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Characterizations of classes of left Ext-reproduced groups


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CHARACTERIZATIONS OF CLASSES OF LEFT EXT-REPRODUCED GROUPS

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1. Introduction.

In a previous paper [4], we considered classes \( \mathfrak{M} \) of abelian groups \( G \), which are maximal with respect to the property: There exists a reduced group \( X \) such that

\[
G \cong \text{Ext}(G, X) \quad \text{for all } G \in \mathfrak{M}.
\]

Such classes are called \textit{maximal classes of left Ext-reproduced groups}. We found two such classes, viz. \( \mathfrak{F} \) and \( \mathfrak{A} \) (see [4], Theorem 2.7 and Theorem 2.9).

In this paper, we give characterizations of two classes of left Ext-reproduced groups, viz. \( \mathfrak{F} \) and \( \mathfrak{G} \), where

- \( \mathfrak{F} \) denotes the class of all groups \( Q^n \oplus T \) where \( n \) is a non-negative integer and \( T \) is a finite group, and

- \( \mathfrak{G} \) denotes the class of all groups \( \prod_{p \in \mathbb{P}} T_p \) where, for all primes \( p \), \( T_p \) is a finite \( p \)-group.

The latter class is a subclass of the class \( \mathfrak{A} \), and it is therefore not a maximal class of left Ext-reproduced groups. We mention, in passing, the fact that all groups in \( \mathfrak{A} \) are reduced and adjusted cotorsion groups (see [5], p. 373), and that \( \mathfrak{G} \) contains all finite groups. The classes \( \mathfrak{F} \) and \( \mathfrak{G} \) are linked together in a very special way. Let us consider a number of properties of a class \( \mathfrak{M} \) of left Ext-reproduced groups, i.e. a class of groups for which there exists a reduced group \( X \) such that

\[
G \cong \text{Ext}(G, X) \quad \text{for all } G \in \mathfrak{M}.
\]
(I) If \( G \in \mathcal{M} \), then every direct summand \( U \) of \( G \) belongs to \( \mathcal{M} \).

(II) If \( G \in \mathcal{M} \) and \( H \in \mathcal{M} \), then \( \text{Hom}(G, H) \in \mathcal{M} \).

(III) If \( G \in \mathcal{M} \) and \( H \in \mathcal{M} \), and if \( \varphi : G \to H \) is a homomorphism, then \( \text{Ker}\varphi \in \mathcal{M} \) and \( \text{Coker}\varphi \in \mathcal{M} \).

(IV) If \( G \in \mathcal{M} \) and \( H \in \mathcal{M} \), then \( \text{Ext}(G, H) \in \mathcal{M} \).

(V) If \( G \in \mathcal{M} \), then \( \text{Hom}(G, X) = 0 \).

Now the following statement summarizes the main results:

\( \mathcal{F} \) and \( \mathcal{G} \) are classes of left Ext-reproduced groups which are maximal with respect to each of the properties (I) to (V). Conversely, if \( \mathcal{M} \) is a class of left Ext-reproduced groups which is maximal with respect to any one of the properties (I) to (V), then either \( \mathcal{M} = \mathcal{F} \) or \( \mathcal{M} = \mathcal{G} \), the latter being the case if \( \mathcal{M} \) contains only reduced groups.

**Notation.**

\( A \oplus B, \bigoplus_{i \in I} A_i, A^{(m)} \), direct sum;

\( \prod_{i \in I} A_i, A^m \), direct product;

\( A \otimes B \), tensor product of \( A \) and \( B \);

\( tG \), maximal torsion subgroup of \( G \);

\( G_p \), \( p \)-component of \( G \);

\( Z \), additive group of integers;

\( Q \), additive group of rational numbers;

\( Z(p) \), additive group of \( p \)-adic integers;

\( C(n) \), cyclic group of order \( n \);

\( C(p') \), quasi-cyclic group;

\( \mathcal{N} \), the power of the continuum;

\( P \), the set of all prime numbers;

cotorsion group, a group \( X \) such that \( \text{Ext}(Q, X) = 0 \); adjusted reduced cotorsion group, a reduced cotorsion group \( G \) such that \( G/tG \) is divisible.

All groups under consideration are additively written abelian groups.

2. **Characterizations of classes of left Ext-reproduced groups.**

Let \( \mathcal{M} \) be a class of left Ext-reproduced groups. Throughout this paper, \( X \) will denote a reduced group such that

\( \text{Ext}(G, X) \cong G \) for all \( G \in \mathcal{M} \).

Recall that \( \mathcal{F} \) is the class of all groups \( \mathbb{Q}^n \oplus T \) where \( n \) is a non-negative integer and \( T \) is a finite group. For this class of groups, we have that

\( \prod_{p \in \mathbb{P}} Z(p)/X \cong Q \) ([4], Example 2.3). Let \( \mathcal{G} \) denote the class of all
groups $\prod_{p \in P} T_p$ where $T_p$ is a finite $p$-group for all primes $p$. Note that $\mathcal{G}$ is a proper subclass of $\mathcal{A}$, consequently $\mathcal{G}$ is also a class of left Ext-reproduced groups ([4], Example 2.8).

The classes $\mathfrak{S}$ and $\mathcal{G}$ are quite remarkable, and we give characterizations of them in this paragraph. First, we prove several lemmas which we shall need subsequently.

**Lemma 1.** — Let $\mathfrak{S}$ be a class of left Ext-reproduced groups such that $\mathfrak{S} \supseteq \mathcal{G}$. Then $X \cong \prod_{p \in P} Z(p)$.

**Proof.** — Since $C(p^k) \in \mathfrak{S}$ for all primes $p$ and all natural numbers $k$, it follows from [4] (Lemma 2.4) that $X$ is isomorphic to a pure subgroup of $\prod_{p \in P} Z(p)$ and that $X/pX \cong C(p)$ for all primes $p$. Note that $\text{Ext}(Q, X) = 0$, this follows from the exact sequences

$$0 \rightarrow \bigoplus_{p \in P} C(p) \rightarrow \prod_{p \in P} C(p) \rightarrow Q(\mathcal{S}) \rightarrow 0$$

and

$$0 \rightarrow \text{Ext}(Q, X) \rightarrow \text{Ext} \left( \prod_{p \in P} C(p), X \right) \cong \prod_{p \in P} C(p)$$

since $\prod_{p \in P} C(p)$ is reduced and $\prod_{p \in P} C(p) \in \mathfrak{S}$. Hence ([5], p. 372)

$$X \cong \prod_{p \in P} \text{Ext}(C(p^*), X) \cong \prod_{p \in P} \text{Hom}(C(p^*), X \otimes C(p^*))$$

$$\cong \prod_{p \in P} \text{Hom}(C(p^*), C(p^*)) \cong \prod_{p \in P} Z(p),$$

which completes the proof.

**Lemma 2.** — Let $\mathfrak{M}$ be a class of left Ext-reproduced groups and let $G \in \mathfrak{M}$.

1° If $G_p \neq 0$, then $X/pX \neq 0$.

2° If the reduced part of $G$ contains an unbounded $p$-component for some prime $p$ then

(i) $X_p = 0$;

(ii) $G$ has a direct summand $Z(p)^{\mathfrak{M}} \oplus \prod_{i=1}^\infty ((C(p^i))^{m_{pi}})$ where $m_{pi}$ is finite and $m_{pi} \neq 0$ for an infinite number of $i$'s;
(iii) $\text{Hom}(G, G)$ is not left Ext-reproduced;
(iv) $\text{Ext}(G, G)$ is not left Ext-reproduced.

Proof.

