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CHARACTERIZATIONS OF CLASSES OF LEFT EXT-REPRODUCED GROUPS

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1. Introduction.

In a previous paper [4], we considered classes $\mathfrak{M}$ of abelian groups G, which are maximal with respect to the property : There exists a reduced group X such that

$$G \cong \operatorname{Ext}(G, X) \quad \text{for all} \quad G \in \mathfrak{M}.$$

Such classes are called *maximal classes of left* Ext-*reproduced groups.* We found two such classes, viz. $\mathfrak{F}$ and $\mathfrak{A}$ (*see* [4], Theorem 2.7 and Theorem 2.9).

In this paper, we give characterizations of two classes of left Ext-reproduced groups, viz. $\mathfrak{F}$ and $\mathfrak{G}$, where

$\mathfrak{F}$ denotes the class of all groups $Q^n \oplus T$ where n is a non-negative integer and T is a finite group, and

$\mathfrak{G}$ denotes the class of all groups $\prod_{p \in P} T_p$ where, for all primes p, T_p is a finite p-group.

The latter class is a subclass of the class $\mathfrak{A}$, and it is therefore not a maximal class of left Ext-reproduced groups. We mention, in passing, the fact that all groups in $\mathfrak{A}$ are reduced and adjusted cotorsion groups (*see* [5], p. 373), and that $\mathfrak{G}$ contains all finite groups. The classes $\mathfrak{F}$ and $\mathfrak{G}$ are linked together in a very special way. Let us consider a number of properties of *a class* $\mathfrak{M}$ *of left* Ext-*reproduced groups*, i. e. a class of groups for which there exists a reduced group X such that

$$G \cong \operatorname{Ext}(G, X) \quad \text{for all} \quad G \in \mathfrak{M}.$$

(I) If $G \in \mathfrak{M}$, then every direct summand U of G belongs to $\mathfrak{M}$.

(II) If $G \in \mathfrak{M}$ and $H \in \mathfrak{M}$, then $\mathrm{Hom}(G, H) \in \mathfrak{M}$.

(III) If $G \in \mathfrak{M}$ and $H \in \mathfrak{M}$, and if $\varphi : G \to H$ is a homomorphism, then $\mathrm{Ker}\,\varphi \in \mathfrak{M}$ and $\mathrm{Coker}\,\varphi \in \mathfrak{M}$.

(IV) If $G \in \mathfrak{M}$ and $H \in \mathfrak{M}$, then $\mathrm{Ext}\,(G, H) \in \mathfrak{M}$.

(V) If $G \in \mathfrak{M}$, then $\mathrm{Hom}(G, X) = 0$.

Now the following statement summarizes the main results :

$\mathfrak{F}$ and $\mathfrak{G}$ are classes of left Ext-reproduced groups which are maximal with respect to each of the properties (I) to (V). Conversely, if $\mathfrak{M}$ is a class of left Ext-reproduced groups which is maximal with respect to any one of the properties (I) to (V), then either $\mathfrak{M} = \mathfrak{F}$ or $\mathfrak{M} = \mathfrak{G}$, the latter being the case if $\mathfrak{M}$ contains only reduced groups.

Notation.

$A \oplus B$, $\bigoplus_{i \in I} A_i$, $A^{(m)}$, direct sum;

$\prod_{i \in I} A_i$, A^m, direct product;

$A \otimes B$, tensor product of A and B;

tG, maximal torsion subgroup of G;

G_p, p-component of G;

Z, additive group of integers;

Q, additive group of rational numbers;

$Z(p)$, anditive group of p-adic integers;

$C(n)$, cyclic group of order n;

$C(p^\infty)$, quasi-cyclic group;

$\aleph$, the power of the continuum;

P, the set of all prime numbers;

cotorsion group, a group X such that $\mathrm{Ext}(Q, X) = 0$; adjusted reduced cotorsion group, a reduced cotorsion group G such that G/tG is divisible.

All groups under consideration are additively written abelian groups.

2. Characterizations of classes of left Ext-reproduced groups.

Let $\mathfrak{M}$ be a class of left Ext-reproduced groups. Throughout this paper, X will denote a reduced group such that

$$\mathrm{Ext}(G, X) \cong G \quad \text{for all } G \in \mathfrak{M}.$$

Recall that $\mathfrak{F}$ is the class of all groups $Q^n \oplus T$ where n is a non-negative integer and T is a finite group. For this class of groups, we have that $\prod_{p \in P} Z(p)/X \cong Q$ ([4], Example 2.3). Let $\mathfrak{G}$ denote the class of all groups $\prod_{p \in P} T_p$ where T_p is a finite p-group for all primes p. Note that $\mathfrak{G}$ is a proper subclass of $\mathfrak{A}$, consequently $\mathfrak{G}$ is also a class of left Ext-reproduced groups ([4], Example 2.8).

The classes $\mathfrak{F}$ and $\mathfrak{G}$ are quite remarkable, and we give characterizations of them in this paragraph. First, we prove several lemmas which we shall need subsequently.

LEMMA 1. — *Let* $\mathfrak{H}$ *be a class of left* Ext-*reproduced groups such that* $\mathfrak{H} \supseteq \mathfrak{G}$. *Then* $X \cong \prod_{p \in P} Z(p)$.

Proof. — Since $C(p^k) \in \mathfrak{H}$ for all primes p and all natural numbers k, it follows from [4] (Lemma 2.4) that X is isomorphic to a pure subgroup of $\prod_{p \in P} Z(p)$ and that $X/pX \cong C(p)$ for all primes p. Note that $\mathrm{Ext}(Q, X) = 0$, this follows from the exact sequences

$$0 \to \bigoplus_{p \in P} C(p) \to \prod_{p \in P} C(p) \to Q^{(\aleph)} \to 0$$

and

$$0 \to (\mathrm{Ext}(Q, X))^{\aleph} \to \mathrm{Ext}\left(\prod_{p \in P} C(p), X\right) \cong \prod_{p \in P} C(p)$$

since $\prod_{p \in P} C(p)$ is reduced and $\prod_{p \in P} C(p) \in \mathfrak{H}$. Hence ([5], p. 372)

$$X \cong \prod_{p \in P} \mathrm{Ext}(C(p^\infty), X) \cong \prod_{p \in P} \mathrm{Hom}(C(p^\infty), X \otimes C(p^\infty))$$
$$\cong \prod_{p \in P} \mathrm{Hom}(C(p^\infty), C(p^\infty)) \cong \prod_{p \in P} Z(p),$$

which completes the proof.

LEMMA 2. — *Let* $\mathfrak{M}$ *be a class of left* Ext-*reproduced groups and let* $G \in \mathfrak{M}$.

1° *If* $G_p \neq 0$, *then* $X/pX \neq 0$.

2° *If the reduced part of* G *contains an unbounded* p-*component for some prime* p *then*

(i) $X_p = 0$;

(ii) G *has a direct summand* $Z(p)^{\aleph} \oplus \prod_{i=1}^{\infty} \left((C(p^i))^{\mathfrak{m}_{p_i}}\right)$ *where* $\mathfrak{m}_{p_i}$ *is finite and* $\mathfrak{m}_{p_i} \neq 0$ *for an infinite number of* i's;

(iii) $\mathrm{Hom}(G, G)$ *is not left* Ext-*reproduced*;

(iv) $\mathrm{Ext}(G, G)$ *is not left* Ext-*reproduced.*

Proof.

1° Let $G_p \neq 0$, and suppose that $X = pX$. Then, by [1] (p. 245), we have $pG = G$ since $G \cong \mathrm{Ext}(G, X)$, consequently $G = (C(p^\infty))^{(\mathfrak{n})} \oplus G'$, where $G'_p = 0$ and $pG' = G'$. Hence

$$(1) \qquad G \cong \mathrm{Ext}(G, X) \cong (\mathrm{Ext}(C(p^\infty), X))^{\mathfrak{n}} \oplus \mathrm{Ext}(G', X)$$

and since X is reduced and $pX = X$ it follows that $X_p = 0$. The exact sequences

$$0 \to tX \to X \to X/tX \to 0$$

and

$$\mathrm{Ext}(C(p^\infty), tX) = 0 \to \mathrm{Ext}(C(p^\infty), X) \to \mathrm{Ext}(C(p^\infty), X/tX) \to 0$$

show that

$$(2) \qquad \mathrm{Ext}(C(p^\infty), X) \cong \mathrm{Ext}(C(p^\infty), X/tX) \cong \mathrm{Hom}(C(p^\infty), (X/tX) \otimes (Q/Z)) = 0$$

since $(X/tX) \otimes C(p^\infty) = 0$ ([1], p. 251). In addition, $G'_p = 0$ and $pG' = G'$ imply (*see* [1], p. 245)

$$(3) \qquad \mathrm{Ext}(G', X)_p = 0$$

and hence it follows from (1), (2) and (3) that $G_p = 0$, contrary to the assumption $G_p \neq 0$. We conclude that $X/pX \neq 0$. This proves 1°.

