

# BULLETIN DE LA S. M. F.

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*Bulletin de la S. M. F.*, tome 99 (1971), p. 247-258

[<http://www.numdam.org/item?id=BSMF\\_1971\\_\\_99\\_\\_247\\_0>](http://www.numdam.org/item?id=BSMF_1971__99__247_0)

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## SOME REMARKS ON THE FORMAL POWER SERIES RING

BY

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Let  $R$  be an integral domain with identity, let  $X$  be an indeterminate over  $R$ , let  $S$  be the formal power series ring  $R[[X]]$ , and let  $G$  be a finite group of  $R$ -automorphisms of  $S$ . If  $R$  is a local ring (that is, a noetherian ring with unique maximal ideal  $M$ ), and if  $R$  is complete in the  $M$ -adic topology, then P. SAMUEL shows the existence of  $f \in S$  such that the ring  $S^G$  of invariants of  $G$  is  $R[[f]]$  [9].

This paper was motivated by an attempt to generalize this result. Specifically, we will prove that the same conclusion holds if  $R$  is any noetherian integral domain with identity whose integral closure is a finite  $R$ -module.

In our efforts to obtain this result, we have had to make strong use of the results of [7] and theorem (2.6) of [6]. The notion of topological completeness is essential. In paragraph 2, we develop the needed topological results and we make some additional comments concerning ideal-adic topologies under which  $R$  and  $R[[X_1, \dots, X_n]]$  are complete. Paragraph 3 extends the result of Samuel.

All rings in this paper are assumed to be commutative and, except for one brief mention in paragraph 2, to contain an identity element. The symbols  $\omega$  and  $\omega_0$  are used throughout the paper to denote the sets of positive and nonnegative integers, respectively.

The author wishes to express his sincere thanks to Professor Robert GILMER for his advice and encouragement during the preparation of this paper.

### 1. Notation and terminology.

Throughout the paper, we denote by  $R$  a commutative ring with identity, and by  $S_n$  the formal power series ring  $R[[X_1, \dots, X_n]]$  in  $n$ -indeterminates over  $R$ . If  $\{a_\alpha\}_{\alpha \in \Lambda}$  is a collection of elements of  $R$ ,

then  $(\{a_\alpha\}_{\alpha \in \Lambda})$  will denote the ideal of  $R$  generated by  $\{a_\alpha\}_{\alpha \in \Lambda}$ . If  $A$  is an ideal of  $R$  and if  $T$  is a unitary overring of  $R$ , then  $AT$  will denote the extension of  $A$  to  $T$ . In particular, if  $a \in R$ , then  $(aT)$  will be the ideal of  $T$  generated by  $a$ . We write  $\chi_n$  to denote the ideal of  $S_n$  generated by  $\{X_i\}_{i=1}^n$ , and if there is no ambiguity, we will simply write  $\chi$ . If  $A$  is an ideal of  $R$ , then we use  $(A, \chi_n)$  [or  $(A, \chi)$ ] to denote the ideal  $AS_n + \chi_n$  of  $S_n$ . Moreover, we will write  $A[[X_1, \dots, X_n]]$  to denote the ideal of  $S_n$  consisting of those power series, each coefficient of which is in  $A$ . We remark that in general  $A[[X_1, \dots, X_n]]$  is distinct from  $AS_n$ , but, if  $A$  is finitely generated, equality holds [3].

A collection of ideals  $\{A_i\}_{i \in \omega}$  of the ring  $R$  will be called a *d-sequence* of ideals provided that for any  $n, m \in \omega$ , there exists a  $k \in \omega$ , depending on  $n$  and  $m$ , such that  $A_k \subseteq A_n \cap A_m$ . If  $\Omega$  is the topology induced on  $R$  by the *d-sequence*  $\{A_i\}_{i \in \omega}$ , then  $R$  is a topological ring under the topology  $\Omega$ . It is well known that  $(R, \Omega)$  is Hausdorff if, and only if,  $\bigcap_{i \in \omega} A_i = (0)$ . We say that  $(R, \Omega)$  is *complete* if each Cauchy sequence of  $(R, \Omega)$  converges to an element of  $R$ . If there exists an ideal  $A$  of  $R$  such that  $A^i = A_i$  for each  $i \in \omega$ , then  $\Omega$  will be called the *A-adic topology*, and we write  $(R, A)$  to denote the topological ring  $R$  under this topology.

