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DIAGONALIZABLY LINEARIZED COHERENT SHEAVES

BY

H. ANDREAS NIELSEN

SUMMARY. — Let X denote a smooth projective scheme with an action of the smooth diagonalizable group D . The Grothendieck group $K_D(X)$ on the category of D -linearized coherent sheaves on X is studied.

The main result is a localization theorem for K_D , an algebraic analogue of the Atiyah-Segal theorem.

Applications are given to Lefschetz formulas of various types.

RÉSUMÉ. — Soient X un schéma projectif lisse, muni d'une action du groupe diagonalisable, lisse, D . On fait une étude du groupe de Grothendieck, $K_D(X)$ sur la catégorie des faisceaux cohérent, D -linéarisé sur X .

Notre résultat principal est un théorème de localisation pour le foncteur K_D , variante algébrique de celui de Atiyah-Segal.

Comme application des formules de Lefschetz de types variés sont données.

The paper is concerned with equivariant K -theory of a smooth projective scheme X , equipped with an action of a smooth diagonalizable group D .

Our main result is a localization theorem for the equivariant K -functor, K_D . Namely, the inclusion $i : X^D \rightarrow X$ induces a map

$$i^! : K_D(X) \rightarrow K_D(X^D)$$

which, considered as a linear map over the representation ring of D , becomes an isomorphism after a suitable localization.

The localization theorem combined with the Riemann-Roch formula yields a Lefschetz fixed point formula of the type,

$$\sum_i (-1)^i \operatorname{Tr} H^i(X, \mathcal{F}) = \int_{X^D} \frac{\operatorname{ct}(i^* \mathcal{F}) \operatorname{Todd}(X^D)}{\operatorname{ct}(\lambda_{-1} N)}$$

valid in a localization of the representation ring of D , see (4.10). By various specializations of the coefficients, we obtain results of more

classical type, among others the Woods Hole fixed point formula and those of [2], [4], [5], [9].

It should be mentioned that the localization theorem is inspired by a similar topological theorem, *see* [1] for reference.

In case of a torus action, the above form of the Lefschetz fixed point formula was conjectured by Birger IVERSEN, whom I thank for indispensable guidance not only in this subject.

CONTENTS :

- § 1 : Equivariant K -theory.
- § 2 : The Gysin morphism.
- § 3 : The localization theorem.
- § 4 : Applications.

NOTATION. — Throughout we fix an algebraically closed field k and a smooth diagonalizable k -group scheme D . Δ denotes the character group of D , and we put $R(D) = \mathbb{Z}[\Delta]$. For $\kappa \in \Delta$, we let e^κ denote the corresponding element in $R(D)$.

For a k -linear representation E of D , we put

$$\mathrm{tr}(E) = \sum_{\kappa \in \Delta} (\mathrm{rank}_k E_\kappa) e^\kappa,$$

where E_κ is the space of semi-invariants of D of weight κ in E .

As is well known tr induces an isomorphism from the representation ring of D to $R(D)$.

Let $S \subseteq R(D)$ be the multiplicative subset generated by elements of the form $1 - e^\kappa$, κ a non-trivial character of D . An easy consideration shows $0 \notin S$.

1. Equivariant K -theory

DEFINITION 1.1. — Let X be a scheme ⁽¹⁾ with a D -action. Then we let $K_D(X)$ denote the Grothendieck ring of the category of D -linearized [12] locally free sheaves on X , the multiplication being induced by \otimes . The image of a D -linearized locally free sheaf \mathcal{F} in $K_D(X)$ is denoted $\mathrm{cl} \mathcal{F}$. Let **D-Sch** denote the category of k -schemes with D -actions, the morphisms

⁽¹⁾ Scheme = k -scheme throughout the paper.

being D -equivariant morphisms of k -schemes. The pullback functor makes K_D into a functor

$$K_D : \mathbf{D-Sch}^{\text{op}} \rightarrow \mathbf{Rings}.$$

As is well known the trace tr gives an isomorphism $K_D(\text{Spec}(k)) \xrightarrow{\sim} R(D)$. In the following, we shall always view

$$K_D : \mathbf{D-Sch}^{\text{op}} \rightarrow \mathbf{R(D)-Rings}.$$

Put $f^! = K_D(f)$, f a D -equivariant scheme morphism.