2° Let $G \in \mathfrak{M}$ be such that $G = D \oplus G'$, where D is divisible and G' is reduced. Suppose that G'_p is unbounded for some prime p and let $B^{(p)}$ denote a basic subgroup of G'_p. Then $B^{(p)}$ is unbounded, and if we put $B^{(p)} = B_1 \oplus \ldots \oplus B_i \oplus \ldots$, where $B_i = (C(p^i))^{(\mathfrak{m}_{p_i})}$, then each $\mathfrak{m}_{p_i}$ is finite ($i = 1, 2, \ldots$) (*see* [3], p. 136), and $\mathfrak{m}_{p_i} \neq 0$ for an infinite number of i's. Now $B_1 \oplus \ldots \oplus B_i$ is a direct summand of G'_p and hence of G, that is ([1], p. 243)

$$\mathrm{Ext}(B_1 \oplus \ldots \oplus B_i, X) \cong \mathrm{Ext}(B_1, X) \oplus \ldots \oplus \mathrm{Ext}(B_i, X) \cong (X/pX)^{(\mathfrak{m}_{p_1})} \oplus \ldots \oplus (X/p^i X)^{(\mathfrak{m}_{p_i})}$$

is a direct summand of Ext $(G, X) \cong G$. Bearing in mind the fact that $B_1 \oplus \ldots \oplus B_i$ is a maximal p^i-bounded direct summand of G for every i ([1], p. 99), and that $X/pX \neq 0$, we conclude that, for every i,

$$\mathrm{Ext}(B_1 \oplus \ldots \oplus B_i, X) \cong B_1 \oplus \ldots \oplus B_i.$$

(i) Suppose that $X_p \neq 0$, then $X = C(p^k) \oplus X'$ where k is a natural number ([1], p. 80), and

$$\mathrm{Ext}(B_{k+j}, X) \cong (X/p^{k+j} X)^{(\mathfrak{m}_{p_{k+j}})} \cong (C(p^k) \oplus X'/p^{k+j} X')^{(\mathfrak{m}_{p_{k+j}})}$$

contradicts the fact that

$$\mathrm{Ext}(B_1 \oplus \ldots \oplus B_{k+j}, X) \cong \mathrm{Ext}(B_1, X) \oplus \ldots \oplus \mathrm{Ext}(B_{k+j}, X)$$

is isomorphic to $B_1 \oplus \ldots \oplus B_{k+j}$ for all $j \geqq 1$. Hence we conclude that $X_p = 0$. This proves (i).

(ii) Consider the exact sequences

$$0 \to tG \xrightarrow{\iota} G \xrightarrow{\nu} G/tG \to 0$$

and

(4) $$\mathrm{Ext}(G/tG, X) \xrightarrow{\nu^*} \mathrm{Ext}(G, X) \cong G \to \mathrm{Ext}(tG, X) \to 0.$$

It follows from (4) that

(5) $$G \cong \nu^*(\mathrm{Ext}(G/tG, X)) \oplus \prod_{p \in P} \mathrm{Ext}((tG)_p, X)$$

since $\mathrm{Ext}(G/tG, X)$ is divisible. Recall that $X_p = 0$, and hence the exact sequences

$$0 \to tX \to X \to X/tX \to 0$$

and

$$\mathrm{Ext}((tG)_p, tX) = 0 \to \mathrm{Ext}((tG)_p, X) \to \mathrm{Ext}((tG)_p, X/tX) \to 0$$

show that

(6) $$\mathrm{Ext}((tG)_p, X) \cong \mathrm{Ext}((tG)_p, X/tX) \cong \mathrm{Hom}((tG)_p, (X/tX) \otimes C(p^\infty)).$$

Now, $X_p = 0$ and $X/pX \neq 0$ imply $(X/tX)/p(X/tX) \neq 0$ and hence

$$(X/tX) \otimes C(p^\infty) \cong (C(p^\infty))^{(\mathfrak{n}_p)},$$

where

$$\mathfrak{n}_p = r((X/tX)/p(X/tX)) = r(X/pX)$$

([1], p. 255). Hence $\mathrm{Ext}((tG)_p, X)$ has a direct summand $\mathrm{Hom}((tG')_p, C(p^\infty))$ and, by [2] (p. 137),

$$\mathrm{Hom}((tG')_p, C(p^\infty)) \cong Z(p)^{\aleph} \oplus \prod_{i=1}^{\infty} (C(p^i)^{\mathfrak{m}_{p_i}}).$$

However, the latter group is a direct summand of G by virtue of (5) and (6). This proves (ii).

(iii) By making use of the result in (ii), we see that G contains direct summands $Z(p)^{\aleph}$ and $C(p^i)$ for a suitable i. Hence $\mathrm{Hom}(G, G)$ has a direct summand $\mathrm{Hom}(Z(p)^{\aleph}, C(p^i))$. However,

$$V = \mathrm{Hom}(Z(p)^{\aleph}, C(p^i)) \cong \mathrm{Hom}(Z(p)^{\aleph}/p^i(Z(p)^{\aleph}), C(p^i))$$

and since $Z(p)^{\aleph}/p^i(Z(p)^{\aleph})$ is isomorphic to the direct sum of an infinite number of copies of $C(p^i)$, it follows that V is isomorphic to the direct sum of an infinite number of copies of $C(p^i)$. Hence, by [3] (p. 136), $\mathrm{Hom}(G, G)$ is not left Ext-reproduced. This proves (iii).

(iv) By (ii), G contains direct summands $\prod_{i=1}^{\infty}((C(p^i))^{\mathfrak{m}_{p^i}})$ and $Z(p)^{\aleph}$, and hence $\mathrm{Ext}(G, G)$ has a direct summand

$$\mathrm{Ext}\left(\prod_{i=1}^{\infty}((C(p^i))^{\mathfrak{m}_{p^i}}),\ Z(p)^{\aleph}\right).$$

Now, by [2] (p. 137),

$$\begin{aligned}\mathrm{Ext}&\left(\prod_{i=1}^{\infty}((C(p^i))^{\mathfrak{m}_{p^i}}),\ Z(p)^{\aleph}\right)\\ &\cong\left(\mathrm{Ext}\left(\prod_{i=1}^{\infty}((C(p^i))^{\mathfrak{m}_{p^i}}),\ Z(p)\right)^{\aleph}\right)\\ &\cong\left(\mathrm{Hom}\left(t\prod_{i=1}^{\infty}((C(p^i))^{\mathfrak{m}_{p^i}}),\ Z(p)\otimes C(p^{\infty})\right)^{\aleph}\right)\\ &\cong\left(Z(p)^{\aleph}\oplus\prod_{i=1}^{\infty}((C(p^i))^{\mathfrak{m}_{p^i}})\right)^{\aleph}.\end{aligned}$$

The latter group, and consequently $\mathrm{Ext}(G, G)$ as well, contains a bounded direct summand of power $2^{\aleph}$. By [3] (p. 136), $\mathrm{Ext}(G, G)$ is not left Ext-reproduced. This proves (iv), and the proof of the lemma is complete.

LEMMA 3. — *Let*

$$G = K\oplus\prod_{p\in P}T_p,$$

where T_p is a finite p-group and $T_p\neq 0$ for an infinite number of primes p and where K has no reduced torsion direct summand. Then G does not belong to any class $\mathfrak{M}$ of left Ext-*reproduced groups for which* $\mathrm{Ext}(Q, X)\neq 0$.