## 2. Topological aspects.

It follows, from [6], lemma (1.1), that if  $A$  is an ideal of  $R$  such that  $R$  is a complete Hausdorff space in the *A-adic topology*, then  $(R(a))$  is a complete Hausdorff space for each  $a \in A$ . Conversely, observe that if  $A = (a_1, \dots, a_n)$ , then  $A^{kn} \subseteq (a_1^k, \dots, a_n^k)$  for any  $k \in \omega$ . It follows, therefore, that if  $B_k = (a_1^k, \dots, a_n^k)$  for each  $k \in \omega$ , and if  $\Omega$  is the topology induced on  $R$  by the sequence of ideals  $\{B_k\}_{k \in \omega}$ , then  $\Omega$  is equivalent to the *A-adic topology*. Hence,  $(R, A)$  is complete if, and only if,  $(R, \Omega)$  is complete. Extending [7], theorem (3.4), to an ideal with a finite basis, it is straightforward to show that if  $(R, (a_i))$  is complete for each  $i$ , then  $(R, \Omega)$ , and hence  $(R, A)$ , is complete. Thus, if  $(R, A)$  is Hausdorff, we have the converse of [6], lemma (1.1), in case  $A$  is finitely generated.

It should be remarked here that the assumption that  $(R, A)$  is Hausdorff is necessary as even the hypothesis that  $(R, (a_i))$  is a complete Hausdorff space for each  $i$  does not imply that  $(R, A)$  is Hausdorff. In [6], lemma (1.1), it is shown that if  $(R, (a_i))$  is a complete Hausdorff space, then  $a_i$  belongs to the Jacobson radical  $J$  of  $R$ ; hence, if  $\bigcap_{k \in \omega} J^k = (0)$ ,

then  $(R, A)$  is Hausdorff. In particular, if  $R$  is noetherian, then this is true ([5], p. 12). Therefore, we have proved the following theorem.

**THEOREM 2.1.** — *Let  $A$  be a finitely generated ideal of the ring  $R$ , and suppose that  $(R, A)$  is Hausdorff. Then  $(R, A)$  is complete if, and only if,  $(R, (a))$  is complete for each  $a \in A$ . Thus, if  $R$  is noetherian, and if  $A$  is an ideal of  $R$ , then  $(R, A)$  is a complete Hausdorff space if, and only if,  $(R, (a))$  is a complete Hausdorff space for each  $a \in A$ .*

As a corollary to theorem 2.1, we have the following generalization of [11] (theorem 14, p. 275).

**COROLLARY 2.2.** — *Let  $A$  be an ideal of the ring  $R$ , such that  $(R, A)$  is a complete Hausdorff space. Then, for any finitely generated ideal  $B \subseteq A$ ,  $(R, B)$  is a complete Hausdorff space.*

It might be noted here that one direction of theorem 2.1 can be proved directly for the case when  $R$  is not assumed to have an identity element. But the proof of lemma 1.1 of [6] depends rather strongly on the assumption that  $R$  has an identity element. Nevertheless, lemma 1.1 of [6] and hence theorem 2.1 and corollary 2.2 of this paper, are true for the case when  $R$  does not possess an identity. For, let  $T = R[e]$ , the ring obtained by the canonical adjunction of an identity to  $R$ . (Choose  $e$  so that  $T$  has characteristic zero). Then any ideal  $A$  of  $R$  is also an ideal of  $T$ , and it can be shown that  $(R, A)$  is complete (Hausdorff) if, and only if,  $(T, A)$  is complete (Hausdorff). From this fact, it follows easily that the preceding results of this section (and lemma 1.1 of [6]) extend to the case when  $R$  does not have an identity element.

For the remainder of the paper, we will assume that  $R$  possesses an identity. We now turn our attention to the question of completeness in  $S_n = R[[X_1, \dots, X_n]]$ . Let  $\{B_k\}_{k \in \omega}$  be a  $d$ -sequence of ideals of  $R$  and let  $\Omega$  denote the topology induced on  $R$  by  $\{B_k\}_{k \in \omega}$ . For each  $k \in \omega$ , let  $U_k$  and  $V_k$  denote the ideals  $B_k[[X_1, \dots, X_n]]$  and  $B_k S_n + \chi^k$  of  $S_n$ , respectively, and let  $\Lambda_1$  and  $\Lambda_2$  denote the topologies induced on  $S_n$  by the  $d$ -sequences  $\{U_k\}_{k \in \omega}$  and  $\{V_k\}_{k \in \omega}$ , respectively.

