BULLETIN DE LA S. M. F.

L. FUCHS
K.M. RANGASWAMY

Quasi-projective abelian groups

Bulletin de la S. M. F., tome 98 (1970), p. 5-8
<http://www.numdam.org/item?id=BSMF_1970__98__5_0>

© Bulletin de la S. M. F., 1970, tous droits réservés.

L'accès aux archives de la revue « Bulletin de la S. M. F. » (http://smf.emath.fr/Publications/Bulletin/Presentation.html) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM
Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

Bull. Soc. math. France,
98, 1970, p. 5 à 8.

QUASI-PROJECTIVE ABELIAN GROUPS

BY

LASZLO FUCHS AND KULUMANI M. RANGASWAMY.

A module M over a ring R is called *quasi-projective* (*see* [3]) if for every submodule N of M and for every R-homomorphism $\varphi : M \to M/N$ there is an R-endomorphism ψ of M making the diagram

$$\begin{array}{ccc} & & M \\ & \swarrow^{\psi} & \downarrow^{\varphi} \\ M & \xrightarrow{\eta} & M/N \end{array}$$

commute where η denotes the natural map. The aim of this note is to describe explicitly the quasi-projective abelian groups.

It is relatively easy to list the quasi-injective abelian groups, since they are exactly the fully invariant subgroups of injective, i. e. divisible groups, and hence either divisible or torsion groups each p-component of which is the direct sum of isomorphic cyclic or quasicyclic groups $Z(p^n)$ $(n \leqq \infty)$. JANS and WU [3] described the finitely generated quasi-projective abelian groups; the general case seems to be unsettled so far. We shall show that the expected structure theorem holds : an abelian group is quasi-projective exactly if it is either free or a torsion group each p-component of which is the direct sum of isomorphic cyclic groups of orders p^n for some n (which may depend on p).

We shall need a couple of lemmas which we formulate for arbitrary unital R-modules M.

LEMMA 1. — *Every direct summand of a quasi-projective module is quasi-projective.*

LEMMA 2. — *If M is quasi-projective and N is a fully invariant submodule of M, then M/N is likewise quasi-projective.*

For these two lemmas, we refer to JANS and WU [3].

LEMMA 3. — *If M_i $(i \in I)$ are quasi-projective R-modules such that, for every submodule N of the direct sum $M = \oplus M_i$, $N_i = \oplus (N \cap M_i)$ holds, then M is again quasi-projective.*

Hypothesis implies that every quotient module M/N of M is of the form $\oplus (M_i/N_i)$ with $N_i \subseteq M_i$. Every homomorphism $M_i \to M_j/N_j$ with $i \neq j$ must be trivial, because otherwise there exist submodules N'_i and N'_j such that $M_i/N'_i \simeq N'_j/N_j$ are non-zero modules, and so there is a subdirect sum of M_i and N'_j which is not their direct sum. Thus every $\varphi : \oplus M_i \to \oplus (M_i/N_i)$ acts coordinate-wise whence the quasi-projectivity of M is obvious.

LEMMA 4. — *If N is a submodule of a quasi-projective module M such that M/N is isomorphic to a direct summand of M, then N itself is a summand of M.*

Let A be a summand of M with $\pi : M \to A$, $\rho : A \to M$ as projection and injection maps, and let $\alpha : A \to M/N$ be an isomorphism. For the natural map $\eta : M \to M/N$, there exists a $\psi : M \to M$ rendering

$$\begin{array}{ccc} M & \xrightarrow{\pi} & A \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\alpha} \\ M & \xrightarrow{\eta} & M/N \end{array}$$

commutative, i. e. $\eta\psi = \alpha\pi$. Define $M/N \to M$ as $\psi\rho\alpha^{-1}$; then $\eta\psi\rho\alpha^{-1} = \alpha\pi\rho\alpha^{-1}$ is the identity map of M/N. Hence the sequence $0 \to N \to M \xrightarrow{\eta} M/N \to 0$ splits.

LEMMA 5. — *Let N be a submodule of the quasi-projective module M such that there exists an epimorphism $\varepsilon : N \to M$. Then M is isomorphic to a direct summand of N.*

Write $K = \operatorname{Ker} \varepsilon$. Let $\bar{\varepsilon} : N/K \to M$ be the isomorphism induced by ε, α the injection $M \to M/K$ with $\alpha\bar{\varepsilon}$ the identity on N/K, and $\eta : M \to M/K$ the natural map. By quasi-projectivity, some $\psi : M \to M$ satisfies $\eta\psi = \alpha$ where $\psi(M) \subseteq \eta^{-1}(N/K) = N$. For $\psi\bar{\varepsilon} : N/K \to N$, $\eta\psi\bar{\varepsilon} = \alpha\bar{\varepsilon}$ acts identically on N/K, therefore $0 \to K \to N \xrightarrow{\eta} N/K \to 0$ is splitting.

