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A NOTE ON 4-DIMENSIONAL HANDLEBODIES

BY

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1. Introduction

We prove the following theorem :

THEOREM A. — *Let* X^p, Y^p *be the following smooth* 4-*manifolds* :

$$X^p = p \# (S_2 \times D_2), \qquad Y^p = p \# (S_1 \times D_3).$$

Consider a diffeomorphism $h : \partial X^p \to \partial Y^p$ *and the smooth closed* 4-*manifold obtained by gluing* X^p *and* Y^p *along* $h : X^p \cup_h Y^p$.

$X^p \cup_h Y^p$ *is diffeomorphic to* S_4.

Theorem A is clearly equivalent to the following :

THEOREM A'. — *Let* X^p *be as before, and consider p handles of index* 3, *attached successively to* X^p :

$$\varphi_3^i : \quad S_2^i \times D_1^i \hookrightarrow \partial (X^p + (\varphi_3^1) + \ldots + (\varphi_3^{i-1})),$$

where $S_2^i \times D_1^i = \partial D_3^i \times D_1^i \subset \partial (D_3^i \times D_1^i)$ *and* $i = 1, \ldots, p$.

Assume that $\partial (X^p + (\varphi_3^1) + \ldots + (\varphi_3^p)) = S_3$, *and consider a handle of index* 4 :

$$\varphi_4 : \quad \partial D_4 \underset{\approx}{\hookrightarrow} \partial (X^p + (\varphi_3^1) + \ldots + (\varphi_3^p)),$$

attached to $X^p + (\varphi_3^1) + \ldots + (\varphi_3^p)$. *One has* :

$$X^p + (\varphi_3^1) + \ldots + (\varphi_3^p) + (\varphi_4) = S_4 \quad (\textit{diffeomorphism}).$$

This result implies the following :

COROLLARY B. — *Let X^p be as before, and consider p handles of index 3, attached successively to X^p :*

$$\psi_3^i : \quad S_2^i \times D_1^i \hookrightarrow \partial (X^p + (\psi_3^1) + \ldots + (\psi_3^{i-1})) \qquad (i = 1, \ldots, p).$$

If

$$H_2 (X^p + (\psi_3^1) + \ldots + (\psi_3^p), Z) = 0$$

then

$$X^p + (\psi_3^1) + \ldots + (\psi_3^p) = D_4 \quad (\textit{diffeormophism}).$$

2. The proof of theorem A

One has « canonical » identifications :

$$\begin{array}{ll} & \partial X^p \xrightarrow[\approx]{\alpha} \\ (0) & \qquad (S_2^1 \times S_1^1) \# \ldots \# (S_2^p \times S_1^p) = p \# (S_2 \times S_1), \\ & \partial Y^p \xrightarrow[\approx]{\beta} \end{array}$$

which will be given, once for all. It is obvious that

$$X^p \cup_{\beta^{-1}\alpha} Y^p = S_4.$$

LEMMA 1. — *The following two statements are equivalent :*

(i) $X^p \cup_h Y^p = S_4$.

(ii) *There exist diffeomorphisms*: $G : X^p \to X^p$, $H : Y^p \to Y^p$, *such that* :

$$(1) \qquad \beta^{-1} \alpha = (H \mid \partial Y^p) \circ h \circ (G \mid \partial X^p).$$

Proof. — If f_1, f_2 are two differentiable embeddings $f_i : Y^p \to S_4$, it is obvious that the pairs $(S_4, f_1 Y^p)$, $(S_4, f_2 Y^p)$ are diffeomorphic. Hence, if $X^p \cup_h Y^p = S_4 = X^p \cup_{\beta^{-1}\alpha} Y^p$, there exists a diffeomorphism : $X^p \cup_h Y^p \to X^p \cup_{\beta^{-1}\alpha} Y^p$ sending X^p onto X^p and Y^p onto Y^p. This shows that (i) $\Rightarrow$ (ii).

