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HYPER-DIFFERENTIAL OPERATORS IN COMPLEX SPACE (*)

BY

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CONTENTS.

Introduction.

In recent years, the pseudo-differential operators in real space (that is, on real $\mathcal{C}^\infty$ manifolds) have attracted a certain amount of interest, essentially for their possible applications to partial differential equations (*see* for instance the references [1], [2], [3]). Similar reasons, but pertinent to the complex space, have motivated the present work. In an earlier article, I have shown that the Cauchy problem for a determined system of linear partial differential equations with *analytic* cœfficients, relative

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to a noncharacteristic hyperplane, with data valued in the space of analytic functionals with respect to the tangential variables (and depending in a more „ orthodox " manner on the normal variable, for instance, distributions with respect to the latter), always admits a unique solution, the values of which also lie in those spaces of analytic functionals (*see* [6]). The system of PDEs under study is looked upon as an *evolution equation* and, as such, it possesses a *resolvent*. The resolvent enables us to express the solution of the Cauchy problem in terms of the data, that is, of the initial data and of the right hand sides of the equations. This result is to serve, so I hope, as foundation for a theory of solvability of linear PDEs with analytic coefficients. Here solvability is meant in a " classical " sense : in the sense of functions or at least of distributions; and primarily, it is solvability of the Cauchy problem (I also hope that the present approach will throw some light on the apparently more difficult problem of local solvability, when no initial or boundary conditions are imposed). It ought to be recalled that any distribution with compact support K in $\mathbf{R}^n$ can be identified, canonically, with an analytic functional in $\mathbf{C}^n$ (carried by K). Then the solvability question, for a given system of linear PDEs with analytic coefficients, can be stated as follows : is it true, or is it not, that whenever the values of the data in the Cauchy problem are distributions with compact support (with respect to the tangential variables), the same can be said of the solution (which we know to be unique)? Now, even a superficial examination of this question is enough to convince one that the properties of the resolvent will be crucial to the obtention and to the form of the answer. In view of this fact, the investigation of these properties should be given priority. And it would be greatly facilitated if we had at our disposal an *integral representation* of the resolvent, with a kernel which is a function.

This is precisely the main result of this article, and the justification for writing it. It is rigourously stated and proved (1) in the final section (Section 8). The kernel in the integral representation of the resolvent is called the *symbol* of the resolvent; it is a holomorphic function of two sets z, λ of complex variables. Operators acting on analytic functionals (or, by transposition, on holomorphic functions) which admit similar integral representations, i. e., which have this type of symbols, are called here *hyper-differential operators*. They could as well have been called *differential operators of infinite order with analytic coefficients :* roughly speaking, they are infinite series of the kind

$$\sum_p h_p(z)\left(\frac{\partial}{\partial z}\right)^p,$$

(1) In truth, only first order systems are considered. But more general systems can always be reduced to first order ones by introducing additional unknowns.

where p ranges over the set $\mathbf{N}^n$ of all n-tuples of integers $\geqq 0$ and where, for each p, $h_p(z)$ is a holomorphic function in a given open set (of course, we require these series to converge in an appropriate manner). Differential operators with holomorphic coefficients are particular cases of hyper-differential operators — provided that the differentiations be applied first, and multiplication by the coefficients afterwards only (*cf.* p. 201). The reason for the definition of hyper-differential operators given here (in Section 2) is that these are exactly the operators for which it is possible to develop a *symbolic calculus.* Under favourable circumstances, the compose of two hyper-differential operators is also one — although such operators cannot be said to form an algebra (here again, *cf.* p. 201). The symbol of hyper-differential operators is introduced in Section 6, where the reader will also find the formula, (6.5), (6.8), for the symbol of a product. This formula is formally identical with the one in the real case. Section 7 presents the integral formula expressing the operator in terms of its symbol. It is somewhat similar to the formula in the real case, but whereas in real space, one makes use of the Fourier transformation, in complex space, the basic tools are the Laplace and the Fourier-Borel transformations, combined with the Cauchy representation of analytic functionals (*see* Section 3). At this point the reader should be warned that all the reasonings, or at least the most important ones, assume the geometrical configuration to be of the simplest kind, so as to avoid the usual topological difficulties in the theory of several complex variables. Throughout the main sections of this article, the assumption is that the sets which carry the analytic functionals, on which the operators act, are *products* of the form $A = A_1 \times \ldots \times A_n$, where the A_j are *convex* subsets of the complex plane. All the results valid in this rather primitive case can be extended to the case of arbitrary *convex* subsets of $\mathbf{C}^n$, with the help of the projective Fantappiè transformation and of the Fantappiè-Leray formula (*see* [4]). But this is a rather technical theory and it would have been probably unwise to inject it into what, after all, wants only to be an introductory article.

1. Spaces of holomorphic functions. Spaces of analytic functionals.

Let E be a locally convex Hausdorff topological vector space (over the complex numbers, like all the vector spaces considered in this work). Let Ω be a complex analytic manifold, countable at infinity (like all the manifolds considered here). We denote by $H(\Omega; E)$ the space of holomorphic mappings $\Omega \to E$, equipped with the topology of uniform convergence on the compact subsets of Ω. When $E = \mathbf{C}$, the field of complex numbers, we write simply $H(\Omega)$.

If F is another locally convex Hausdorff space, we denote by $L(E; F)$ the space of continuous linear mappings of E into F, equipped with the topology of uniform convergence on the bounded subsets of E. We set

$$H'(\Omega; E) = L(H(\Omega); E).$$

The elements of $H'(\Omega; E)$ are called the *analytic functionals* in Ω valued in E. When $E = \mathbf{C}$ we write simply $H'(\Omega)$: this is the strong dual of $H(\Omega)$.

