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A NOTE ON 4-DIMENSIONAL HANDLEBODIES

BY

FRANÇOIS LAUDENBACH AND VALENTIN POÉNARU

1. Introduction

We prove the following theorem:

THEOREM A. — Let $X^p$, $Y^p$ be the following smooth $4$-manifolds:

$$X^p = p \# (S^3 \times D_3), \quad Y^p = p \# (S^3 \times D_3).$$

Consider a diffeomorphism $h : \partial X^p \to \partial Y^p$ and the smooth closed $4$-manifold obtained by gluing $X^p$ and $Y^p$ along $h : X^p \cup_h Y^p$. $X^p \cup_h Y^p$ is diffeomorphic to $S^4$.

Theorem A is clearly equivalent to the following:

THEOREM A'. — Let $X^p$ be as before, and consider $p$ handles of index $3$, attached successively to $X^p$:

$$\varphi^i : S^2_s \times D_i^i \hookrightarrow \partial (X^p + (\varphi^3) + \ldots + (\varphi^{i-1})),$$

where $S^2_s \times D_i^i = \partial D_i^i \times D_i^i \subset \partial (D_i^i \times D_i^i)$ and $i = 1, \ldots, p$.

Assume that $\partial (X^p + (\varphi^3) + \ldots + (\varphi^i)) = S^3$, and consider a handle of index $4$:

$$\varphi^4 : \partial D_i^i \hookrightarrow \partial (X^p + (\varphi^3) + \ldots + (\varphi^i)).$$

attached to $X^p + (\varphi^3) + \ldots + (\varphi^i)$. One has:

$$X^p + (\varphi^3) + \ldots + (\varphi^i) + (\varphi^4) = S^4 \quad \text{(diffeomorphism)}.$$
This result implies the following:

**Corollary B.** — Let $X^p$ be as before, and consider $p$ handles of index 3, attached successively to $X^p$:

\[ \psi_i : S_i^1 \times D_i^1 \to \partial (X^p + (\psi_1^p) + \ldots + (\psi_i^p)) \quad (i = 1, \ldots, p). \]

If

\[ H_i (X^p + (\psi_1^p) + \ldots + (\psi_i^p), Z) = 0 \]

then

\[ X^p + (\psi_1^p) + \ldots + (\psi_p^p) = D_i \quad \text{(diffeomorphism)}. \]

2. The proof of theorem A

One has « canonical » identifications:

\begin{align*}
\partial X^p & \cong S^3 \\
(0) \quad (S_1^i \times S_1^i) & \neq \ldots \neq (S_p^i \times S_p^i) = p \neq (S_1^i \times S_1^i), \\
\partial Y^p & \cong \beta
\end{align*}

which will be given, once for all. It is obvious that

\[ X^p \cup_{\beta - 1} Y^p = S_i. \]

**Lemma 1.** — The following two statements are equivalent:

(i) $X^p \cup_h Y^p = S_i$.

(ii) There exist diffeomorphisms: $G : X^p \to X^p$, $H : Y^p \to Y^p$, such that:

\[ \beta^{-1} \alpha = (H | \partial Y^p) \circ h \circ (G | \partial X^p). \]

**Proof.** — If $f_1$, $f_2$ are two differentiable embeddings $f_i : Y^p \to S_i$, it is obvious that the pairs $(S_i, f_1 Y^p), (S_i, f_2 Y^p)$ are diffeomorphic. Hence, if $X^p \cup_h Y^p = S_i = X^p \cup_{\beta - 1} Y^p$, there exists a diffeomorphism $X^p \cup_h Y^p \to X^p \cup_{\beta - 1} Y^p$ sending $X^p$ onto $X^p$ and $Y^p$ onto $Y^p$. This shows that (i) $\Rightarrow$ (ii).

On the other hand, the equality (1) tells us that $G$ and $H$ can be patched together so as to give diffeomorphism:

\[ X^p \cup_h Y^p \leftrightarrow X^p \cup_{\beta - 1} Y^p. \]

Hence (ii) $\Rightarrow$ (i).

**Remark.** — The implication (ii) $\Rightarrow$ (i) holds whenever we glue two $n$-manifolds along their (diffeomorphic) boundaries, while (i) $\Rightarrow$ (ii) is very exceptional.
We consider now \( \pi_1 Y^p = \) the free group with \( p \) generators, and we remark that, if \( i : \partial Y^p \rightarrow Y^p \) is the natural inclusion, then:

\[
i_* : \pi_1 \partial Y^p \rightarrow \pi_1 Y^p
\]

is bijective. Let \( \text{Diff} Y^p \) be the group of diffeomorphisms of \( Y^p \) and \( \text{Aut} (\pi_1 Y^p) \) the group of automorphisms of \( \pi_1 Y^p \). We have a commutative triangle of natural homomorphisms:

\[
\begin{array}{ccc}
\pi_0 (\text{Diff} Y^p) & \xrightarrow{\text{Aut} (\pi_1 Y^p)} & \pi_0 (\text{Diff} Y^p) \\
\downarrow & & \downarrow \text{id} \\
\text{Aut} (\pi_1, \partial Y^p) & \xrightarrow{\text{Aut} (\pi_1 Y^p)} & \text{Aut} (\pi_1, \partial Y^p)
\end{array}
\]