On the other hand, the equality (1) tells us that G and H can be patched together so as to give diffeomorphism :

$$X^p \cup_h Y^p \leftrightarrow X^p \cup_{\beta^{-1}\alpha} Y^p.$$

Hence (ii) $\Rightarrow$ (i).

REMARK. — The implication (ii) $\Rightarrow$ (i) holds whenever we glue two n-manifolds along their (diffeomorphic) boundaries, while (i) $\Rightarrow$ (ii) is very exceptional.

We consider now $\pi_1 Y^p$ = the free group with p generators, and we remark that, if $i : \partial Y^p \hookrightarrow Y^p$ is the natural inclusion, then :

$$i_* : \quad \pi_1 \partial Y^p \to \pi_1 Y^p$$

is bijective. Let Diff Y^p be the group of diffeomorphisms of Y^p and Aut $(\pi_1 Y^p)$ [resp. Aut $(\pi_1 \partial Y^p)$] the group of automorphisms of $\pi_1 Y^p$ (resp. $\pi_1 \partial Y^p$). We have a commutative triangle of natural homomorphisms :

$$\begin{array}{ccc} & \overset{A}{\nearrow} & \mathrm{Aut}\,(\pi_1 Y^p) \\ \pi_0\,(\mathrm{Diff}\ Y^p) & & \downarrow \mathrm{id} \\ & \underset{B}{\searrow} & \mathrm{Aut}\,(\pi_1 \partial Y^p) \end{array}$$

LEMMA 2. — A *and* B *are surjective.*

Proof. — We consider a handle-decomposition, given once for all :

$$(2) \qquad Y^p = D_4 + (\varphi_1^1) + \dots + (\varphi_1^p)$$

where (φ_1^i) corresponds to the handle $D_1^i \times D_3^i$.

We orient the D_1^i's and we chose a base-point

$$x_0 \in \partial D_4 - \cup_i \text{Image}\,(\varphi_1^i).$$

The spines $[D_1^i]$ will determine then a basis $x^1, \dots, x^p$ for $\pi = \pi_1(Y^p, x_0)$. We define $\Phi_1, \Phi_2, \Phi_3 \in \mathrm{Aut}\,(\pi)$ by :

(i) $\Phi_1(x^i) = x^i$ if $i \neq l, k$, and $\Phi_1(x^l) = x^k$, $\Phi_1(x^k) = x^l$;

(ii) $\Phi_2(x^i) = x^i$ if $i \neq 1$, $\Phi_2(x^1) = (x^1)^{-1}$;

(iii) $\Phi_3(x^i) = x^i$ if $i \neq 1$, $\Phi_3(x^1) = x^1 x^2$.

In order to prove our lemma, it suffices to exhibit three diffeomorphisms $H_i : (Y^p, x_0) \to (Y^p, x_0)$ $(i = 1, 2, 3)$ such that $(H_i)_* = \Phi_i$.

[In the new handle-decomposition for Y^p, induced by H_i, the $[D_1^j]$'s will determine the basis $\Phi_i(x^j)$ of π.]

The construction of H_1, H_2 is an elementary exercise. In order to define H_3, we start by considering :

$$\overline{Y}^p = (Y^p \cup \partial Y^p \times [0, 1])/x_0 \times [0, 1],$$

where the notation means that we glue $\partial Y^p \times [0, 1]$ to Y^p, along $\partial Y^p \equiv \partial Y^p \times 0$ and afterwards we contract the fiber $x_0 \times [0, 1]$ to a point. $\overline{Y}^p$ collapses onto Y^p, but on the other hand $\overline{Y}^p$ and Y^p can be identified by a (more or less) canonical diffeomorphism leaving x_0 fixed.