Observe that $H(\Omega)$ is a Fréchet-Montel space. In particular, it is reflexive, and as a matter of fact, nuclear (*see* e. g. [5], coroll. of Th. 51.5). We have, if the locally convex Hausdorff space E is complete,

$$H(\Omega; E) \cong H(\Omega) \hat{\otimes} E. \tag{1.2}$$

The dual of a nuclear Fréchet space is also nuclear ([5], prop. 50.5) : thus $H'(\Omega)$ is nuclear. Therefore, still assuming that E is complete and applying Proposition 50.5 in [5] :

$$L(H(\Omega); E) \cong H'(\Omega) \hat{\otimes} E. \tag{1.3}$$

Let U and V be two open subsets of Ω such that $U \subset V$. The restriction of functions from V to U defines a continuous linear map of $H(V)$ into $H(U)$, which we shall denote by ρ_{UV}. Its transpose ${}^t\rho_{UV}$ is a continuous linear map $H'(U) \to H'(V)$. In general, this transpose is not injective. It is injective if and only if the image of ρ_{UV} is dense, i. e. if every holomorphic function in U is a limit of (restrictions of) holomorphic functions in V, in which case one says that U is a *Runge subset of* V; when $V = \Omega$, one says that U is a *Runge open set*. In any case, an analytic functional on V which belongs to the image of ${}^t\rho_{UV}$ is said to be *carried* by U. Let K be a compact subset of Ω; an analytic functional α in Ω is said to be *carried by* K if α is carried by every open neighborhood of K.

Let K be as before a compact subset of Ω, and let $\mathcal{H}(K)$ denote the disjoint union of the linear spaces $H(U)$, where U ranges over the collection of all open subsets of Ω containing K. We say that two functions f and g, defined and holomorphic in open neighborhoods of K, U and V respectively, are *equivalent near* K if there exists a third open neighborhood $W \subset U \cap V$ of K such that the restrictions of f and g to W are equal.

We denote by $H(K)$ the quotient of $\mathcal{H}(K)$ modulo this equivalence relation. We refer to the elements of $\tilde{H}(K)$ as the *germs* of holomorphic functions near K. For every open set $U \supset K$ there is a canonical map $\rho_U : H(U) \to H(K)$: the mapping which to any holomorphic function in U assigns its germ near K. We equip $H(K)$ with the finest locally convex topology such that all the mappings ρ_U are continuous. Then

$H(K)$ is a *countable inductive limit* of Fréchet spaces. But this inductive limit is not *strict* : outside of trivial cases, the mappings ρ_U are not isomorphisms (as a matter of fact, often they are not even injective).

Let now $\mathcal{H}'(K)$ denote the product of the spaces $H'(U)$, where U ranges over the family of all open subsets of Ω containing K. We denote by $H'(K)$ the linear subspace of $\mathcal{H}'(K)$ consisting of the elements (α_U) such that, given any two open sets U, V such that $K \subset U \subset V$, ${}^t\rho_{UV}(\alpha_U) = \alpha_V$. We equip $H'(K)$ with the topology induced by the product space topology on $\mathcal{H}'(K)$ [one says then that $H'(K)$ is the *projective limit* of the spaces $H'(U)$]. We shall refer to the elements of $H'(K)$ as the *analytic functionals* on K (or, if there is any danger of confusion, as the *local* analytic functionals on K). It should be kept in mind that, in general, *these cannot be identified with the analytic functionals in Ω carried by K* Indeed, two analytic functionals in some open neighborhood U of K might have the same restriction on the set of functions which are extendable as holomorphic functions in the whole of Ω without being equal in U. There is an important case where we can identify $H'(K)$ with the subspace of $H'(\Omega)$ consisting on the analytic functionals carried by K [2] : this is when K is a Runge compact set, which means that there is a basis of open neighborhoods of K consisting of Runge open sets. We recall that in the case where Ω is the complex plane, this amounts to saying that K is *simply connected.* Of course, no such simple characterization is valid in higher dimensions. But every *convex* compact set in $\mathbf{C}^n$ is a Runge compact set.

Let α be a linear map of $H(K)$ into some locally convex Hausdorff space E. The map α is continuous if and only if, for every open set $U \supset K$, the compose $\alpha \circ \rho_U : H(U) \to E$ is continuous. Let us assume that $E = \mathbf{C}$ and set $\alpha_U = \alpha \circ \rho_U$. If V is another open set containing U, we have $\alpha_U \circ \rho_{UV} = \alpha_V$. Now α_U and α_V are analytic functionals (on U and on V respectively), and the preceding relation means that ${}^t\rho_{UV}(\alpha_U) = \alpha_V$. In other words, the collection (α_U) defines an element of $H'(K)$. And conversely, every element of $H'(K)$ defines a continuous linear functional on $H(K)$; this is evident. We may identify $H'(K)$ with the dual of $H(K)$; this identification extends to the topologies [the strong dual topology ont the dual of $H(K)$]. The spaces $H(K)$ and $H'(K)$ are barrelled and complete. As a compact subset of a manifold has a countable basis of neighborhoods, in virtue of (50.6) and (50.8) in [5], they are both nuclear. We apply Proposition 50.5 in [5]. If E is any *complete* locally convex Hausdorff space,

$$L(H(K); E) \cong H'(K) \hat{\otimes} E. \tag{1.4}$$

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(2) Equivalent with saying that the complement of K is connected.

Let U range over the family of open neighborhoods of K. Going to the *projective* limit with respect to U commutes with the completion of the tensor product (when dealing with nuclear spaces) and therefore [*cf.* (1.3)]

$$(1.5)\quad H'(K)\hat{\otimes} E = \operatorname{proj}\lim_U H'(U)\hat{\otimes} E = \operatorname{proj}\lim_U L(H(U); E),$$

hence, recalling that by definition $H'(U; E) = L(H(U); E)$,

$$(1.6)\quad H'(K; E) = \operatorname{proj}\lim_U H'(U; E) = L(\tilde{H}(K)\ ;E) = H'(K)\hat{\otimes} E.$$

Let Ω^i ($i = 1, 2$) be two complex analytic manifolds and for each i, A^i a subset of Ω^i, either compact or open. From (1.3) and (1.4), we derive

$$(1.7)\qquad L(H(A^1); H(A^2)) \cong H'(A^1)\hat{\otimes} H(A^2);$$

$$(1.8)\qquad L(H(A^1); H'(A^2)) \cong H'(A^1)\hat{\otimes} H'(A^2).$$

On the other hand, because of the reflexivity of all the spaces involved, transposition of continuous linear mappings establishes canonical isomorphisms

$$(1.9)\qquad L(H(A^1); H(A^2)) \cong L(H'(A^2); H'(A^1));$$

$$(1.10)\qquad L(H(A^1); H'(A^2)) \cong L(H(A^2); H'(A^1)).$$

In particular, we have [*cf.* (1.6)] :

$$(1.11)\quad L(H(A^1); H(A^2)) \cong L(H'(A^2); H'(A^1)) = H'(A^1)\hat{\otimes} H(A^2) = H'(A^1; H(A^2)).$$

Remark 1.1. — When both A^1 and A^2 are compact, it is *not* true (outside trivial cases) that the topological vector spaces in (1.11) are isomorphic with

$$H(A^2; H'(A^1)),$$

the space of germs of holomorphic functions near A^2 valued in $H'(A^1)$. This is due to the fact that the topological tensor product, the way we are assuming it to be defined, does not commute with the inductive limit. For instance, when $A^1 = A^2 = K$, the identity mapping of $H(K)$ cannot be canonically identified with a germ of holomorphic function near K valued in the space of analytic functionals on K.