**PROPOSITION 2.3.** — *Using the notation of the preceding paragraph, the following are equivalent :*

- (i)  $(R, \Omega)$  is complete (Hausdorff);
- (ii)  $(S_n, \Lambda_1)$  is complete (Hausdorff);
- (iii)  $(S_n, \Lambda_2)$  is complete (Hausdorff).

While verification of these statements is rather detailed, it is nevertheless straightforward, and we omit the proof here (see also [4], lemma 2).

Observe that if  $\Omega$  is the  $A$ -adic topology for some ideal  $A$  of  $R$ , then, since  $(A, \chi)^{2k} \subseteq A^k S_n + \chi^k \subseteq (A, \chi)^k$  for each  $k \in \omega$ , it follows that  $R$  is complete in the  $A$ -adic topology if, and only if,  $S_n$  is complete in the  $(A, \chi)$ -adic topology. Moreover, if  $A$  is finitely generated, then  $(AS_n)^k = A^k[[X_1, \dots, X_n]]$ , and hence, in this case,  $(R, A)$  is complete if, and only if,  $(S_n, AS_n)$  is complete.

REMARK 2.4. — Let  $V = (f_1, \dots, f_k)$  be a finitely generated ideal of  $S_n$ , and for each  $i = 1, \dots, k$ , let

$$f_i = \sum_{j=0}^{\infty} c_j^{(i)},$$

where  $c_j^{(i)}$  is zero or a form of degree  $j$  for each  $j \in \omega$ . If

$$A = (c_0^{(1)}, c_0^{(2)}, \dots, c_0^{(k)}),$$

and if  $(R, A)$  is a complete Hausdorff space, then  $(S_n, (A, \chi))$  is a complete Hausdorff space, and hence, by corollary 2.2, since  $V$  is finitely generated,  $(S_n, V)$  is a complete Hausdorff space.

The converse of the above is not true in general. For  $(R, A)$  need not be Hausdorff even if  $(S_n, V)$  is a complete Hausdorff space. To observe this, we make use of an example of GILMER [2]. GILMER establishes the existence of a ring  $R$ , admitting an  $R$ -automorphism  $\varphi$  of  $S = R[[X]]$ , such that  $\varphi(X) = a_0 - X$ , where  $\bigcap_{n \in \omega} (a_0)^n \neq (0)$ .

If  $V = (a_0 - X)$ , then  $S$  is a complete Hausdorff space in the  $V$ -adic topology ([8], theorem 4.5), but  $(R, (a_0))$  is not Hausdorff, since

$$\bigcap_{n \in \omega} (a_0)^n \neq (0).$$

However, if  $(S_n, V)$  is a complete Hausdorff space, then  $(R, A)$  is complete. To see this, we observe that if  $(S_n, V)$  is a complete Hausdorff space, then by theorem 2.3,  $(S_n, (f_i))$  is complete for each  $i = 1, \dots, k$ , and hence, since  $(S_n, (X_j))$  is complete for  $j = 1, \dots, n$ , it follows that  $S_n$  is complete in the  $(f_1, \dots, f_k, X_1, \dots, X_n)$ -adic topology. But

$$(f_1, \dots, f_k, X_1, \dots, X_n) = (c_0^{(1)}, \dots, c_0^{(k)}, X_1, \dots, X_n) = (A, \chi),$$

and thus,  $(S_n, (A, \chi))$  is complete. Therefore, by the comment following proposition 2.3,  $(R, A)$  is complete.

Finally, we observe that if  $(S_n, V)$  is a complete Hausdorff space, then  $V \subseteq \bar{J}$ , the Jacobson radical of  $S_n$ . But

$$\bar{J} = \left\{ \sum_{j=0}^{\infty} c_j \in S_n \mid c_0 \in J, \text{ the Jacobson radical of } R \right\},$$

and hence  $A \subseteq J$ . Thus, if  $\bigcap_{n \in \omega} J^n = (0)$ ,  $(R, A)$  is Hausdorff. In

particular, if  $R$  is noetherian, then  $\bigcap_{n \in \omega} J^n = (0)$ . Thus, in this case,

it follows that  $(S_n, V)$  is a complete Hausdorff space if, and only if,  $(R, A)$  is a complete Hausdorff space.