Proof. — Suppose, to the contrary, that there exists a class $\mathfrak{M}$ of left Ext-reproduced groups with $\mathrm{Ext}(Q, X)\neq 0$, and which contains a group $G = K\oplus\prod_{p\in P}T_p$ satisfying the above-mentioned properties. The exact sequence

$$0\to\bigoplus_{p\in P}T_p\to G\to Q^{(\aleph)}\oplus K\to 0$$

gives rise to the exact sequence

$$(7)\qquad \mathrm{Hom}\Big(\bigoplus_{p\in P} T_p,\ X\Big) \to \mathrm{Ext}\big(Q^{(\aleph)}\oplus K,\ X\big) \to \mathrm{Ext}(G,\ X) \cong G$$

and we have that

$$(8)\qquad \Big|\prod_{p\in P} \mathrm{Hom}(T_p,\ X)\Big| \leqq \aleph.$$

This follows from the following observation : If $T_p \neq 0$ then by Lemma 2, $X/pX \neq 0$ and if we put $T_p = B_1 \oplus \ldots \oplus B_n$ where $B_i = (C(p^i))^{m_i}$, then we deduce from

$$\mathrm{Ext}(T_p,\ X) \cong (X/pX)^{m_1} \oplus \ldots \oplus (X/p^n X)^{m_n}$$

that $\mathrm{Ext}(T_p,\ X) \cong T_p$. This implies in particular that X_p is finite for all primes p for which $T_p \neq 0$. This proves (8).
However,

$$\mathrm{Ext}\big(Q^{(\aleph)}\oplus K,\ X\big) \cong (\mathrm{Ext}(Q, X))^{\aleph} \oplus \mathrm{Ext}(K,\ X)$$

and our initial assumption implies that

$$\big|(\mathrm{Ext}(Q,\ X))^{\aleph}\big| \geqq 2^{\aleph}.$$

Hence we deduce from (7) that G contains a torsion-free divisible subgroup of infinite rank, and this contradiction completes the proof.

LEMMA 4. — *Let $\mathfrak{M}$ be a class of left* Ext*-reproduced groups for which* $\mathrm{Ext}(Q,\ X) \neq 0$. *Suppose that $G \in \mathfrak{M}$ is not reduced, and let*

$$G = D \oplus \Big(\bigoplus_{p\in P} D_p\Big) \oplus G'$$

where $D = Q^n$ (n a non-negative integer), $D_p = (C(p^\infty))^{(n_p)}$, and where G' is reduced.

1° *If* $\bigoplus_{p\in P} D_p \neq 0$, *then*

(i) $G' = L' \oplus tG'$, *where $L' \neq 0$ is a reduced and torsion-free cotorsion group and where tG' is finite*;

(ii) $X_p = 0$ *for all primes p for which $D_p \neq 0$*;

(iii) $\mathrm{Ext}\big(\bigoplus_{p\in P} D_p,\ X\big) \cong L'$, $\mathrm{Ext}(L',\ X) \cong \bigoplus_{p\in P} D_p$.

2° *If* $\mathrm{Ext}(Z(p),\ X) \neq 0$ *for some prime p, then* $D_p = 0$.

Proof. — Let $G \in \mathfrak{M}$ and let

$$G = D \oplus \Big(\bigoplus_{p\in P} D_p\Big) \oplus G'$$

where D, D_p and G' are defined as above.

1° Let us assume that $\bigoplus_{p \in P} D_p \neq 0$, then it is clear that G' is not torsion. Moreover, $(tG')_p$ is bounded for all primes p. In fact, if $(tG')_p$ is unbounded for some prime p then by Lemma 2, $X_p = 0$, and G has a direct summand

$$U = \prod_{i=1}^{\infty} ((C(p^i))^{\mathfrak{m}_{p_i}}),$$

where $\mathfrak{m}_{p_i}$ is finite for all i, and $\mathfrak{m}_{p_i} \neq 0$ for an infinite number of i's. By [4] (Lemma 2.5), $U/tU \cong Q^{(\mathfrak{m})} \oplus K$, where $\mathfrak{m}$ is infinite and K is reduced. The exact sequences

$$0 \to tU \to U \to Q^{(\mathfrak{m})} \oplus K \to 0$$

and

$$\operatorname{Hom}(tU, X) = 0 \to \operatorname{Ext}(Q^{(\mathfrak{m})} \oplus K, X) \to \operatorname{Ext}(U, X)$$

show that $\operatorname{Ext}(U, X)$ contains a direct summand $(\operatorname{Ext}(Q, X))^{\mathfrak{m}} \cong Q^{(2^{\mathfrak{m}})}$. However, $\operatorname{Ext}(U, X)$ is a direct summand of G, and hence G contains a torsion-free divisible subgroup of infinite rank. This contradiction proves that $(tG')_p$ is bounded and therefore finite for all primes p.

Now G', being a direct summand of a cotorsion group, is also cotorsion and hence the exact sequences ([5], p. 372),

$$0 \to tG' \to G' \to G'/tG' \to 0 \tag{9}$$

and

$$0 \to \operatorname{Ext}(Q/Z, tG') \cong \prod_{p \in P} (tG')_p \to \operatorname{Ext}(Q/Z, G') \cong G' \to \operatorname{Ext}(Q/Z, G'/tG') = L' \to 0 \tag{10}$$

imply

$$G' \cong \prod_{p \in P} (tG')_p \oplus L'$$

since $\prod_{p \in P} (tG')_p$ is cotorsion and L' is torsion-free cotorsion. Moreover, by Lemma 3, $(tG')_p \neq 0$ for at most a finite number of primes p and hence

$$G' \cong tG' \oplus L',$$

where $L' \neq 0$ since, by assumption, $\bigoplus_{p \in P} D_p \neq 0$, and where tG' is finite. This proves (i).

In order to prove (ii), put $tG' = \bigoplus_{i=1}^{n} (tG')_{p_i}$ where

$$(tG')_{p_i} = B_1^{(p_i)} \oplus \ldots \oplus B_{r_i}^{(p_i)} \text{ and } B^{(p_i)} = (C(p_i^j))^{m_{ij}},$$

m_{ij} a non-negative integer. Then

$$\mathrm{Ext}(tG', X) \cong \bigoplus_{i=1}^{n} \left((X/p_i X)^{m_{i1}} \oplus \dots \oplus (X/p_i^{r_i} X)^{m_{ir_i}}\right)$$

and hence it follows from Lemma 2 that $\mathrm{Ext}(tG', X) \cong tG'$. Furthermore, it is clear that $\mathrm{Ext}(D, X) \cong D$. Hence, if $X_p \neq 0$ for some prime p for which $D_p \neq 0$ then $\mathrm{Ext}(C(p^\infty), X)$ will contain a finite p-group and therefore, if we consider $\mathrm{Ext}(G, X) \cong G$ then tG' is no longer the maximal reduced torsion subgroup of G. Consequently $X_p = 0$ for all primes p for which $D_p \neq 0$ and this proves (ii).

We have $G = D \oplus \left(\bigoplus_{p \in P} D_p\right) \oplus L' \oplus tG'$ and, hence, if we consider $\mathrm{Ext}(G, X) \cong G$, then $\mathrm{Ext}(D, X) \cong D$ and $\mathrm{Ext}(tG', X) \cong tG'$ imply

$$\mathrm{Ext}\left(\left(\bigoplus_{p \in P} D_p\right) \oplus L', X\right) \cong \left(\bigoplus_{p \in P} D_p\right) \oplus L'.$$

However,

$$\mathrm{Ext}\left(\left(\bigoplus_{p \in P} D_p\right) \oplus L', X\right) \cong \prod_{p \in P} \mathrm{Ext}(D_p, X) \oplus \mathrm{Ext}(L', X)$$

and by (ii)

$$\prod_{p \in P} \mathrm{Ext}(D_p, X) \cong \prod_{p \in P} \mathrm{Hom}(D_p, (X/tX) \otimes C(p^\infty)) \tag{11}$$

and the latter group is a reduced torsion-free cotorsion group. Moreover, $\mathrm{Ext}(L', X)$ is divisible since L' is torsion-free and hence assertion (iii) follows.

