated by the ordered \( r \)-simplexes of \( K \).
We introduce a bigraded linear space $\Phi = \{ \Phi^q_r \}$, where $\Phi^q_r = \Delta_{r-1} (K) \otimes \Omega^q$ for $r > 0$, $\Phi^q_0 = \Omega^q$, and $\Phi^q_r = 0$ for $r < 0$. Elements of $\Phi^q_r$ are said to be of bidegree $(q, r)$. The exterior differential in $\Omega$ is extended to a bigraded linear mapping $d = \{ d^q \}$ of bidegree $(1, 0)$ by the formula

$$d^q((\alpha_1, \ldots, \alpha_r) \otimes \mu) = (\alpha_1, \ldots, \alpha_r) \otimes d\mu,$$

where $(\alpha_1, \ldots, \alpha_r)$ is an ordered $r+1$-simplex and $\mu \in \Omega^q$. A bigraded linear mapping $\partial = \{ \partial^q_r \}$ of bidegree $(0, -1)$ is defined by

$$\partial^q_r((\alpha_1, \ldots, \alpha_r) \otimes \mu) = \sum_{i} (-1)^{i-1} (\alpha_1, \ldots, \alpha_i, \ldots, \alpha_r) \otimes d\tau_i \mu.$$

For each fixed $r$, $\{ \Phi^q_r, \partial^q_r \}$ is a cochain complex, and for each fixed $q$, $\{ \Phi^q_r, \partial^q_r \}$ is a chain complex. Since $\partial^q_{r+1} d^q_r = d^q_{r-1} \partial^q_r$, for each fixed $r$ the collection $\{ \partial^q_r : \Phi^q_r \to \Phi^q_{r-1} \}$ is a cochain mapping, and for each fixed $q$ the collection $\{ d^q_r : \Phi^q_r \to \Phi^q_{r+1} \}$ is a chain mapping.

**Proposition 4.1.** For each fixed $q > 0$ the chain complex $\{ \Phi^q_r, \partial^q_r \}$ is acyclic for $r > 0$.

**Proof.** For each $\alpha = 1, \ldots, r$, let a graded linear mapping

$$\sigma^q_\alpha = \{ \sigma^q_\alpha : \Omega^q \to \Omega^q \}$$

be defined by $\sigma^q_\alpha = 0$ and

$$\sigma^q_\alpha = -\frac{1}{q} \sum_{m \in I, \beta > 0} (\alpha_1, \ldots, \alpha_r) \otimes \sigma^q_\alpha \mu,$$

$I_\alpha = \{ m \in \mathbb{N}^p; m_\beta > 0, m_\beta = 0 \text{ for } \beta > \alpha \}$ and the summation range is governed by a convention similar to the one used in the definition of $\tau$ in Section 3. From Proposition 2.1, it follows easily for $q > 0$ that $\sigma^q_\alpha d\tau_\alpha = 0$ if $\beta < \alpha$, $\sigma^q_\alpha d\tau_\beta = 1 - \sum \sigma^q_\alpha d\tau_\beta$ if $\beta > \alpha$. A bigraded linear mapping $D = \{ D^q_r \}$ is defined by $D^q_r \mu = \sum_{\beta < \alpha} (\beta) \otimes \sigma^q_\beta \mu$ and

$$D^q_r((\alpha_1, \ldots, \alpha_r) \otimes \mu) = \sum_{\alpha_1 < \alpha_2 < \ldots < \alpha_r} (\alpha_1, \ldots, \alpha_r) \otimes \sigma^q_\beta \mu,$$

where $\mu \in \Omega^q$ and $\alpha_1 < \alpha_2 < \ldots < \alpha_r$. Relations $\partial^q_{r+1} D^q_r + D^q_{r-1} \partial^q_r = 1$ for $r > 0$, $q > 0$ are readily verified using the above stated properties of $\sigma^q_\alpha$. It follows that for each fixed $q > 0$ the graded mapping $D^q = \{ D^q_r \}$ defines a chain contraction of $\{ \Phi^q_r, \partial^q_r \}$ for $r > 0$. Hence $\{ \Phi^q_r, \partial^q_r \}$ is acyclic for $r > 0$.

**Proposition 4.2.** For each $q > 0$, the mapping $\tau^q : \Phi^q_0 \to \Lambda^q$ is an augmentation of the chain complex $\{ \Phi^q_r, \partial^q_r \}$ and the sequence

$$\ldots \to \Phi^q_r \to \Phi^q_{r-1} \to \ldots \to \Phi^q_0 \to \Lambda^q \to 0$$

is a resolution of $\Lambda^q$. 