1.2. λ -operations. — We have naturel equivariant operations

$$\lambda^i : K_D(X) \rightarrow K_D(X), \quad i \geq 0,$$

satisfying

(★) For \mathcal{F} a D -linearized locally free sheaf on X

$$\lambda^i(\text{cl } \mathcal{F}) = \text{cl } \Lambda^i \mathcal{F}.$$

(★★) $\lambda_t : K_D(X) \rightarrow 1 + t K_D(X)[[t]]$,

$$x \mapsto 1 + \sum_{i=0}^{\infty} \lambda^i(x) t^i,$$

is a group homomorphism.

$K_D(X)$ is actually a λ -ring in the sense of SGA 6 ([14], V, 2.4).

1.3. *The trivial action.* — Suppose D acts trivially on X . A D -linearized locally free sheaf \mathcal{F} decomposes $\mathcal{F} = \bigoplus_{\kappa \in \Delta} \mathcal{F}_{\kappa}$, where D acts on \mathcal{F}_{κ} through κ , see [3]. $\mathcal{F} \mapsto \sum_{\kappa \in \Delta} \text{cl}(\mathcal{F}_{\kappa}) \otimes e^{\kappa}$ induces a natural map

$$\text{tr}_X : K_D(X) \rightarrow K(X) \otimes_{\mathbf{Z}} R(D)$$

which is an $R(D)$ -linear isomorphism (*loc. cit.*).

1.4. *Linear action on projective space.* — Let E be a rank $r+1$ k -linear representation of D . Put

$$E = \bigoplus_{\kappa \in \Delta} E_{\kappa}, \quad \text{rank}_k E_{\kappa} = n_{\kappa}, \quad \sum_{\kappa \in \Delta} n_{\kappa} = r+1.$$

The action of D on E induces an action of D on $\mathbf{P}(E) \xrightarrow{\Pi} \text{Spec } k$ together with a linearization of $\mathcal{O}_{\mathbf{P}}(1)$.

THEOREM 1.5 ⁽²⁾. — *We have an $R(D)$ -linear isomorphism*

$$K_D(\mathbf{P}(E)) \xleftarrow{\sim} R(D)[T]/\prod_{\kappa \in \Delta} (T - e^*)^{\kappa}, \\ \text{cl}(\mathcal{O}_{\mathbf{P}}(1)) \hookleftarrow T.$$

Proof. — Fix notation for the proof

$$l = \text{cl}(\mathcal{O}_{\mathbf{P}}(1)), \\ w = \text{cl}(\text{Ker } \Pi^*(E) \rightarrow \mathcal{O}_{\mathbf{P}}(1)), \\ v = \text{cl}(\Pi^*(E)).$$

The proof consists in three steps (1.6), (1.7), (1.8).

(1.6) $K_D(\mathbf{P}(E))$ is generated over $R(D)$ by $\{l^n; n \in \mathbf{Z}\}$.

Let us first make some considerations over graded modules.

D acts on $A = \text{Sym}_k E$ through E . By a graded D - A -module we understand a graded A -module M together with a k -linear action of D on each graded piece of M subjected to

$$\sigma(am) = (\sigma a)(\sigma m); \quad \sigma \in D, \quad a \in A, \quad m \in M.$$

The morphisms in the category of graded D - A -modules are graded of degree 0 and as well A - as D -linear.

If $\kappa \in \Delta$ and M is a graded D - A -module, then M_{κ} denotes the graded D - A -module obtained from M by twisting the D -action as follows :

$$\sigma m : = \kappa(\sigma) \sigma m; \quad \sigma \in D, \quad m \in M.$$

Note that if N is a graded D - A -module $\kappa \in \Delta$ and $n \in \mathbf{Z}$, then

$$\text{Hom}_{\text{gr-}D\text{-}A}(A_{\kappa}(-n), N) \simeq (N_n)_{\kappa}$$

where $(N_n)_{\kappa}$ denotes the semi-invariants of D of weight κ in N_n .

Let us call a graded D - A -module free if it is a finite direct sum of graded D - A -modules of the form $A_{\kappa}(-n)$, $\kappa \in \Delta$, $n \in \mathbf{Z}$.

In virtue of the above remark, it is clear that if M is a finitely generated D - A -module then there exists a surjective morphism of graded D - A -modules $L \rightarrow M$ with L a free graded D - A -module.