We turn our attention to 2°, and we suppose that $\mathrm{Ext}(Z(p), X) \neq 0$. We shall prove that $D_p = 0$. Assume, to the contrary, that $D_p \neq 0$. Then it follows from (11) that G has a direct summand

$$\mathrm{Hom}(D_p, (X/tX) \otimes C(p^\infty)).$$

Let $D_p = (C(p^\infty))^{(\mathfrak{m})}$. We assert that $\mathfrak{m}$ is finite. In fact, if $\mathfrak{m} \geqq \aleph_0$, then $\mathrm{Hom}(D_p, (X/tX) \otimes C(p^\infty))$ has a direct summand

$$\mathrm{Hom}\left(C(p^\infty)^{(\mathfrak{m})}, C(p^\infty)\right) \cong (\mathrm{Hom}(C(p^\infty), C(p^\infty))) \cong Z(p)^{\mathfrak{m}}$$

and hence the exact sequences

$$0 \to Z(p)^{(\mathfrak{m})} \to Z(p)^{\mathfrak{m}} \quad \text{and} \quad \mathrm{Ext}\left(Z(p)^{\mathfrak{m}}, X\right) \to (\mathrm{Ext}(Z(p), X))^{\mathfrak{m}} \to 0,$$

and $\mathrm{Ext}(Z(p), X) \neq 0$, show that $\mathrm{Ext}(Z(p)^{\mathfrak{m}}, X)$ has a torsion-free divisible subgroup of infinite rank. This is a contradiction since $\mathrm{Ext}(Z(p)^{\mathfrak{m}}, X)$ is a direct summand of G. Hence $\mathfrak{m}$ is finite, i. e.

$$D_p = (C(p^\infty))^{m_p}, \quad m_p \text{ a non-negative integer.}$$

Next, we assert that $r((X/tX)/p(X/tX)) = r(X/pX)$ is finite. Indeed, if $r(X/pX) = \mathfrak{n} \geqq \aleph_0$, then $\mathrm{Hom}(D_p, (X/tX) \otimes C(p^\infty))$ has a direct summand

$$\mathrm{Hom}(C(p^\infty), (X/tX) \otimes C(p^\infty)) \cong \mathrm{Hom}\big(C(p^\infty), C(p^\infty)^{(\mathfrak{n})}\big) = V,$$

and the latter group contains a subgroup

$$(\mathrm{Hom}(C(p^\infty), C(p^\infty)))^{(\mathfrak{n})} \cong Z(p)^{(\mathfrak{n})}.$$

Now the exact sequences $0 \to Z(p)^{(\mathfrak{n})} \to V$ and

$$\mathrm{Ext}(V, X) \to (\mathrm{Ext}(Z(p), X))^{\mathfrak{n}} \to 0 \qquad \text{and} \qquad \mathrm{Ext}(Z(p), X) \neq 0,$$

imply that $\mathrm{Ext}(V, X)$ has a torsion-free divisible subgroup of infinite rank. This gives rise to a contradiction since $\mathrm{Ext}(V, X)$ is a direct summand of G, and we conclude that $\mathfrak{n}$ is finite. Consequently, G has a direct summand

$$\mathrm{Hom}(D_p, (X/tX) \otimes C(p^\infty)) \cong Z(p)^{n_p}, \qquad n_p \text{ a natural number.}$$

Moreover, it follows from (iii) and [1] (p. 245) that

$$(12) \qquad \mathrm{Ext}(Z(p)^{n_p}, X) \cong D_p \cong (C(p^\infty))^{m_p}.$$

We assert that (12) is impossible. Indeed, suppose that (12) holds, and consider the exact sequences (*see* [1], p. 250 and p. 252)

$$0 \to Z(p)^{n_p} \otimes Z \cong Z(p)^{n_p} \to (Z(p) \otimes Q)^{n_p} \to Z(p)^{n_p} \otimes (Q/Z) \cong (C(p^\infty))^{n_p} \to 0$$

and

$$(13) \qquad \mathrm{Ext}((C(p^\infty))^{n_p}, X) \to (\mathrm{Ext}(Q, X))^{\aleph} \to \mathrm{Ext}(Z(p)^{n_p}, X) \to 0.$$

We have that

$$(a) \qquad |\mathrm{Ext}((C(p^\infty))^{n_p}, X)| = |(\mathrm{Hom}(C(p^\infty), (X/tX) \otimes C(p^\infty)))^{n_p}| = \aleph$$

since $r((X/tX)/p(X/tX))$ is finite;

$$(b) \qquad |(\mathrm{Ext}(Q, X))^{\aleph}| = 2^{\aleph}$$

since, by assumption, $\mathrm{Ext}(Q, X) \neq 0$;

$$(c) \qquad |\mathrm{Ext}(Z(p)^{n_p}, X)| = \aleph_0,$$

on account of (12), and the exact sequence (13) implies that this is clearly impossible. Hence (12) leads to a contradiction, and we conclude that $D_p = 0$. This proves 2°, and completes the proof of the lemma.

We now derive properties which are characteristic of the classes $\mathfrak{F}$ and $\mathfrak{G}$.

THEOREM 5.

(i) *If* $G \in \mathfrak{F}$, *then for every direct summand* U *of* G *we have* $U \in \mathfrak{F}$.

(ii) $\mathfrak{G}$ *is a class of left* Ext-*reproduced groups which is maximal with respect to the property* :

If $G \in \mathfrak{G}$, *then for every direct summand* U *of* G *we have* $U \in \mathfrak{G}$.

Proof.

(i) This statement is an immediate consequence of [4] (Theorem 2.14).

(ii) Let $G = G' \oplus G'' \in \mathfrak{G}$ where $G' \neq 0$, $G'' \neq 0$. Then $tG' \neq 0$, $tG'' \neq 0$ and recall that $G \cong \operatorname{Ext}(Q/Z, tG)$ since G is adjusted ([5], p. 375). Now, both G' and G'' are adjusted and hence

$$\begin{aligned} G = G' \oplus G'' &\cong \operatorname{Ext}(Q/Z, tG') \oplus \operatorname{Ext}(Q/Z, tG'') \\ &\cong \prod_{p \in P} \operatorname{Ext}(C(p^\infty), (tG')_p) \oplus \prod_{p \in P} \operatorname{Ext}(C(p^\infty), (tG'')_p) \\ &\cong \prod_{p \in P} (tG')_p \oplus \prod_{p \in P} (tG'')_p \end{aligned}$$

whence it follows that

$$G' \cong \prod_{p \in P} (tG')_p \in \mathfrak{G}, \qquad G'' \cong \prod_{p \in P} (tG'')_p \in \mathfrak{G}.$$

Let $\mathfrak{H}$ be a class of left Ext-reproduced groups such that if $G \in \mathfrak{H}$ then for every direct summand U of G we have $U \in \mathfrak{H}$, and let $\mathfrak{H} \supseteq \mathfrak{G}$. Then it follows from Lemma 1 that $X \cong \prod_{p \in P} Z(p)$. Let $G \in \mathfrak{H}$, then G_p is bounded for all primes p. Indeed, if G_p is unbounded for some prime p then by Lemma 2, G has a direct summand $Z(p)^{\aleph}$ and by assumption $Z(p)^{\aleph} \in \mathfrak{H}$, which is clearly impossible. Hence G_p is bounded and consequently finite for all primes p, and we have (*see* [4], Example 2.8)

$$G \cong \operatorname{Ext}\left(tG, \prod_{p \in P} Z(p)\right) \cong \operatorname{Hom}(tG, Q/Z) \cong \prod_{p \in P} (tG)_p.$$

Hence $G \in \mathfrak{H}$ implies $G \in \mathfrak{G}$ and consequently $\mathfrak{H} \subseteq \mathfrak{G}$. This completes the proof.

THEOREM 6. — *Let* $\mathfrak{M}$ *be a class of left* Ext-*reproduced groups which is maximal with respect to the property* :

If $G \in \mathfrak{M}$ *then for all direct summands* U *of* G, *we have* $U \in \mathfrak{M}$. *Then either* $\mathfrak{M} = \mathfrak{F}$ *or* $\mathfrak{M} = \mathfrak{G}$.

Proof. — Let $\mathfrak{M}$ be a class of left Ext-reproduced groups which is maximal with respect to the above mentioned property. Then there are two possibilities, viz. either

(i) there exists a group $G \in \mathfrak{M}$ which is not reduced, or

(ii) all groups G in $\mathfrak{M}$ are reduced.

