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FOURIER ANALYSIS AND PATHS OF BROWNIAN MOTION

by

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RÉSUMÉ. — Le mouvement brownien transforme presque sûrement un ensemble fermé de dimension $> 1/2$ en ensemble linéaire à intérieur non vide. La preuve se fonde sur les inégalités de Burkholder pour la norme dans L^p d'une martingale, et sur l'inversion des transformées de Fourier.

Let μ be a probability measure of compact support E on the line, satisfying a Lipschitz condition in exponent $b > 1/2 : \mu(T) \ll (\text{diam } T)^b$ for all measurable sets T. The transform of E by a Brownian motion X, with continuous sample paths, has positive Lebesgue measure, almost surely. Taking a planar process, $Y(t) = (X_1(t), X_2(t))$, we have the same conclusion for each projection $X_1 \cos\theta + X_2 \sin\theta$, by a theorem on Fourier-Stieltjes coefficients ([3], p. 165), but it has not been observed that the projected path has non-empty interior, and this seems beyond the reach of the method of estimating individual Fourier coefficients.

In order to treat a more general problem, we write h for a function of class $C^\beta(R^2)$, $1 < \beta < 2$, whose gradient never vanishes. (By $C^\beta(R^2)$, we denote the space of functions defined on R^2, whose first partial derivatives are subjects to a Lipschitz condition in exponent $\beta - 1$, uniformly on each bounded subset of R^2.) S_θ denotes the rotation of R^2 through an angle θ.

THEOREM. — *With probability* 1, *all composite mappings $h \circ S_\theta \circ Y$ transform E onto a linear set of non-empty interior; in fact, these mappings transform μ to a measure with a continuous density.*

In proving that a finite measure λ has a continuous density, we use its Fourier-Stieltjes transform $\hat{\lambda}(u) = \int e(-ut)\,\lambda(dt)$, where $e(a) \equiv e^{ia}$. To recover λ from $\hat{\lambda}$, we choose and fix a function φ of class

$C^\infty(R)$, with support in $(-2, 2)$, equal to 1 on $(-1, 1)$ and write (for $-\infty < x < \infty, k > 1$)

$$I(x, k) = \int \varphi(k^{-1} u) e(ux) \hat{\lambda}(u) \, du.$$

Then $I(x, k) \to 2\pi\lambda(dx)$ in the weak* topology of measures, so that λ has a continuous density if some subsequence converges uniformly on compact subsets of the x-axis. A closer look yields the formula

$$I(x, k) = \int k \hat{\varphi}(kt - kx) \lambda(dt);$$

now if $\eta > 0$ is arbitrary but fixed, and $|x - t| > k^{\eta - 1}$, then

$$|k \hat{\varphi}(kt - kx)| < k^{-L} \quad \text{for any constant } L \text{ and } k > k(L, \eta).$$

Thus our method leads us to investigate the total λ-measure of intervals of length $k^{\eta-1}$. Another stage in the estimation of $I(x, k) - I(x, 2k)$ uses a Fourier-type integral, arising as an expected value. The final step of the proof is a reduction of $I(x, k) - I(x, 2k)$ to a martingale and application of L^p-inequalities about the square function S of a martingale ([1], [2]). I thank D. L. Burkholder for help with the theory of martingales and distribution function inequalities.

1. In the program outlined above, it is expedient to eliminate all values of Y outside some ball in R^2. Therefore we choose a function ψ of class $C^\infty(R^2)$, $0 \leqslant \psi \leqslant 1$, with compact support. We then work with the transforms of $\mu_0 = \psi(Y).\mu$, but by using an appropriate sequence of test-functions ψ, we obtain all our assertions for the measure μ itself. We write g for any of the composites $h \circ S_\theta$, and denote by $M(x, r)$ the μ_0-measure of the t-set defined by

$$|g \circ Y(t) - x| \leqslant r, \qquad 0 < r < 1, \quad -\infty < x < \infty.$$

The analysis in the lemmas below is used extensively in [3], and in [4], to obtain bounds very similar to those needed here.

Lemma 1. — *Each L^p-norm* $\| M(x, r) \|_p \leqslant B_p r, p = 1, 2, 3, \ldots$.

In the proof, we operate with μ-measure, adding the inequality $\| Y(t) \| < C(\psi)$, since ψ has compact support. To bound the p-th moment of $M(x, r)$, we integrate the p-fold product measure of the set in R^p :

$$|g \circ Y(t_n) - x| \leqslant r, \qquad |Y(t_n)| < C, \qquad 1 \leqslant n \leqslant p.$$

We can adjoin the inequalities $t_1 \leqslant t_2 \leqslant \ldots \leqslant t_p$, because this decreases the product measure by a factor $p!$. The event so obtained is a subset of the event

$$|g \circ Y(t_1) - x| \leqslant r, \qquad |g \circ Y(t_{n+1}) - g \circ Y(t_n)| \leqslant 2r, \qquad 1 \leqslant n < p,$$
$$|Y(t_n)| < C, \qquad 1 \leqslant n \leqslant p.$$

