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ON A ζ FUNCTION RELATED TO THE CONTINUED FRACTION TRANSFORMATION (1)

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ABSTRACT. — We consider Ruelle's generalized ζ function for the continued fraction transformation T of the unit interval. In spite of the fact that the number N_m of periodic points of period m of the transformation T is infinite for every m, it is shown that the function ζ can be defined for a certain class of complex valued functions φ and can be extended to a meromorphic function in the whole complex plane. It further turns out to be meromorphic in the function φ.

Introduction

In two papers ([5], [6]), RUELLE introduced a generalized ζ function for a transformation f of a set M. This ζ function has as its argument a function $\tilde{\varphi}$ defined on the set M and is defined as follows: let M be any set and $f : M \to M$ a transformation of M, let further $\tilde{\varphi} : M \to \mathbf{C}$ be a complex valued function on M. Then consider the formal expression

$$Z(\tilde{\varphi}) = \exp \sum_{n=1}^{\infty} \frac{1}{n} \sum_{\xi \in \mathrm{Fix}\, f^n} \prod_{k=0}^{n-1} \tilde{\varphi}(f^k \xi)$$

where $\mathrm{Fix}\, f^n$ is the set of fixed points of the mapping f^n. Taking for $\tilde{\varphi}$ the constant function $\tilde{\varphi} = z$, $z \in \mathbf{C}$, we get the well known Artin-Mazur [1] ζ function

$$\zeta(z) = \exp \sum_{n=1}^{\infty} \frac{z^n}{n} N_n(f), \tag{1}$$

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where $N_n(f)$ denotes the number of fixed points of the map f^n. Setting on the other hand $\tilde{\varphi} = z\varphi$, where $\varphi : M \to \mathbf{C}$ is another complex valued function on M we get

$$(2) \qquad \zeta(z, \varphi) = \exp \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{\xi \in \mathrm{Fix}\, f^n} \prod_{k=0}^{n-1} \varphi(f^k \xi).$$

In the above mentioned papers ([5], [6]) meromorphy of this function in the whole complex z-plane and in φ for real analytic expanding endomorphisms of a compact manifold M has been proven.

It is clear from the definition (1) of the Artin-Mazur function that it is defined only for mappings f such that $N_m(f) < \infty$ for all m. On the other hand, it turns out that Ruelle's ζ function (2) can be defined also in the case where $N_m(f) = \infty$ for some m or even for all m, as we will show in the case of the continued fraction transformation. This arises from the fact that one can choose the function φ in such a way that the infinite sums over the set of fixed points of f^n exist and do not increase to rapidly. To be more precise we show the following:

Let $[0, 1)$ be the half open unit interval and consider the continued fraction transformation $T : [0, 1) \to [0, 1)$ defined by [2] (chap. 1, sect. 4):

$$(3) \qquad Tx = \begin{cases} \dfrac{1}{x} - \left[\dfrac{1}{x}\right] & \text{for} \quad x \neq 0, \\ 0 & \text{for} \quad x = 0. \end{cases}$$

The numbers $N_m(T)$ for this transformation are infinite for every $m \geqslant 1$, because the set $\mathrm{Fix}(T^m)$ is given by

$$\mathrm{Fix}(T^m) = \{0, [i_1, \ldots, i_m], i_k \in \mathbf{Z}^+, \forall k = 1, \ldots, m\},$$

where $[i_1, \ldots, i_m]$ denotes the infinite periodic continued fraction

$$x = \frac{1|}{|i_1} + \frac{1|}{|i_2} + \ldots + \frac{1|}{|i_m} + \frac{1|}{|i_1} + \ldots$$

with entries $i_1, \ldots, i_m$. Then we can show the following:

Let $[i_1, \ldots, i_n]$ be the infinite periodic continued fraction with entries $i_1, \ldots, i_n$. Let φ be a complex valued analytic function in the domain $D_2 = \{z \in \mathbf{C}; |z-1| < 3/2\}$ which is continuous on clos D_2 and which vanishes at $z = 0$ like z^2. Define the numbers

$$a_n = \sum_{i_1=1}^{\infty} \ldots \sum_{i_n=1}^{\infty} \prod_{k=1}^{n} \varphi([i_k, \ldots, i_n, i_1, \ldots, i_{k-1}]).$$

Then the function

$$\zeta(z, \varphi) = \exp \textstyle\sum_{n=1}^{\infty} \dfrac{z^n}{n} a_n$$

extends to a meromorphic function in the whole z plane, the poles of which if arranged by increasing modulus and according to multiplicity tend to infinity exponentially fast. $\zeta(z, \varphi)$ is also meromorphic in φ from the complex Banach space $B_2(D_2)$.