1° Let $G_p \neq o$, and suppose that $X = pX$. Then, by [1] (p. 245), we have $pG = G$ since $G \cong \text{Ext}(G, X)$, consequently $G = (C(p^\infty)(^n) \oplus G'$, where $G_p = o$ and $pG' = G'$. Hence

$$G \cong \text{Ext}(G, X) \cong (\text{Ext}(C(p^\tau), X))^n \oplus \text{Ext}(G', X)$$

and since $X$ is reduced and $pX = X$ it follows that $X_p = o$. The exact sequences

$$0 \rightarrow tX \rightarrow X \rightarrow X/tX \rightarrow 0$$

and

$$\text{Ext}(C(p^\tau), tX) \cong 0 \rightarrow \text{Ext}(C(p^\tau), X) \rightarrow \text{Ext}(C(p^\tau), X/tX) \rightarrow 0$$

show that

$$\text{Ext}(C(p^\tau), X) \cong \text{Ext}(C(p^\tau), X/tX) \cong \text{Hom}(C(p^\tau), (X/tX) \otimes (Q/Z)) = o$$

since $(X/tX) \otimes C(p^\tau) = o$ ([1], p. 251). In addition, $G_p = o$ and $pG' = G'$ imply (see [1], p. 246)

$$\text{Ext}(G', X)_p = o$$

and hence it follows from (1), (2) and (3) that $G_p = o$, contrary to the assumption $G_p \neq o$. We conclude that $X/pX \neq o$. This proves 1°.

2° Let $G \in \mathfrak{D}$ be such that $G = D \oplus G'$, where $D$ is divisible and $G'$ is reduced. Suppose that $G_p$ is unbounded for some prime $p$ and let $B^{(p)}$ denote a basic subgroup of $G_p$. Then $B^{(p)}$ is unbounded, and if we put $B^{(p)} = B_1 \oplus \ldots \oplus B_i \oplus \ldots$, where $B_i = (C(p^i))^{(m_{p^i})}$, then each $m_{p^i}$ is finite ($i = 1, 2, \ldots$) (see [3], p. 136), and $m_{p_i} \neq o$ for an infinite number of $i$'s. Now $B_1 \oplus \ldots \oplus B_i$ is a direct summand of $G_p$ and hence of $G$, that is ([1], p. 243)

$$\text{Ext}(B_1 \oplus \ldots \oplus B_i, X) \cong \text{Ext}(B_1, X) \oplus \ldots \oplus \text{Ext}(B_i, X) \cong (X/pX)^{(m_{p^i})} \oplus \ldots \oplus (X/p^iX)^{(m_{p^i})}$$

is a direct summand of Ext $(G, X) \cong G$. Bearing in mind the fact that $B_1 \oplus \ldots \oplus B_i$ is a maximal $p'$-bounded direct summand of $G$ for every $i$ ([1], p. 99), and that $X/pX \neq o$, we conclude that, for every $i$,

$$\text{Ext}(B_1 \oplus \ldots \oplus B_i, X) \cong B_i \oplus \ldots \oplus B_i$$

(i) Suppose that $X_p \neq o$, then $X = C(p^k) \oplus X'$ where $k$ is a natural number ([1], p. 80), and

$$\text{Ext}(B_{k+i}, X) \cong (X/p^kX)^{(m_{p^i})} \cong (C(p^i) \oplus X'/p^{k+i}X')^{(m_{p^i})}$$
contradicts the fact that
\[ \text{Ext}(B_1 \oplus \ldots \oplus B_{k+j}, X) \cong \text{Ext}(B_1, X) \oplus \ldots \oplus \text{Ext}(B_{k+j}, X) \]
is isomorphic to \( B_1 \oplus \ldots \oplus B_{k+j} \) for all \( j \geq 1 \). Hence we conclude
that \( X_p = 0 \). This proves (i).

(ii) Consider the exact sequences

\[ 0 \to tG \to G \to G/tG \to 0 \]

and

\[ \text{Ext}(G/tG, X) \to \text{Ext}(G, X) \cong G \to \text{Ext}(tG, X) \to 0. \]

It follows from (4) that

\[ G \cong \nu^*(\text{Ext}((G/tG, X)) \oplus \prod_{p \in \mathbb{P}} \text{Ext}((tG)^p, X)) \]

since \( \text{Ext}(G/tG, X) \) is divisible. Recall that \( X_p = 0 \), and hence the exact sequences

\[ 0 \to tX \to X \to X/tX \to 0 \]

and

\[ \text{Ext}((tG)^p, tX) = 0 \to \text{Ext}((tG)^p, X) \to \text{Ext}((tG)^p, X/tX) \to 0 \]

show that

\[ \text{Ext}((tG)^p, X) \cong \text{Ext}((tG)^p, X/tX) \cong \text{Hom}((tG)^p, (X/tX) \otimes C(p^*))). \]

Now, \( X_p = 0 \) and \( X/pX \neq 0 \) imply \( (X/tX)/p(X/tX) \neq 0 \) and hence

\[ (X/tX) \otimes C(p^*) \cong (C(p^*)), \]

where

\[ n_p = r((X/tX)/p(X/tX)) = r(X/pX) \]

(1), p. 255). Hence \( \text{Ext}((tG)^p, X) \) has a direct summand \( \text{Hom}((tG)^p, C(p^*)) \) and, by [2] (p. 137),

\[ \text{Hom}((tG)^p, C(p^*)) \cong Z(p)^N \oplus \prod_{i=1}^\infty (C(p)^{m_i}). \]

However, the latter group is a direct summand of \( G \) by virtue of (5) and (6). This proves (ii).

(iii) By making use of the result in (ii), we see that \( G \) contains direct summands \( Z(p)^N \) and \( C(p)^i \) for a suitable \( i \). Hence \( \text{Hom}(G, G) \) has a
direct summand \( \text{Hom}(Z(p)^N, C(p^i)) \). However,

\[ V = \text{Hom}(Z(p)^N, C(p^i)) \cong \text{Hom}(Z(p)^N/p^i(Z(p)^N), C(p^i)) \]
and since $Z(p)^N/p^i(Z(p)^N)$ is isomorphic to the direct sum of an infinite number of copies of $C(p^i)$, it follows that $V$ is isomorphic to the direct sum of an infinite number of copies of $C(p^i)$. Hence, by [3] (p. 136), $\text{Hom}(G, G)$ is not left Ext-reproduced. This proves (iii).

(iv) By (ii), $G$ contains direct summands $\prod_{i=1}^{\infty} ((C(p^i))^{m_{pi}})$ and $Z(p)^N$, and hence $\text{Ext}(G, G)$ has a direct summand

$$\text{Ext} \left( \prod_{i=1}^{\infty} ((C(p^i))^{m_{pi}}), Z(p)^N \right).$$

Now, by [2] (p. 137),

$$\text{Ext} \left( \prod_{i=1}^{\infty} ((C(p^i))^{m_{pi}}), Z(p)^N \right) \cong \left( \text{Ext} \left( \prod_{i=1}^{\infty} ((C(p^i))^{m_{pi}}), Z(p) \right)^N \right) \cong \left( \text{Hom} \left( \prod_{i=1}^{\infty} ((C(p^i))^{m_{pi}}), Z(p) \otimes C(p^*) \right)^N \right) \cong \left( Z(p)^N \oplus \prod_{i=1}^{\infty} ((C(p^i))^{m_{pi}}) \right)^N.$$

The latter group, and consequently $\text{Ext}(G, G)$ as well, contains a bounded direct summand of power $2^N$. By [3] (p. 136), $\text{Ext}(G, G)$ is not left Ext-reproduced. This proves (iv), and the proof of the lemma is complete.

**Lemma 3.** — Let

$$G = K \oplus \prod_{p \in \mathfrak{p}} T_p,$$

where $T_p$ is a finite $p$-group and $T_p \neq 0$ for an infinite number of primes $p$ and where $K$ has no reduced torsion direct summand. Then $G$ does not belong to any class $\mathfrak{M}$ of left Ext-reproduced groups for which $\text{Ext}(Q, X) \neq 0$.

**Proof.** — Suppose, to the contrary, that there exists a class $\mathfrak{M}$ of left Ext-reproduced groups with $\text{Ext}(Q, X) \neq 0$, and which contains a group $G = K \oplus \prod_{p \in \mathfrak{p}} T_p$ satisfying the above-mentioned properties. The exact sequence

$$0 \rightarrow \bigoplus_{p \in \mathfrak{p}} T_p \rightarrow G \rightarrow Q^{(N)} \oplus K \rightarrow 0$$
gives rise to the exact sequence

\[(7) \quad \text{Hom} \left( \bigoplus_{p \in \mathcal{P}} T_p, X \right) \rightarrow \text{Ext} \left( Q^{(n)} \oplus K, X \right) \rightarrow \text{Ext} \left( G, X \right) \cong G'
\]

and we have that

\[(8) \quad \left| \prod_{p \in \mathcal{P}} \text{Hom} (T_p, X) \right| \leq \aleph.
\]

This follows from the following observation: If \( T_p \neq o \) then by Lemma 2, \( X/pX \neq o \) and if we put \( T_p = B_1 \oplus \ldots \oplus B_n \) where \( B_i = (C(p^i))^m \), then we deduce from

\[\text{Ext} (T_p, X) \cong (X/pX)^{m_1} \oplus \ldots \oplus (X/p^nX)^{m_n}\]

that \( \text{Ext} (T_p, X) \cong T_p \). This implies in particular that \( X_p \) is finite for all primes \( p \) for which \( T_p \neq o \). This proves (8).