⁽²⁾ This is a particular case of the theorem giving the structure of $K(P(E))$ for E a locally free sheaf on a ringed topos ([14], VII.1.4).

We are going to prove :

A finitely generated graded D - A -module M has a resolution

$$(\star) \quad 0 \rightarrow L_{r+1} \rightarrow L_r \rightarrow \dots \rightarrow L_0 \rightarrow M \rightarrow 0,$$

where the L_i 's are free graded D - A -modules.

First it is clear from the preceding remarks that we can find a resolution as above where L_0, L_1, \dots, L_r are free graded D - A -modules. By Hilbert's syzygy theorem, L_{r+1} is a free graded A -Module. Thus it suffices to prove.

($\star\star$) A finitely generated graded D - A -module M which is free as a graded A -module is free as a graded D - A -module.

Proof. — Pick a family (m_i) of semi-invariant homogeneous elements of M such that $(m_i \otimes 1_k)$ form a basis for $M \otimes_A k$. Let m_i have degree d_i and weight κ_i , and put $L = \otimes A_{\kappa_i}(-d_i)$. We have a morphism $f: L \rightarrow M$ whose reduction mod A_+ is an isomorphism. From this and the assumption that M is a free graded A -module follows that f is an isomorphism (see [8], Lemma 2.2).

Returning to the proof of (1.6). The sheafification functor lifts to a functor $(\tilde{})$ from the category of finitely generated D - A -modules to D -linearized coherent sheaves on $\mathbf{P}(E)$. $(\tilde{})$ is exact and onto objects. If L is a free graded D - A -module then $\text{cl } \tilde{L}$ is an $R(D)$ -linear combination of $\{l^n; n \in \mathbf{Z}\}$. Now (1.6) follows from (\star),

$$(1.7) \quad \prod_{x \in \Delta} (1 - e^x)^{n_x} = 0.$$

The sequence $0 \rightarrow \text{Ker} \rightarrow \Pi^*(E) \rightarrow \mathcal{O}_{\mathbf{P}}(1) \rightarrow 0$ gives, with the introduced notation, $v \cdot l^{-1} = w \cdot l^{-1} + 1$. Applying λ_t gives

$$\lambda_t(vl^{-1}) = (1+t)\lambda_t(wl^{-1}).$$

Now substitute $v = \sum_{x \in \Delta} n_x e^x$, and use (1.2) ($\star\star$), then

$$\prod_{x \in \Delta} \lambda_t(e^x l^{-1})^{n_x} = (1+t)\lambda_t(wl^{-1}).$$

For $t = -1$, we get the relation

$$\prod_{x \in \Delta} (1 - e^x l^{-1})^{n_x} = 0.$$

(1.7) follows after multiplication with l^{r+1} :

$$(1.8) \quad 1, l, \dots, l^r \text{ are linearly independent over } R(D).$$

Let $\chi_D(\mathbf{P}, x) : K_D(\mathbf{P}(E)) \rightarrow R(D)$ denote the Lefschetz trace, *see* (4.2) for details.

Suppose $\sum_{i=0}^r a_i l^i = 0$. Let a_s be the biggest non-trivial coefficient. $a_s = \sum_{i=0}^{s-1} -a_i l^{i-s}$. By Serre's calculations [13], $\chi_D(\mathbf{P}, l^{i-s}) = 0, i = 0, \dots, s-1$. Now apply $\chi_D(\mathbf{P}, x)$ to the above relation, use that $\chi_D(\mathbf{P}, l)$ is $R(D)$ -linear and conclude $a_s = 0$.

2. The Gysin morphism

In this paragraph, we introduce a Gysin morphism $(i_!)$ for equivariant K -theory, and give three formulas interrelating $i_!$ and $i^!$.

PROPOSITION 2.1. — *Let D act on the smooth projective scheme X . Then the natural map of $K_D(X)$ into the Grothendieck group of the category of D -linearized coherent sheaves on X is an isomorphism.*

Proof. — The category of D -linearized locally free sheaves on X is a full subcategory of the abelian category of D -linearized coherent sheaves on X . So by standard theory, e. g. [2] or [6], we are easily reduced to prove the following lemma.