For the special case $\varphi(z) = z^2$, we can show that the first pole of $\zeta(z, \varphi)$ is at the point $z = 1$ and that this is a simple pole and that there is no other pole in the disk $|z| \leqslant 1$.

To prove this, we apply the techniques developed by RUELLE in [6].

1. The Transfer Matrix $\mathscr{L}$ for the mapping T

Let be $x \in (0, 1)$. Then we have

$$T^{-1} x = \left\{ y \in (0, 1);\ y = \frac{1}{x+n},\ n \in \mathbf{Z}^+ \right\}.$$

Denote by T_n, $n \in \mathbf{Z}^+$, the mapping $T_n : (0, 1) \to (0, 1)$ defined by

$$T_n x = \frac{1}{x+n}.$$

This mapping can be extended to the domain $D_2 = \{ z \in \mathbf{C};\ |z-1| < 3/2 \}$ and is there a holomorphic mapping. If one defines the domain D_1 by

$$D_1 = \{ z \in \mathbf{C};\ |z-1| < 1 \}$$

then one finds

$$T_n : D_2 \to D_1 \tag{4}$$

and $\operatorname{clos} T_n D_2 \subseteq \operatorname{clos} D_1$.

Consider next the Banach space $B(D_2)$ of all holomorphic functions on the domain D_2 which are continuous on $\operatorname{clos} D_2$ together with the natural topology defined by the sup norm. Let be $B_2(D_2)$ the following subspace of $B(D_2)$:

$$B_2(D_2) = \{ f \in B(D_2);\ \exists g \in B(D_2) : f(z) = z^2 g(z) \}.$$

If φ is now an element of the space $B_2(D_2)$, we define an operator $\mathscr{L}$ on the space $B_2(D_2)$:

$$(\mathscr{L}f)(z) := \varphi(z)\sum_{n=1}^{\infty} f(T_n z). \tag{5}$$

In fact it is easy to show that $\mathscr{L}$ can even be extended to a linear operator from the space $\mathscr{H}_2(D_2)$ into he space $B_2(D_2)$, where

$$\mathscr{H}_2(D_2) = \{f \in \mathscr{H}(D_2);\ \exists\, g \in \mathscr{H}(D_2) : f(z) = z^2 g(z)\}$$

and $\mathscr{H}(D_2)$ is the space of holomorphic functions on D_2 together with the compact open topology which makes it a Fréchet space. Because $\mathscr{H}_2(D_2)$ with the induced topology is a closed subspace of the nuclear space $\mathscr{H}(D_2)$ [3] it is also a nuclear space [7]. One can then show the following lemma.

LEMMA. — *$\mathscr{L} : \mathscr{H}_2(D_2) \to B_2(D_2)$ is a bounded linear operator and therefore nuclear.*

Proof. — We only have to show that there exists a neighbourhood V of zero in $\mathscr{H}_2(D_2)$ which is mapped onto a bounded set in $B_2(D_2)$. Let

$$K = \left\{ z \in \mathbf{C};\ |z-1| \leqslant \frac{5}{4} \right\}.$$

Then define

$$V_M(0) := \{f \in \mathscr{H}_2(D_2)\ ;\ \sup_{z \in K} |f(z)| < M\}.$$

For $f \in \mathscr{H}_2(D_2)$ there exists a $g \in \mathscr{H}(D_2)$ such that $f(z) = z^2 g(z)$. If $f \in V_M(0)$ then because of the maximum principle we get

$$\sup_{z \in K} |g(z)| \leqslant 16 M.$$

Therefore

$$\sup_{z \in \bar{D}_2} |(\mathscr{L}f)(z)| \leqslant \sup_{z \in \bar{D}_2} |\varphi(z)| \sum_{n=1}^{\infty} \left(\frac{1}{n-1/2} \right)^2 16 M$$

for all $f \in V_M(0)$ and this shows that $\mathscr{L}\, V_M(0)$ is a bounded set in $B_2(D_2)$. This then shows that $\mathscr{L}$ is a nuclear operator of order 0 ([6], [3]).