We conclude this section with an observation concerning the results of [6]. For any element

$$h = \sum_{i=0}^{\infty} a_i X^i \in R[[X]],$$

all of the results of [6], § 2, hold for  $h$ , provided there exists  $k \in \omega$  such that  $a_k$  is a unit of  $R$ , while the ideal  $(a_0, \dots, a_{k-1})$  generates a complete Hausdorff topology on  $R$ . Thus, if

$$f = f(X_1, \dots, X_n) = \sum_{i=0}^{\infty} c_i X_n^i \in S_{n-1}[[X_n]] = S_n,$$

then the results of [6], § 2, apply to  $f$ , if there exists  $k \in \omega$  such that  $c_k$  is a unit of  $S_{n-1}$ , while the ideal  $C = (c_0, \dots, c_{k-1})$  generates a complete Hausdorff topology on  $S_{n-1}$ .

However, if  $g = f(0, \dots, 0, X_n)$ , then

$$g = \sum_{i=0}^{\infty} a_i X_n^i \in R[[X_n]],$$

and it follows that, for each  $i \in \omega_0$ ,

$$c_i = a_i + \sum_{j=1}^{\infty} u_j^{(i)},$$

where  $u_j^{(i)}$  is either 0 or a form of degree  $j$  in  $S_{n-1}$  for each  $j \in \omega_0$ . Thus,  $c_k$  is a unit of  $S_{n-1}$  if, and only if,  $a_k$  is a unit of  $R$  ([11], p. 131), and, by remark 2.4,  $(S_{n-1}, C)$  is a complete Hausdorff space, if  $R$  is complete and Hausdorff in the  $A = (a_0, \dots, a_{k-1})$ -adic topology. The converse, is true, if  $\bigcap_{m \in \omega} J^m = (0)$ , where  $J$  is the Jacobson radical of  $R$ .

Thus, if  $R$  is a ring such that  $\bigcap_{m \in \omega} J^m = (0)$  (in particular, if  $R$  is noetherian), then the results of [6], § 2, apply to an element  $f = f(X_1, \dots, X_n)$

of  $S_n$  if, and only if, the results of [6], § 2 hold for the element  $g = f(o, \dots, o, X_n)$  of  $R[[X_n]]$ .

### 3. Finite groups of $R$ -automorphisms of $R[[X]]$ .

Throughout this section, we write  $S$  to denote the formal power series ring  $R[[X]]$ . The main purpose of this section will be to prove the following generalization of a result of SAMUEL [9].

**THEOREM 3.1.** — *Let  $R$  be a noetherian integral domain with identity, and suppose that the integral closure of  $R$  is a finite  $R$ -module. Let  $G$  be a finite group of  $R$ -automorphisms of  $S$ , and let  $S^G$  denote the invariant subring of  $G$  on  $S$ . Then there exists  $f \in S$  such that  $S^G = R[[f]]$ .*

We require several preliminary results before proving theorem 3.1. Our proof will essentially follow that of SAMUEL, but we will need to make strong use of the results of [7] and theorem 2.6 of [6]. We begin by making the following definition.

**DEFINITION 3.2.** — Let  $f \in S$ , and suppose that  $S$  is a complete Hausdorff space in the  $(f)$ -adic topology. By the results of [8], there exists a unique  $R$ -endomorphism  $\varphi_f$  of  $S$  mapping  $X$  onto  $f$ . We write  $R[[f]]$  to denote the range of  $\varphi_f$ .

We remark that it follows from Remark 2.4 that the above definition is a more general definition of  $R[[f]]$  than that given in [6]. We mean this in the sense that there exist rings  $R$  and elements  $f \in S$  such that  $R[[f]]$  is defined by definition 3.2, but not by the definition given in [6].

The following lemma is a generalization of [7], lemma 4.9; its proof is trivial.