**Lemma 2.** — \( A \) and \( B \) are surjective.

**Proof.** — We consider a handle-decomposition, given once for all:

\[
Y^p = D_1 + (\varphi_1^p) + \ldots + (\varphi_i^p)
\]

where \( (\varphi_i^p) \) corresponds to the handle \( D_i^p \times D_i^p \).

We orient the \( D_i^p \)'s and we chose a base-point \( x_0 \in \partial D_i = \cup_i \text{Image} (\varphi_i^p) \).

The spines \([D_i^p]\) will determine then a basis \( x^p, \ldots, x^p \) for \( \pi = \pi_1 (Y^p, x_0) \).

We define \( \Phi_1, \Phi_2, \Phi_3 \in \text{Aut} (\pi) \) by:

(i) \( \Phi_1 (x^p) = x^p \) if \( i \neq l, k \), and \( \Phi_1 (x^p) = x^p \); \( \Phi_1 (x^p) = x^p \);

(ii) \( \Phi_2 (x^p) = x^p \) if \( i \neq 1 \), \( \Phi_2 (x^p) = x^p \); \( \Phi_2 (x^p) = x^p \);

(iii) \( \Phi_3 (x^p) = x^p \) if \( i \neq 1 \), \( \Phi_3 (x^p) = x^p \).

In order to prove our lemma, it suffices to exhibit three diffeomorphisms \( H_i : (Y^p, x_0) \rightarrow (Y^p, x_0) \) \( (i = 1, 2, 3) \) such that \( (H_i)_* = \Phi_i \).

In the new handle-decomposition for \( Y^p \), induced by \( H_i \), the \([D_i^p]\)'s will determine the basis \( \Phi_i (x^p) \) of \( \pi \).

The construction of \( H_i, H_3 \) is an elementary exercise. In order to define \( H_3 \), we start by considering:

\[
\overline{Y}^p = (Y^p \cup \partial Y^p \times (0, 1))/x_0 \times (0, 1),
\]

where the notation means that we glue \( \partial Y^p \times (0, 1) \) to \( Y^p \), along \( \partial Y^p = \partial Y^p \times 0 \) and afterwards we contract the fiber \( x_0 \times (0, 1) \) to a point. \( \overline{Y}^p \) collapses onto \( Y^p \), but on the other hand \( \overline{Y}^p \) and \( Y^p \) can be identified by a (more or less) canonical diffeomorphism leaving \( x_0 \) fixed.
Inside $\overline{Y}^p$ we can slide the handle $D_1^i \times D_2^i$ (of $Y^p$) along $D_1^i \times D_2^i$ (using the positive orientation of $D_1^i$), without touching $x_o$. This changes $Y^p$ into a new subset : $Y^p \subset \overline{Y}^p$, diffeotopic to $Y^p \subset \overline{Y}^p$. $Y^p$ has a natural handle-decomposition [induced by (2) and by the slide] and since $\overline{Y}^p$ collapses onto $Y^p$ one gets a handle-decomposition of $\overline{Y}^p$ (hence of $Y^p$):

\[(3)\quad Y^p = D_1 + (\psi_1^p) + \ldots + (\psi_2^p)\]

[where $x_o \in \partial D_1 = \cup_l \text{Image } (\psi_i^p)$]. Since

\[
\cup_l \text{Image } (\psi_i^p) (\cup_l \text{Image } (\psi_i^p))
\]

is just a collection of disjoint disks in $\partial D_1 = S_0$, we can find a diffeomorphism :

\[
C : D_1 + (\psi_1^p) + \ldots \rightarrow D_1 + (\psi_1) + \ldots
\]

such that $C (D_1, x_o) = (D_1, x_o)$, $C (D_1^i \times D_2^i) = D_1^i \times D_2^i$, $C$ respects the orientations of the 1-handles. Combining (2), (3) and $C$, we get our $H_*$.

**Remark.** — The same argument holds for $p \neq (S_1 \times D_0) (n \geq 2)$. On the other hand, using a similar proof, we can show that $\text{Aut } (H_1(p \neq (S_1 \times D_0)))$ where $H_* = \text{integral homology}$, $n \geq 2$, is generated by $\pi_0 (\text{Diff } (p \neq (S_1 \times D_0)))$.