LEMMA 2.2. — *Let $X \xrightarrow{\Pi} \text{Spec } k$ be a smooth projective scheme on which D acts.*

(2.3) *There exists a D -linearized ample sheaf \mathcal{L} on X .*

(2.4) *Every D -linearized coherent sheaf \mathcal{F} on X is an equivariant quotient of a D -linearized locally free sheaf on X .*

Proof. — (2.3) is contained in the results of Kambayashi (*see* [10]). For (2.4) choose m so large that $\mathcal{F} \otimes \mathcal{L}^m$ is generated by its global sections. $V = H^0(X, \mathcal{F} \otimes \mathcal{L}^m)$ is a k -linear representation of D , (4.1), hence we have a D -equivariant surjection $\Pi^* V \rightarrow \mathcal{F} \otimes \mathcal{L}^m$ and therefore \mathcal{F} is a quotient of $\Pi^* V \otimes \mathcal{L}^{-m}$.

DEFINITION 2.5. — *Let $i : Y \rightarrow X$ be a D -equivariant closed immersion of smooth projective schemes with D -action. By (2.1), the direct image functor i_* induces an Abelian group homomorphism*

$$i_! : K_D(Y) \rightarrow K_D(X)$$

such that for $Z \xrightarrow{i} Y \xrightarrow{j} X$, we have

$$(i \circ j)_! = i_! \circ j_!,$$

Three formulas. — Notation as in (2.5). $i : Y \rightarrow X$.

(2.6) *The projection formula* : For every $x \in K_D(X)$, $y \in K_D(Y)$:

$$i_!(y \cdot i^!(x)) = i_!(y) \cdot x.$$

(2.7) *The self-intersection formula* : Put $N = \text{cl}(\mathcal{N}_{Y/X})$, $\mathcal{N}_{Y/X}$ being the conormal bundle on Y with its canonical linearization. For every $y \in K_D(Y)$:

$$i^!(i_!(y)) = y \cdot \lambda_{-1}(N).$$

(2.8) *The cartesian formula* : Let

$$\begin{array}{ccc} & j' & \\ T & \xrightarrow{\quad} & Y \\ i' \downarrow & & \downarrow i \\ Z & \xrightarrow[\quad j]{} & X \end{array}$$

be a cartesian square of D equivariant closed immersions between smooth projective schemes with D -action. Then there exists $\gamma_T \in K_D(T)$ such that for every $y \in K_D(Y)$:

$$j^!(i_!(y)) = i'_!(\gamma_T \cdot j'^!(y)), \quad y \in K_D(Y).$$

Remark on proof. — (2.6) follows from a natural isomorphism. (2.7) follows from a closer look at the “unlinearized” proof (see MANIN [11] or SGA 6 ([14], VII, 2.7)). (2.8) is proved as follows :

Let t denote the inclusion $T \rightarrow X$. Put (**Tor** is short for **Tor** ^{\mathcal{O}_X}) :

$$\gamma_T = \sum (-1)^i \text{cl } t^* \text{Tor}_i(\mathcal{O}_Y, \mathcal{O}_Z).$$

Let now $y = \text{cl}(\mathcal{F})$, where \mathcal{F} is a locally free sheaf on Y :

$$j^! i_! y = \sum (-1)^i \text{cl } j^* \text{Tor}_i(\mathcal{O}_Z, i_* \mathcal{F}).$$

Now

$$\text{Tor}_i(\mathcal{O}_Z, i_* \mathcal{F}) = \text{Tor}_i(\mathcal{O}_Z, \mathcal{O}_Y) \otimes i_* \mathcal{F},$$

gives

$$\begin{aligned} j^! i_! y &= \sum (-1)^i \text{cl } j^* \text{Tor}_i(\mathcal{O}_Z, \mathcal{O}_Y) \otimes j^* i_* \mathcal{F} \\ &= \sum (-1)^i \text{cl } i'_*(t^* \text{Tor}_i(\mathcal{O}_Z, \mathcal{O}_Y) \otimes j'^* \mathcal{F}) \\ &= i'_!(\sum (-1)^i \text{cl } t^* \text{Tor}_i(\mathcal{O}_Z, \mathcal{O}_Y) \cdot j'^! y) \\ &= i'_!(\gamma_T \cdot j'^! y). \end{aligned}$$

3. The localization theorem

Let D act on the smooth projective scheme X . The fixed point scheme X^D is smooth [9]. The inclusion $i : X^D \rightarrow X$ induces an $R(D)$ -linear map

$$i^! : K_D(X) \rightarrow K_D(X^D).$$

We show that this map becomes an isomorphism after localization with respect to the multiplicative subset $S \subseteq R(D)$ generated by elements of the form $1 - e^\chi$, χ a nontrivial character of D . Note $0 \notin S$.

