

# BULLETIN DE LA S. M. F.

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*Bulletin de la S. M. F.*, tome 99 (1971), p. 241-245

[http://www.numdam.org/item?id=BSMF\\_1971\\_\\_99\\_\\_241\\_0](http://www.numdam.org/item?id=BSMF_1971__99__241_0)

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## A METRIC PROPERTY OF SOME RANDOM FUNCTIONS

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0. **Introduction.** — Let  $E_1$  and  $E_2$  be compact linear sets of Hausdorff dimensions  $d_1 > 0$  and  $d_2 > 0$ , and suppose that  $d = d_1 + d_2 < 1$ . For real numbers  $t > 0$ ,  $F_t$  is the set  $E_1 + tE_2$ . It is proved in [3], that  $\dim F_t \geq d$  excepting a  $t$ -set of dimension at most  $d$ ; in [4], the exceptional set has positive dimension. It is thus natural to search for sets “complementary” to a given set  $E_2$ ; namely, sets  $E_1$  for which the exceptional  $t$ -set is void. This particular problem is solved by some theorems in ([1], chap. XV), founded on harmonic analysis. However, the analysis of that work uses special properties of linear mappings, while the method of [3] applies without change to the following non-linear variant. Consider a continuously differentiable function  $h(u, v)$  defined in the plane, whose partial derivatives satisfy inequalities  $1 \leq \frac{\partial h}{\partial u}, \frac{\partial h}{\partial v} \leq C$ . Setting  $h_t(u, v) = h(u, tv)$  and  $F_t = h_t(E_1 \times E_2)$ , we see that the result of [3] holds for this definition of  $F_t$ . In the next statement,  $X$  denotes linear Brownian motion ([1], chap. XI) on  $[0, \infty)$  and  $d = d_1 + d_2 < 1$ .

**THEOREM.** — *Let  $h_t$  be as defined above and  $E \subseteq [0, \infty)$  a compact set of dimension  $1/2 d_1$ , while  $\dim E_2 = d_2$ . Then it is almost sure that*

$$\dim h_t(X(E) \times E_2) \geq d \text{ for all } t > 0.$$

Because it is almost sure that  $\dim X(E) = d_1$  ([1], p. 143), the set  $X(E)$  is “complementary” to  $E_2$  (with respect to the transformation  $h$ ).

After the proof of this theorem, the special choice  $h = u + v$  is briefly considered; then it will be explained how this is indeed a simple case.

1. — Suppose that  $\mu$  and  $\lambda$  are probability measures in  $E$  and  $E_2$ , respectively; then  $F_t$  carries a probability measure  $\sigma_t$  defined by the formula

$$\int f(u) \sigma_t(du) = \iint f \circ h(X(x), ty) \mu(dx) \lambda(dy).$$

Let  $V_k[u] = 1$  when  $|u| < 2^{-k}$ ,  $V_k[u] = 0$  otherwise ( $k = 1, 2, 3, \dots$ ) Then by the method of ([2], chap. III), it is easy to see that

$$\dim F_t \geq e_3 \quad \text{provided} \quad \iint V_k[u - u'] \sigma_t(du) \sigma_t(du') = O(2^{-ke_3}).$$

In order to handle all values  $t$  in an interval,  $1 \leq t \leq 2$ , we choose an integer  $k_0$  so large that

$$V_k[h(a, ty) - h(b, ty)] \leq V_{k-k_0}[h(a, sy) - h(b, sy)],$$

whenever  $y, y_1 \in E_2$  and  $|t - s| \leq 2^{-k}$ . Thus we can obtain a lower bound for  $\dim F_t$ , uniform over  $1 \leq t \leq 2$ , by majorizing

$$\max(t \in S_k) \iint V_k[h(X(x), ty) - h(X(x'), ty')] \mu(dx) \dots \lambda(dy'),$$

where  $S_k = \{1, 1 + 2^{-k}, \dots, 2\}$  has  $1 + 2^k$  elements. In the following paragraph,  $t$  occurs innocuously as a parameter, and so it is suppressed until the end of the proof.

2. — To each number  $e_1 \in (0, 1/2 d_1)$ , there is a probability measure  $\mu$  in  $E$ , fulfilling a Lipschitz condition to exponent  $e_1$ , and similarly, for each number  $e_2 \in (0, d_2)$ , a measure  $\lambda$  in  $E_2$  (see [2], chap. II). For any numbers  $a \geq 0, b$ , we form the double integrals

$$\begin{aligned} I_k &= \iint V_k[h(X(x), y) - h(X(a), b)] \mu(dx) \lambda(dy) \\ &= \int V_k[u - h(X(a), b)] \sigma(du). \end{aligned}$$

LEMMA. — The  $r$ th moment of  $I_k$  is  $O(k 2^{-e_1 k - e_2 k})^r$ , for each  $r = 1, 2, 3, \dots$ , uniformly with respect to  $a$  and  $b$ .