We will also need the following

**Lemma 3.** — Let

\[
f : p \neq (S_1 \times S_2) \rightarrow p \neq (S_1 \times S_2)
\]

be an orientation-preserving homeomorphism inducing :

\[
f_{\ast, \ast} : \pi_1(p \neq (S_1 \times S_2)) \rightarrow \pi_1(p \neq (S_1 \times S_2)).
\]

If $f_{\ast, \ast}$ is the identity then $f_{\ast, \ast}$ is also the identity.

**Proof.** — $f$ lifts to the universal covering space :

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{f} & \tilde{X} \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & X
\end{array}
\]
where $X = \rho \neq (S_1 \times S_2)$. One has a commutative diagram:

$$
\begin{array}{ccc}
H_\delta (\tilde{X}) & \xrightarrow{f_*} & H_\delta (\tilde{X}) \\
\cong \downarrow & & \cong \downarrow \\
\pi_2 (X) & \xrightarrow{f_*} & \pi_2 (X)
\end{array}
$$

(Hurewicz)

Lemma 3 follows now from:

**Lemma 4.** — Let $X_n$ be a closed orientable topological manifold and $f : X_n \to X_n$ an orientation preserving homeomorphism, such that $f_1 : \pi_1 (X_n) \to \pi_1 (X_n)$ is the identity map. Then

$$
\tilde{f}_{\#} : H_{n-1} (\tilde{X}_n, Z) \to H_{n-1} (\tilde{X}_n, Z)
$$

is also the identity map.

**Proof.** — Since $H^1 (\tilde{X}_n, Z) = 0$ one has a canonical isomorphism $H^1_c (\tilde{X}_n, Z) \cong H^1 (\pi, Z [\pi])$, where $\pi = \pi_1 (X_n)$. This isomorphism is functorial, hence the following diagramm is commutative:

$$
\begin{array}{ccc}
H^1_c (\tilde{X}_n, Z) & \cong & H^1 (\pi, Z [\pi]) \\
\downarrow \tilde{f}^* & & \downarrow \tilde{f}^* \\
H^1_c (\tilde{X}_n, Z) & \cong & H^1 (\pi, Z [\pi])
\end{array}
$$

Since $f^* = \text{identity}$, it follows that $\tilde{f}^*$ is the identity too. On the other hand, one has an isomorphism (the Poincaré duality):

$$
H^1_c (\tilde{X}_n, Z) \cong H_{n-1} (\tilde{X}_n, Z),
$$

which is functorial for maps preserving the fundamental class. Now one deduces easily that $\tilde{f}_{\#}$ is the identity map.

**Remark.** — Let $b \tilde{X}_n$ be the space of ends of $\tilde{X}_n$ (which is a compact totally discontinuous space). Any homeomorphism $g : X_n \to X_n$ induces a homeomorphism $\tilde{g} : b \tilde{X}_n \to b \tilde{X}_n$. If $f$ is like in lemma 4, $\tilde{f} : b \tilde{X}_n \to b \tilde{X}_n$ is the identity.

Now we can prove our theorem $\Lambda$. We consider the identifications $\alpha$, $\beta$ from the beginning of this section. Lemma 2 tells us that we can
always find \( H \in \text{Diff} (Y^p) \) such that, if \( H_1 = H \mid \partial Y^p \), the following diagramm is commutative:

\[
\begin{array}{ccc}
\pi_1 (\partial X^p) & \xrightarrow{\approx} & \pi_1 (p \neq (S_1 \times S_1)) \\
\downarrow h_* & & \uparrow \beta_* \\
\pi_1 (\partial Y^p) & \xrightarrow{(H_1)_*} & \pi_1 (\partial Y^p)
\end{array}
\]

(4)

Let us assume for the time being that

\[ \beta \circ H_1 \circ h \circ x^{-1} : p \neq (S_1 \times S_1) \rightarrow p \neq (S_1 \times S_1) \]

is orientation-preserving. Consider \( x_1 \in S'_1 \) [see formula (0)] and the embedded 2-spheres:

\[ \Sigma' = S'_1 \times x_1 \subset p \neq (S_1 \times S_1). \]

From lemma 3, it follows that \( \Sigma' \) and \( \beta \circ H_1 \circ h \circ x^{-1} (\Sigma') \) are homotopic. From [1], section 5, it follows now that there exists an isotopy \( H' \in \text{Diff} (Y^p) (t \in (0, 1)) \) such that \( H' = H \), and

\[ \Sigma' = \beta \circ H'_1 \circ h \circ x^{-1} (\Sigma') \subset p \neq (S_1 \times S_1). \]

One can also remark that the diffeomorphisms \( H_2 (x) \) from [1], section 5.3, extend to elements of \( \text{Diff} (p \neq (D_1 \times S_1)) \). Hence, by [1], 5.4, there exists an \( L \in \text{Diff} (Y^p) \) such that : \( \beta \circ L \circ H'_1 \circ h \circ x^{-1} = \text{id} \). Since this means \( \beta^{-1} x = (L \circ H'_1 \mid \partial Y^p) \circ h \), lemma 1 tells us that \( X^p \cup_h Y^p = S_1 \) (mark that no diffeomorphism of \( X^p \) was needed here).