2. Definition of the hyper-differential operators.

Let Ω^i ($i = 1, 2$) be two open subsets of $\mathbf{C}^n$, K a compact subset of $\mathbf{C}^n$ such that

$$(2.1)\qquad K + \Omega^1 \subset \Omega^2.$$

Let then α and φ be arbitrary elements of $H'(K)$ and $H(\Omega^2)$ respectively. For arbitrary h in $H(\Omega^2)$ and w in Ω^1, we set

$$\langle \varphi(z)\,\alpha_{z-w}, h(z)\rangle = \langle \alpha_t, \varphi(t+w)\,h(t+w)\rangle. \tag{2.2}$$

That this makes sense follows easily from (2.1). Indeed, let Ω' be any relatively compact open subset of Ω^1. In view of (2.1), we can find an open neighborhood U of K such that $U+\Omega'\subset\Omega^2$. When w ranges over Ω', $t\mapsto\varphi(t+w)\,h(t+w)$ is a holomorphic function in U. Not only does the right hand side of (2.2) makes sense, but it is also a holomorphic function of w in Ω', and therefore in Ω^1, as Ω' is arbitrary. Furthermore, the linear map

$$h\mapsto(w\mapsto\langle \alpha_t, \varphi(t+w)\,h(t+w)\rangle),$$

from $H(\Omega^2)$ into $H(\Omega^1)$, is evidently continuous. In other words, $\varphi(z)\,\alpha_{z-w}$ as defined by (2.2) is an element of $H(\Omega^1)\,\hat{\otimes}\,H'(\Omega^2)$. Another assertion not difficult to check is that the bilinear map

$$(\alpha, \varphi)\rightarrow\varphi(z)\,\alpha_{z-w}, \tag{2.3}$$

from $H'(K)\times H(\Omega^2)$ into $H(\Omega^1)\,\hat{\otimes}\,H'(\Omega^2)$, is continuous. The universal property of the topological tensor product (*see* e. g. [5], Prop. 43.4) associates with (2.3) a continuous linear map

$$H'(K)\times H(\Omega^2)\rightarrow H(\Omega^1)\,\hat{\otimes}\,H'(\Omega^2)$$

for which we shall use the notation

$$G_t(z)\mapsto G_{z-w}(z). \tag{2.4}$$

By using the canonical isomorphisms (1.11), in particular

$$H(\Omega^1)\,\hat{\otimes}\,H'(\Omega^2)\cong L(H'(\Omega^1);\,H'(\Omega^2)),$$

we may regard $G_{z-w}(z)$ as the kernel associated with a continuous linear map

$$G:\ H'(\Omega^1)\rightarrow H'(\Omega^2).$$

DEFINITION 2.1. — *A continuous linear map* $G: H'(\Omega^1)\rightarrow H'(\Omega^2)$ *will be called a hyper*-differential operator *if there is a compact subset* K *of* $\mathbf{C}^n$ *satisfying* (2.1) *such that the kernel associated with* G *belongs to the image of the mapping* (2.4).

We shall sometimes refer to the mapping (2.4) as the « defining mapping »; we shall say that the compact set K *carries the translations* of the hyper-differential operators whose kernel lies in the image of (2.4). Observe that it is not clear whether the defining mapping (2.4) is injective, i. e., wheter the kernel $G_t(z) \in H'(K) \otimes H(\Omega^2)$ corresponding to some hyper-differential operator $G : H'(\Omega^1) \to H'(\Omega^2)$ is unique.

We give now a few very simple (but basic) examples. By δ, we denote the Dirac measure. In the first two examples, we take $K = \{0\}$; then (2.1) means $\Omega^1 \subset \Omega^2$.

EXAMPLE 2.1. — Suppose $K = \{0\}$ and $\Omega^1 \subset \Omega^2$. Take

$$G_t(z) = \delta_t \otimes 1(z).$$

Then G is the natural « *extension* mapping » from Ω^1 to Ω^2, the transpose of the restriction mapping from $H(\Omega^2)$ to $H(\Omega^1)$.

EXAMPLE 2.2. — Same assumption as in Example 2.1. Take $G_t(z) = \delta_t \otimes h(z)$, where h is any function belonging to $H(\Omega^2)$. Then G is the extension of analytic functionals from Ω^1 to Ω^2 followed by *multiplication* by h.

EXAMPLE 2.3. — Let now K be any compact set satisfying (2.1) and let α be any analytic functional in K; take $G_t(z) = \alpha_t \otimes 1(z)$. Then G is the convolution of analytic functionals in Ω^1 with α.

The hyper-differential operators of the kind considered in Examples 2.2 and 2.3 are the « building blocks » out of which all the hyper-differential operators are made. Let α^j $(j = 1, \ldots, N)$ be N analytic functionals in K, h^j an equal number of functions belonging to $H(\Omega^2)$. Take

$$G_t(z) = \sum_{j=1}^{N} \alpha_t^j \otimes h^j(z) \tag{2.5}$$

[in other words, $G_t(z)$ is a « typical » element of $H'(K) \otimes H(\Omega^2)$]. Then G is the operator from $H'(\Omega^1)$ to $H'(\Omega^2)$ defined by

$$G\beta = \sum_{j=1}^{N} h_j(z)\,(\alpha^j \star \beta). \tag{2.6}$$

As a matter of fact, if we allow the integer N to go to $+\infty$, formulae like (2.6) would define any hyper-differential operator $H'(\Omega^1) \to H'(\Omega^2)$ with translations carried by K. Indeed, any element $G_t(z)$ of $H'(K) \hat{\otimes} H(\Omega^2)$ admits a series representations of the kind (2.5) with $N = +\infty$, converging in the space $H'(K) \hat{\otimes} H(\Omega^2)$.