LEMMA 3.1. — *Let $N = \text{cl}(\mathcal{N}_{X^D/X})$ the class in $K_D(X^D)$ of the conormal bundle of X^D in X . Then $\lambda_{-1} N$ becomes a unit in $S^{-1} K_D(X^D)$.*

Proof. — It is enough to prove that $\lambda_{-1} N$ is a unit when we restrict to every connected component Z of X^D . Now choose a closed point $z \in Z$ and let $j_z : \{z\} \rightarrow Z$ be the inclusion. By MANIN ([11], § 8 and 9) :

$$K(Z) = \mathbf{Z} \oplus \text{Ker}(j_z^*)$$

and $\text{Ker}(j_z^*)$ is nilpotent. Tensoring this with $R(D)$ and using (1.3), we obtain a decomposition

$$K_D(Z) \simeq R(D) \oplus \text{Ker}(j_z^!),$$

with $\text{Ker}(j_z^!)$ nilpotent. Clearly it suffices to prove that the component of $\lambda_{-1} N$ after $R(D)$ belongs to S . Now the component of $\lambda_{-1} N$ after $R(D)$ equals

$$\text{tr}_{\{z\}} j_z^! (\lambda_{-1} N) = \text{tr}_{\{z\}} \lambda_{-1} (j_z^! N).$$

All weights of D in the fibre of $\mathcal{N}_{X^D/X}$ at z are nontrivial as it follows from the fact that the fixed point scheme is smooth [9], hence we can write $\text{tr}_{\{z\}} j_z^! N = \sum_{\chi \neq 0} m_\chi e^\chi$. By (1.2) (★★),

$$\text{tr}_{\{z\}} j_z^! \lambda_{-1} N = \prod_{\chi \neq 0} (1 - e^\chi)^{m_\chi}$$

which belongs to S .

THEOREM 3.2. — *The inclusion $i : X^D \rightarrow X$ induces an $R(D)$ -linear map*

$$i^! : K_D(X) \rightarrow K_D(X^D)$$

which is an isomorphism after localization with respect to S . The inverse map is given by

$$y \mapsto S^{-1} i_! (y \cdot (\lambda_{-1} N)^{-1}), \quad y \in S^{-1} K_D(X^D).$$

Proof. — Localizing the formulas (2.6) and (2.7), we get

$$\begin{aligned} S^{-1} i_! (y \cdot S^{-1} i^! (x)) &= S^{-1} i_! (y) \cdot x, \\ S^{-1} i^! (S^{-1} i_! (y)) &= y \cdot \lambda_{-1} (N) \end{aligned}$$

for $x \in S^{-1} K_D(X)$, $y \in S^{-1} K_D(X^D)$.

Using (3.1) and these formulas, it remains to prove the following two equivalent statements :

$$(3.3) \quad S^{-1} i^! : S^{-1} K_D(X) \rightarrow S^{-1} K_D(X^D) \text{ is injective,}$$

$$(3.4) \quad S^{-1} i_! : S^{-1} K_D(X^D) \rightarrow S^{-1} K_D(X) \text{ takes the value 1.}$$

We proceed by two lemmas.

LEMMA 3.5. — (3.3) is true for a linear action on a projective space $\mathbf{P}(E)$ [cf. (1.4) for notation].

Proof. — By the calculations in (1.4), we get $\mathbf{P}(E)^D = \bigsqcup_{\kappa \in \Delta} \mathbf{P}(E_\kappa)$ and

$$S^{-1} i^! : (S^{-1} R(D)[T] / \prod_{\kappa \in \Delta} (T - e^\kappa)^{n_\kappa}) \rightarrow (\prod_{\kappa \in \Delta} S^{-1} R(D)[T] / (T - e^\kappa)^{n_\kappa}) \\ (T \mapsto \prod T).$$

Using that

$$(T - e^\kappa) = e^{\kappa'} (1 - e^{\kappa - \kappa'}) + (T - e^{\kappa'})$$

is a unit in $S^{-1} R(D)[T] / (T - e^{\kappa'})^{n_{\kappa'}}$ for $\kappa' \neq \kappa$ this map is easily seen to be injective.