Inside $\overline{Y}^p$ we can slide the handle $D_1^1 \times D_3^1$ (of Y^p) along $D_1^2 \times D_3^2$ (using the positive orientation of D_1^2), without touching x_0. This changes Y^p into a new subset : ${}_1Y^p \subset \overline{Y}^p$, diffeotopic to $Y^p \subset \overline{Y}^p$. ${}_1Y^p$ has a natural handle-decomposition [induced by (2) and by the slide] and since $\overline{Y}^p$ collapses onto ${}_1Y^p$ one gets a handle-decomposition of $\overline{Y}^p$ (hence of Y^p!):

$$(3) \qquad Y^p = D_4 + (\psi_1^1) + \ldots + (\psi_1^p)$$

[where $x_0 \in \partial \mathrm{D}_4 - \cup_i \operatorname{Image}(\psi_1^i)$]. Since

$$\cup_i \operatorname{Image}(\varphi_1^i) \; (\cup_i \operatorname{Image}(\psi_1^i))$$

is just a collection of disjoint disks in $\partial \mathrm{D}_4 = S_3$, we can find a diffeomorphism :

$$C: \; D_4 + (\varphi_1^1) + \ldots \to D_4 + (\psi_1^1) + \ldots$$

such that $C(D_4, x_0) = (D_4, x_0)$, $C(D_1^i \times D_3^i) = D_1^i \times D_3^i$, C respects the orientations of the 1-handles. Combining (2), (3) and C, we get our H_3.

Remark. — The same argument holds for $p \# (S_1 \times D_n)$ $(n \geq 2)$. On the other hand, using a similar proof, we can show that $\operatorname{Aut}(\mathrm{H}_\lambda(p \# (\mathrm{S}_\lambda \times D_n))$ where $H_* =$ integral homology, $n \geq 2$, is generated by $\pi_0(\operatorname{Diff}(p \# (S_\lambda \times D_n)))$.

We will also need the following

Lemma 3. — *Let*

$$f: \; p \# (S_1 \times S_2) \to p \# (S_1 \times S_2)$$

be an orientation-preserving homeomorphism inducing :

$$f_{i,*}: \; \pi_i(p \# (S_1 \times S_2)) \to \pi_i(p \# (S_1 \times S_2)).$$

If $f_{1,*}$ is the identity then $f_{2,*}$ is also the identity.

Proof. — f lifts to the universal covering space :

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{X} \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & X \end{array}$$

where $X = p \# (S_1 \times S_2)$. One has a commutative diagramm :

$$\begin{array}{ccc} H_2(\tilde{X}) & \xrightarrow{\tilde{f}_*} & H_2(\tilde{X}) \\ \approx\downarrow & & \approx\downarrow \\ \pi_2(X) & \xrightarrow{f_{2,*}} & \pi_2(X) \end{array} \qquad \text{(Hurewicz)}$$

Lemma 3 follows now from :

LEMMA 4. — *Let X_n be a closed orientable topological manifold and $f : X_n \to X_n$ an orientation preserving homeomorphism, such that $f_{1,*} : \pi_1(X_n) \to \pi_1(X_n)$ is the identity map. Then*

$$\tilde{f}_* : \; H_{n-1}(\tilde{X}_n, Z) \to H_{n-1}(\tilde{X}_n, Z)$$

is also the identity map.

Proof. — Since $H^1(\tilde{X}_n, Z) = 0$ one has a canonical isomorphism $H^1_c(\tilde{X}_n, Z) = H^1(\pi, Z[\pi])$, where $\pi = \pi_1(X_n)$. This isomorphism is functorial, hence the following diagramm is commutative :

$$\begin{array}{ccc} H^1_c(\tilde{X}_n, Z) & \xrightarrow[\approx]{} & H^1(\pi, Z[\pi]) \\ \downarrow \tilde{f}^* & & \downarrow f^*_* \\ H^1_c(\tilde{X}_n, Z) & \xrightarrow[\approx]{} & H^1(\pi, Z[\pi]) \end{array}$$

Since $f^*_* =$ identity, it follows that $\tilde{f}^*$ is the identity too. On the other hand, one has an isomorphism (the Poincaré duality) :

$$H^1_c(\tilde{X}_n, Z) \xrightarrow[D]{} \mathrm{H}_{n-1}(\tilde{X}_n, Z),$$

which is functorial for maps preserving the fundamental class. Now one deduces easily that $\tilde{f}_*$ is the identity map.