However,

\[\text{Ext} \left( Q^{(n)} \oplus K, X \right) \cong (\text{Ext} (Q, X))^n \oplus \text{Ext} (K, X)\]

and our initial assumption implies that

\[| (\text{Ext} (Q, X))^n| \geq 2^n\]

Hence we deduce from (7) that \( G \) contains a torsion-free divisible subgroup of infinite rank, and this contradiction completes the proof.

**Lemma 4.** — Let \( \mathfrak{M} \) be a class of left Ext-reproduced groups for which \( \text{Ext} (Q, X) \neq o \). Suppose that \( G \in \mathfrak{M} \) is not reduced, and let

\[G = D \oplus \left( \bigoplus_{p \in \mathcal{P}} D_p \right) \oplus G'\]

where \( D = Q^n \) (\( n \) a non-negative integer), \( D_p = (C(p^e))^{(n_p)} \), and where \( G' \) is reduced.

1° If \( \bigoplus_{p \in \mathcal{P}} D_p \neq o \), then

(i) \( G' = L' \oplus tG' \), where \( L' \neq o \) is a reduced and torsion-free cotorsion group and where \( tG' \) is finite;

(ii) \( X_p = o \) for all primes \( p \) for which \( D_p \neq o \);

(iii) \( \text{Ext} \left( \bigoplus_{p \in \mathcal{P}} D_p, X \right) \cong L', \text{Ext} (L', X) \cong \bigoplus_{p \in \mathcal{P}} D_p \).

2° If \( \text{Ext} (Z(p), X) \neq o \) for some prime \( p \), then \( D_p = o \).

**Proof.** — Let \( G \in \mathfrak{M} \) and let

\[G = D \oplus \left( \bigoplus_{p \in \mathcal{P}} D_p \right) \oplus G'\]

where \( D, D_p \) and \( G' \) are defined as above.
Let us assume that \( \bigoplus_{p \in P} D_p \neq 0 \), then it is clear that \( G' \) is not torsion. Moreover, \( (tG')_p \) is bounded for all primes \( p \). In fact, if \( (tG')_p \) is unbounded for some prime \( p \) then by Lemma 2, \( X_p = 0 \), and \( G \) has a direct summand

\[
U = \prod_{i=1}^{\infty} ((C(p_i))^{m_{p_i}}),
\]

where \( m_{p_i} \) is finite for all \( i \), and \( m_{p_i} \neq 0 \) for an infinite number of \( i \)'s.

By [4] (Lemma 2.5), \( U/tU \cong Q^{(m)} \oplus K \), where \( m \) is infinite and \( K \) is reduced. The exact sequences

\[
o \to tU \to U \to Q^{(m)} \oplus K \to o
\]

and

\[
\text{Hom}(tU, X) = o \to \text{Ext}(Q^{(m)} \oplus K, X) \to \text{Ext}(U, X)
\]

show that \( \text{Ext}(U, X) \) contains a direct summand \( (\text{Ext}(Q, X))^{m} \cong Q^{(m)} \). However, \( \text{Ext}(U, X) \) is a direct summand of \( G \), and hence \( G \) contains a torsion-free divisible subgroup of infinite rank. This contradiction proves that \( (tG')_p \) is bounded and therefore finite for all primes \( p \).

Now \( G' \), being a direct summand of a cotorsion group, is also cotorsion and hence the exact sequences ([5], p. 373),

\[
o \to tG' \to G' \to G'/tG' \to o
\]

and

\[
o \to \text{Ext}(Q/Z, tG') \cong \prod_{p \in P} (tG')_p \to \text{Ext}(Q/Z, G') \cong G'
\]

\[
\quad \to \text{Ext}(Q/Z, G'/tG') = L' \to o
\]

imply

\[
G' \cong \prod_{p \in P} (tG')_p \oplus L'
\]

since \( \prod_{p \in P} (tG')_p \) is cotorsion and \( L' \) is torsion-free cotorsion. Moreover, by Lemma 3, \( (tG')_p \neq 0 \) for at most a finite number of primes \( p \) and hence

\[
G' \cong tG' \oplus L',
\]

where \( L' \neq 0 \) since, by assumption, \( \bigoplus_{p \in P} D_p \neq 0 \), and where \( tG' \) is finite. This proves (i).

In order to prove (ii), put \( tG' = \bigoplus_{i=1}^{n} (tG')_{p_i} \), where

\[
(tG')_{p_i} = B_{(p_i)}^{(p_i)} \oplus \ldots \oplus B_{(p_i)}^{(p_i)} \text{ and } B_{(p_i)}^{(p_i)} = (C(p_i))^{m_{p_i}},
\]
$m_{ij}$ a non-negative integer. Then

$$\text{Ext}(tG', X) \simeq \bigoplus_{i=1}^{n} (\bigoplus \ldots \bigoplus \bigoplus \bigoplus (X/p_{i}, X)^{m_{ij}})$$

and hence it follows from Lemma 2 that $\text{Ext}(tG', X) \cong tG'$. Furthermore, it is clear that $\text{Ext}(D, X) \cong D$. Hence, if $X_{p} \neq 0$ for some prime $p$ then $\text{Ext}(C(p^\ast), X)$ will contain a finite $p$-group and therefore, if we consider $\text{Ext}(G, X) \cong G$ then $tG'$ is no longer the maximal reduced torsion subgroup of $G$. Consequently $X_{p} = 0$ for all primes $p$ for which $D_{p} \neq 0$ and this proves (ii).

We have $G = \bigoplus \bigoplus D_{p} \bigoplus L' \bigoplus tG'$ and, hence, if we consider $\text{Ext}(G, X) \cong G$, then $\text{Ext}(D, X) \cong D$ and $\text{Ext}(tG', X) \cong tG'$ imply

$$\text{Ext}(\bigoplus \bigoplus D_{p} \bigoplus L', X) \cong \bigoplus \bigoplus \text{Ext}(D_{p}, X) \bigoplus \text{Ext}(L', X).$$

However,

$$\prod_{p \in \mathcal{P}} \text{Ext}(D_{p}, X) \cong \prod_{p \in \mathcal{P}} \text{Hom}(D_{p}, (X/tX) \otimes C(p^\ast))$$

and by (ii)

$$(\ast) \prod_{p \in \mathcal{P}} \text{Ext}(D_{p}, X) \cong \prod_{p \in \mathcal{P}} \text{Hom}(D_{p}, (X/tX) \otimes C(p^\ast))$$

and the latter group is a reduced torsion-free cotorsion group. Moreover, $\text{Ext}(L', X)$ is divisible since $L'$ is torsion-free and hence assertion (iii) follows.