Let us first consider case (i). If $G \in \mathfrak{M}$ is not reduced, then $G = Q^n \oplus G'$ where n is a natural number, and G' is reduced. By assumption, $Q \in \mathfrak{M}$ and hence, by [4] (Theorem 2.2), $H/X \cong Q$ where H is a reduced cotorsion group. Moreover, we also have that $G' \in \mathfrak{M}$. Now $\mathfrak{M}$, being maximal, cannot contain only divisible groups and hence there exists a non-zero reduced group $G' \in \mathfrak{M}$. We contend that

(*a*) *G' is finite.*

In order to prove this, consider the exact sequences

$$0 \to tG' \to G' \to G'/tG' \to 0$$

and

$$\mathrm{Ext}(G'/tG', X) \to \mathrm{Ext}(G', X) \cong G' \to \mathrm{Ext}(tG', X) \to 0.$$

We conclude that

$$G' \cong \mathrm{Ext}(tG', X) \cong \prod_{p \in P} \mathrm{Ext}((tG')_p, X)$$

since G' is reduced. In the first instance, $(tG')_p$ is bounded for all primes p. In fact, if $(tG')_p$ is unbounded for some prime p then it follows from Lemma 2 that G' has a direct summand $Z(p)^{\aleph}$ and hence our initial assumption implies that $Z(p)^{\aleph} \in \mathfrak{M}$. This contradiction shows that $(tG')_p$ is bounded and hence finite for all primes p. Moreover, by Lemma 3, $(tG')_p \neq 0$ for only a finite number of primes p whence it follows that tG', and consequently $G' \cong \mathrm{Ext}(tG', X)$ as well, is finite. This proves (*a*).

To recapitulate, if (i) holds and if $G \in \mathfrak{M}$ then $G \cong Q^n \oplus T$ where n is a non-negative integer and T is a finite group, that is to say $G \in \mathfrak{F}$. Hence $\mathfrak{M} \subseteq \mathfrak{F}$ and the maximality of $\mathfrak{M}$ implies that $\mathfrak{M} = \mathfrak{F}$. This settles the first case.

Next we consider case (ii). In this case, $tG \neq 0$ for all $G \in \mathfrak{M}$. Again, if $G \in \mathfrak{M}$, then $(tG)_p$ is bounded for all primes p, for if $(tG)_p$ is unbounded for some prime p then by Lemma 2, $Z(p)^{\aleph} \in \mathfrak{M}$. This contradiction shows that $(tG)_p$ is bounded and hence finite for all primes p. Since $(tG)_p$ is finite and pure it follows from [1] (p. 80) that $(tG)_p \in \mathfrak{M}$ for all primes p whence

$$G \cong \prod_{p \in P} \mathrm{Ext}((tG)_p, X) \cong \prod_{p \in P} (tG)_p. \tag{14}$$

We maintain that $\mathrm{Ext}(Q, X) = 0$.

Assume, to the contrary, that $\operatorname{Ext}(Q, X) \neq 0$. The maximality of $\mathfrak{M}$ implies that tG cannot be finite for all $G \in \mathfrak{M}$ and hence there exists an $H \in \mathfrak{M}$ with tH an infinite torsion group. This implies that $(tH)_p \neq 0$ for an infinite number of primes p and hence

$$H \cong \operatorname{Ext}\Big(\bigoplus_{p \in P} (tH)_p,\ X\Big) \cong \prod_{p \in P} (tH)_p.$$

By Lemma 3, $H \notin \mathfrak{M}$. This contradiction shows that $\operatorname{Ext}(Q, X) = 0$.

The latter fact and (14) show that if $G \in \mathfrak{M}$ then $G \in \mathfrak{G}$, that is, $\mathfrak{M} \subseteq \mathfrak{G}$ and hence $\mathfrak{M} = \mathfrak{G}$ since $\mathfrak{M}$ is maximal. This completes the proof of the theorem.

THEOREM 7.

(i) *If* $G \in \mathfrak{F}$ *and* $H \in \mathfrak{F}$, *then* $\operatorname{Hom}(G, H) \in \mathfrak{F}$.

(ii) $\mathfrak{G}$ *is a class of left* Ext-*reproduced groups which is maximal with respect to the property* :

If $G \in \mathfrak{G}$ *and* $H \in \mathfrak{G}$, *then* $\operatorname{Hom}(G, H) \in \mathfrak{G}$.

Proof.

(i) If $G \in \mathfrak{F}$ and $H \in \mathfrak{F}$ then $G = Q^n \oplus S$, $H = Q^m \oplus T$ where m and n are non-negative integers and S and T are finite groups. Consequently,

$$\operatorname{Hom}(G, H) \cong Q^{mn} \oplus \operatorname{Hom}(S, T) \in \mathfrak{F}.$$

This proves (i).

(ii) Let $G \in \mathfrak{G}$ and $H \in \mathfrak{G}$, then $G \cong \prod_{p \in P} G_p$, $H \cong \prod_{p \in P} H_p$ where G_p and H_p are finite p-groups for all primes p. Then we have

$$\operatorname{Hom}(G, H) \cong \operatorname{Hom}(tG, H) \cong \operatorname{Hom}(tG, tH) \cong \prod_{p \in P} \operatorname{Hom}((tG)_p, (tH)_p) \in \mathfrak{G}.$$

Let $\mathfrak{H}$ be a class of left Ext-reproduced groups such that if $G \in \mathfrak{H}$, $H \in \mathfrak{H}$ then $\operatorname{Hom}(G, H) \in \mathfrak{H}$, and suppose that $\mathfrak{H} \supseteq \mathfrak{G}$. Then, by Lemma 1, $X \cong \prod_{p \in P} Z(p)$. All groups $G \in \mathfrak{H}$ are reduced since

$$G \cong \operatorname{Ext}(G, X) \cong \operatorname{Ext}(tG, X),$$

and hence if $0 \neq G \in \mathfrak{H}$, then $tG \neq 0$.

Let $G \in \mathfrak{H}$. Then G_p is bounded for all primes p, for if G_p is unbounded for some prime p then it follows from Lemma 2 that $\operatorname{Hom}(G, G)$ is not left Ext-reproduced. Hence G_p is bounded and therefore finite for all primes p. Consequently,

$$G \simeq \mathrm{Ext}(tG, X) \simeq \prod_{p \in P} \mathrm{Ext}((tG)_p, X) \simeq \prod_{p \in P} \mathrm{Hom}((tG)_p, Q/Z) \simeq \prod_{p \in P} (tG)_p$$

and hence $G \in \mathfrak{G}$. Therefore $\mathfrak{H} \subseteq \mathfrak{G}$, and the proof is complete.

THEOREM 8. — *Let* $\mathfrak{M}$ *be a class of left* Ext-*reproduced groups which is maximal with respect to the property :*

If $G \in \mathfrak{M}$ *and* $H \in \mathfrak{M}$, *then* $\mathrm{Hom}(G, H) \in \mathfrak{M}$.

Then either $\mathfrak{M} = \mathfrak{F}$ *or* $\mathfrak{M} = \mathfrak{G}$.

Proof. — There are two possibilities, viz. either

(i) $\mathfrak{M}$ contains a group G wich is not reduced, or

(ii) all groups G in $\mathfrak{M}$ are reduced.

Consider case (i), and let $G = D \oplus G'$ where $D \neq 0$ is divisible and G' is reduced. Then D contains no subgroup $C(p^\infty)$, for if $D = C(p^\infty) \oplus D'$, then $\mathrm{Hom}(C(p^\infty), C(p^\infty)) \simeq Z(p)$ is a direct summand of $\mathrm{Hom}(G, G) \in \mathfrak{M}$ and, by assumption, $\mathrm{Hom}(\mathrm{Hom}(G, G), G) \in \mathfrak{M}$. However, the latter group has a direct summand $\mathrm{Hom}(Z(p), C(p^\infty)) \simeq Q^{(2^{\aleph})} \oplus C(p^\infty)$ ([2], p. 136), and this gives rise to a contradiction since any left Ext-reproduced group contains at most a finite number of copies of Q. This proves that $G = Q^n \oplus G'$ where n is a natural number and G' is reduced. It is also clear that $G' \in \mathfrak{M}$.

Now G'_p is bounded for all primes p, for if G'_p is unbounded for some prime p then it follows from Lemma 2 that $\mathrm{Hom}(G, G)$ is not left Ext-reproduced. This shows that G'_p is bounded and consequently finite for all primes p. Furthermore, $(tG')_p \neq 0$ for at most a finite number of primes p, for if $(tG')_p \neq 0$ for an infinite number of primes p, then by Lemma 3,

$$G' \simeq \prod_{p \in P} \mathrm{Ext}((tG')_p, X) \simeq \prod_{p \in P} (tG')_p \notin \mathfrak{M}.$$

[We mention in passing the fact that if $(tG')_p \neq 0$ for an infinite number of primes p, then it can also easily be shown that $\mathrm{Hom}(G, G)$ contains a torsion-free divisible subgroup of infinite rank.] This contradiction shows that $(tG')_p \neq 0$ for at most a finite number of primes p whence we deduce that G' is finite. The proof thus far shows in fact that the reduced part of each group $G \in \mathfrak{M}$ is finite.