**LEMMA 3.3.** — *Let  $\varphi$  be an  $R$ -endomorphism of  $S$ , and let  $\alpha \in S$  be such that  $\varphi(\alpha) = \alpha$ . Then  $\varphi$  is continuous in the  $(\alpha)$ -adic topology on  $S$ .*

**LEMMA 3.4.** — *Let  $\alpha \in S$ , and suppose that  $S$  is a complete Hausdorff space in the  $(\alpha)$ -adic topology. If  $\varphi$  is any  $R$ -endomorphism of  $S$  such that  $\varphi(\alpha) = \alpha$ , then  $\varphi$  is the identity on  $R[[\alpha]]$ .*

*Proof.* — By [8], theorem 2.3, if  $h \in R[[\alpha]]$ , then  $h$  is the unique limit of a sequence of the form  $\left\{ \sum_{i=0}^n r_i \alpha^i \right\}_{n \in \omega}$ , where  $r_i \in R$  for each  $i$ , in the  $(\alpha)$ -adic topology on  $S$ . Therefore, since  $\varphi$  is continuous from

$(S, (\alpha))$  into itself, we have that

$$\varphi(h) = \varphi\left(\lim_n \left(\sum_{i=0}^n r_i \alpha^i\right)\right) = \lim_n \varphi\left(\sum_{i=0}^n r_i \alpha^i\right) = \lim_n \left(\sum_{i=0}^n r_i \alpha^i\right) = h.$$

We observe that if  $\varphi$  is an  $R$ -automorphism of  $S$  such that  $\varphi(X) = \beta$ , then  $S$  is complete and Hausdorff in the  $(\beta)$ -adic topology ([8], theorem 4.5). Hence, by theorem 2.1, for any element  $g \in (\beta)$ ,  $S$  is a complete Hausdorff space in the  $(g)$ -adic topology. In particular, if  $G$  is a finite group

of  $R$ -automorphisms of  $S$ , and if  $f = \prod_{\varphi \in G} \varphi(X)$ , then  $S$  is a complete

Hausdorff space in the  $(f)$ -adic topology, and  $R[[f]]$  is defined. Moreover, since  $\varphi(f) = f$  for each  $\varphi \in G$ , it follows, from lemma 3.4, that  $G$  is the identity on  $R[[f]]$ . Thus, we have proved the following.

**COROLLARY 3.5.** — *Let  $G$  be a finite group of  $R$ -automorphisms of  $S$ . If  $f = \prod_{\varphi \in G} \varphi(X)$ , then  $G$  is the identity on  $R[[f]]$ .*

Corollary 3.5 shows that  $R[[f]] \subseteq S^G$  for any ring  $R$  and finite group  $G$  of  $R$ -automorphisms of  $S$ . The remainder of the paper will be concerned with showing that if  $R$  satisfies the hypothesis of theorem 3.1, then the reverse containment holds.

The following theorem is a restatement of several of the results of [7].

**THEOREM 3.6.** — *Let  $R$  be a noetherian ring and let  $\beta = \sum_{i=0}^{\infty} b_i X^i \in S$ .*

*Then there exists an  $R$ -endomorphism  $\varphi_\beta$  of  $S$  such that  $\varphi_\beta(X) = \beta$  if, and only if,  $(R, (b_0))$  is a complete Hausdorff space. Furthermore, if such a  $\varphi_\beta$  exists, then*

- (i)  $b_0 \in J$ , the Jacobson radical of  $R$ ;
- (ii)  $\varphi_\beta$  is the unique  $R$ -endomorphism of  $S$  mapping  $X$  onto  $\beta$ ; and
- (iii)  $\varphi_\beta$  is an automorphism if, and only if,  $b_1$  is a unit of  $R$ .

Suppose now that  $R$  is a noetherian ring, and let  $R'$  be a unitary overring of  $R$ , which is a finite  $R$ -module. If  $\beta = \sum_{i=0}^{\infty} b_i X^i \in S$  and

if  $(R, (b_0))$  is a complete Hausdorff space, then  $R'$  is a noetherian ring complete and Hausdorff in its  $(b_0 R')$ -adic topology ([11], theorem 15, p. 276). Hence, by theorem 3.6, there exist unique  $R$  and  $R'$ -endomorphisms  $\varphi_\beta$  and  $\varphi'_\beta$  of  $S$  and  $S' = R'[[X]]$ , respectively, such that  $\varphi_\beta(X) = \varphi'_\beta(X) = \beta$ . Furthermore, if  $\varphi_\beta$  is onto [and hence, an auto-



morphism ([8], theorem 4.7)], then  $b_1$  is a unit of  $R$ . Therefore, since  $b_1$  must also be a unit of  $R'$ , it follows that  $\varphi_\beta^*$  is onto (and hence, an automorphism). Our next result shows that  $\varphi_\beta^*$  is the unique extension of  $\varphi_\beta$  to  $S'$ . Since  $\varphi_\beta^*$  is the unique  $R'$ -endomorphism of  $S'$  mapping  $X$  onto  $\beta$ , it suffices to show that  $\varphi_\beta^*$ , restricted to  $S$ , is equal to  $\varphi_\beta$ . We shall have need of a precise definition of  $\varphi_\beta^*$ .