It is not difficult to give examples of continuous linear mappings $H'(\Omega^1) \to H'(\Omega^2)$ which are *not* hyper-differential operators : for instance, when $\Omega^1 = \Omega^2 = \mathbf{C}^n$, no such mapping is one if it has a finite dimensional image (*see* [6], coroll. 1 of Th. 23.2).

Another example of a continuous linear operator $H'(\Omega^1) \to H'(\Omega^2)$ which is not a hyper-differential operator is obtained as follows. Suppose that Ω^i is the disk in the complex plane centered at the origin and with radius R^i $(i = 1, 2)$ and that $R^1 \leqq R^2 \leqq \frac{3}{2} R^1$. Let h be a holomorphic function in Ω^1 which cannot be extended as a holomorphic function in Ω^2. Then multiplication of analytic functionals in Ω^1 by this function h, followed by extension to Ω^2, is *not* a hyper-differential operator (*see* [6], Th. 23.3). This last exemple, compared with Examples 2.1 and 2.2, shows that *the compose of two hyper-differential operators may very well not be a hyper-differential operator.*

Nevertheless, under favourable circumstances, it might happen that the compose of two hyper-differential operators is also one. Let us consider two additional subsets of $\mathbf{C}^n$, one open, U, one compact, C. We assume the following inclusion relations, in addition to (2.1) :

$$U - C \subset \Omega^2; \tag{2.7}$$

$$K + C + \Omega^1 \subset U. \tag{2.8}$$

Consider then any two kernels

$$F_s(z) \quad \text{in } H'(C) \hat{\otimes} H(U), \qquad G_t(z') \quad \text{in } H'(K) \hat{\otimes} H(\Omega^2).$$

Applying the mapping « restriction of holomorphic functions » from Ω^2 to $\Omega^1 + K$, we may view $G_t(z')$ as an element of $H'(K) \hat{\otimes} H(K + \Omega^1)$. Then the kernel $G_{z'-w}(z')$ is associated with a hyper-differential operator

$$G: \quad H'(\Omega^1) \to H'(\Omega^1 + K).$$

In view of (2.8), the kernel $F_{z-w}(z)$ is associated with a hyper-differential operator

$$F: \quad H'(\Omega^1 + K) \to H'(U).$$

We may therefore form the compose :

$$F \circ G: \quad H'(\Omega^1) \to H'(U),$$

which is a continuous linear map.

Proposition 2.1. — *Under the preceding hypotheses, the compose $F \circ G$ is a hyper-differential operator.*

Proof. — We begin by defining a quadrilinear mapping

$$Q: \quad H'(K) \times H(\Omega^2) \times H'(C) \times H(U) \to H'(K + C) \hat{\otimes} H(U)$$

by the formula

$$Q(\alpha, \varphi, \beta, f) = \langle \alpha_{t-s}, \varphi(z - s) \beta_s \rangle f(z), \tag{2.9}$$

meaning that, for an arbitrary function h in $H(K+C)$,

$$\langle Q(\alpha, \varphi, \beta, f), h\rangle = f(z)\langle \alpha_x \otimes \beta_y, \varphi(z-y)\, h(x+y)\rangle.$$

Note that this makes sense, since $\varphi(z-y)$ is a holomorphic function of (y, z) in an open neighborhood of $C \times U$ in view of (2.7), whereas $h(x+y)$ is a holomorphic function of (x, y) in $K \times C$. One also sees easily that Q is continuous. Once more the universal property of the topological tensor product comes to our help, and associates with Q a continuous bilinear map

$$\tilde{Q}: \quad (H'(K) \hat{\otimes} H(\Omega^2)) \times (H'(C) \hat{\otimes} H(U)) \to H'(K+C) \hat{\otimes} H(U).$$

We may write

$$\tilde{Q}(G_t(z'), F_s(z)) = \langle G_{t-s}(z-s), F_s(z)\rangle.$$

Next we take the image of $Q(G_t(z'), F_s(z))$ under the " defining mapping "

$$H'(K+C) \hat{\otimes} H(U) \to H(\Omega^1) \hat{\otimes} H'(U),$$

the analog of (2.4) with $K+C$ substituted for K and U for Ω^2; this image turns out to be

$$\langle G_{z'-w}(z'), F_{z-z'}(z)\rangle$$

and this is seen at once to be the kernel associated with $F \circ G$.

Q. E. D.

Remark 2.1. — The proof of Proposition 2.1 shows that the compact set $K+C$ carries the translations of the compose $F \circ G$.

3. Cauchy representation of analytic functionals in one variable Links with the Fourier-Borel transformation and with the Laplace transformation.

Let K be a compact subset of the complex plane, α any (local) analytic functional on K. Given any point ζ in the complement of K, $(\zeta - z)^{-1}$ is a holomorphic function of z in some open neighborhood of K and therefore defines a germ of holomorphic function near K, that is, an element of $H(K)$. We may form

$$(3.1) \qquad \langle \alpha_z, (2i\pi)^{-1}(\zeta - z)^{-1}\rangle.$$

It is convenient to look upon (3.1) as a function of ζ in the complement of K *with respect to the Riemann sphere* $\mathbf{S}^1$: this complement is open and (4.1) is a holomorphic function there, vanishing at infinity. We denote (4.1) by $\Phi\alpha(\zeta)$ and call it the *Cauchy representation* of α (sometimes also the *Fantappiè transform*). Let us denote by $H_0(\mathbf{S}^1 \setminus K)$ the space of holomorphic functions in $\mathbf{S}^1 \setminus K$ which are equal to zero at the point ∞. The following is wellknown :

THEOREM 3.1. — *The Cauchy representation* $\alpha \mapsto \Phi\alpha$ *is a linear bijection of* $H'(K)$ *onto* $H_0(\mathbf{S}^1 \setminus K)$.