Hence, if condition (i) holds and $G \in \mathfrak{M}$ then $G = Q^n \oplus T$ where n is a non-negative integer and T is finite, that is to say, $G \in \mathfrak{F}$ and hence $\mathfrak{M} \subseteq \mathfrak{F}$. The maximality of $\mathfrak{M}$ implies that $\mathfrak{M} = \mathfrak{F}$.

We turn our attention to case (ii). Since all $G \in \mathfrak{M}$ are reduced, it follows that if $0 \neq G \in \mathfrak{M}$, then $tG \neq 0$ for we have

$$G \cong \operatorname{Ext}(tG, X) \text{ for all } G \in \mathfrak{M},$$

recall (4) and (5) in the proof of Lemma 2. Moreover, Lemma 2 implies that $(tG)_p$ is bounded and hence finite for all primes p. Consequently, if $G \in \mathfrak{M}$ then

$$G \cong \prod_{p \in P} \operatorname{Ext}((tG)_p, X) \cong \prod_{p \in P} (tG)_p$$

and hence $G \in \mathfrak{G}$. This proves that $\mathfrak{M} \subseteq \mathfrak{G}$ and since $\mathfrak{M}$ is maximal it follows that $\mathfrak{M} = \mathfrak{G}$. This completes the proof.

THEOREM 9.

(i) *Let* $G \in \mathfrak{F}$ *and* $H \in \mathfrak{F}$, *and let* $\varphi : G \to H$ *be a homomorphism. Then* $\operatorname{Ker}\varphi \in \mathfrak{F}$ *and* $\operatorname{Coker} \varphi \in \mathfrak{F}$.

(ii) $\mathfrak{G}$ *is a class of left* Ext-*reproduced groups which is maximal with respect to the property :*

If $G \in \mathfrak{G}$ *and* $H \in \mathfrak{G}$, *and if* $\varphi : G \to H$ *is a homomorphism, then* $\operatorname{Ker}\varphi \in \mathfrak{G}$ *and* $\operatorname{Coker}\varphi \in \mathfrak{G}$.

Proof.

(i) Let $G \in \mathfrak{F}$, $H \in \mathfrak{F}$, and let $\varphi : G \to H$ be a homomorphism. Then $G = Q^n \oplus S$, $H = Q^m \oplus T$ where m and n are non-negative integers, and S and T are finite groups. It is clear that $\varphi(Q^n) \subseteq Q^m$ and that $\varphi(S) \subseteq T$. Hence $\operatorname{Ker}\varphi = (\operatorname{Ker}\varphi \cap Q^n) \oplus (\operatorname{Ker}\varphi \cap S)$ and since $\operatorname{Ker}\varphi \cap Q^n$ is divisible and $\operatorname{Ker} \varphi \cap S$ is finite, our assertion follows.

(ii) Let $G \in \mathfrak{G}$, $H \in \mathfrak{G}$, and let $\varphi : G \to H$ be a homomorphism. $\operatorname{Ker}\varphi$ and $\operatorname{Im}\varphi$ are reduced, and hence it follows from the exact sequences

$$0 \to \operatorname{Ker}\varphi \to G \to \operatorname{Im}\varphi \to 0$$

and

$$0 \to \operatorname{Ext}(Q, \operatorname{Ker}\varphi) \to \operatorname{Ext}(Q, G) = 0 \to \operatorname{Ext}(Q, \operatorname{Im}\varphi) \to 0$$

that $\operatorname{Ker}\varphi$ and $\operatorname{Im}\varphi$ are reduced cotorsion groups.

Note that if $tG \subseteq \operatorname{Ker} \varphi$, then $\varphi = 0$ for then we have

$$G/\operatorname{Ker}\varphi \cong (G/tG)/(\operatorname{Ker}\varphi/tG) \subseteq H$$

and G/tG is divisible, whence $G/\operatorname{Ker}\varphi$ is divisible and hence 0. Hence $\operatorname{Ker}\varphi = G$, that is, $\varphi = 0$. Note further that if $\operatorname{Ker}\varphi$ is a torsion group then it is necessarily finite and then our assertion is obvious. We may therefore assume that $\operatorname{Ker}\varphi$ is infinite and that $\varphi \neq 0$. We then have

$$\operatorname{Ker}\varphi \cap tG \neq tG, \qquad (\operatorname{Ker}\varphi)_p \subseteq (tG)_p \quad \text{for all primes } p.$$

We maintain that $\operatorname{Ker}\varphi$ is adjusted. In order to prove this, assume to the contrary that $\operatorname{Ker}\varphi$ is not adjusted. Then it follows from [5] (p. 373-374) that

$$\operatorname{Ker}\varphi \cong \prod_{p\in P} (\operatorname{Ker}\varphi)_p \oplus L,$$

where $L \neq 0$ is a reduced torsion-free cotorsion group, and hence L has a direct summand $Z(p)$ for some prime p([5], p. 372). Hence $\operatorname{Ker}\varphi$, and therefore G as well, contains a subgroup $Z(p)$. That is to say, G contains elements of infinite order which are divisible by arbitrarily high powers of primes $q(q \neq p)$ since $qZ(p) = Z(p)$ for all primes $q \neq p$. This is evidently impossible, and we conclude that $\operatorname{Ker}\varphi$ is adjusted, i. e. $\operatorname{Ker}\varphi \cong \prod_{p\in P} (\operatorname{Ker}\varphi)_p \in \mathfrak{G}$.

We have the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & t\operatorname{Ker}\varphi & \longrightarrow & tG & \longrightarrow & \bigoplus_{p\in P}((tG)_p/(t\operatorname{Ker}\varphi)_p) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \lambda} & & \\
0 & \longrightarrow & \operatorname{Ker}\varphi & \longrightarrow & G & \longrightarrow & \operatorname{Im}\varphi & \longrightarrow & 0
\end{array}$$

where ι, μ and λ are the obvious mappings. This gives rise to the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \to & \operatorname{Ext}(Q/Z, t\operatorname{Ker}\varphi) & \to & \operatorname{Ext}(Q/Z, tG) & \to & \operatorname{Ext}(Q/Z, \bigoplus_{p\in P}((tG)_p/(t\operatorname{Ker}\varphi)_p) & \to & 0 \\
 & & \downarrow{\scriptstyle \iota_*} & & \downarrow{\scriptstyle \mu_*} & & \downarrow{\scriptstyle \lambda_*} & & \\
0 & \to & \operatorname{Ext}(Q/Z, \operatorname{Ker}\varphi) & \to & \operatorname{Ext}(Q/Z, G) & \to & \operatorname{Ext}(Q/Z, \operatorname{Im}\varphi) & \to & 0
\end{array}$$

and since ι_* and μ_* are isomorphisms ([5], p. 375), it follows that λ_* is also an isomorphism. Hence

$$\operatorname{Im}\varphi \cong \operatorname{Ext}\Big(Q/Z, \bigoplus_{p\in P}((tG)_p/(t\operatorname{Ker}\varphi)_p)\Big) \cong \prod_{p\in P}((tG)_p/(t\operatorname{Ker}\varphi)_p) \in \mathfrak{G}.$$

The exact sequences

$$0 \to \operatorname{Im}\varphi \to H \to \operatorname{Coker}\varphi \to 0 \tag{15}$$

and

$$\operatorname{Hom}(Q, H) = 0 \to \operatorname{Hom}(Q, \operatorname{Coker}\varphi) \to \operatorname{Ext}(Q, \operatorname{Im}\varphi) = 0$$

show that $\operatorname{Coker}\varphi$ is a reduced cotorsion group. If we consider the exact sequence (15), then we obtain a situation entirely similar to that in the above commutative diagrams, and we conclude that $\operatorname{Coker}\varphi \in \mathfrak{G}$.