We turn our attention to $\omega$, and we suppose that $\text{Ext}(Z(p), X) \neq 0$. We shall prove that $D_{p} = 0$. Assume, to the contrary, that $D_{p} \neq 0$. Then it follows from $(\ast)$ that $G$ has a direct summand

$$\text{Hom}(D_{p}, (X/tX) \otimes C(p^\ast)).$$

Let $D_{p} = (C(p^\ast))^{(m)}$. We assert that $m$ is finite. In fact, if $m \geq \aleph_{0}$, then $\text{Hom}(D_{p}, (X/tX) \otimes C(p^\ast))$ has a direct summand

$$\text{Hom}(C(p^\ast)^{(m)}, C(p^\ast)) \cong (\text{Hom}(C(p^\ast), C(p^\ast))) \cong Z(p)^{m}$$

and hence the exact sequences

$$0 \rightarrow Z(p)^{(m)} \rightarrow Z(p)^{m} \rightarrow Z(p)_{p}$$

and $\text{Ext}(Z(p)^{m}, X) \rightarrow (\text{Ext}(Z(p), X))^{m} \rightarrow 0,$

and $\text{Ext}(Z(p), X) \neq 0$, show that $\text{Ext}(Z(p)^{m}, X)$ has a torsion-free divisible subgroup of infinite rank. This is a contradiction since $\text{Ext}(Z(p)^{m}, X)$ is a direct summand of $G$. Hence $m$ is finite, i.e.

$$D_{p} = (C(p^\ast))^{m_{p}}, \quad m_{p} \text{ a non-negative integer}.$$
Next, we assert that \( r((X/tX)/p(X/tX)) = r(X/pX) \) is finite. Indeed, if \( r(X/pX) = n \geq \aleph_0 \), then \( \text{Hom}(D_p, (X/tX) \otimes C(p^*)) \) has a direct summand
\[
\text{Hom}(C(p^*), (X/tX) \otimes C(p^*)) \cong \text{Hom}(C(p^*), C(p^*)^{(n)}) = V,
\]
and the latter group contains a subgroup
\[
(\text{Hom}(C(p^*), C(p^*))^{(n)}) \cong Z(p)^{(n)}.
\]
Now the exact sequences \( 0 \to Z(p)^{(n)} \to V \) and
\[
\text{Ext}(V, X) \to (\text{Ext}(Z(p), X))^n \to 0 \quad \text{and} \quad \text{Ext}(Z(p), X) \neq 0,
\]
imply that \( \text{Ext}(V, X) \) has a torsion-free divisible subgroup of infinite rank. This gives rise to a contradiction since \( \text{Ext}(V, X) \) is a direct summand of \( G \), and we conclude that \( n \) is finite. Consequently, \( G \) has a direct summand
\[
\text{Hom}(D_p, (X/tX) \otimes C(p^*)) \cong Z(p)^{n_p}, \quad n_p \text{ a natural number}.
\]
Moreover, it follows from (iii) and [1] (p. 245) that
\[
(12) \quad \text{Ext}(Z(p)^{n_p}, X) \cong D_p \cong (C(p^*))^{n_p}.
\]
We assert that (12) is impossible. Indeed, suppose that (12) holds, and consider the exact sequences (see [1], p. 250 and p. 252)
\[
o \to Z(p)^{n_p} \otimes Z \cong Z(p)^{n_p} \to (Z(p) \otimes Q)^{n_p} \to Z(p)^{n_p} \otimes (Q/Z) \cong (C(p^*))^{n_p} \to 0
\]
and
\[
(13) \quad \text{Ext}((C(p^*))^{n_p}, X) \to (\text{Ext}(Q, X))^\mathbb{N} \to \text{Ext}(Z(p)^{n_p}, X) \to 0.
\]
We have that
\[
(a) \quad |\text{Ext}((C(p^*))^{n_p}, X)| = |\text{Hom}(C(p^*), (X/tX) \otimes C(p^*))^{n_p}| = \aleph_0
\]
since \( r((X/tX)/p(X/tX)) \) is finite;
\[
(b) \quad |(\text{Ext}(Q, X))^\mathbb{N}| = 2^\aleph_0
\]
since, by assumption, \( \text{Ext}(Q, X) \neq 0 \);
\[
(c) \quad |\text{Ext}(Z(p)^{n_p}, X)| = \aleph_0,
\]
on account of (12), and the exact sequence (13) implies that this is clearly impossible. Hence (12) leads to a contradiction, and we conclude that \( D_p = \emptyset \). This proves \( 2^\aleph_0 \), and completes the proof of the lemma.

We now derive properties which are characteristic of the classes \( \overline{Y} \) and \( \overline{G} \).
Theorem 5.
(i) If \( G \in \mathcal{G} \), then for every direct summand \( U \) of \( G \) we have \( U \in \mathcal{Y} \).
(ii) \( \mathcal{G} \) is a class of left Ext-reproduced groups which is maximal with respect to the property:
If \( G \in \mathcal{G} \), then for every direct summand \( U \) of \( G \) we have \( U \in \mathcal{G} \).

Proof.
(i) This statement is an immediate consequence of [4] (Theorem 2.14).
(ii) Let \( G = G' \oplus G'' \in \mathcal{G} \) where \( G' \neq 0, G'' \neq 0 \). Then \( tG' \neq 0, tG'' \neq 0 \) and recall that \( G \cong \text{Ext}(Q/Z, tG) \) since \( G \) is adjusted ([5], p. 375). Now, both \( G' \) and \( G'' \) are adjusted and hence

\[
G = G' \oplus G'' \cong \text{Ext}(Q/Z, tG') \oplus \text{Ext}(Q/Z, tG'')
\]

\[
\cong \prod_{p \in \mathbb{P}} \text{Ext}(C(p^\infty), (tG')_p) \oplus \prod_{p \in \mathbb{P}} \text{Ext}(C(p^\infty), (tG'')_p)
\]

whence it follows that

\[
G' \cong \prod_{p \in \mathbb{P}} (tG')_p \in \mathcal{G}, \quad G'' \cong \prod_{p \in \mathbb{P}} (tG'')_p \in \mathcal{G}.
\]

Let \( \mathcal{H} \) be a class of left Ext-reproduced groups such that if \( G \in \mathcal{H} \) then for every direct summand \( U \) of \( G \) we have \( U \in \mathcal{H} \), and let \( \mathcal{H} \supseteq \mathcal{G} \). Then it follows from Lemma 1 that \( X \cong \prod_{p \in \mathbb{P}} Z(p) \). Let \( G \in \mathcal{H} \), then \( G_p \) is bounded for all primes \( p \). Indeed, if \( G_p \) is unbounded for some prime \( p \) then by Lemma 2, \( G \) has a direct summand \( Z(p)^\mathbb{N} \) and by assumption \( Z(p)^\mathbb{N} \in \mathcal{H} \), which is clearly impossible. Hence \( G_p \) is bounded and consequently finite for all primes \( p \), and we have (see [4], Example 2.8)

\[
G \cong \text{Ext} \left( tG, \prod_{p \in \mathbb{P}} Z(p) \right) \cong \text{Hom}(tG, Q/Z) \cong \prod_{p \in \mathbb{P}} (tG)_p.
\]

Hence \( G \in \mathcal{H} \) implies \( G \in \mathcal{G} \) and consequently \( \mathcal{H} \subseteq \mathcal{G} \). This completes the proof.

Theorem 6. — Let \( \mathcal{M} \) be a class of left Ext-reproduced groups which is maximal with respect to the property:
If \( G \in \mathcal{M} \) then for all direct summands \( U \) of \( G \), we have \( U \in \mathcal{M} \). Then either \( \mathcal{M} = \mathcal{Y} \) or \( \mathcal{M} = \mathcal{G} \).
Proof. — Let \( \mathcal{M} \) be a class of left Ext-reproduced groups which is maximal with respect to the above mentioned property. Then there are two possibilities, viz. either

(i) there exists a group \( G \in \mathcal{M} \) which is not reduced, or
(ii) all groups \( G \in \mathcal{M} \) are reduced.

Let us first consider case (i). If \( G \in \mathcal{M} \) is not reduced, then \( G = Q^n \oplus G' \) where \( n \) is a natural number, and \( G' \) is reduced. By assumption, \( Q \in \mathcal{M} \) and hence, by [4] (Theorem 2.2), \( H/X \cong Q \) where \( H \) is a reduced cotorsion group. Moreover, we also have that \( G' \in \mathcal{M} \). Now, \( \mathcal{M} \), being maximal, cannot contain only divisible groups and hence there exists a non-zero reduced group \( G' \in \mathcal{M} \). We contend that

(a) \( G' \) is finite.

In order to prove this, consider the exact sequences

\[
0 \to tG' \to G' \to G/\mathcal{H} \to 0
\]

and

\[
\text{Ext}(G/\mathcal{H}, X) \to \text{Ext}(G, X) \cong G' \to \text{Ext}(tG', X) \to 0.
\]

We conclude that

\[
G' \cong \text{Ext}(tG', X) \cong \prod_{p \in \mathbb{P}} \text{Ext}((tG')_p, X)
\]

since \( G' \) is reduced. In the first instance, \((tG')_p\) is bounded for all primes \( p \). In fact, if \((tG')_p\) is unbounded for some prime \( p \) then it follows from Lemma 2 that \( G' \) has a direct summand \( Z(p)^N \) and hence our initial assumption implies that \( Z(p)^N \in \mathcal{M} \). This contradiction shows that \((tG')_p\) is bounded and hence finite for all primes \( p \). Moreover, by Lemma 3, \((tG')_p\) is compact for only a finite number of primes \( p \) whence it follows that \( tG' \), and consequently \( G' \cong \text{Ext}(tG', X) \) as well, is finite. This proves (a).