Let  $\alpha = \sum_{j=0}^{\infty} c_j X^j \in S$ , and suppose that  $(R, (c_0))$  is a complete Hausdorff space. Then there exists a unique  $R$ -endomorphism  $\varphi_\alpha$  of  $S$  such that  $\varphi_\alpha(X) = \alpha$  ([7], theorems 4.2 and 4.3). For any element  $h = \sum_{i=0}^{\infty} h_i X^i \in S$ ,  $\varphi_\alpha(h)$  is defined to be  $\sum_{k=0}^{\infty} p_k(h) X^k$ , where

$$p_k(h) = \lim_n \pi_k \left( \sum_{i=0}^n h_i \alpha^i \right) \quad \text{for each } k \in \omega_0.$$

The limit is taken in  $(R, (c_0))$ , and  $\pi_k(g) = g_k$  for any element  $g = \sum_{i=0}^{\infty} g_i X^i \in S$  [7].

We can now prove the following.

**LEMMA 3.7.** — *Let  $R$  be a noetherian ring,  $R'$  a unitary overring of  $R$  that is a finite  $R$ -module, and  $\varphi_\beta$  the unique  $R$ -endomorphism of  $S$  mapping  $X$  onto  $\beta = \sum_{i=0}^{\infty} b_i X^i$ . Then  $\varphi_\beta$  can be extended to a unique  $R'$ -endomorphism  $\varphi_\beta^*$  of  $S' = R'[[X]]$ . Furthermore, if  $\varphi_\beta$  is an automorphism, then  $\varphi_\beta^*$  is an automorphism.*

*Proof.* — It only remains to show that  $\varphi_\beta^*$  extends  $\varphi_\beta$ . Let

$$h = \sum_{i=0}^{\infty} h_i X^i \in S;$$

then, by definition,

$$\varphi_\beta^*(h) = \sum_{k=0}^{\infty} p_k(h) X^k, \quad \text{where} \quad p_k(h) = \lim_n \pi_k \left( \sum_{i=0}^n h_i \beta^i \right)$$

for each  $k \in \omega_0$ , and the limit is taken in  $(R', (b_0 R'))$ . However, since  $h$  and  $\beta \in S$ , it follows from [7], lemma 4.1 that, for each  $k \in \omega_0$ ,

$\left\{ \pi_k \left( \sum_{i=0}^n h_i \beta^i \right) \right\}_{n \in \omega_0}$  is a Cauchy sequence of  $(R, (b_0))$ , and hence,

converges to a unique element  $d_k$  of  $R$ . But since each Cauchy sequence of  $(R, (b_0))$  is Cauchy in  $(R', (b_0 R'))$  and since  $(R', (b_0 R'))$  is Hausdorff, it follows that  $p_k(h) = d_k \in R$  for each  $k \in \omega_0$ . Therefore,  $\varphi_{\beta}^*(S) \subseteq S$ , and thus  $\varphi_{\beta}^*$ , restricted to  $S$ , is an  $R$ -endomorphism of  $S$  mapping  $X$  onto  $\beta$ . But  $\varphi_{\beta}$  is the unique  $R$ -endomorphism of  $S$  mapping  $X$  onto  $\beta$ , and therefore,  $\varphi_{\beta}^*$ , restricted to  $S$ , is equal to  $\varphi_{\beta}$ .

Our next result shows that if the coefficients of  $\beta$  satisfy certain conditions, then, if  $g \in S'$  and if  $\varphi_{\beta}^*(g) \in S$ , then  $g$  itself is an element of  $S$ . This fact will be applicable to the proof of theorem 3.1. The method of proof used here is the same as that used in [9].