If $\mathfrak{H}$ is a class of left Ext-reproduced groups such that if $K \in \mathfrak{H}$, $M \in \mathfrak{H}$, and if $\psi : K \to M$ is a homomorphism, then $\operatorname{Ker}\psi \in \mathfrak{H}$, $\operatorname{Coker}\psi \in \mathfrak{H}$, and if

$\mathfrak{H} \supseteq \mathfrak{G}$, then $\mathfrak{H} = \mathfrak{G}$. In fact, by Lemma 1, $X \cong \prod_{p \in P} Z(p)$ whence it follows that all groups in $\mathfrak{H}$ are reduced. Hence if $K \in \mathfrak{H}$ then, obviously, every direct summand of K belongs to $\mathfrak{H}$ and hence, by Theorem 6, $K \in \mathfrak{G}$. We have therefore shown that $\mathfrak{H} \supseteq \mathfrak{G}$, and the proof is complete.

Remark. — We turn our attention to the converse of the above theorem. Let $\mathfrak{M}$ be a class of left Ext-reproduced groups which is maximal with respect to the property :

If $G \in \mathfrak{M}$, $H \in \mathfrak{M}$ and if $\varphi : G \to H$ is a homomorphism then $\operatorname{Ker} \varphi \in \mathfrak{M}$. $\operatorname{Coker} \varphi \in \mathfrak{M}$.

Then either $\mathfrak{M} = \mathfrak{F}$ or $\mathfrak{M} = \mathfrak{G}$.

Indeed, if $G \in \mathfrak{M}$ then, manifestly, every direct summand of G belongs to $\mathfrak{M}$ and hence Theorem 6 implies our assertion.

THEOREM 10.

(i) *If* $G \in \mathfrak{F}$ *and* $H \in \mathfrak{F}$, *then* $\operatorname{Ext}(G, H) \in \mathfrak{F}$.

(ii) $\mathfrak{G}$ *is a class of left* Ext-*reproduced groups which is maximal with respect to the property* :

If $G \in \mathfrak{G}$ *and* $H \in \mathfrak{G}$, *then* $\operatorname{Ext}(G, H) \in \mathfrak{G}$.

Proof.

(i) Let $G \in \mathfrak{F}$, $H \in \mathfrak{F}$, then $G = Q^n \oplus S$, $H = Q^m \oplus T$ where m and n are non-negative integers and S and T are finite groups. It is clear that $\operatorname{Ext}(G, H) \cong \operatorname{Ext}(S, T)$ and since the latter group is finite it follows that $\operatorname{Ext}(G, H) \in \mathfrak{F}$.

(ii) Let $G \in \mathfrak{G}$, $H \in \mathfrak{G}$, then $G \cong \prod_{p \in P} (tG)_p$, $H \cong \prod_{p \in P} (tH)_p$ where $(tH)_p$ and $(tG)_p$ are finite p-groups for all primes p. The exact sequences

$$0 \to \bigoplus_{p \in P} (tG)_p \to \prod_{p \in P} (tG)_p \to Q^{(\aleph)} \to 0$$

and

$$\operatorname{Ext}(Q^{(\aleph)}, H) = 0 \to \operatorname{Ext}\left(\prod_{p \in P} (tG)_p, H\right) \to \operatorname{Ext}\left(\bigoplus_{p \in P} (tG)_p, H\right) \to 0$$

imply

$$\operatorname{Ext}(G, H) \cong \prod_{p \in P} \operatorname{Ext}((tG)_p, H) \cong \prod_{p \in P} \operatorname{Ext}((tG)_p, (tH)_p)$$

and Ext $((tG)_p, (tH)_p)$ is a finite p-group for all primes p so that Ext $(G, H) \in \mathfrak{G}$.

Let $\mathfrak{H}$ be a class of left Ext-reproduced groups such that $\text{Ext}(G, H) \in \mathfrak{H}$ whenever $G \in \mathfrak{H}$ and $H \in \mathfrak{H}$, and suppose that $\mathfrak{H} \supseteq \mathfrak{G}$. Then, by Lemma 1, $X \cong \prod_{p \in P} Z(p)$ and hence all groups in $\mathfrak{H}$ are reduced, by virtue of the fact that $G \cong \text{Ext}(tG, X)$ for all $G \in \mathfrak{H}$.

Let $0 \neq G \in \mathfrak{H}$, then $tG \neq 0$. Then Lemma 2 implies that $(tG)_p$ is bounded and hence finite for all primes p. Thus we have (*see* [4], Example 2.8)

$$G \cong \text{Ext}(tG, X) \cong \text{Hom}(tG, X \otimes (Q/Z))$$
$$\cong \prod_{p \in P} \text{Hom}((tG)_p, Q/Z) \cong \prod_{p \in P} (tG)_p,$$

in other words, $G \in \mathfrak{G}$. Hence $\mathfrak{H} \subseteq \mathfrak{G}$, and the proof is complete.

THEOREM 11. — *Let $\mathfrak{M}$ be a class of left* Ext-*reproduced groups which is maximal with respect to the property* :

If $G \in \mathfrak{M}$ and $H \in \mathfrak{M}$ then $\text{Ext}(G, H) \in \mathfrak{M}$.

Then either $\mathfrak{M} = \mathfrak{F}$ or $\mathfrak{M} = \mathfrak{G}$.

Proof. — The method of approach is basically the same as in Theorems 6 and 8. There are two possibilities, viz. either

(i) all groups G in $\mathfrak{M}$ are reduced, or

(ii) $\mathfrak{M}$ contains a group which is not reduced.

If (i) holds then for all $G \in \mathfrak{M}$, we have

$$G \cong \text{Ext}(tG, X) \cong \prod_{p \in P} \text{Ext}((tG)_p, X).$$

By Lemma 2, $(tG)_p$ is bounded and hence finite for all primes p. Consequently,

$$G \cong \prod_{p \in P} \text{Ext}((tG)_p, X) \cong \prod_{p \in P} (tG)_p$$

and hence $G \in \mathfrak{M}$ implies $G \in \mathfrak{G}$. The maximality of $\mathfrak{M}$ implies that $\mathfrak{M} = \mathfrak{G}$.

Let (ii) be valid, then it is clear that $\text{Ext}(Q, X) \neq 0$. Suppose that $G \in \mathfrak{M}$ is not reduced and let

$$G = D \oplus \Big(\bigoplus_{p \in P} D_p\Big) \oplus G'$$

where $D = Q^n$, n a non-negative integer, $D_p = (C(p^\infty))^{(n)}$ and where G' is reduced. We assert that $\bigoplus_{p \in P} D_p = 0$. In fact, if this is not the case then it follows from Lemma 4 that

$$G = D \oplus \Big(\bigoplus_{p \in P} D_p\Big) \oplus L' \oplus tG'$$

where $L' \neq 0$ is a reduced torsion-free cotorsion group and tG' is finite. By assumption, $\operatorname{Ext}(G, G) \in \mathfrak{M}$. However,

$$\operatorname{Ext}(G, G) \cong \operatorname{Ext}\Big(\bigoplus_{p \in P} D_p, L'\Big) \oplus \operatorname{Ext}\Big(\bigoplus_{p \in P} D_p, tG'\Big) \oplus \operatorname{Ext}(tG', L') \oplus \operatorname{Ext}(tG', tG')$$

and we have

(a) $\operatorname{Ext}\Big(\bigoplus_{p \in P} D_p, L'\Big) \neq 0$ is a torsion-free and reduced cotorsion group [this follows from Lemma 4, (ii)];

(b) $\operatorname{Ext}\Big(\bigoplus_{p \in P} D_p, tG'\Big)$ is a finite group or else $\operatorname{Ext}(G, G) \notin \mathfrak{M}$;

(c) $\operatorname{Ext}(tG', L')$ is a finite group for the same reason as in (b);

(d) $\operatorname{Ext}(tG', tG')$ is a finite group — this is obvious.

In other words, $\operatorname{Ext}(G, G)$ is the direct sum of a reduced and non-zero torsion-free cotorsion group, and a finite group, and hence it is not left Ext-reproduced, contrary to $\operatorname{Ext}(G, G) \in \mathfrak{M}$.

This contradiction shows that $\bigoplus_{p \in P} D_p = 0$ and hence

$$G = D \oplus G' = Q^n \oplus G'$$

where n is a natural number and G' is a finite group. It follows from Lemma 2 and Lemma 3 that the reduced part of every group $G \in \mathfrak{M}$ is finite and hence if $G \in \mathfrak{M}$ then $G \in \mathfrak{F}$. Hence $\mathfrak{M} = \mathfrak{F}$ and the proof is complete.