To recapitulate, if (i) holds and if \( G \in \mathcal{M} \) then \( G \cong Q^n \oplus T \) where \( n \) is a non-negative integer and \( T \) is a finite group, that is to say \( G \in \mathcal{N} \). Hence \( \mathcal{M} \subseteq \mathcal{N} \) and the maximality of \( \mathcal{M} \) implies that \( \mathcal{M} = \mathcal{N} \). This settles the first case.

Next we consider case (ii). In this case, \( tG \neq 0 \) for all \( G \in \mathcal{M} \). Again, if \( G \in \mathcal{M} \), then \((tG)_p\) is bounded for all primes \( p \), for if \((tG)_p\) is unbounded for some prime \( p \) then by Lemma 2, \( Z(p)^N \in \mathcal{M} \). This contradiction shows that \((tG)_p\) is bounded and hence finite for all primes \( p \). Since \((tG)_p\) is finite and pure it follows from [1] (p. 80) that \((tG)_p \in \mathcal{M} \) for all primes \( p \) whence

\[
G \cong \prod_{p \in \mathbb{P}} \text{Ext}((tG)_p, X) \cong \prod_{p \in \mathbb{P}} (tG)_p.
\]

We maintain that \( \text{Ext}(Q, X) = 0 \).
Assume, to the contrary, that Ext(\(Q, X\)) \(\neq o\). The maximality of \(M\) implies that \(tG\) cannot be finite for all \(G \in M\) and hence there exists an \(H \in M\) with \(tH\) an infinite torsion group. This implies that \((tH)_p \neq o\) for an infinite number of primes \(p\) and hence

\[ H \cong \text{Ext}\left(\bigoplus_{p \in \mathbb{P}} (tH)_p, X\right) \cong \prod_{p \in \mathbb{P}} (tH)_p. \]

By Lemma 3, \(H \notin M\). This contradiction shows that Ext(\(Q, X\)) = o.

The latter fact and (14) show that if \(G \in M\) then \(G \in \mathfrak{G}\), that is, \(M \subseteq \mathfrak{G}\) and hence \(M = \mathfrak{G}\) since \(M\) is maximal. This completes the proof of the theorem.

**Theorem 7.**

(i) If \(G \in \mathfrak{G}\) and \(H \in \mathfrak{G}\), then \(\text{Hom}(G, H) \in \mathfrak{G}\).

(ii) \(\mathfrak{G}\) is a class of left Ext-reproduced groups which is maximal with respect to the property:

If \(G \in \mathfrak{G}\) and \(H \in \mathfrak{G}\), then \(\text{Hom}(G, H) \in \mathfrak{G}\).

**Proof.**

(i) If \(G \in \mathfrak{G}\) and \(H \in \mathfrak{G}\) then \(G = Q^n \oplus S\), \(H = Q^m \oplus T\) where \(m\) and \(n\) are non-negative integers and \(S\) and \(T\) are finite groups. Consequently,

\[ \text{Hom}(G, H) \cong Q^{mn} \oplus \text{Hom}(S, T) \in \mathfrak{G}. \]

This proves (i).

(ii) Let \(G \in \mathfrak{G}\) and \(H \in \mathfrak{G}\), then \(G \cong \prod_{p \in \mathbb{P}} G_p\), \(H \cong \prod_{p \in \mathbb{P}} H_p\) where \(G_p\) and \(H_p\) are finite \(p\)-groups for all primes \(p\). Then we have

\[ \text{Hom}(G, H) \cong \text{Hom}(tG, tH) \cong \prod_{p \in \mathbb{P}} \text{Hom}((tG)_p, (tH)_p) \in \mathfrak{G}. \]

Let \(\mathfrak{S}\) be a class of left Ext-reproduced groups such that if \(G \in \mathfrak{S}\), \(H \in \mathfrak{S}\) then \(\text{Hom}(G, H) \in \mathfrak{S}\), and suppose that \(\mathfrak{S} \supseteq \mathfrak{G}\). Then, by Lemma 1,

\[ X \cong \prod_{p \in \mathbb{P}} Z(p). \]

All groups \(G \in \mathfrak{S}\) are reduced since

\[ G \cong \text{Ext}(G, X) \cong \text{Ext}(tG, X), \]

and hence if \(o \neq G \in \mathfrak{S}\), then \(tG \neq o\).

Let \(G \in \mathfrak{S}\). Then \(G_p\) is bounded for all primes \(p\), for if \(G_p\) is unbounded for some prime \(p\) then it follows from Lemma 2 that \(\text{Hom}(G, G)\) is not left Ext-reproduced. Hence \(G_p\) is bounded and therefore finite for all
primes \( p \). Consequently,

\[
G \cong \text{Ext}(tG, X) \cong \prod_{p \in P} \text{Ext}((tG)_p, X) \cong \prod_{p \in P} \text{Hom}((tG)_p, Q/Z) \cong \prod_{p \in P} (tG)_p
\]

and hence \( G \in \mathcal{G} \). Therefore \( \mathcal{S} \subseteq \mathcal{G} \), and the proof is complete.

**Theorem 8.** — Let \( M \) be a class of left Ext-reproduced groups which is maximal with respect to the property:

If \( G \in M \) and \( H \in M \), then \( \text{Hom}(G, H) \in M \).

Then either \( M = \mathcal{S} \) or \( M = \mathcal{G} \).

**Proof.** — There are two possibilities, viz. either

(i) \( M \) contains a group \( G \) which is not reduced, or
(ii) all groups \( G \) in \( M \) are reduced.

Consider case (i), and let \( G = D \oplus G' \) where \( D \neq 0 \) is divisible and \( G' \) is reduced. Then \( D \) contains no subgroup \( C(p^{*}) \), for if \( D = C(p^{*}) \oplus D' \), then \( \text{Hom}(C(p^{*}), C(p^{*})) \cong Z(p) \) is a direct summand of \( \text{Hom}(G, G) \in M \) and, by assumption, \( \text{Hom}(\text{Hom}(G, G), G) \in M \). However, the latter group has a direct summand \( \text{Hom}(Z(p), C(p^{*})) \cong Q(\mathbb{N}) \oplus C(p^{*}) \) ([2], p. 136), and this gives rise to a contradiction since any left Ext-reproduced group contains at most a finite number of copies of \( Q \). This proves that \( G = Q^{\oplus} \oplus G' \) where \( n \) is a natural number and \( G' \) is reduced. It is also clear that \( G' \in M \).

Now \( G' \) is bounded for all primes \( p \), for if \( G' \) is unbounded for some prime \( p \) then it follows from Lemma 2 that \( \text{Hom}(G, G) \) is not left Ext-reproduced. This shows that \( G' \) is bounded and consequently finite for all primes \( p \). Furthermore, \( (tG')_p \neq 0 \) for at most a finite number of primes \( p \), for if \( (tG')_p \neq 0 \) for an infinite number of primes \( p \), then by Lemma 3,

\[
G' \cong \prod_{p \in P} \text{Ext}((tG')_p, X) \cong \prod_{p \in P} (tG')_p \in M.
\]

[We mention in passing the fact that if \( (tG')_p \neq 0 \) for an infinite number of primes \( p \), then it can also easily be shown that \( \text{Hom}(G, G) \) contains a torsion-free divisible subgroup of infinite rank.] This contradiction shows that \( (tG')_p \neq 0 \) for at most a finite number of primes \( p \) whence we deduce that \( G' \) is finite. The proof thus far shows in fact that the reduced part of each group \( G \in M \) is finite.