REMARK. — Let $b\tilde{X}_n$ be the space of ends of $\tilde{X}_n$ (which is a compact totally discontinuous space). Any homeomorphism $g : X_n \to X_n$ induces a homeomorphism $\tilde{\tilde{g}} : b\tilde{X}_n \to b\tilde{X}_n$. If f is like in lemme 4, $\tilde{\tilde{f}} : b\tilde{X}_n \to b\tilde{X}_n$ is the identity.

Now we can prove our theorem A. We consider the identifications α, β from the beginning of this section. Lemma 2 tells us that we can always find $H \in \mathrm{Diff}\,(Y^p)$ such that, if $H_1 = H \mid \partial Y^p$, the following diagramm is commutative :

$$(4) \qquad \begin{array}{ccc} \pi_1(\partial X^p) & \xrightarrow{\alpha_*} & \pi_1(p \# (S_2 \times S_1)) \\ \downarrow h_* & & \uparrow \beta_* \\ \pi_1(\partial Y^p) & \xrightarrow{(H_1)_*} & \pi_1(\partial Y^p) \end{array}$$

Let us assume for the time being that

$$\beta \circ H_1 \circ h \circ \alpha^{-1} : \quad p \# (S_2 \times S_1) \to p \# (S_2 \times S_1)$$

is orientation-preserving. Consider $x_i \in S_1^i$ [*see* formula (0)] and the embedded 2-spheres :

$$\Sigma^i = S_2^i \times x_i \subset p \# (S_2 \times S_1).$$

From lemma 3, it follows that Σ^i and $\beta \circ H_1 \circ h \circ \alpha^{-1}(\Sigma^i)$ are homotopic. From [1], section 5, it follows now that there exists an isotopy $H^t \in \mathrm{Diff}\,(Y^p)$ $(t \in [0, 1])$ such that $H^0 = H$, and

$$\Sigma^i = \beta \circ H_1^1 \circ h \circ \alpha^{-1}(\Sigma^i) \subset p \# (S_2 \times S_1).$$

One can also remark that the diffeomorphisms $H_\Sigma(\alpha)$ from [1], section 5.3, extend to elements of $\mathrm{Diff}\,(p \# (D_3 \times S_1))$. Hence, by [1], 5.4, there exists an $L \in \mathrm{Diff}\,(Y^p)$ such that : $\beta \circ \mathrm{L}_1 \circ H_1^1 \circ h \circ \alpha^{-1} =$ identity. Since this means $\beta^{-1}\alpha = ((\mathrm{L} \circ H^1) \mid \partial Y^p) \circ h$, lemma 1 tells us that $X^p \cup_h Y^p = S_4$ (mark that no diffeomorphism of X^p was needed here !).

If $\beta \circ H_1 \circ h \circ \alpha^{-1}$ is not orientation-preserving, we can change (4) into :

$$(5) \qquad \begin{array}{ccccc} \pi_1(\partial X^p) & \xrightarrow{(F_1)_*} & \pi_1(\partial X^p) & \xrightarrow{\alpha_*} & \pi_1(p \# (S_2 \times S_1)) \\ \downarrow h_* & & & & \uparrow \beta_* \\ \pi_1(\partial Y^p) & & \xrightarrow{(H_1)_*} & & \pi_1(\partial Y^p) \end{array}$$

where $F_1 = F \mid \partial X^p$, $F \in \mathrm{Diff}\,(X^p)$ with F orientation-reversing and $(F_1)_* =$ the identity. From here on the proof continues as before.