LEMMA 3.6. — (3.4) is true for any X .

Proof. — By (2.3), we can find a k -linear representation E of D and a D -equivariant closed immersion $j : X \rightarrow \mathbf{P}(E)$. The following diagram

$$\begin{array}{ccc} X^D & \xrightarrow{\quad} & \mathbf{P}(E)^D \\ \downarrow i & j^D & \downarrow i_P \\ X & \xrightarrow{\quad} & \mathbf{P}(E) \\ & j & \end{array}$$

is cartesian as it follows from the definition of the fixed point scheme.

According to (3.5), we can find $z_p \in S^{-1} K_D(\mathbf{P}(E)^D)$ such that $S^{-1} i_{p_1} z_p = 1$. Put

$$z_X = S^{-1} j^{D!}(z_p) \cdot \gamma_{XD}, \quad S^{-1} i_1(z_X) = 1$$

as it follows from the localized version of (2.8).

4. Applications

The applications of the localization theorem we are going to discuss are based on the Lefschetz trace :

$$\chi_D(X, x) : K_D(X) \rightarrow R(D),$$

X a projective scheme with D -action.

4.1. *Construction of k -linear representations.* — Let us recall that if \mathcal{F} is a D -linearized sheaf on X , then we have a canonical action of D on the cohomology groups $H^i(X, \mathcal{F})$. Namely for $\sigma \in D(k)$ the D -linearization of \mathcal{F} provides a morphism $\sigma^* \mathcal{F} \rightarrow \mathcal{F}$ which induces a linear map $H^i(X, \sigma^* \mathcal{F}) \rightarrow H^i(X, \mathcal{F})$. Composing this and the canonical map $H^i(X, \mathcal{F}) \rightarrow H^i(X, \sigma^* \mathcal{F})$ gives the action of σ on $H^i(X, \mathcal{F})$.

4.2. *The Lefschetz trace.* — Let X be a smooth projective scheme with a D -action. The functor

$$\mathcal{F} \mapsto \sum_i (-1)^i \operatorname{tr} H^i(X, \mathcal{F})$$

from the category of D -linearized locally free sheaves to $R(D)$ is additive. This functor induces the Lefschetz trace :

$$\chi_D(X, x) : K_D(X) \rightarrow R(D),$$

$\chi_D(X, x)$ is an $R(D)$ -linear map satisfying :

(4.3) For a D -equivariant closed immersion $j : Y \rightarrow X$:

$$\chi_D(Y, y) = \chi_D(X, x) \circ j_!(y).$$

(4.4) If D acts trivially on X then the following diagram commutes :

$$\begin{array}{ccc} & \operatorname{tr}_X & \\ K_D(X) & \longrightarrow & K(X) \otimes_{\mathbf{Z}} R(D) \\ & \searrow \quad \swarrow & \\ \chi_D(X, x) & \longrightarrow & \chi(X, x) \otimes \operatorname{id}_{R(D)} \\ & & R(D) \end{array}$$

The next proposition shows how to compute the total χ_D by means of χ_D on the fixed point scheme. Moreover (4.4) shows that χ_D on the fixed point scheme may be computed from the unlinearized χ .

PROPOSITION 4.5. — *Let $i : X^D \rightarrow X$, $x \in K_D(X)$, then*

$$\chi_D(X, x) = S^{-1} \chi_D(X^D, i^!(x) \cdot (\lambda_{-1} N)^{-1}) \text{ in } S^{-1} R(D).$$

Proof. — (3.2), (4.3).

Example 4.6. — From the exact sequence

$$0 \rightarrow \mathcal{N}_{X^D/X} \rightarrow i^* \Omega_X \rightarrow \Omega_{X^D} \rightarrow 0,$$

we get $\lambda_{-1} \text{cl}(i^* \Omega_X) = \lambda_{-1} N \cdot \lambda_{-1} (\text{cl } \Omega_{X^D})$ in $K_D(X^D)$. By (4.5),

$$\chi_D(X, \lambda_{-1} \text{cl } \Omega_X) = \chi_D(X^D, \lambda_{-1} \text{cl } \Omega_{X^D}) \text{ in } S^{-1} R(D).$$

For $D = T$ an algebraic torus, the equality above holds in $R(D)$. So we may specialize the characters to 1, and get

$$\chi(X, \lambda_{-1} \Omega_X) = \chi(X^T, \lambda_{-1} \Omega_{X^T})$$

in \mathbf{Z} proved by Birger IVERSEN [9].