Hence, if condition (i) holds and \( G \in M \) then \( G = Q^{n} \oplus T \) where \( n \) is a non-negative integer and \( T \) is finite, that is to say, \( G \in \mathcal{S} \) and hence \( M \subseteq \mathcal{S} \). The maximality of \( M \) implies that \( M = \mathcal{S} \).
We turn our attention to case (ii). Since all $G \in \mathcal{M}$ are reduced, it follows that if $\varphi \neq G \in \mathcal{M}$, then $tG \neq o$ for we have

$$G \cong \text{Ext}(tG, X)$$

for all $G \in \mathcal{M}$, recall (4) and (5) in the proof of Lemma 2. Moreover, Lemma 2 implies that $(tG)_p$ is bounded and hence finite for all primes $p$. Consequently, if $G \in \mathcal{M}$ then

$$G \cong \prod_{p \in P} \text{Ext}(tG_p, X) \cong \prod_{p \in P} (tG)_p$$

and hence $G \in \mathcal{G}$. This proves that $\mathcal{M} \subseteq \mathcal{G}$ and since $\mathcal{M}$ is maximal it follows that $\mathcal{M} = \mathcal{G}$. This completes the proof.

**Theorem 9.**

(i) Let $G \in \mathcal{G}$ and $H \in \mathcal{G}$, and let $\varphi : G \rightarrow H$ be a homomorphism. Then $\ker \varphi \in \mathcal{G}$ and $\operatorname{coker} \varphi \in \mathcal{G}$.

(ii) $\mathcal{G}$ is a class of left Ext-reproduced groups which is maximal with respect to the property:

If $G \in \mathcal{G}$ and $H \in \mathcal{G}$, and if $\varphi : G \rightarrow H$ is a homomorphism, then $\ker \varphi \in \mathcal{G}$ and $\operatorname{coker} \varphi \in \mathcal{G}$.

**Proof.**

(i) Let $G \in \mathcal{G}$, $H \in \mathcal{G}$, and let $\varphi : G \rightarrow H$ be a homomorphism. Then $G = Q^m \oplus S$, $H = Q^n \oplus T$ where $m$ and $n$ are non-negative integers, and $S$ and $T$ are finite groups. It is clear that $\varphi(S) \leq Q^n$ and that $\varphi(S) \leq T$. Hence $\ker \varphi = (\ker \varphi \cap Q^n) \oplus (\ker \varphi \cap S)$ and since $\ker \varphi \cap Q^n$ is divisible and $\ker \varphi \cap S$ is finite, our assertion follows.

(ii) Let $G \in \mathcal{G}$, $H \in \mathcal{G}$, and let $\varphi : G \rightarrow H$ be a homomorphism. $\ker \varphi$ and $\operatorname{Im} \varphi$ are reduced, and hence it follows from the exact sequences

$$0 \rightarrow \ker \varphi \rightarrow G \rightarrow \operatorname{Im} \varphi \rightarrow 0$$

and

$$0 \rightarrow \text{Ext}(Q, \ker \varphi) \rightarrow \text{Ext}(Q, G) = 0 \rightarrow \text{Ext}(Q, \operatorname{Im} \varphi) \rightarrow 0$$

that $\ker \varphi$ and $\operatorname{Im} \varphi$ are reduced cotorsion groups.

Note that if $tG \subseteq \ker \varphi$, then $\varphi = o$ for then we have

$$G/\ker \varphi \cong (G/tG)/(\ker \varphi/tG) \subseteq H$$

and $G/tG$ is divisible, whence $G/\ker \varphi$ is divisible and hence $o$. Hence $\ker \varphi = G$, that is, $\varphi = o$. Note further that if $\ker \varphi$ is a torsion group then it is necessarily finite and then our assertion is obvious. We may therefore assume that $\ker \varphi$ is infinite and that $\varphi \neq o$. We then have

$$\ker \varphi \cap tG \neq tG, \quad (\ker \varphi)_p \subseteq (tG)_p$$

for all primes $p$. 


We maintain that $\text{Ker} \varphi$ is adjusted. In order to prove this, assume to the contrary that $\text{Ker} \varphi$ is not adjusted. Then it follows from [5] (p. 373-374) that

$$\text{Ker} \varphi \cong \prod_{p \in \mathfrak{P}} (\text{Ker} \varphi)_p \oplus L,$$

where $L \neq 0$ is a reduced torsion-free cotorsion group, and hence $L$ has a direct summand $Z(p)$ for some prime $p([5]$, p. 372). Hence $\text{Ker} \varphi$, and therefore $G$ as well, contains a subgroup $Z(p)$. That is to say, $G$ contains elements of infinite order which are divisible by arbitrarily high powers of primes $q(q \neq p)$ since $qZ(p) = Z(p)$ for all primes $q \neq p$. This is evidently impossible, and we conclude that $\text{Ker} \varphi$ is adjusted, i.e. $\text{Ker} \varphi \cong \prod_{p \in \mathfrak{P}} (\text{Ker} \varphi)_p \in \mathfrak{S}$.

We have the commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 & \rightarrow & t \text{Ker} \varphi & \rightarrow & tG & \rightarrow & \bigoplus_{p \in \mathfrak{P}} ((tG)_p/(t(\text{Ker} \varphi)_p)) & \rightarrow & 0 \\
& & v & & & & \lambda & & \\
0 & \rightarrow & \text{Ker} \varphi & \rightarrow & G & \rightarrow & \text{Im} \varphi & \rightarrow & 0
\end{array}$$

where $\iota$, $\mu$ and $\lambda$ are the obvious mappings. This gives rise to the commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 & \rightarrow & \text{Ext}(Q/Z, t \text{Ker} \varphi) & \rightarrow & \text{Ext}(Q/Z, tG) & \rightarrow & \prod_{p \in \mathfrak{P}} ((tG)_p/(t(\text{Ker} \varphi)_p)) & \rightarrow & 0 \\
& & \iota_* & & \mu_* & & \gamma_* & & \\
0 & \rightarrow & \text{Ext}(Q/Z, \text{Ker} \varphi) & \rightarrow & \text{Ext}(Q/Z, G) & \rightarrow & \text{Ext}(Q/Z, \text{Im} \varphi) & \rightarrow & 0
\end{array}$$

and since $\iota_*$ and $\mu_*$ are isomorphisms ([5], p. 375), it follows that $\gamma_*$ is also an isomorphism. Hence

$$\text{Im} \varphi \cong \text{Ext} \left( Q/Z, \prod_{p \in \mathfrak{P}} ((tG)_p/(t(\text{Ker} \varphi)_p)) \right) \cong \prod_{p \in \mathfrak{P}} ((tG)_p/(t(\text{Ker} \varphi)_p)) \in \mathfrak{S}.$$

The exact sequences

$$(15) \quad 0 \rightarrow \text{Im} \varphi \rightarrow H \rightarrow \text{Coker} \varphi \rightarrow 0$$

and

$$\text{Hom}(Q, H) = 0 \rightarrow \text{Hom}(Q, \text{Coker} \varphi) \rightarrow \text{Ext}(Q, \text{Im} \varphi) = 0$$

show that $\text{Coker} \varphi$ is a reduced cotorsion group. If we consider the exact sequence (15), then we obtain a situation entirely similar to that in the above commutative diagrams, and we conclude that $\text{Coker} \varphi \in \mathfrak{S}$.

If $\mathfrak{S}$ is a class of left Ext-reproduced groups such that if $K \in \mathfrak{S}$, $M \in \mathfrak{S}$, and if $\psi : K \rightarrow M$ is a homomorphism, then $\text{Ker} \psi \in \mathfrak{S}$, $\text{Coker} \psi \in \mathfrak{S}$, and if
§3. Let $\mathfrak{G} \supseteq \mathfrak{G}$, then $\mathfrak{G} = \mathfrak{G}$. In fact, by Lemma 1, $X \cong \prod_{p \in \mathcal{P}} Z(p)$ whence it follows that all groups in $\mathfrak{G}$ are reduced. Hence if $K \in \mathfrak{G}$ then, obviously, every direct summand of $K$ belongs to $\mathfrak{G}$ and hence, by Theorem 6, $K \in \mathfrak{G}$. We have therefore shown that $\mathfrak{G} \supseteq \mathfrak{G}$, and the proof is complete.

Remark. — We turn our attention to the converse of the above theorem. Let $\mathfrak{M}$ be a class of left Ext-reproduced groups which is maximal with respect to the property:

If $G \in \mathfrak{M}$, $H \in \mathfrak{M}$ and if $\varphi : G \to H$ is a homomorphism then $\text{Ker}\varphi \in \mathfrak{M}$, $\text{Coker}\varphi \in \mathfrak{M}$.