If we now compose $\mathscr{L}$ with the continuous injection of $B_2(D_2)$ into $\mathscr{H}_2(D_2)$ we see that $\mathscr{L}$ is a nuclear mapping of order 0 of the Banach space $B_2(D_2)$ [6], and corresponds to a unique Fredholm kernel of order 0 because every nuclear space has the approximation property [7]. Therefore one has [6] :

$$\operatorname{Tr} \mathscr{L} = \sum_{\{i\}} \lambda_i \quad \text{and} \quad \det(1 - z\mathscr{L}) = \prod_{\{i\}} (1 - z\lambda_i)$$

where λ_i are the eigenvalues of $\mathscr{L}$ repeated according to their algebraic multiplicity. One furthermore knows that the Fredholm determinant $\det(1-z\mathscr{L})$ is an entire function of order 0 in z [6].

2. The spectrum of the operators $\mathscr{L}_n$

Let us write the operator $\mathscr{L}$ in the following form $\mathscr{L}=\sum_{n=1}^{\infty}\mathscr{L}_n$, where $(\mathscr{L}_n f)(z)=\varphi(z)f(T_n z)$ with $f,\ \varphi\in B_2(D_2)$.

We want to determine the spectrum of the nuclear operator $\mathscr{L}_n$ in the space $B_2(D_2)$. To do this, we apply a method used by KAMOWITZ [4] to determine the spectrum of certain composition operators in the disc algebra $B(D)$.

Because $T_n:\bar{D}_2\to\bar{D}_1$ and $\bar{D}_1\subset D_2$, we get from a lemma of RUELLE [6] that T_n has exactly one fixed point z^* in D_2 and that $|T_n'(z^*)|<1$. Consider then the eigenvalue equation

$$(\mathscr{L}_n f)(z)=\lambda f(z)=\varphi(z)f(T_n z).$$

At the point $z=z^*$ this gives $\lambda f(z^*)=\varphi(z^*)f(z^*)$.

If therefore $f(z^*)\neq 0$, we get $\lambda=\varphi(z^*)$. If $f(z^*)=0$ one looks at the differentiated equation

$$\lambda f'(z)=\varphi'(z)f(T_n z)+\varphi(z)\,T_n'(z)f'(T_n z).$$

For $z=z^*$, we then get $\lambda f'(z^*)=\varphi(z^*)\,T_n'(z^*)f'(z^*)$ and therefore $\lambda=\varphi(z^*)\,T_n'(z^*)$ if $f'(z^*)\neq 0$. Continuing this way one finds

$$\lambda=\varphi(z^*)(T_n'(z^*))^k \text{ for some } k\in\mathbf{Z}^+.$$

Therefore $\sigma(\mathscr{L}_n)\subset\{0,\ \varphi(z^*)\,T_n'(z^*)^k,\ k=0,1,\ldots\}$.

To see that every point different from 0 of the above set is really an eigenvalue one only has to show that the mapping

$$\mathscr{L}_n-\lambda_k 1,\qquad \lambda_k=\varphi(z^*)\,T_n'(z^*)^k$$

has no continuous inverse in the space $B_2(D_2)$ because then $\lambda_k\in\sigma(\mathscr{L}_n)$ and because $\mathscr{L}_n$ is compact λ_k is also an eigenvalue. Now one can see very easily that the equations $(\mathscr{L}_n-\lambda_k)f(z)=(z-z^*)^k z^2$ do not have a solution in the space $B_2(D_2)$. This also gives at once that the algebraic multiplicity of λ_k is always one [4]. Therefore the trace of the nuclear operator $\mathscr{L}_n$ is given by

$$\operatorname{Tr}\mathscr{L}_n=\sum_{k=0}^{\infty}\varphi(z^*)\,T_n'(z^*)^k=\frac{\varphi(z^*)}{1-T_n'(z^*)}$$

because $|T_n'(z^*)| < 1$. This is exactly the result we would have got applying a formula due to RUELLE [6]. By using this result, we get for the trace of the operator $\mathscr{L}$:

$$\operatorname{tr}\mathscr{L} = \sum_{n=1}^{\infty} \frac{\varphi([n])}{1+[n]^2},$$

where $[n] = \frac{1|}{|n} + \frac{1|}{|n} + \ldots$ is the unique fixed point of T_n in the domain D_2.