**LEMMA 3.8.** — *Under the same hypothesis as that of lemma 3.7, if*

$\beta = \sum_{i=0}^{\infty} b_i X^i$  *has the property that there exists*  $n \in \omega$  *such that*  $b_n$  *is a unit of*  $R$ , *and such that*  $B = (b_0, b_1, \dots, b_{n-1}) \subseteq J$ , *then, for any element*  $g \in S'$ , *if*  $\varphi_{\beta}^*(g) \in S$ ,  $g \in S$ . *As a consequence, we have :*

- (i)  $\varphi_{\beta}^*$  *is one-to-one if, and only if,*  $\varphi_{\beta}$  *is one-to-one; and*
- (ii)  $\varphi_{\beta}^*$  *is onto if, and only if,*  $\varphi_{\beta}$  *is onto.*

*Proof.* — Let  $g = \sum_{j=0}^{\infty} g_j X^j \in S'$ , and suppose that

$$\varphi_{\beta}^*(g) = \sum_{k=0}^{\infty} c_k X^k \in S.$$

By definition of  $\varphi_{\beta}^*$ ,

$$c_k = \lim_i \pi_k \left( \sum_{j=0}^i g_j \beta^j \right) \quad \text{for each } k \in \omega_0,$$

where the limit is taken in  $(R', (b_0 R'))$ . Let  $E$  be the  $R$ -submodule of  $R'$  generated by  $1$  and  $\{g_j\}_{j \in \omega_0}$ . We show that  $E = R$ .

We first make some observations. Since  $B \subseteq J$  and since  $R'$  is a finite  $R$ -module, we have that  $(R', BR')$  is Hausdorff, and, furthermore, since  $BE$  is an  $R$ -submodule of  $R'$ ,  $BE$  is closed in  $(R', BR')$  ([11], theorem 9, p. 262). Moreover, since  $(b_0 R') \subseteq BR'$ , and since  $(R', (b_0 R'))$  is complete, we have that each Cauchy sequence of  $(R', (b_0 R'))$  converges in  $(R', BR')$  also. Thus, in particular, since  $BE$  is closed in  $(R', BR')$ , if  $\{u_j\}_{j \in \omega}$  is a sequence of elements of  $BE$ , Cauchy in  $(R', (b_0 R'))$ , then  $\{u_j\}_{j \in \omega}$  converges to a unique element  $u \in BE$  in  $(R', BR')$ . We make strong use of this fact.

We prove by induction on  $j$  that  $g_j \in R + BE$  for each  $j \in \omega_0$ . Now,

$$\begin{aligned} c_0 &= \lim_i \pi_0 \left( \sum_{j=0}^i g_j \beta^j \right) = \lim_i \left( \sum_{j=0}^i g_j b_0^j \right) \\ &= g_0 + \lim_i \left( \sum_{j=1}^i g_j b_0^j \right). \end{aligned}$$

Therefore, since  $\left\{ \sum_{j=1}^i g_j b_0^j \right\}_{i \in \omega}$  is a Cauchy sequence of  $(R', (b_0 R'))$ , and since  $\sum_{j=1}^i g_j b_0^j \in BE$  for each  $i$ , we have, by the above observation, that

$$g_0 = c_0 - \lim_i \left( \sum_{j=1}^i g_j b_0^j \right) \in R + BE.$$

Suppose we have shown that  $g_j \in R + BE$  for  $j < k$ . Then

$$\begin{aligned} c_{nk} &= \lim_i \pi_{nk} \left( \sum_{j=0}^i g_j \beta^j \right) = \lim_i \left( \sum_{j=0}^i g_j \pi_{nk}(\beta^j) \right) \\ &= \sum_{j=0}^{k-1} g_j \pi_{nk}(\beta^j) + g_k \pi_{nk}(\beta^k) + \lim_i \left( \sum_{j=k+1}^i g_j \pi_{nk}(\beta^j) \right). \end{aligned}$$

Now,  $\pi_{nk}(\beta^j) \in R$  for each  $j$ , and hence, in particular, by the induction hypothesis,

$$\sum_{j=0}^{k-1} g_j \pi_{nk}(\beta^j) = u \in (R + BE)R = R + BE.$$

Moreover,  $\pi_{nk}(\beta^k)$  has the form  $b_n^k + d$  where  $d \in B \subseteq J$ , and hence, since  $b_n$  is a unit of  $R$ ,  $b_n^k + d$  is a unit of  $R$  ([10], p. 206). Finally,

we observe that for  $j > k$ ,  $\pi_{nk}(\beta^j) \in B$ , and hence  $\sum_{j=k+1}^i g_j \pi_{nk}(\beta^j) \in BE$  for each  $i$ . Therefore, as in the case for  $j = 0$ , we have that

$$\lim_{i \rightarrow \infty} \left( \sum_{j=k+1}^i g_j \pi_{nk}(\beta^j) \right) = v \in BE.$$

Thus, since  $b_n^k + d$  is a unit of  $R$ , we have

$$g_k = [c_{nk} - (u + v)](b_n^k + d)^{-1} \in R + BE.$$

Therefore, by induction, we have that  $E \subseteq R + BE$ . But, since  $E$  is an  $R$ -module containing 1 and since  $B$  is an ideal of  $R$ , we have the reverse containment, and hence  $E = R + BE$ .