Remark. — If we consider $\mathfrak{F}$ then X satisfies $\prod_{p \in P} Z(p)/X \cong Q$ ([4], Example 2.3) and it is clear that $\operatorname{Hom}(G, X) = 0$ for all $G \in \mathfrak{F}$. For the class $\mathfrak{G}$, we have $X \cong \prod_{p \in P} Z(p)$ and we also have that $\operatorname{Hom}(G, X) = 0$ for all $G \in \mathfrak{G}$. Moreover, $\mathfrak{G}$ is a class of left Ext-reproduced groups which is maximal with respect to the property : $\operatorname{Hom}(G, X) = 0$ for all $G \in \mathfrak{G}$. In fact, if $\mathfrak{H}$ is a class of left Ext-reproduced groups which contains $\mathfrak{G}$ and which is such that $\operatorname{Hom}(G, X) = 0$ for all $G \in \mathfrak{H}$, then $\mathfrak{H} = \mathfrak{G}$. This follows from following : If $G \in \mathfrak{H}$ then G_p is bounded for all primes p, for if G_p is unbounded for some prime p then G contains a direct summand $Z(p)^{\aleph}$ (Lemma 2), and since $X \cong \prod_{p \in P} Z(p)$ (Lemma 1), we deduce that $\operatorname{Hom}(G, X) \neq 0$. Hence G_p is bounded and therefore finite for all primes p, consequently

$$G \cong \prod_{p \in P} \operatorname{Ext}((tG)_p, X) \cong \prod_{p \in P} (tG)_p \in \mathfrak{G}$$

whence $\mathfrak{H} \subseteq \mathfrak{G}$.

This property can also be used to characterize the classes $\mathfrak{F}$ and $\mathfrak{G}$.

THEOREM 12. — *Let* $\mathfrak{M}$ *be a class of left* Ext-*reproduced groups which is maximal with respect to the property* :

$\mathrm{Hom}(G, X) = 0$ *for all* $G \in \mathfrak{M}$.
Then either $\mathfrak{M} = \mathfrak{F}$ *or* $\mathfrak{M} = \mathfrak{G}$.

Proof. — There are two alternatives, viz. either

(i) all groups G in $\mathfrak{M}$ are reduced, or

(ii) $\mathfrak{M}$ contains a group G which is not reduced.

Let us consider case (i) and let $G \in \mathfrak{M}$. The exact sequence

$$0 \to tG \to G \to G/tG \to 0$$

yields the exact sequence

$$0 \to \mathrm{Hom}(tG, X) \to \mathrm{Ext}(G/tG, X) \to \mathrm{Ext}(G, X) \cong G \to \mathrm{Ext}(tG, X) \to 0$$

and since G and $\mathrm{Hom}(tG, X)$ are reduced it follows that

$$\mathrm{Hom}(tG, X) = 0 = \mathrm{Ext}(G/tG, X) \tag{16}$$

and hence $G \cong \prod_{p \in P} \mathrm{Ext}((tG)_p, X)$ shows that if $0 \neq G \in \mathfrak{M}$ then $tG \neq 0$.

If $G \in \mathfrak{M}$, then $(tG)_p$ is bounded for all primes p. Assume, to the contrary, that $(tG)_p$ is unbounded for some prime p, then by Lemma 2, G has a direct summand $Z(p)^{\aleph}$, and hence $\mathrm{Hom}(G, X) = 0$ and (16) imply

$$\mathrm{Hom}(Z(p), X) = 0 = \mathrm{Ext}(Z(p), X).$$

The exact sequence (*see* [1], p. 252 and 255)

$$0 \to Z(p) \otimes Z \cong Z(p) \to Z(p) \otimes Q \to Z(p) \otimes (Q/Z) \cong C(p^\infty) \to 0$$

leads to the exact sequence

$$0 \to \mathrm{Ext}(C(p^\infty), X) \to (\mathrm{Ext}(Q, X))^{\aleph} \to \mathrm{Ext}(Z(p), X) = 0$$

and hence

$$\mathrm{Ext}(C(p^\infty), X) = 0 = \mathrm{Ext}(Q, X). \tag{17}$$

However, $\mathrm{Hom}(Z(p), X) = 0$ implies $X_p = 0$ and, by Lemma 2, $(tG)_p \neq 0$ implies $X/pX \neq 0$. Hence

$$\mathrm{Ext}(C(p^\infty), X) \cong \mathrm{Ext}(C(p^\infty), X/tX) \cong \mathrm{Hom}(C(p^\infty), (X/tX) \otimes C(p^\infty)) \neq 0$$

since $X/tX \neq p(X/tX)$ ([1], p. 255). This is however contrary to (17) and hence we conclude that $(tG)_p$ is bounded and consequently finite for all primes p. This implies that

$$G \cong \prod_{p \in P} \mathrm{Ext}((tG)_p, X) \cong \prod_{p \in P} (tG)_p$$

or alternatively, $G \in \mathfrak{M}$ implies $G \in \mathfrak{G}$ and hence $\mathfrak{M} \subseteq \mathfrak{G}$ so that $\mathfrak{M} = \mathfrak{G}$.

We now turn our attention to (ii), and we notice that $\mathrm{Ext}(Q, X) \neq 0$. Let $G \in \mathfrak{M}$ be a group which is not reduced and let

$$G = D \oplus \left(\bigoplus_{p \in P} D_p \right) \oplus G'$$

where $D = Q^n$, n a non-negative integer, $D_p = (C(p^\infty))^{(n)}$ and where G' is reduced. We assert that G'_p is bounded for all primes p. In fact, if G'_p is unbounded for some prime p, then by Lemma 2, G contains a direct summand $Z(p)^{\aleph}$ and hence $\mathrm{Hom}(Z(p), X) = 0$. The exact sequence

$$(18) \qquad 0 \to Z(p) \to Z(p) \otimes Q \to C(p^\infty) \to 0$$

yields the exact sequence

$$(19) \qquad 0 \to \mathrm{Ext}(C(p^\infty), X) \to (\mathrm{Ext}(Q, X))^{\aleph} \to \mathrm{Ext}(Z(p), X) \to 0$$

and hence it follows from (19) that $\mathrm{Ext}(Z(p), X) \neq 0$ since $\mathrm{Ext}(C(p^\infty), X)$ is reduced and $\mathrm{Ext}(Q, X) \neq 0$. Now, $\mathrm{Ext}\left(Z(p)^{\aleph}, X\right)$ is a direct summand of $G \cong \mathrm{Ext}(G, X)$ and hence the exact sequences

$$0 \to Z(p)^{(\aleph)} \to Z(p)^{\aleph} \text{ and } \mathrm{Ext}\left(Z(p)^{\aleph}, X\right) \to (\mathrm{Ext}(Z(p), X))^{\aleph} \to 0$$

show that $\mathrm{Ext}\left(Z(p)^{\aleph}, X\right)$, and consequently G as well, contains a torsion-free divisible subgroup of infinite rank. This contradiction shows that G'_p is bounded and therefore finite for all primes p.

We contend that $\bigoplus_{p \in P} D_p = 0$. Indeed, if $\bigoplus_{p \in P} D_p \neq 0$ then by Lemma 4, we have $G = D \oplus \left(\bigoplus_{p \in P} D_p \right) \oplus L' \oplus tG'$, where $L' \neq 0$ is a reduced and torsion-free cotorsion group and where tG' is finite. Moreover, it follows from Lemma 4, (iii), that for some prime p for which $D_p \neq 0$, L' contains a direct summand $Z(p)$. Hence $\mathrm{Hom}(Z(p), X) = 0$, and we deduce from the exact sequences (18) and (19) that $\mathrm{Ext}(Z(p), X) \neq 0$. Now it follows from Lemma 4, 2°, that $D_p = 0$. This contradiction shows that $\bigoplus_{p \in P} D_p = 0$ and hence

$$G = Q^n \oplus G'$$

where n is a natural number and G' is reduced. Moreover, the finiteness of G'_p for all primes p implies that

$$G' \cong \mathrm{Ext}(tG', X) \cong \prod_{p \in P} \mathrm{Ext}((tG')_p, X) \cong \prod_{p \in P} (tG')_p$$

and by Lemma 3, $(tG')_p \neq 0$ for only a finite number of primes p, that is to say, G' is finite. It is also clear that the reduced part of each group $G \in \mathfrak{M}$ is finite. Hence if $G \in \mathfrak{M}$ then $G \in \mathfrak{F}$ and hence $\mathfrak{M} \subseteq \mathfrak{F}$, Consequently $\mathfrak{M} = \mathfrak{F}$, and the proof is complete.

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