Now h is of class $C^1(R^2)$ and has a gradient vanishing nowhere; by independence of increments, we can conclude that the p-th moment has a magnitude comparable with the p-th power of

$$\sup_s \int \min(1, r|t-s|^{-1/2}) \mu(dt) \leqslant r \sup_s \int |t-s|^{-1/2} \mu(dt) \ll r.$$

LEMMA 2. — *Let E_j be disjoint closed sets, and $m = \max \mu(E_j)$. Let $M_j(x, r)$ be the μ_0-measure of the set defined by $|g \circ Y(t) - x| \leqslant r$, $t \in E_j$, and put*

$$M^*(x, r) = \sup M_j(x, r).$$

Then $\| M^*(x, r) \|_p \leqslant B(p, q)\, r m^q$ for any $q < (2b-1)/2b$.

First we majorize the moments of each $M_j(x, r)$, adding the condition $t_n \in E_j$ $(1 \leqslant n \leqslant p)$ in the product set used in the proof of lemma 1. Hence we obtain p factors $\sup \int_F |t-s|^{-1/2} \mu(dt)$, with $F = E_j$. Now this integral is $\ll \mu(F)^q$ for each q specified. Indeed, the Lipschitz condition imposed on μ yields $\int |s-t|^{-f} \mu(dt) < C(f)$ for each $f < b$, so we can use Hölder's inequality to obtain the factor $\mu(F)^q$, q being the conjugate index to $f/2$. We apply this bound with $F = E_j$, finding that $M^*(x, r)$ has p-th moment $\ll r^p \sum \mu(E_j)^{pq} \leqslant r^p m^{pq-1}$; $\| M^*(x, r) \|_p \ll r m^q m^{-1/p}$. This yields our lemma because $q - p^{-1}$ can be made arbitrarily close to $(2b-1)/2b$, and the L^p-norm increases with p.

2. In this paragraph, we investigate the integral $I(x, k)$ formed from μ_0, namely

$$\iint \varphi(k^{-1} u)\, e(ux - ug \circ Y(t))\, \psi(Y)\, \mu(dt)\, du.$$

We approximate $I(x, k)-I(x, 2k)$ by a martingale sum and use the estimates of M and M^* already found. There remain some estimates whose derivation is the most technical point in the paper. Let α be fixed once and for all in the interval $(0, \beta-1)$ and T_j be the interval $(jk^{-\alpha}, (j+1)k^{-\alpha}]$; then $I(x, k)-I(x, 2k)$ is correspondingly divided into integrals, over T_j, which we name. Because Γ_{2j} is measurable over the σ-field F_{2j} of the variables $\{X(s) : s \leqslant (2j+1)k^{-\alpha}\}$, the variables $\Gamma_{2j}-E(\Gamma_{2j} \mid F_{2j-2})$ form a sequence of martingale differences. We proceed to a bound of $E(\Gamma_{2j} \mid F_{2j-2})$. Γ_{2j} is the integral with respect to μ, over T_{2j}, of

$$\int [\varphi(k^{-1}u)-\varphi(2^{-1}k^{-1}u)]\,\psi(Y(t))\,e(ux-ug\circ Y(t))\,du.$$

We shall give a uniform bound for the expectation, for $t > 2jk^{-\alpha}$, of this integral. To bound $E(\Gamma_{2j} \mid F_{2j-2})$, we have only to multiply by $\mu(T_{2j})$.

By the Markoff property, the conditioning depends only on

$$Y((2j-1)k^{-\alpha}) = Y(v),$$

say, and we have the inequality $t-v \geqslant k^{-\alpha}$. Thus $Y(t)$ has a conditional distribution represented by $Y(v) + |t-s|^{1/2}\,Y(1)$, which we write as $y^0+\sigma\,Y(1)$, $\sigma^2 \geqslant k^{-\alpha}$. At each point in the ball $\|y\| \leqslant C(\psi)$ in R^2, there is a direction τ so that $\partial h/\partial\tau > 0$; consequently, there is a finite covering $\bigcup V_n$ of the support of ψ by convex open sets, and directions τ_n, so that $\partial h/\partial\tau_n \geqslant a > 0$ on V_n. Let $\psi = \sum \psi_n$ be a C^∞-partition of ψ, wherein ψ_n vanishes outside V_n. It will be enough to obtain a bound for the integral containing $\psi_n(Y)$ in place of $\psi(Y)$, and to take $\theta = 0$, $g = h$ (in view of the symmetry of the normal law).