We can proceed in the same way to determine the trace of the nuclear operator $\mathscr{L}^n$, and get

$$\text{(6)}\quad \operatorname{tr}\mathscr{L}^n = \sum_{i_1=1}^{\infty}\ldots\sum_{i_n=1}^{\infty} \frac{\prod_{k=1}^{n}\varphi([i_k, \ldots, i_n, i_1, \ldots, i_{k-1}])}{1-(-1)^n\prod_{k=1}^{n}[i_k, \ldots, i_n, i_1, \ldots, i_{k-1}]^2},$$

where $[i_1, \ldots, i_n]$ is the infinite periodic continued fraction with entries $i_1, \ldots, i_n$.

What we have done above for the operator $\mathscr{L}$ we can repeat in the same way for the operator $\mathscr{L}_1$, where

$$(\mathscr{L}_1 f)(z) = \varphi(z)\sum_{n=1}^{\infty} T_n'(z) f(T_n z), \quad \text{where } T_n'(z) = -\left(\frac{1}{z+n}\right)^2.$$

We get for the trace of $\mathscr{L}_1^n$:

$$\text{(7)}\quad \operatorname{tr}\mathscr{L}_1^n = (-1)^n\sum_{i_1=1}^{\infty}\ldots\sum_{i_n=1}^{\infty}$$
$$\times\frac{\prod_{k=1}^{n}\varphi([i_k, \ldots, i_n, i_1, \ldots, i_{k-1}])[i_k, \ldots, i_n, i_1, \ldots, i_{k-1}]^2}{1-(-1)^n\prod_{k=1}^{n}[i_k, \ldots, i_n, i_1, \ldots, i_{k-1}]^2}.$$

Combining (6) and (7) gives

$$\operatorname{tr}(\mathscr{L}^n - \mathscr{L}_1^n) = \sum_{i_1=1}^{\infty}\ldots\sum_{i_n=1}^{\infty}\prod_{k=1}^{n}\varphi([i_k, \ldots, i_n, i_1, \ldots, i_{k-1}]) = a_n.$$

Applying then the general theory of Fredholm determinants for nuclear operators in a Banach space as RUELLE used it in [6], we get

$$\det(1-z\mathscr{L}) = \exp - \sum_{n=1}^{\infty}\frac{z^n}{n}\operatorname{tr}\mathscr{L}^n$$

and $\det(1-z\mathscr{L})$ is an entire function in z of order 0.

It also follows from the results of RUELLE in [6] that the Fredholm determinant is an entire function of $z\varphi$ where $\varphi \in B_2(D_2)$.

We therefore have the following theorem:

THEOREM:

(a) *Let* $\varphi \in B_2(D_2)$ *and*

$$a_n = \sum_{i_1=1}^{\infty} \cdots \sum_{i_n=1}^{\infty} \prod_{k=1}^{n} \varphi([i_k, \ldots, i_n, i_1, \ldots, i_{k-1}]).$$

Then the function

$$\zeta(z, \varphi) = \exp \sum_{n=1}^{\infty} \frac{z^n}{n} a_n$$

extends to a meromorphic function in the whole complex z plane. ζ can also be extended to a meromorphic function in φ in the Banach space $B_2(D_2)$.

(b) *If*

$$b_n = \sum_{i_1=1}^{\infty} \cdots \sum_{i_n=1}^{\infty} \frac{\prod_{k=1}^{n} [i_k, \ldots, i_n, i_1, \ldots, i_{k-1}]^2}{1-(-1)^n \prod_{k=1}^{n} [i_k, \ldots, i_n, i_1, \ldots, i_{k-1}]^2},$$

the function $\zeta(z) = \exp \sum_{n=1}^{\infty} (z^n/n)\, b_n$ *is holomorphic for all* $|z| < 1$ *and has a simple pole at* $z = 1$ *and no other pole for* $|z| \leqslant 1$.