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Proof. — The mapping \( \tau^q : \Omega^q \to \Lambda^q \) is an epimorphism, and \( \tau^q \partial^q_1 = 0 \) follows from Proposition 3.1. Further \( \tau^q + \partial^q_1 D^q_0 = 1 \), where \( D^q_0 \) is the mapping defined in the proof of Proposition 4.1. Hence \( \tau^q \mu = 0 \) implies \( \mu = \partial^q_1 D^q_0 \mu \) for each \( \mu \in \Omega^q \). It follows that \( \ker \tau^q = \text{im} \partial^q_1 \).

Proof of Theorems 3.1 and 3.2. — We define a nonnegative graded linear space \( C = \{ C_r \} \) by \( C_0 = \mathbb{R} \) and \( C_r = \Delta_{r-1}(K) \otimes \mathbb{R} \) for \( r > 0 \), and a collection \( \eta = \{ \eta_r : C_r \to \Phi^0_r \} \) by \( \eta_r = 1 \otimes \eta_0 \), where \( \eta_0 : \mathbb{R} \to \Omega^0 \) is the canonical injection of the space \( \mathbb{R} \subset \Omega^0 \) of equivalence classes of constant functions identified with the field \( \mathbb{R} \) of constants. If the manifold \( M \) is contractible, then all rows except the bottom row of the commutative diagram

\[
0 \to C_p \overset{\eta_p}{\longrightarrow} \Phi^0_p \overset{d^0_p}{\longrightarrow} \Phi^1_p \overset{d^1_p}{\longrightarrow} \cdots \overset{d^q_p}{\longrightarrow} \Phi^q_p \overset{d^q_p}{\longrightarrow} \cdots
\]

are known to be exact and all columns for \( q > 0 \) are exact. For each \( q > 0 \), the top statement in the sequence

\[
\ker (\partial^q_{p+1} d^q_p + \partial^q_{p+1} d^q_p) = \text{im} \partial^q_{p+1} d^q_p + \partial^q_{p+1} d^q_p,
\]

is true, and each of the remaining statements follows from the one immediately above. Hence the bottom statement is true. The same holds for \( q = 0 \) if the bottom statement is replaced by

\[
\ker (\tau^q + 1 d^q_0) = \eta_0 \otimes \partial^q_0.
\]
If \( q > 0 \) and \( \mu \) is an element of \( \Lambda^q \subset \Omega^q \), then \( \tau^q \mu = \mu \), and \( \delta^q \mu = \tau^{q+1} d_0 \mu \).

If \( \delta^q \mu = 0 \), then there are elements \( \kappa \in \Phi^{q-1}_0 \) and \( \lambda \in \Phi^q_1 \) such that

\[
\mu = d_0^{q-1} \kappa + \partial_1^q \lambda.
\]

It follows that

\[
\mu = \tau^q \mu = \tau^q d_0^{q-1} \kappa = \tau^q d_0^{q-1} \tau^{q-1} q = \delta^{q-1} \tau^{q-1} q.
\]

Hence \( \text{ker} \, \delta^q = \text{im} \, \delta^{q-1} \) and the Lagrange complex is acyclic for \( q > 0 \).

We note that \( \delta^0 = \tau^1 d_0^0 \) and

\[
G = R \otimes (d_{\tau_1}(\Omega^0) + \ldots + d_{\tau_p}(\Omega^0)) = \text{im} \chi_0 \otimes \text{im} \delta^0.
\]

Hence \( \text{ker} \, \delta^0 = G \). It follows that the sequence

\[
0 \to G \xrightarrow{\gamma} \Lambda^0 \xrightarrow{\delta^0} \Lambda^1 \xrightarrow{\delta^1} \ldots \xrightarrow{\Lambda^q \delta^q} \ldots
\]

is exact.

5. Applications of the \( \delta \)-Poincaré lemma in the calculus of variations

A smooth mapping \( \chi : R^{p+1} \to M : (s, t_1, \ldots, t_p) \mapsto \chi(s, t_1, \ldots, t_p) \) will be called a homotopy. For each \( s \in R \), we denote by \( \chi_s \) the mapping

\[
\chi_s : R^p \to M : (t_1, \ldots, t_p) \mapsto \chi(s, t_1, \ldots, t_p).
\]