As a matter of fact, Φ is an isomorphism for the topological-vector space structures if we equip $H_0(\mathbf{S}^1 \setminus K)$ with its natural topology, the one of uniform convergence on the compact subsets of $\mathbf{S}^1 \setminus K$.

The proof of Theorem 3.1 consists essentially in establishing the *inversion formula* for the Cauchy representation :

$$\langle \alpha, h \rangle = \int_c h(\zeta)\, \Phi\alpha(\zeta)\, d\zeta, \tag{3.2}$$

where h is an arbitrary (germ of) holomorphic function near K and c a certain homology class in $\mathbf{C}^1 \setminus K$. For simplicity let us suppose that the complement of K, as well as K itself are connected — although we could dispense with both these assumptions. Suppose that h has a representative, also denoted by h, in an open neighborhood U of K which we may take to be connected and simply connected. Then the homology class c has a representative in U which is an oriented cycle „ around "K, i.e., a simple closed rectifiable curve, oriented counterclockwise, whose „ inner region " contains K. The right hand side of (3.2) might then just be interpreted as the integral along this curve.

Let now Ω be an open subset of $\mathbf{C}^1$. A compact subset K of Ω is *holomorphically convex* with respect to Ω if no connected component of the complement of K in $\mathbf{C}^1$ is a relatively compact subset of Ω. Then K has the Runge property with respect to Ω (*cf.* p. 197). Of course, every compact subset of Ω is contained in some other compact subset of Ω which is holomorphically convex. If K is holomorphically convex, $H'(K)$ can be regarded as a subspace of $H'(\Omega)$ and obviously the latter is equal to the union of such subspaces where K ranges over the collection of all compact subsets of Ω which are holomorphically convex. By appling Theorem 3.1 to these subspaces $H'(K)$ we see that *the Cauchy representation is a linear bijection of* $H'(\Omega)$ *onto* $H_0(\mathbf{S}^1 \setminus \Omega)$, *the space of germs of holomorphic functions nears* $\mathbf{S}^1 \setminus \Omega$ *which vanish at infinity.*

Let now α be any analytic functional in $\mathbf{C}^n$ (for the moment, we may assume n to be any integer > 0). The *Fourier-Borel transform* of α is the function of λ in $\mathbf{C}^n$,

$$\hat{\alpha}(\lambda) = \langle \alpha_z, e^{\langle \lambda, z \rangle} \rangle. \tag{3.3}$$

This is easily seen (e. g., by the Cauchy-Riemann criterion) to be an *entire* function of λ. Suppose then that α is carried by a compact subset

K of $\mathbf{C}^n$, which means that to every $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that, for all h in $H(\mathbf{C}^n)$,

$$(3.4) \qquad |\langle \alpha, h \rangle| \leq C_\varepsilon \sup_{d(z, K) \leq \varepsilon} |h(z)|.$$

Taking $h(z) = \exp \langle \lambda, z \rangle$ in (3.4) yields at once

$$(3.5) \qquad |\hat{\alpha}(\lambda)| \leq C_\varepsilon \exp \{ I_K(\lambda) + \varepsilon |\lambda| \},$$

where we have set

$$(3.6) \qquad I_K(\lambda) = \sup_{z \in K} \operatorname{Re} \langle \lambda, z \rangle.$$

[Conversely, one can prove that if (3.5) holds for every $\varepsilon > 0$ and all λ in $\mathbf{C}^n$, the analytic functional α is carried by the *convex hull* of the compact set K. Estimate (3.5) implies that the entire function $\hat{\alpha}$ is of exponential type. Let us denote by exp $(\mathbf{C}^n)$ the space of entire functions in $\mathbf{C}^n$ of exponential type. We have (*see* for example [5], Theor. 22.2)

THEOREM 3.2. — *The Fourier-Borel transformation is a linear bijection of $H'(\mathbf{C}^n)$ onto* exp $(\mathbf{C}^n)$.

Let us now go back to the case $n = 1$, and let α be an analytic functional in $\mathbf{C}^1$ carried by a *convex compact* set K. For every $\zeta \notin K$, let us denote by ζ_K the orthogonal projection of ζ onto K, and by $L(\zeta)$ the half straight line in the complex plane, joining 0 to ∞ and passing through $(\bar{\zeta} - \bar{\zeta}_K)$. An easy argument, based on inequality (3.5), shows that $\hat{\alpha}(z) \exp(-z\zeta)$ is an integrable function with respect to the measure $|dz|$ over $L(\zeta)$ (*see* proof of Lemma 7.1). One calls the integral

$$(3.7) \qquad \mathcal{L}\hat{\alpha}(\zeta) = \int_{L(\zeta)} e^{-z\zeta} \hat{\alpha}(\zeta)\, dz$$

the *Laplace transform* of $\hat{\alpha}$. One ought to say that many a path of integration from 0 to ∞ can be substituted for $L(\zeta)$ — indeed, certain problems might require a different choice from ours. At any rate, $\mathcal{L}\hat{\alpha}(\zeta)$ is seen at once to be a holomorphic function of ζ in a neighborhood of ∞ and to vanish at ∞. As a matter of fact, an immediate computation shows that, in such a neighborhood of ∞,

$$(3.8) \qquad \Phi \alpha = \mathcal{L}\hat{\alpha}.$$

In other words, *the Cauchy representation of the analytic functional α can be viewed as an analytic continuation of the Laplace transform of its Fourier-Borel transform.*

4. Extension to polydomains.

Let $\mathbf{S}$ or $\mathbf{S}^1$ denote the Riemann sphere; as usual, the complex plane $\mathbf{C}^1$ is identified with the complement of the point at infinity ∞. Given any product $A = A_1 \times \ldots \times A_n$ of subsets A_j of $\mathbf{S}$, we set

$$\mathbf{S}^n \ominus A = (\mathbf{S} \setminus A_1) \times \ldots \times (\mathbf{S} \setminus A_n).$$