4.7. *Isolated fixed points.* — Assume X^D finite. Then (4.5) gives for a D -linearized sheaf \mathcal{F} on X :

$$\sum_i (-1)^i \text{tr } H^i(X, \mathcal{F}) = \sum_{z \in X^D} \frac{\text{tr } \mathcal{F}_z}{\sum_i (-1)^i \text{tr } (\Lambda^i T_z(X)^V)}.$$

4.8. *H. Weyl's character formula.* — An interesting application of (4.7) is the case where $X = G/B$, B is a Borel subgroup of the reductive linear algebraic group G and $D = T$ a maximal torus contained in B (see [2] and [4]). In characteristic zero, this leads to a proof of Weyl's character formula (*loc. cit.*).

4.9. *The Woods Hole formula.* — Let $\sigma \in D(k)$. The evaluation map $\Delta \rightarrow k^*$, $\kappa \mapsto \kappa(\sigma)$ gives rise to a ring homomorphism $\text{ev}_\sigma : R(D) \rightarrow k$ such that for a k -linear representation E we have $\text{ev}_\sigma(\text{tr } E) = \text{Tr}(\sigma, E) \in k$ the usual trace for the operation of σ on E .

Let $\sigma \in D(k)$ be a dense element, i. e. $\kappa(\sigma) \neq 1$ for all nontrivial characters $\kappa \in \Delta$, then ev_σ factors through $R(D) \rightarrow S^{-1} R(D)$.

Applying ev_σ to the formula (4.7) gives the following formula in k :

$$\sum_i (-1)^i \text{Tr}(\sigma, H^i(X, \mathcal{F})) = \sum_{z \in X^D} \frac{\text{Tr}(\sigma, \mathcal{F}_z)}{\text{Det}(1 - d_z \sigma)}.$$

4.10. *The cohomological formula.* — Assume we have a cohomology theory in the sense of Grothendieck [7] such that its Chern-character satisfies the Riemann-Roch theorem

$$\chi(X, \mathcal{F}) = \int_X \text{ch } \mathcal{F} \cdot \text{Todd}(X).$$

For the trivial action of D on X put

$$\begin{aligned} \text{ct}_D : K_D(X) &\xrightarrow{\text{tr}_X} K(X) \otimes R(D) \xrightarrow{\text{ch} \otimes \text{id}_{R(D)}} A(X) \otimes \mathbf{Q} \otimes R(D), \\ \text{Todd}_D &= \text{Todd} \otimes 1_{R(D)}, \\ &\int_X \text{“} = \text{”} \int_X \otimes \text{id}_{R(D)}. \end{aligned}$$

Now (4.4), (4.5) gives the formula

$$\chi_D(X, \mathcal{F}) = \int_{X^D} \frac{\text{ct}_D(i^* \mathcal{F}) \cdot \text{Todd}_D(X^D)}{\text{ct}_D(\lambda_{-1} N)}$$

in $\mathbf{Q} \otimes S^{-1} R(D)$ for \mathcal{F} a D -linearized coherent sheaf on X .

4.11. *Specialization to the Witt ring.* — Assume $\text{char}(k) = p \neq 0$. For an element $\sigma \in D(k)$ the composite of the evaluation map (4.9) $\text{ev}_\sigma : \Delta \rightarrow k^*$ and the Teichmüller lifting $w : k^* \rightarrow W(k)$ gives the map $b_\sigma : R(D) \rightarrow W(k)$ such that for a k -linear representation E , we have $b_\sigma(\text{tr } E) = B \text{Tr}(\sigma, E)$, the Brauer trace for the operation of σ on E .

Now assume D to be finite cyclic with generator $d \in D(k)$. d is “dense” (4.9), so b_d factors

$$R(D) \rightarrow S^{-1} R(D) \rightarrow W(k).$$

The cohomological formula (4.10) specializes through this map to the formula of Donovan ([2] and [5]).

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