3. The proof of corollary B

Corollary B follows from theorem A′ and the following :

LEMMA 5. — *Let X^p, (ψ_3^i) be as in the statement of corollary B. Then :*

$$\partial\,(X^p + (\psi_3^1) + \ldots + (\psi_3^p)) = S_3 \quad (\textit{diffeomorphism}).$$

Proof. — The condition on H_2 implies that $X^p + (\psi_3^i) + \ldots$ is contractible. ψ_3^i stands for the attaching map :

$$\psi_3^i : \quad S_2 \times I \hookrightarrow \partial (X^p + (\psi_3^1) + \ldots + (\psi_3^{i-1})).$$

For each i, $\psi_3^i\left(S_2 \times \frac{1}{2}\right)$ is a 2-cycle of $\partial (X^p + (\psi_3^1) + \ldots + (\psi_3^{i-1}))$ *not* homologous to 0. [Otherwise (ψ_3^i) would introduce a 3-cycle in $X^p + (\psi_3^1) + \ldots + (\psi_3^i)$ which could never be killed by adding 3-cells, only.]

Hence, the $\psi_3^i\left(S_2 \times \frac{1}{2}\right) \hookrightarrow \partial X^p = p \# (S_2 \times S_1)$ are embedded, disjoined, homologically independant. It follows easily that

$$\left(p \# (S_2 \times S_1), \cup \psi_3^i\left(S_2 \times \frac{1}{2}\right)\right)$$

is diffeomorphic to $(p \# (S_2 \times S_1), \bigcup_i S_2^i \times x^i)$ a. s. o. (Caution : This diffeomorphism does *not* necessarily extend to X^p.)

4. Final remarks

We will place now corollary B, which is the starting point of our investigation, in its proper context.

If $n \geqq 4$, $n - 2 \geqq \lambda \geqq 1$, let $C_{n,\lambda}$ denote the class of smooth manifolds of the form

$$X = D_n + (\varphi_\lambda^1) + \ldots + (\varphi_\lambda^p) + (\varphi_{\lambda+1}^1) + \ldots + (\varphi_{\lambda+1}^p)$$

such that X is contractible.

The h-cobordism theorem of Smale implies that : $X \in C_{n,\lambda} \Rightarrow X = D_n$ provided that : $n \geqq 6$, $n - 3 > \lambda$. On the other hand, $C_{4,1}$ (and in general $C_{n,n-3}$) contains elements with non-simply-connected boundary.

Here are some conjectures for the cases which are not settled :

$$\begin{aligned}
C(1) &: \quad X \in C_{4,2} \Rightarrow X = D_4, \\
C(2) &: \quad X \in C_{4,2} \Rightarrow \pi_1 \partial X = 0, \\
C(3) &: \quad X \in C_{5,1} \Rightarrow X = D_5, \\
C(4) &: \quad X \in C_{5,1} \Rightarrow \partial X = S_4.
\end{aligned}$$

$C(2)$ is a very modest version of $C(1)$, while $C(3)$ and $C(4)$ are clearly equivalent. Our corollary B is just the simplest case where we can hope to check $C(1)$. From [2], [3] and very easy arguments, it follows that $C(1) \Rightarrow$ the Poincaré conjecture in dimensions 3 *and* 4. Also $C(2)$

and the Poincaré conjecture in dimensions 3 and 4 $\Rightarrow C(1)$. $C(3)$ ($C(4)$) implies the following weak version of the Poincaré conjecture in dimension 4 : if Σ_4 is a smooth oriented homotopy 4-sphere, then :

$$(6) \qquad \Sigma_4 \# (-\Sigma_4) = S_4 \quad \textit{(diffeomorphism)}.$$

The Poincaré conjecture in dimension 4 $\Leftrightarrow$ (6) and the smooth 4-dimensional Schoenfliess conjecture.

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