We shall suppose that each one of the sets A_j is either open or closed (which means compact). If A_j is closed, we denote by $H_0(\mathbf{S} \setminus A_j)$ the space of holomorphic functions in the open set $\mathbf{S} \setminus A_j$ which vanish at infinity, equipped with the topology of uniform convergence on the compact subsets of $\mathbf{S} \setminus A_j$. If A_j is open, we mean by $H_0(\mathbf{S} \setminus A_j)$ the space of germs of holomorphic functions near the compact set $\mathbf{S} \setminus A_j$ which moreover vanish at infinity; this space is provided with its "natural" inductive limit topology (*cf.* p. 197). With these meanings we set

$$(4.1) \qquad H_0(\mathbf{S}^n \ominus A) = H_0(\mathbf{S} \setminus A_1) \hat{\otimes} \ldots \hat{\otimes} H_0(\mathbf{S} \setminus A_n).$$

It is not difficult to describe the elements of $H_0(\mathbf{S}^n \ominus A)$ except insofar as the distinction between functions and germs of functions must be preserved. If we forget about it for a moment, we may say that the elements of $H_0(\mathbf{S}^n \ominus A)$ are the holomorphic functions of $z = (z_1, \ldots, z_n)$ in $\mathbf{S}^n \ominus A$ which vanish whenever anyone of the variables z_j takes the value ∞.

Let α be any (local) analytic functional on a product $K = K_1 \times \ldots \times K_n$ of n compact subsets of $\mathbf{C}^1$. We set, for any ζ in $\mathbf{S}^n \ominus K$,

$$\Phi \alpha(\zeta) = (2 i \pi)^{-n} \langle \alpha_z, (\zeta_1 - z_1)^{-1} \ldots (\zeta_n - z_n)^{-1} \rangle,$$

and refer to $\Phi \alpha$ as the Cauchy representation (or the Fantappiè transform) of α [*cf.* (3.1)]. We have here the analog of Theorem 3.1 :

THEOREM 4.1. — *The Cauchy representation is a linear bijection of* $H'(K)$ *onto* $H_0(\mathbf{S}^n \ominus K)$.

Here also, like for Theorem 3.1, the proof consists essentially in the establishing of the inversion formula, that is, of the analog of (3.2) (*see* [6], Appendix) :

$$(4.2) \qquad \langle \alpha, h \rangle = \int_c h(\zeta) \Phi \alpha(\zeta) \, d\zeta,$$

where $d\zeta = d\zeta_1 \wedge \ldots \wedge d\zeta_n$ and c is a homology class of degree n in $\mathbf{C}^n \ominus K$ $c = c_1 \times \ldots \times c_n$ with c_j a homology class in $\mathbf{C}^1 \setminus K_j$ suitably adapted to K_j $(1 \leq j \leq n)$ in the manner indicated in p. 203.

Theorem 4.1 has an immediate extension to the case where some of the K_j are open sets (*cf.* the remarks on p. 203).

Let now α be an analytic functional in $\mathbf{C}^n$, carried by a compact set of the kind $K = K_1 \times \ldots \times K_n$ with the K_j *convex.* Let $\zeta_j \notin K_j$ and $\zeta = (\zeta_1, \ldots, \zeta_n)$ thus belong to $\mathbf{S}^n \ominus K$. For each j, let ζ_j^0 denote the orthogonal projection of ζ_j onto the closed convex set K_j and call $L_j(\zeta_j)$ the half straight line in $\mathbf{C}^1$, joining 0 to ∞ and passing through $(\bar{\zeta}_j - \bar{\zeta}_j^0)$. Setting $L(\zeta) = L_1(\zeta_1) \times \ldots \times L_n(\zeta_n)$ and calling $\hat{\alpha}$ the Fourier-Borel transform of α, we may set, like in the case of a single variable :

$$\mathcal{L}\,\hat{\alpha}(\zeta) = \int_{L(\zeta)} e^{-\langle \zeta, \lambda \rangle}\,\hat{\alpha}(\lambda)\,d\lambda.$$

Then, when all the ζ_j are close enough to ∞ in $\mathbf{S}^1$,

(4.3) $$\Phi\,\alpha(\zeta) = \mathcal{L}\,\hat{\alpha}(\zeta) \quad [cf.\ (3.8)].$$

Here also we may view $\Phi\alpha$ as an analytic continuation of $\mathcal{L}\hat{\alpha}$.

5. An injectivity result.

We return now to the situation in Section 2, where we were dealing with two open sets Ω^i ($i = 1, 2$) and a compact set K, all in $\mathbf{C}^n$, satisfying (2.1), that is,

$$K + \Omega^1 \subset \Omega^2.$$

But now we assume that the Ω^i and K are products of n subsets of the complex plane :

(5.1) $$\Omega^i = \Omega_1^i \times \ldots \times \Omega_n^i \quad (i = 1, 2); \qquad K = K_1 \times \ldots \times K_n.$$

For the sake of simplicity, we shall make the following assumption

(5.2) Ω^i ($i = 1, 2$) *and K are connected.*

Of course (5.2) is equivalent with saying that all the Ω_j^i and K_j are connected. We also shall assume that :

(5.3) *For every $j = 1, \ldots, n$, the complement of K_j in $\mathbf{C}^1$ is connected and Ω_j^i ($i = 1, 2$) is simply connected.*

Assumption (5.3) implies that K and the Ω^i are Runge sets.

Let then $G_t(z)$ be an arbitrary kernel belonging to $H'(K) \hat{\otimes} H(\Omega^2)$. We perform the Cauchy representation of $G_t(z)$ with respect to the variable t. We denote it by $\Phi\,G(z, \tau)$; it is a function belonging to $H_0(\mathbf{S}^n \setminus K) \hat{\otimes} H(\Omega^2)$.

Let now $\Omega' = \Omega_1' \times \ldots \times \Omega_n'$ be an arbitrary polydomain whose closure is compact and contained in Ω^1. In view of (2.1), $K + \Omega'$ is a relatively compact subset of Ω^2. In view of (5.3), it is simply connected. For each

$j = 1, \ldots, n$, we select a simple closed rectifiable curve c_j in Ω^2 „ encircling " $K_j + \overline{\Omega}'_j$ and we set $c = c_1 \times \ldots \times c_n$.

LEMMA 5.1. — *Let β be any analytic functional in Ω^1 carried by Ω', h be any holomorphic function in Ω^2. We have*

$$(5.4) \qquad \langle G\beta, h \rangle = \int_c \langle \beta_w, \Phi\, G(\zeta, \zeta - w) \rangle h(\zeta)\, d\zeta.$$

Proof. — It follows at once from the inversion formula (4.2) that (5.4) holds when $G_t(z)$ belongs to $H'(K) \otimes H(\Omega^2)$, hence for any $G_t(z)$ in $H'(K) \hat{\otimes} H(\Omega^2)$ by continuity.