Proof. — The  $r$ th power of  $I_k$  is a multiple integral

$$\int \dots \int \prod_{j=1}^r V_k[h(X(x_j), y_j) - h(X(a), b)] \mu(dx_1) \dots \lambda(dy_r).$$

We can suppose that  $x_1 \leq x_2 \leq \dots \leq x_r$ , and then divide the integral into subsets depending upon the relative position of  $a$ ; here we shall suppose  $a \leq x_1$ . The inequality  $\prod V_k \neq 0$  implies the system

$$\begin{aligned} |h(X(x_1), y_1) - h(X(a), b)| &< 2^{-k}, \\ |h(X(x_{j+1}), y_{j+1}) - h(X(x_j), y_j)| &< 2^{1-k}. \end{aligned}$$

To these, we can adjoin the inequalities

$$|X(x_1) - X(a)| < k|x - a|^{\frac{1}{2}}, \quad |X(x_{j+1}) - X(x_j)| < k|x_{j+1} - x_j|^{\frac{1}{2}}$$

for the set on which one of these fails has probability  $< e^{-\frac{1}{2}k^2}$ . These systems of inequalities yield

$$|y_1 - b| \leq 2^{-k} + Ck|x - a|^{\frac{1}{2}}, \quad |y_{j+1} - y_j| \leq 2^{1-k} + Ck|x_{j+1} - x_j|^{\frac{1}{2}}$$

where  $C$  depends also on  $t$ , but is bounded for  $1 \leq t \leq 2$ . Because  $\frac{\partial h}{\partial u} \geq 1$ , the probability of the first system of inequalities is

$$\leq \inf_{1, 2^{-k}} |x_1 - a|^{-\frac{1}{2}} \prod_1^{r-1} \inf_{1, 2^{1-k}} |x_{j+1} - x_j|^{-\frac{1}{2}}.$$

To each determination of  $x_1, \dots, x_r$  a domain of values  $(y_1, \dots, y_r)$  is defined, of measure

$$O\left(2^{-k} + k|x - a|^{\frac{1}{2}}\right)^{e_2} \prod_1^{r-1} \left(2^{-k} + k|x_{j+1} - x_j|^{\frac{1}{2}}\right)^{e_2}.$$

To majorize the  $r$ th moment, we perform iterated integration; we require a bound for integrals of the type

$$\int \inf_{1, 2^{-k}} |x - c|^{-\frac{1}{2}} \left(2^{-k} + k|x - c|^{\frac{1}{2}}\right)^{e_2} \mu(dx).$$

For the intervals  $|x - c| < 4^{-k}$ , a contribution  $O(4^{-e_1 k} \cdot k^{e_2} 2^{-e_2 k})$  is obtained. For the intervals  $4^{-n} \leq |x - c| < 4^{1-n}$ , the magnitude does not exceed

$$2^{n-k} \cdot 4^{-ne_1} \cdot k^{e_2} 2^{-e_2 n} = 2^{-k} k^{e_2} 2^{n(1-2e_1-e_2)}.$$

As  $2e_1 + e_2 \leq d_1 + d_2 < 1$ , the sum, for  $k \geq n$ , of all partial integrals, is of magnitude  $2^{-e_1 k - e_2 k} k^{e_2}$ , and from this the required estimate follows.

Now, setting

$$J_k = \int \cdots \int V_k[h(X(x), y) - h(X(x'), y')] \mu(dx) \dots \lambda(dy'),$$

we find by Jensen's inequality the same estimate for the  $r$ th moment of  $J_k$ .

Let now  $e_3 < 2e_1 + e_2$ . Then

$$P\{J_k \geq 2^{-ke_3}\} = O(2^{ke_3 - 2ke_1 - ke_2})^r k^r.$$

At this point, we restore the parameter  $t$ , and find

$$P\{\max J_k(t) \geq 2^{-ke_3}, t \in S_k\} = O(2^{ke_3 - 2ke_1 - ke_2})^r 2^k k^r.$$

Choosing  $r$  so large that  $r(e_3 - 2e_1 - e_2) + 1 < 0$ , we find  $\max J_k(t) = o(2^{-ke_3})$  almost surely. Since  $e_3$  is arbitrarily close to  $d = d_1 + d_2$ , we have finally  $\dim F_t \geq d$  for  $1 \leq t \leq 2$ , almost surely.

In the proof, we allowed  $t$  to operate on the coordinate  $v$ , since  $E_2$  is bounded. But  $X(E)$  is almost surely bounded, and so  $t$  might also operate on  $u$ . In fact, the proof is valid for functions  $h(u, v, t)$  continuously differentiable in  $u, v, t$  provided  $\frac{\partial h}{\partial u} > 0, \frac{\partial h}{\partial v} > 0$  everywhere.

3. — In the special case  $h = u + v$ , we use harmonic analysis of Fourier-Stieltjes transforms. Let  $\mu$  and  $\lambda$  be the measures introduced in the beginning of paragraph 2, and  $\nu$  the transform of  $\mu$  by the trajectory  $X$ :

$$\hat{\nu}(s) \equiv \int e^{-isX(x)} \mu(dx) \equiv \int e^{-isx'} \nu(dx').$$

By Theorem 1 of ([1], chap. XV),

$$\hat{\nu}(s) = o(|s|^{-e}) \text{ for every } e < \frac{1}{2}e_1.$$

Also,

$$\int_{|s|>1} |\hat{\lambda}(s)|^2 |s|^f ds < \infty \text{ for every } f < e_2 - 1.$$

Hence

$$\int_{|s|>1} |\hat{\lambda}(s)|^2 |\hat{\nu}(s)|^2 |s|^f ds < \infty \text{ for every } g < e_2 + e_1 - 1.$$

Now the measure  $\nu \star \lambda$  has Fourier transform  $\hat{\nu} \hat{\lambda}$ , and is supported by  $X(E) + E_2$ , so that  $X(E) + E_2$  has dimension  $\geq e_1 + e_2$ . (For this paragraph, see [2], chap. III.) The set  $X(E)$  is therefore complementary

to every set  $E_2$ , but the proof of this fact seems to have little relation to the non-linear problem. Observe that when  $e_1 + e_2 > 1$ , the set  $X(E) + E_2$  has positive measure, by the Plancherel formula; it would be extremely interesting to obtain a theorem of this type for non-linear mappings, valid for all  $t > 0$  uniformly; it would also be interesting to find properties of  $h_t$  dependent upon the higher derivatives of  $h$  when these exist.

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(Texte reçu le 5 août 1970.)

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