The mapping \( \gamma = \chi_0 \) will be called the base of the homotopy \( \chi \). We say that the homotopy \( \chi \) is constant on \( A \subset R^p \) if \( \chi(s, t_1, \ldots, t_p) = \chi(0, t_1, \ldots, t_p) \) for each \( s \in R \) and each \((t_1, \ldots, t_p) \in A\). For each mapping

\[
\varphi : R^p \to M : (t_1, \ldots, t_p) \mapsto \varphi(t_1, \ldots, t_p),
\]

we denote by \( \varphi^{(k)} \) the mapping

\[
\varphi^{(k)} : R^p \to T^{(k)} M : (t_1, \ldots, t_p) \mapsto j^{(k)}(\varphi).
\]

For each homotopy \( \chi \), we denote by \( \chi^{(k)} \) the mapping

\[
\chi^{(k)} : R^p \to TT^{(k)} M : (t_1, \ldots, t_p) \mapsto j^{(1, k)}(\chi),
\]

where \( j^{(1, k)}(\chi) \) is a jet-like object similar to \( j^{(1, k)}_0(\chi) \) defined in terms of partial derivatives at \((0, t_1, \ldots, t_p)\) instead of \((0, 0, \ldots, 0)\) and identified with an element of \( TT^{(k)} M \).

Each element \( L \in \Omega^0_0 \) gives rise to a family of functions

\[
\gamma \mapsto \int_{\nu} L \circ \gamma^{(k)},
\]

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defined on the set of smooth mappings of $\mathbb{R}^p$ into $M$ for each domain $V \subset \mathbb{R}^p$.

**Definition 5.1.** A mapping $\gamma : \mathbb{R}^p \rightarrow M$ is called an extremal of the family of functions

$$\gamma \mapsto \int_V L \circ \gamma^{(k)}$$

if

$$\frac{d}{ds} \int_V L \circ \gamma_s^{(k)} \bigg|_{s=0} = 0,$$

for each domain $V \subset \mathbb{R}^p$ and each homotopy $\chi$ with base $\gamma$ constant on the boundary $\partial V$ of $V$.

**Definition 5.2.** A form $\lambda \in \Omega^1_{(k)}$ is called an Euler-Lagrange form associated with $L \in \Omega^0_{(k)}$ if $i_{F_m} \lambda = 0$ for each $m > 0$ and if

$$\int_V \langle \chi', dL \rangle = \int_V \langle \chi^{(k)}', \lambda \rangle$$

for each domain $V \subset \mathbb{R}^p$ and each homotopy $\chi$ constant on $\partial V$.

It is clear from the definition of $F_m$ that if $\lambda \in \Omega^1_{(k)}$ satisfies $i_{F_m} \lambda = 0$ for each $m > 0$, then $\lambda$ can be interpreted as a mapping $\lambda : T^{(k)} M \rightarrow T^* M$. If $\lambda$ is an Euler-Lagrange form associated with $L$ then

$$\frac{d}{ds} \int_V L \circ \gamma_s^{(k)} \bigg|_{s=0} = \int_V \langle \chi^{(k)}', dL \rangle$$

$$= \int_V \langle \chi^{(k)}', \lambda \rangle$$

$$= \int_V \langle \chi^{(0)}', \lambda \circ \gamma^{(k)} \rangle,$$

for each homotopy $\chi$ with base $\gamma$ constant on $\partial V$. It follows that $\gamma : \mathbb{R}^p \rightarrow M$ is an extremal of the family

$$\gamma \mapsto \int_V L \circ \gamma^{(k)},$$

if, and only if, $\gamma$ satisfies the equation $\lambda \circ \gamma^{(k)} = 0$ called the Euler-Lagrange equation.

We show that $\lambda = \delta^0 L$ is the unique Euler-Lagrange form associated with $L \in \Omega^0$. We also show that $i_{F_m} \lambda = 0$ for each $m > 0$ means that $\lambda \in \Omega^1$ is in $\Lambda^1$. These statements imply applications of the $\delta$-Poincaré lemma. A form $\lambda \in \Omega^1$ is an Euler-Lagrange form if, and only if, $\lambda \in \Lambda^1$ and $\delta^1 \lambda = 0$. Euler-Lagrange forms associated with two elements $L$ and $L'$ of $\Omega^0$ are the same if, and only if, $L' - L \in \mathbb{R} \oplus (d_{T^1} \Omega^0) + \ldots + d_{T^p} \Omega^0)$.  

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PROPOSITION 5.1. - A form \( \lambda \in \Omega^1 \) belongs to \( \Lambda^1 \) if, and only if, \( i_{F_m} \lambda = 0 \) for each \( m > 0 \).