Then either $\mathfrak{M} = \mathfrak{G}$ or $\mathfrak{M} = \mathfrak{G}$.

Indeed, if $G \in \mathfrak{M}$ then, manifestly, every direct summand of $G$ belongs to $\mathfrak{M}$ and hence Theorem 6 implies our assertion.

Theorem 10.

(i) If $G \in \mathfrak{G}$ and $H \in \mathfrak{G}$, then $\text{Ext}(G, H) \in \mathfrak{G}$.

(ii) $\mathfrak{G}$ is a class of left Ext-reproduced groups which is maximal with respect to the property:

If $G \in \mathfrak{G}$ and $H \in \mathfrak{G}$, then $\text{Ext}(G, H) \in \mathfrak{G}$.

Proof.

(i) Let $G \in \mathfrak{G}$, $H \in \mathfrak{G}$, then $G = Q^m \oplus S$, $H = Q^n \oplus T$ where $m$ and $n$ are non-negative integers and $S$ and $T$ are finite groups. It is clear that $\text{Ext}(G, H) \cong \text{Ext}(S, T)$ and since the latter group is finite it follows that $\text{Ext}(G, H) \in \mathfrak{G}$.

(ii) Let $G \in \mathfrak{G}$, $H \in \mathfrak{G}$, then $G \cong \prod_{p \in \mathcal{P}} (tG)_p$, $H \cong \prod_{p \in \mathcal{P}} (tH)_p$ where $(tH)_p$ and $(tG)_p$ are finite $p$-groups for all primes $p$. The exact sequences

$$0 \to \bigoplus_{p \in \mathcal{P}} (tG)_p \to \prod_{p \in \mathcal{P}} (tG)_p \to \mathcal{Q}^{(\mathcal{P})} \to 0$$

and

$$\text{Ext}(\mathcal{Q}^{(\mathcal{P})}, H) = 0 \to \text{Ext}\left(\prod_{p \in \mathcal{P}} (tG)_p, H\right) \to \text{Ext}\left(\bigoplus_{p \in \mathcal{P}} (tG)_p, H\right) \to 0$$

imply

$$\text{Ext}(G, H) \cong \prod_{p \in \mathcal{P}} \text{Ext}((tG)_p, H) \cong \prod_{p \in \mathcal{P}} \text{Ext}((tG)_p, (tH)_p)$$

and $\text{Ext}((tG)_p, (tH)_p)$ is a finite $p$-group for all primes $p$ so that $\text{Ext}(G, H) \in \mathfrak{G}$. 

BULL. SOC. MATH. — T. 98, FASC. 4.
Let $\mathfrak{S}$ be a class of left Ext-reproduced groups such that $\text{Ext}(G, H) \in \mathfrak{S}$ whenever $G \in \mathfrak{S}$ and $H \in \mathfrak{S}$, and suppose that $\mathfrak{S} \supseteq \mathfrak{G}$. Then, by Lemma 1, $X \cong \prod_p Z(p)$ and hence all groups in $\mathfrak{S}$ are reduced, by virtue of the fact that $G \cong \text{Ext}(tG, X)$ for all $G \in \mathfrak{S}$.

Let $\mathfrak{G} \nsubseteq G \in \mathfrak{S}$, then $tG \neq o$. Then Lemma 2 implies that $(tG)_p$ is bounded and hence finite for all primes $p$. Thus we have (see [4], Example 2.8)

$$G \cong \text{Ext}(tG, X) \cong \text{Hom}(tG, X \otimes (Q/Z)) \cong \prod_p \text{Hom}((tG)_p, Q/Z) \cong \prod_p (tG)_p,$$

in other words, $G \in \mathfrak{G}$. Hence $\mathfrak{S} \subseteq \mathfrak{G}$, and the proof is complete.

**Theorem 11.** — Let $\mathfrak{M}$ be a class of left Ext-reproduced groups which is maximal with respect to the property:

If $G \in \mathfrak{M}$ and $H \in \mathfrak{M}$ then $\text{Ext}(G, H) \in \mathfrak{M}$.

Then either $\mathfrak{M} = \mathfrak{S}$ or $\mathfrak{M} = \mathfrak{G}$.

**Proof.** — The method of approach is basically the same as in Theorems 6 and 8. There are two possibilities, viz. either

(i) all groups $G$ in $\mathfrak{M}$ are reduced, or

(ii) $\mathfrak{M}$ contains a group which is not reduced.

If (i) holds then for all $G \in \mathfrak{M}$, we have

$$G \cong \text{Ext}(tG, X) \cong \prod_p \text{Ext}((tG)_p, X).$$

By Lemma 2, $(tG)_p$ is bounded and hence finite for all primes $p$. Consequently,

$$G \cong \prod_p \text{Ext}((tG)_p, X) \cong \prod_p (tG)_p,$$

and hence $G \in \mathfrak{M}$ implies $G \in \mathfrak{G}$. The maximality of $\mathfrak{M}$ implies that $\mathfrak{M} = \mathfrak{G}$.

Let (ii) be valid, then it is clear that $\text{Ext}(Q, X) \neq o$. Suppose that $G \in \mathfrak{M}$ is not reduced and let

$$G = D \oplus (\bigoplus_{p \in P} D_p) \oplus G'$$

where $D = Q^n$, $n$ a non-negative integer, $D_p = (C(p^\infty))^{(n)}$ and where $G'$ is reduced. We assert that $\bigoplus_{p \in P} D_p = o$. In fact, if this is not the case then it follows from Lemma 4 that

$$G = D \oplus (\bigoplus_{p \in P} D_p) \oplus L' \oplus tG'$$
where $L' \neq o$ is a reduced torsion-free cotorsion group and $tG'$ is finite. By assumption, $\text{Ext}(G, G) \in \mathcal{M}$. However,
\[
\text{Ext}(G, G) \cong \text{Ext}\left(\bigoplus_{p \in \mathcal{P}} D_p, L'\right) \oplus \text{Ext}\left(\bigoplus_{p \in \mathcal{P}} D_p, tG'\right)
\oplus \text{Ext}(tG', L') \oplus \text{Ext}(tG', tG')
\]
and we have
\begin{enumerate}[(a)]  
\item $\text{Ext}\left(\bigoplus_{p \in \mathcal{P}} D_p, L'\right) \neq o$ is a torsion-free and reduced cotorsion group [this follows from Lemma 4, (ii)];  
\item $\text{Ext}\left(\bigoplus_{p \in \mathcal{P}} D_p, tG'\right)$ is a finite group or else $\text{Ext}(G, G) \notin \mathcal{M}$;  
\item $\text{Ext}(tG', L')$ is a finite group for the same reason as in (b);  
\item $\text{Ext}(tG', tG')$ is a finite group — this is obvious.
\end{enumerate}

In other words, $\text{Ext}(G, G)$ is the direct sum of a reduced and non-zero torsion-free cotorsion group, and a finite group, and hence it is not left Ext-reproduced, contrary to $\text{Ext}(G, G) \in \mathcal{M}$. This contradiction shows that $\bigoplus_{p \in \mathcal{P}} D_p = o$ and hence
\[
G = D \oplus G' = Q' \oplus G'
\]
where $n$ is a natural number and $G'$ is a finite group. It follows from Lemma 2 and Lemma 3 that the reduced part of every group $G \in \mathcal{M}$ is finite and hence if $G \in \mathcal{M}$ then $G \in \mathcal{F}$. Hence $\mathcal{M} = \mathcal{F}$ and the proof is complete.