Formula (5.4) has a " dual " equivalent. Let ${}^tG : H(\Omega^2) \to H(\Omega^1)$ be the transpose of the continuous linear operator G. Then, for any holomorphic function h in Ω^2 and any point w in Ω',

$$(5.5) \qquad {}^tG\, h(w) = \int_c h(\zeta)\, \Phi\, G(\zeta, \zeta - w)\, d\zeta.$$

We recall that c is an n-cycle in $\Omega^2 \cap (S^n \ominus (K + \overline{\Omega}'))$ „ encircling " $K + \overline{\Omega}'$.

Now we may state and prove :

THEOREM 5.1. — *Suppose that Ω^i $(i = 1, 2)$ and K are product sets, like in (5.1), satisfying $K + \Omega^1 \subset \Omega^2$. Suppose moreover that (5.2) and (5.3) hold.*

We make the following additional hypothesis :

(I) *For each $j = 1, \ldots, n$, there is a point z_j of Ω^2_j such that*

$$K_j \subset z_j - \Omega^1_j.$$

Then the mapping (2.4) is injective.

We recall that (2.4) is the „ defining map " $G_t(z) \rightsquigarrow G_{z-w}(z)$ from $H'(K) \hat{\otimes} H(\Omega^2)$ into $H(\Omega^1) \hat{\otimes} H'(\Omega^2)$.

Proof. — Suppose $G_{z-w}(z) = 0$ or equivalently [by (1.11)] that the hyper-differential operator G defined by $G_t(z)$ vanishes; then its transpose ${}^tG$ vanishes also. We apply (5.5) with $\Omega' = D(w^0)$, any open polydisk centered at an arbitrary point $w^0 = (w^0_1, \ldots, w^0_n)$ of Ω^1 whose closure $\overline{D}(w^0)$ is compact and contained in Ω^1. For all h in $H(\Omega^2)$ and all w in $D(w^0)$,

$$(5.6) \qquad \int_c h(\zeta)\, \Phi\, G(\zeta, \zeta - w)\, d\zeta = 0.$$

We shall apply the following elementary (and well-known) one-dimensional lemma.

LEMMA 5.2. — *Let U be an open bounded subset of the complex plane, connected and simply connected. Let C be a compact subset of U and f a holomorphic function in $U \setminus C$ having the following property :*

(5.7) *Given any simple closed rectifiable curve in U encircling C and any holomorphic function h in U,*

$$\oint_c f(\zeta)\, h(\zeta)\, d\zeta = 0.$$

Then f can be extended as a holomorphic function in the whole of U.

With the help of lemma 5.2, we shall derive from (5.6) that $\Phi\; G(z, \tau) \equiv 0$ for z in Ω^2 and τ in $S^n \ominus K$. We reason by induction on the number n of variables. The result is trivial when $n = 0$. We assume therefore $n > 0$ and adopt the following notation : $z' = (z_1, \ldots, z_{n-1})$, similarly for w', $w^{0\prime}$, ζ', etc. Also

$$\Omega'^i = \Omega_1^i \times \ldots \times \Omega_{n-1}^i \quad (i = 1, 2), \qquad K' = K_1 \times \ldots \times K_{n-1},$$
$$c' = c_1 \times \ldots \times c_{n-1}, \qquad \ldots.$$

From (5.6), we derive

$$(5.8) \qquad \int_{c_n} \int_{c'} \Phi\; G(\zeta', \zeta_n, \zeta' - w', \zeta_n - w_n)\, h_1(\zeta')\, h_2(\zeta_n)\, d\zeta'\, d\zeta_n = 0$$

for all w in $D(w^0)$, all h_1 in $H(\Omega'^2)$, all h_2 in $H(\Omega_n^2)$. We set

$$f(\zeta_n, w_n) = \int_{c'} \Phi\; G(\zeta', \zeta_n, \zeta' - w', \zeta_n - w_n)\, h_1(\zeta')\, d\zeta',$$

so that (5.8) can be rewritten

$$\int_{c_n} f(\zeta_n, w_n)\, h_2(\zeta_n)\, d\zeta_n = 0.$$

We apply Lemma 5.2 with U an open bounded subset of Ω_n^2, connected and simply connected, containing $C = K_n + \bar{D}_n(w^0)$ [$D_n(w^0)$ is the projection of $D(w^0)$ on the plane of the n-th coordinate]. We derive from Lemma 5.2 that $f(\zeta_n, w_n)$ can be extended as a holomorphic function of ζ_n in the whole of U and therefore, in the whole of Ω_n^2. From the Cauchy formula applied to this extension of $f(\zeta_n, w_n)$, we see at once that it depends holomorphically on w_n in $D_n(w_n^0)$. As w_n^0 can be any point in Ω_n^1 we see that $f(\zeta_n, w_n)$ can be extended as a holomorphic function of (ζ_n, w_n) in the whole of $\Omega_n^2 \times \Omega_n^1$. At this point, we set $z_n = \zeta_n$, $\tau_n = \zeta_n - w_n$. We see that $f(z_n, z_n - \tau_n)$ can be extended as a holomorphic function of τ_n in $z_n - \Omega_n^1$. But in view of Condition (I), there is an open subset V of Ω_n^2 such that, for every z_n in V, $K_n \subset z_n - \Omega_n^1$. And

$$f(z_n, z_n - \tau_n) = \int_{c'} \Phi\, G(\zeta', z_n, \zeta' - w', \zeta_n)\, h_1(\zeta')\, d\zeta'$$

is a holomorphic function of (z_n, τ_n) in $\Omega_n^2 \times (\mathbf{S}^1 \setminus K_n)$, vanishing when $\tau_n = \infty$. In particular, for each z_n in V, $f(z_n, z_n - \tau_n)$ is an entire function of τ_n vanishing at ∞, in other words, vanishes identically. But then this must also be true for any z_n in Ω_n^2. This means that we have, whatever h_1 in $H(\Omega'^2)$, whatever $w^{0\prime}$ in Ω'^1 and w' in $D'(w^{0\prime})$ [this is the projection of $D(w^0)$ on the hyperplane $w_n = 0$], whatever z_n in Ω_n^2 and τ_n in the complex plane,

$$\int_{c'} \Phi\, G(\zeta', z_n, \zeta' - w', \tau_n)\, h_1(\zeta')\, d\zeta' = 0,$$

where c' is any $(n-1)$-cycle in Ω'^2 encircling $K' + \overline{D}'(w^{0\prime})$. The induction on n yields then the desired result, namely that $\Phi\, G(z, \tau)$ vanishes identically for z in Ω^2 and τ in $\mathbf{S}^n \ominus K$. Theorem 4.1 implies that the analytic functional with respect to t, $G_t(z)$, vanishes for all z in Ω^2.