Notice that lemma 5 can also be derived from a result of DE ROBERT [2]; it follows that $\operatorname{Hom}_R(M, N) \to \operatorname{Hom}_R(M, M)$ is epic whenever M is quasi-projective and $N \subseteq M$, and it suffices to look at a preimage of 1_M to obtain lemma 5.

By $E(M)$ we denote the ring of all R-endomorphisms of M.

LEMMA 6. — *If N is a submodule in a quasi-projective module M, then the cardinality of $E(M/N)$ does not exceed that of $E(M)$.*

To every $\alpha \in E(M/N)$ there exists a $\psi_\alpha \in E(M)$ such that $\eta\psi_\alpha = \alpha\eta$ where again $\eta : M \rightarrow M/N$ is the natural map. If $\alpha, \beta \in E(M/N)$ are distinct, then $\alpha\eta \neq \beta\eta$ (since η is epic), and hence $\psi_\alpha \neq \psi_\beta$ in $E(M)$.

We are now ready to prove our result (for the needed facts on abelian groups we refer to [1]) :

THEOREM. — *An abelian group A is quasi-projective if, and only if, it is :*

1° *free, or*

2° *a torsion group such that every p-component A_p is a direct sum of cyclic groups of the same order p^n.*

Free groups F are quasi-projective, so by lemma 2, the groups $F/p^n F$ are likewise quasi-projective. By lemma 3, a direct sum of groups $F/p^n F$ with different primes p is quasi-projective. Since $F/p^n F$ is a direct sum of cyclic groups of order p^n, the sufficiency is evident.

Conversely, assume A is quasi-projective. If A is torsion, then by lemma 1, every A_p is quasi-projective. If A_p is not reduced, then it contains a summand of type $Z(p^\infty)$. By lemmas 1 and 4, every proper subgroup of $Z(p^\infty)$ must be a summand of $Z(p^\infty)$ which is absurd, thus A_p is reduced. It cannot have a summand of the form $Z(p^n) \oplus Z(p^m)$ with $n < m$, because this cannot be quasi-projective in view of the existence of an epimorphism $Z(p^m) \rightarrow Z(p^n)$ whose kernel is not a summand. Therefore, the basic subgroups B_p of A_p are direct sums of cyclic groups of the same orders p^n, and so $A_p = B_p$ (namely, B_p is now a summand of A_p, and A_p is reduced).

If A is torsion-free, then we distinguish two cases according as A has finite or infinite rank. If A is of finite rank r, then let F be a free subgroup of rank r in A. Now $E(A)$ is countable, hence $E(A/F)$ is at most countable (lemma 6). Since A/F is torsion, this can happen only if A/F is finite in which case A too is free. If A is of infinite rank, then let F be a free subgroup of A of the same rank as A. The existence of an epimorphism $F \rightarrow A$ and lemma 5 lead us to conclude that A is isomorphic to a summand of F and hence A is free.

Finally, we show that A can not be mixed. If T is the torsion part of A, then A/T is quasi-projective by lemma 2, and hence free by what has been proved, i. e. $A = T \oplus F$ with quasi-projective T and free F. If neither $T = 0$ nor $F = 0$, then there exist a cyclic direct summand $Z(p^n)$ of T and an epimorphism $\varepsilon : F \rightarrow Z(p^n)$ whose kernel is not a summand of F, in contradiction to lemma 4. This completes the proof.

REFERENCES.

[1] Fuchs (L.). — *Abelian groups.* — Budapest, Hungarian Academy of Sciences, 1958.

[2] de Robert (E.). — Sur une propriété des quasi-injectifs et des quasi-projectifs, *C.R. Acad. Sc.*, t. 266, 1968, série A, p. 547-549.

[3] Wu (L. E. T.) and Jans (J. P.). — On quasi-projectives, *Illinois J. Math.*, t. 11, 1967, p. 439-448.

(Texte reçu le 16 septembre 1969.)

Laszlo Fuchs
Department of Mathematics
Tulane University
New Orleans, La 70 118 (États-Unis)
Kulumani M. Rangaswamy
Madurai University
University Buildings
Madurai 2 (Inde)