Remark. — If we consider $\mathcal{F}$ then $X$ satisfies $\prod_{p \in \mathcal{P}} Z(p)/X \cong Q$ ([4], Example 2.3) and it is clear that $\text{Hom}(G, X) = o$ for all $G \in \mathcal{F}$. For the class $\mathcal{G}$, we have $X \cong \prod_{p \in \mathcal{P}} Z(p)$ and we also have that $\text{Hom}(G, X) = o$ for all $G \in \mathcal{G}$. Moreover, $\mathcal{G}$ is a class of left Ext-reproduced groups which is maximal with respect to the property : $\text{Hom}(G, X) = o$ for all $G \in \mathcal{G}$. In fact, if $\mathcal{S}$ is a class of left Ext-reproduced groups which contains $\mathcal{G}$ and which is such that $\text{Hom}(G, X) = o$ for all $G \in \mathcal{S}$, then $\mathcal{S} = \mathcal{G}$. This follows from following : If $G \in \mathcal{S}$ then $G_p$ is bounded for all primes $p$, for if $G_p$ is unbounded for some prime $p$ then $G$ contains a direct summand $Z(p)^\mathcal{S}$ (Lemma 2), and since $X \cong \prod_{p \in \mathcal{P}} Z(p)$ (Lemma 1), we deduce that $\text{Hom}(G, X) \neq o$. Hence $G_p$ is bounded and therefore finite for all primes $p$, consequently
\[
G \cong \prod_{p \in \mathcal{P}} \text{Ext}((tG)_p, X) \cong \prod_{p \in \mathcal{P}} (tG)_p \in \mathcal{G}
\]
whence $\mathcal{S} \subseteq \mathcal{G}$.

This property can also be used to characterize the classes $\mathcal{F}$ and $\mathcal{G}$. 

Theorem 12. — Let $\mathfrak{M}$ be a class of left $\text{Ext}$-reproduced groups which is maximal with respect to the property:

$\text{Hom}(G, X) = 0$ for all $G \in \mathfrak{M}$.

Then either $\mathfrak{M} = \mathfrak{G}$ or $\mathfrak{M} = \mathfrak{O}$.

Proof. — There are two alternatives, viz. either

(i) all groups $G$ in $\mathfrak{M}$ are reduced, or

(ii) $\mathfrak{M}$ contains a group $G$ which is not reduced.

Let us consider case (i) and let $G \in \mathfrak{M}$. The exact sequence

$$0 \rightarrow tG \rightarrow G \rightarrow G/tG \rightarrow 0$$

yields the exact sequence

$$0 \rightarrow \text{Hom}(tG, X) \rightarrow \text{Ext}(G/tG, X) \rightarrow \text{Ext}(G, X) \rightarrow \text{Ext}(G/tG, X) \rightarrow 0$$

and since $G$ and $\text{Hom}(tG, X)$ are reduced it follows that

$$(16) \quad \text{Hom}(tG, X) = 0 = \text{Ext}(G/tG, X)$$

and hence $G \cong \prod_{p \in \mathfrak{P}} \text{Ext}((tG)_p, X)$ shows that if $0 \not\cong G \in \mathfrak{M}$ then $tG \neq 0$.

If $G \in \mathfrak{M}$, then $(tG)_p$ is bounded for all primes $p$. Assume, to the contrary, that $(tG)_p$ is unbounded for some prime $p$, then by Lemma 2, $G$ has a direct summand $Z(p)^\mathfrak{N}$, and hence $\text{Hom}(G, X) = 0$ and (16) imply

$$\text{Hom}(Z(p), X) = 0 = \text{Ext}(Z(p), X).$$

The exact sequence (see [1], p. 252 and 255)

$$0 \rightarrow Z(p) \otimes Z \cong Z(p) \rightarrow Z(p) \otimes Q \rightarrow Z(p) \otimes (Q/Z) \cong C(p^\ast) \rightarrow 0$$

leads to the exact sequence

$$0 \rightarrow \text{Ext}(C(p^\ast), X) \rightarrow (\text{Ext}(Q, X))^\mathfrak{N} \rightarrow \text{Ext}(Z(p), X) = 0$$

and hence

$$(17) \quad \text{Ext}(C(p^\ast), X) = 0 = \text{Ext}(Q, X).$$

However, $\text{Hom}(Z(p), X) = 0$ implies $X_p = 0$ and, by Lemma 2, $(tG)_p \neq 0$ implies $X/pX \neq 0$. Hence

$$\text{Ext}(C(p^\ast), X) \cong \text{Ext}(C(p^\ast), X/tX) \cong \text{Hom}(C(p^\ast), (X/tX) \otimes C(p^\ast)) \neq 0$$

since $X/tX \neq (X/tX)$ ([1], p. 255). This is however contrary to (17) and hence we conclude that $(tG)_p$ is bounded and consequently finite for all primes $p$. This implies that

$$G \cong \prod_{p \in \mathfrak{P}} \text{Ext}((tG)_p, X) \cong \prod_{p \in \mathfrak{P}} (tG)_p$$

or alternatively, $G \in \mathfrak{M}$ implies $G \in \mathfrak{G}$ and hence $\mathfrak{M} \subseteq \mathfrak{G}$ so that $\mathfrak{M} = \mathfrak{G}$. 

We now turn our attention to (ii), and we notice that \( \text{Ext}(Q, X) \neq 0 \).

Let \( G \in \mathcal{M} \) be a group which is not reduced and let

\[
G = D \oplus \left( \bigoplus_{\rho \in \mathbb{P}} D_{\rho} \right) \oplus G'
\]

where \( D = Q^n, n \) a non-negative integer, \( D_{\rho} = (C(p^\rho))^{(n)} \) and where \( G' \) is reduced. We assert that \( G_{\rho} \) is bounded for all primes \( p \). In fact, if \( G_{\rho} \) is unbounded for some prime \( p \), then by Lemma 2, \( G \) contains a direct summand \( Z(p)^\mathcal{N} \) and hence \( \text{Hom}(Z(p), X) = 0 \). The exact sequence

\[
0 \to Z(p)^\mathcal{N} \to Z(p) \otimes Q \to C(p^\rho) \to 0
\]

yields the exact sequence

\[
0 \to \text{Ext}(C(p^\rho), X) \to (\text{Ext}(Q, X))^\mathcal{N} \to \text{Ext}(Z(p), X) \to 0
\]

and hence it follows from (19) that \( \text{Ext}(Z(p), X) \neq 0 \) since \( \text{Ext}(C(p^\rho), X) \) is reduced and \( \text{Ext}(Q, X) \neq 0 \). Now, \( \text{Ext}(Z(p)^\mathcal{N}, X) \) is a direct summand of \( G \cong \text{Ext}(G, X) \) and hence the exact sequences

\[
0 \to Z(p)^{(K)} \to Z(p)^\mathcal{N} \quad \text{and} \quad \text{Ext}(Z(p)^\mathcal{N}, X) \to (\text{Ext}(Z(p), X))^\mathcal{N} \to 0
\]

show that \( \text{Ext}(Z(p)^\mathcal{N}, X) \), and consequently \( G \) as well, contains a torsion-free divisible subgroup of infinite rank. This contradiction shows that \( G_{\rho} \) is bounded and therefore finite for all primes \( p \).

We contend that \( \bigoplus D_{\rho} = 0 \). Indeed, if \( \bigoplus D_{\rho} \neq 0 \) then by Lemma 4, we have \( G = D \oplus \left( \bigoplus_{\rho \in \mathbb{P}} D_{\rho} \right) \oplus L' \oplus tG' \), where \( L' \neq 0 \) is a reduced and torsion-free cotorsion group and where \( tG' \) is finite. Moreover, it follows from Lemma 4, (iii), that for some prime \( p \) for which \( D_{\rho} \neq 0 \), \( L' \) contains a direct summand \( Z(p) \). Hence \( \text{Hom}(Z(p), X) = 0 \), and we deduce from the exact sequences (18) and (19) that \( \text{Ext}(Z(p), X) \neq 0 \). Now it follows from Lemma 4, 2°, that \( D_{\rho} = 0 \). This contradiction shows that \( \bigoplus D_{\rho} = 0 \) and hence

\[
G = Q^n \oplus G'
\]

where \( n \) is a natural number and \( G' \) is reduced. Moreover, the finiteness of \( G_{\rho} \) for all primes \( p \) implies that

\[
G' \cong \text{Ext}(tG', X) \cong \prod_{\rho \in \mathbb{P}} \text{Ext}((tG')_{\rho}, X) \cong \prod_{\rho \in \mathbb{P}} (tG')_{\rho}
\]

and by Lemma 3, \( (tG')_{\rho} \neq 0 \) for only a finite number of primes \( p \), that is to say, \( G' \) is finite. It is also clear that the reduced part of each group \( G \in \mathcal{M} \) is finite. Hence if \( G \in \mathcal{M} \) then \( G \in \mathcal{F} \) and hence \( \mathcal{M} \subseteq \mathcal{F} \). Consequently \( \mathcal{M} = \mathcal{F} \), and the proof is complete.
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