The conditional expectation is given explicitly as an integral involving the normal density $(2\pi)^{-1}\exp(-1/2\,\|y\|^2)$. In this integral, we make an affine change of variable, $z = y^0+\sigma\,Y(1)$ and then integrate on lines in the τ_n-direction. Suppressing the variable of integration in the direction orthogonal to τ_n, we obtain

$$\iint [\varphi(k^{-1}u)-\varphi(2^{-1}k^{-1}u)]\,e(ux-uh(y))\,\psi_n(y)$$
$$\times\exp\left(-\frac{1}{2}\sigma^{-2}(y-y^0)^2\right)dy\,du/\sigma(2\pi)^{1/2}.$$

The integration is extended over an interval $|y| \leqslant C$, and in fact, we can neglect all of this interval except that part on which $|x-h(y)| \leqslant k^{\eta-1}$, for the reason explained in the first paragraph. In case $|x-h(y)| < k^{\eta-1}$ for some y in $[-C, C]$, this inequality defines a subinterval of length $\leqslant k^{\eta-1}$. In the remainder of this argument, we assume that this interval is included entirely in $[-C, C]$, but only minor variations are necessary in other cases. Let us consider the error in replacing $h(y)$ by its tangent line at some point in this interval, say $h_1(y) = h(y_0)+(y-y_0)\,h'(y_0)$. Fisrt, the Lipschitz condition on h', and Taylor's formula, yield $|h_1-h| \leqslant k^{(\eta-1)\beta}$ throughout the interval. Now $|u| \leqslant 2k$, and the integration with respect to u extends over this range at most, introducing a factor $\leqslant k^2$. But $\sigma^2 \geqslant k^{-\alpha}$, and the integration with respect to y is confined to an interval of length $\leqslant k^{\eta-1}$. Thus the error is $\leqslant k^e$, with $e = 2+(\eta-1)(\beta+1)+(1/2)\,\alpha$, approaching

$$1-\beta+\frac{1}{2}\alpha<\frac{1}{2}(1-\beta)$$

as η approaches O^+. Thus we can choose $\eta > 0$ so small that the error is $\leqslant k^{-\delta}$ for some $\delta > 0$.

Next we evaluate the integral in which h has been replaced by the linear function h_1; at the end-points of the domain of integration on the y-axis, $|x-h_1(y)| \approx k^{\eta-1}$. Integration with respect to u gives

$$k\hat{\varphi}(kh_1(y)-kx)-2k\hat{\varphi}(2kh_1(y)-2kx),$$

and our plan now is to integrate by parts several times in succession.

The function $r(s) \equiv \hat{\varphi}(s)-2\hat{\varphi}(2s)$ is represented by a Fourier transform of C^∞ function of compact support, and so are each of its indefinite integrals if they are normalized so as to vanish at infinity. Successive integrations of $kr(kh_1(y)-kx)$ with respect to y therefore bring in factors k^{-1}. The L^1-norm of the p-th derivative of the cofactor is $\leqslant \sigma^{-p}$, and this disposes of the integral obtained in integrating by parts several times. The integrated terms occur at the end-points, where $k|x-h_1(x)| \sim k^\eta$, and the rapid decrease of r and its integrals at infinity enable us to obtain a bound $\leqslant k^{-L}$ for any fixed L. In summary, then, we have

$$|E(\Gamma_{2j}|F_{2j-2})| \leqslant k^{-\delta}\mu(T_{2j}) \quad \text{for a certain } \delta > 0.$$

3. From the properties of $I(x, k)$ mentioned in the first paragraph, we have $|\Gamma_{2j}| \ll k^{1-L}\mu(T_{2j})+k\,M_j(x, k^{\eta-1})$. Here M_j is the partial μ_0 measure of lemma 2, and $E_j = T_{2j}$. Thus $m = \max \mu(T_{2j}) \ll k^{-\alpha b}$, and $\max |\Gamma_{2j}|$ has L^p-norms of magnitude k^{e_1}, with $e_1 = \eta - \alpha\, bq$. Taking $\eta < \alpha\, bq$, we again find $\|\max|\Gamma_{2j}|\|_p \ll k^{-\delta}$ for a certain $\delta > 0$ and every $p = 1, 2, 3, \ldots$. Using lemma 1 instead of lemma 2, we obtain $\|\sum|\Gamma_{2j}|\|_p \ll k^{\eta}$; two applications of Hölder's inequality yield $\|\sum|\Gamma_{2j}|^2\|_p \ll k^{\eta-\delta}$, and the exponent is negative for small $\eta > 0$. In view of the bound on $E(\Gamma_{2j}\,|\,F_{2j-2})$ obtained above, the martingale square function defined by $S^2 = \sum|\Gamma_{2j} - E(\Gamma_{2j}\,|\,F_{2j-2})|^2$ has L^p-norms $\ll k^{-\delta}$ for some $\delta > 0$. By a theorem of BURKHOLDER ([1], [2], theorem 3.2), the sum has L^p-norms of comparable magnitude; but then there is a $\gamma > 0$ so that

$$P(|I(x, k) - I(x, 2k)| > k^{-\gamma}\} \ll k^{-L} \quad \text{for every } L.$$

The integral $I(x, k) - I(x, 2k)$ depends on the parameters x and θ, but has partial derivatives with respect to these variables $\ll k^2$. From this it is easily seen that the probability estimate is valid for the supremum over $0 \leqslant \theta \leqslant 2\pi$ and $|x| \leqslant k$. Choosing now $k = 2^j$, we find that $I(x, 2^j)$ converges uniformly on compact subsets of the x-axis, and even uniformly with respect to θ, with probability 1.

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