Hence, by Nakayama's lemma since  $B \subseteq J$ , we have  $E = R$ , so that  $g \in S$ .

The proofs of (i) and (ii) are straightforward and we omit them.

We make one final observation before proving theorem 3.1. Let  $R$  be a noetherian ring, and let  $G = \{\varphi_i\}_{i=1}^n$  be a finite group of  $R$ -automorphisms of  $S$ . If

$$f = \prod_{i=1}^n \varphi_i(X) = \sum_{i=0}^{\infty} a_i X^i,$$

then it follows that  $a_n$  is a unit of  $R$ , and  $A = (a_0, a_1, \dots, a_{n-1})$  generates a complete Hausdorff topology on  $R$  (theorems 3.6 and 2.1 and corollary 2.2). Thus, the coefficients of  $f$  satisfy the hypothesis of [6], theorem 2.6, and we have that  $\{1, X, \dots, X^{n-1}\}$  is a free module basis for  $S$  over  $R[[f]]$ .

*Proof of theorem 3.1.* — Let  $L$ ,  $K$  and  $F$  denote the quotient fields of  $S$ ,  $S^G$ , and  $R[[f]]$ , respectively  $\left(f = \prod_{i=1}^n \varphi_i(X) = \sum_{i=0}^{\infty} a_i X^i\right)$ . If  $G^*$  denotes the finite group of automorphisms of  $L$  induced by  $G$ , then, since  $\{1, X, \dots, X^{n-1}\}$  is a free module basis for  $S$  over  $R[[f]]$ , it follows that  $G^*$  is the Galois group of  $L$  with respect to  $F$ . Moreover, since  $R[[f]] \subseteq S^G$  (corollary 3.5),  $F \subseteq K$ , and thus, since  $G^*$  is the identity on  $K$ , it follows from Galois theory that  $F = K$ . Therefore, since  $S$  is integral over  $R[[f]]$ , we have that  $S \cap F = S \cap K \subseteq$  integral closure of  $R[[f]]$ . Hence, if  $R$  is integrally closed, we have that  $R[[f]]$  is integrally closed ([1], chap. V, § 1, n° 4, prop. 14) (since  $R$  is noetherian), so that  $S^G \subseteq R[[f]]$ .

In the general case, we suppose that  $R'$ , the integral closure of  $R$ , is a finite  $R$ -module. By lemma 3.7, each  $R$ -automorphism  $\varphi$  of  $S$  can be extended to an  $R'$ -automorphism  $\varphi^*$  of  $S' = R'[[X]]$ . Therefore, if  $G' = \{\varphi_i^* \mid \varphi_i \in G\}$ , then  $G'$  is a finite group of  $R'$ -automorphisms of  $S'$ , and the "integrally closed" case applies. Thus, the invariant subring  $(S')^{G'}$  of  $G'$  on  $S'$  is  $R'[[f]]$ .

Therefore, since  $S^G = R'[[f]] \cap S$ , it suffices to show that

$$R'[[f]] \cap S \subseteq R[[f]].$$

But this is a special case of lemma 3.8. This follows from the fact

that  $f = \sum_{i=0}^{\infty} a_i X^i$  is such that  $a_n$  is a unit of  $R$  and  $(a_0, a_1, \dots, a_{n-1}) \subseteq J$ .

Moreover, if  $\psi$  and  $\psi^*$  denote the unique  $R$  and  $R'$ -endomorphisms of  $S$  and  $S'$ , respectively, which map  $X$  onto  $f$ , then  $R[[f]]$  and  $R'[[f]]$  denote the ranges of  $\psi$  and  $\psi^*$ , respectively. Hence, if  $g \in R'[[f]] \cap S$ , then there exists  $h \in S'$  such that  $\psi^*(h) = g$ . But  $g \in S$  implies that  $h \in S$ , so that  $g = \psi^*(h) = \psi(h) \in \psi(S) = R[[f]]$ . This completes the proof.

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(Texte reçu le 16 décembre 1970.)

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