If \( \beta \circ H_1 \circ h \circ x^{-1} \) is not orientation-preserving, we can change (4) into :

\[
\begin{array}{ccc}
\pi_1 (\partial X^p) & \xrightarrow{\approx} & \pi_1 (\partial X^p) \\
\downarrow h_* & & \uparrow \beta_* \\
\pi_1 (\partial Y^p) & \xrightarrow{(H_1)_*} & \pi_1 (\partial Y^p)
\end{array}
\]

(5)

where \( F_1 = F \mid \partial X^p \), \( F \in \text{Diff} (X^p) \) with \( F \) orientation-reversing and \( (F_1)_* = \text{id} \). From here on the proof continues as before.

3. The proof of corollary B

Corollary B follows from theorem A' and the following :

**Lemma 5.** — Let \( X^p, (\psi_i^p) \) be as in the statement of corollary B. Then :

\[ \partial (X^p + (\psi_1^p) + \ldots + (\psi_5^p)) = S_2 \quad (\text{diffeomorphism}). \]
Proof. — The condition on $H_2$ implies that $X^p + (\psi_1^i) + \ldots$ is contractible. $\psi_2^i$ stands for the attaching map:

$$\psi_2^i : S^i \times I \hookrightarrow \partial (X^p + (\psi_1^i) + \ldots + (\psi_{-1}^i)).$$

For each $i$, $\psi_2^i \left( S^i \times \frac{1}{2} \right)$ is a 2-cycle of $\partial (X^p + (\psi_1^i) + \ldots + (\psi_{-1}^i))$ not homologous to 0. [Otherwise $(\psi_2^i)$ would introduce a 3-cycle in $X^p + (\psi_1^i) + \ldots + (\psi_{-1}^i)$ which could never be killed by adding 3-cells, only.]

Hence, the $\psi_2^i \left( S^i \times \frac{1}{2} \right) \subset \partial X^p = p \neq (S^i \times S^i)$ are embedded, disjointed, homologically independent. It follows easily that

$$(p \neq (S^i \times S^i), \cup \psi_2^i \left( S^i \times \frac{1}{2} \right))$$

is diffeomorphic to $(p \neq (S^i \times S^i), \cup_i S^i \times x^i)$ a.s.o. (Caution: This diffeomorphism does not necessarily extend to $X^p$.)

4. Final remarks

We will place now corollary B, which is the starting point of our investigation, in its proper context.

If $n \geq 4$, $n - 2 > \lambda \geq 1$, let $C_{n, \lambda}$ denote the class of smooth manifolds of the form

$$X = D_n + (\varphi_1^i) + \ldots + (\varphi_\lambda^i) + (\varphi_{n+1}^i) + \ldots + (\varphi_{n+1}^i)$$

such that $X$ is contractible.

The $h$-cobordism theorem of Smale implies that : $X \in C_{n, \lambda} \Rightarrow X = D_n$ provided that : $n \geq 6, n - 3 > \lambda$. On the other hand, $C_{n, 1}$ (and in general $C_{n, n-3}$) contains elements with non-simply-connected boundary.

Here are some conjectures for the cases which are not settled:

- **C (1)**: $X \in C_{4,2} \Rightarrow X = D_4$,
- **C (2)**: $X \in C_{4,2} \Rightarrow \pi_1 \partial X = 0$,
- **C (3)**: $X \in C_{6,1} \Rightarrow X = D_6$,
- **C (4)**: $X \in C_{6,1} \Rightarrow \partial X = S_4$.

$C (2)$ is a very modest version of $C (1)$, while $C (3)$ and $C (4)$ are clearly equivalent. Our corollary B is just the simplest case where we can hope to check $C (1)$. From [2], [3] and very easy arguments, it follows that $C (1) \Rightarrow$ the Poincaré conjecture in dimensions 3 and 4. Also $C (2)$
and the Poincaré conjecture in dimensions 3 and 4 implies the following weak version of the Poincaré conjecture in dimension 4: if \( \Sigma_4 \) is a smooth oriented homotopy 4-sphere, then:

\[
\Sigma_4 \not\cong (\Sigma_4) = S^4 \quad \text{(diffeomorphism).}
\]

The Poincaré conjecture in dimension 4 is equivalent to (6) and the smooth 4-dimensional Schoenflies conjecture.

REFERENCES


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