Proof. - If \( i_{F_m} \lambda = 0 \) for each \( m > 0 \), then

\[ \tau^1 \lambda = \sum_m (-1)^m (m!)^{-1} d_{F_m}^m i_{F_m} \lambda = i_{F_0} \lambda = \lambda. \]

Hence \( \lambda \in \text{im} \tau^1 = \Lambda^1 \). From Proposition 2.1, it follows that

\[ i_{F_{e^a}} d_{F_{e^a}}^m = d_{F_{e^a}}^m i_{F_{e^a}} + (m!/(m-e^a)!) d_{F_{e^a}}^{m-e^a} i_{F_{e^a}} \]

if \( m \geq e^a \) and \( i_{F_{e^a}} d_{F_{e^a}}^m = d_{F_{e^a}}^m i_{F_{e^a}} \) in all other cases. Since \( i_{F_m} \mu = i_{F_{m+n}} \mu \) for each \( \mu \in \Omega^1 \), it follows that

\[ i_{F_{e^a}} \tau^1 = \sum_m (-1)^m (m!)^{-1} i_{F_{e^a}} d_{F_{e^a}}^m i_{F_m} = \sum_m (-1)^m (m!)^{-1} d_{F_{e^a}}^m i_{F_{m+n}} + \sum_{m > e^a} (-1)^m (m-e^a)!^{-1} d_{F_{e^a}}^{m-e^a} i_{F_m} = 0. \]

Consequently, \( i_{F_m} \tau^1 = 0 \) for each \( m > 0 \), and if \( \lambda \in \Lambda^1 \) then \( i_{F_m} \lambda = 0 \) for each \( m > 0 \).

PROPOSITION 5.2. - The space \( \Omega^1 \) is the direct sum of \( \Lambda^1 \) and \( d_{T^1} (\Omega^1) + \ldots + d_{T^P} (\Omega^1) \).

Proof. - Let \( \mu \) be an element of \( \Omega^1 \). Then \( \mu = \lambda + \nu \), where \( \lambda = \tau^1 \mu \in \Lambda^1 \), and

\[ \nu = -\sum_{m > 0} (-1)^m (m!)^{-1} d_{T_{e^a}}^m i_{T_{e^a}} \mu \in d_{T^1} (\Omega^1) + \ldots + d_{T^P} (\Omega^1). \]

It follows from \( \tau^1 \tau^1 = \tau^1 \) and \( d_{T^1} = 0 \) that this decomposition of \( \mu \) into elements of \( \Lambda^1 \) and \( d_{T^1} (\Omega^1) + \ldots + d_{T^P} (\Omega^1) \) is unique.

PROPOSITION 5.3. - Let \( \mu \) be an element of \( \Omega^1_{(k)} \). Then

\[ \int_V \langle \chi^{(k)}, \mu \rangle = 0, \]

for each domain \( V \subset R^n \) and each homotopy \( \chi : R^{p+1} \rightarrow M \) constant on \( \partial V \)

if, and only if, \( \mu \in d_{T^1} (\Omega^1) + \ldots + d_{T^P} (\Omega^1) \).

Proof. - If \( \mu = \sum \omega^a \) then

\[ \int_V \langle \chi^{(k)}, \mu \rangle = \sum \int_V \frac{\partial}{\partial r^a} \langle \chi^{(k)}, \omega^a \rangle = \sum \int_{\partial V} n^a \langle \chi^{(k)}, \omega^a \rangle, \]
where \( n \) are the components of the normal vector. If \( \chi \) is constant on \( \partial V \), then

\[
\int_{\partial V} \langle \chi^{(k)}, \mu \rangle = 0.
\]

Let \( \mu = \lambda + v \) be the unique decomposition of \( \mu \in \Omega^1 \) used in the proof of proposition 5.2. If \( \int_{\partial V} \langle \chi^{(k)}, \mu \rangle = 0 \), then

\[
\int_{\partial V} \langle \chi^{(k)}, \gamma \rangle = \int_{\partial V} \langle \chi^{(0)}, \lambda \circ \gamma^{(k)} \rangle = 0,
\]

where \( \gamma \) is the base of \( \chi \), and \( \lambda \) is interpreted as a mapping \( \lambda : T^{(k)} M \to T^* M \).

It follows that \( \lambda = 0 \) and \( \mu = v \). Hence \( \mu \in d_{T^1} (\Omega^1) + \ldots + d_{T^p} (\Omega^1) \).

**Corollary.** — If \( L \) is an element of \( \Omega^0 \), then \( \lambda = \delta^0 L \) is the unique element of \( \Lambda^1 \) such that \( dL - \lambda \in d_{T^1} (\Omega^1) + \ldots + d_{T^p} (\Omega^1) \). It follows that \( \lambda \) is the unique Euler-Lagrange form associated with \( L \).

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