(c) *The poles of the function $\zeta(z, \varphi)$, if arranged in increasing absolute values and according to their order tend to infinity exponentially fast.*

Proof. – From our discussion above we get

$$\zeta(z, \varphi) = \det(1 - z\mathcal{L})^{-1} \det(1 - z\mathcal{L}_1)$$

and therefore part (*a*) is clear. To see part (*b*), we first remark that b_n is given by the trace of the operator $\mathcal{L}^n$ for the special choice $\varphi(z) = z^2$. Now in this case we know one eigenvalue and the corresponding eigenvector of the operator $\mathcal{L}$: consider namely the function $f_0(z) = z^2(1+z)^{-1}$ then we get $\mathcal{L} f_0(z) = f_0(z)$.

Instead of looking at the operator $\mathcal{L}$ in the space $B_2(D_2)$ we can consider an operator $\tilde{\mathcal{L}}$ in the space $B(D_2)$ given by

$$(\tilde{\mathcal{L}} g)(z) = \sum_{n=1}^{\infty} \left(\frac{1}{z+n}\right)^2 g\left(\frac{1}{z+n}\right).$$

This operator trivially has the same spectrum as $\mathcal{L}$ and therefore also the same trace. Therefore it determines the same ζ function. Now the operator $\tilde{\mathcal{L}}$ can also be written in the form

$$\tilde{\mathcal{L}} g(z) = -\sum_{w \in T^{-1}z} \frac{1}{T'(w)} g(w)$$

if we restrict ourselves to real z, because

$$T^{-1} z = \left\{ w : w = \frac{1}{z+n}, \; n = 1, 2, \ldots \right\}.$$

This class of operators is a special case of a more general class of operators studied by P. WALTERS [8] and *via* a generalized Ruelle-Perron-Frobenius theorem one can show that $\lambda = 1$, which is the eigenvalue of the T invariant Gauss measure $d\mu = 1/(1+x)\, dx$ (if we restrict ourselves to the real interval $(0, 1)$), is the highest uniquely determined eigenvalue of $\mathscr{L}$. This eigenvalue is therefore simple and larger in absolute value than all other eigenvalues of $\mathscr{L}$. Finally, part (*c*) of the theorem follows from the results in [6] when one considers the fact that the operator $\mathscr{L}$ on $B_2(D_2)$ given by (5) has the same spectrum as the operator $\tilde{\mathscr{L}}$ on $B(D_2)$ given by

$$(\tilde{\mathscr{L}} g)(z) = \tilde{\varphi}(z) \sum_{n=1}^{\infty} \left(\frac{1}{z+n} \right)^2 g(T_n z) \qquad \text{when} \quad \varphi(z) = z^2 \tilde{\varphi}(z).$$

Because $\tilde{\mathscr{L}}$ determines via its traces the same ζ function as $\mathscr{L}$, and because the results in [6] apply to the operator $\tilde{\mathscr{L}}$ as an operator acting on the space $\mathscr{H}(D_2)$, we get the result we wanted.

Let us finally add some remarks concerning the operators $\mathscr{L}$ or $\tilde{\mathscr{L}}$. We calculated the trace of these operators *via* the traces of the operators $\mathscr{L}_n$. One could try to determine these traces directly from the spectrum of $\mathscr{L}$ or $\tilde{\mathscr{L}}$ which is the same. This would perhaps give interesting relations between the periodic continued fractions and these numbers of the spectrum. But unfortunately our method, which is similar to the method applied in paragraph 2, namely to characterize the eigenfunctions by their behaviour at certain critical points like $z = -n$, $n \in \mathbf{Z}^+$ or $z = -1/[i_1, \ldots, i_n]$ gives only in the case $z = -1$ a nice solution namely the function $f(z) = z^2 (1+z)^{-1}$, which was already known as the eigenfunction for the eigenvalue $\lambda = 1$. At all other critical $z = -1/[i_1, \ldots, i_n]$ we get a whole set of possible values for λ from which we could not extract the true spectrum of $\mathscr{L}$.

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