Q. E. D.

6. The symbol of a hyper-differential operator.

We continue to deal with two open subsets Ω^i $(i = 1, 2)$ and a compact subset K of $\mathbf{C}^n$, like in Section 2; these sets satisfy (2.1). We do not assume here that they are products like in Section 5, but we do make the following assumptions :

(6.1) *The defining map* (2.4) *is injective;*

(6.2) *K is a Runge set.*

(6.2) allows us to identify $H'(K)$ with the space of analytic functionals in $\mathbf{C}^n$ which are carried by K.

Let then $G_t(z)$ be a kernel belonging to $H'(K) \hat{\otimes} H(\Omega^2)$,

$$G : \quad H'(\Omega^1) \to H'(\Omega^2)$$

the hyper-differential operator defined by $G_t(z)$.

DEFINITION 6.1. — *The Fourier-Borel transform of $G_t(z)$ will be called the symbol of the hyper-differential operator G.*

We shall sometimes denote by $\sigma\, G(z, \lambda)$ the symbol of G :

$$\sigma\, G(z, \lambda) = \langle G_t(z), e^{\langle \lambda, t \rangle} \rangle.$$

In virtue of theorem 3.2, the symbol of a hyper-differential operator like G is uniquely determined [recalling that (6.1) and (6.2) hold].

Examples. — If G is the hyper-differential operator in Example 2.1, its symbol is the function identically equal to one. If G is the operator in Example 2.2 its symbol is $h(z)$; if G is the operator in Example 2.3, its symbol is the Fourier-Borel transform of the analytic functional α, $\hat{\alpha}$. More generally, suppose that

$$G_t(z) = \sum_{j=1}^{N} \alpha_t^j \otimes h^j(z),$$

with $\alpha^j \in H'(K)$, $h^j \in H(\Omega^2)$. Then

$$\sigma\, G(z, \lambda) = \sum_{j=1}^{N} h^j(z)\, \hat{\alpha}(\lambda);$$

this remains valid, under appropriate conditions of convergence, when $N = +\infty$.

The symbol $\sigma\, G(z, \lambda)$ is a holomorphic function of (z, λ) in $\Omega^2 \times \mathbf{C}^n$; it is of exponential type with respect to λ. More precisely, to every $\varepsilon > 0$ and to every compact subset K' of Ω^2 there is a constant $C_\varepsilon(K') > 0$ such that, for all z in K' and all λ in C^n,

$$|\sigma\, G(z, \lambda)| \leq C_\varepsilon(K') \exp(I_K(\lambda) + \varepsilon\, |\lambda|) \tag{6.3}$$

[(*cf.* 3.5)].

It is natural, at this stage, to ask whether there is a formula giving the symbol of the product (i. e., the compose) of two pseudo-differential operators — when this product itself is a hyper-differential operator. In order to take a look at this question, we consider the situation introduced at the end of Section 2. This is the situation in which Proposition 2.1 is valid. Keeping exactly the same notation as there, we use the fact, pointed out at the end of the proof of Proposition 2.1, that the kernel associated with $F \circ G$ is

$$\langle G_{z'-w}(z'), F_{z-z'}(z) \rangle$$

or, equivalently, that the hyper-differential operator $F \circ G$ is defined by the kernel

$$\langle G_{t-s}(z - s), F_s(z) \rangle \in H'(K + C) \hat{\otimes} H(U)$$

[where K, C, U are submitted to the conditions (2.7), (2.8)]. It follows at once from this that the symbol of $F \circ G$ is

$$\begin{aligned} \sigma(F \circ G)(z, \lambda) &= \langle\langle G_{t-s}(z - s), F_s(z) \rangle, e^{\langle \lambda, t \rangle} \rangle \\ &= \langle \sigma\, G(z - s, \lambda)\, e^{\langle \lambda, s \rangle}, F_s(z) \rangle. \end{aligned} \tag{6.4}$$

In the last member, the bracket is the one of the duality between analytic functionals and holomorphic functions with respect to the variable s.

Observe that, in (6.4), z is the variable in U and s is the variable near C [keeping in mind that (2.7) holds].

Suppose for a moment that C is a compact polydisk centered at the origin. In view of (2.7), we may write, whatever z in U,

$$\sigma\, G(z-s, \lambda) = \sum_{p \in \mathbf{N}^n} \frac{(-1)^{|p|}}{p!} \sigma\, G^{(p,0)}(z, \lambda)\, s^p,$$

where we have used the notation $f^{(p,0)}(z, \lambda) = (\partial/\partial z)^p f(z, \lambda)$. We now use the notation $f^{(0,p)}(z, \lambda) = (\partial/\partial \lambda)^p f(z, \lambda)$, and observe that

$$\sigma\, F^{(0,p)}(z, \lambda) = \langle s^p e^{\langle \lambda, s \rangle}, F_s(z) \rangle.$$

We may then write

$$\sigma(F \circ G)(z, \lambda) = \sum_{p \in \mathbf{N}^n} (-1)^{|p|} \frac{1}{p!} \sigma\, G^{(p,0)}(z, \lambda)\, \sigma\, F^{(0,p)}(z, \lambda). \tag{6.5}$$

Let us show rapidly that the series in the right hand side of (6.5) converges normally in $U \times \mathbf{C}^n$. Suppose that the polyradius of C is $r = (r_1, \ldots, r_n)$, $0 < r_j < +\infty$ for every $j = 1, \ldots, n$. Let K' be an arbitrary compact subset of U; then $K - C = K + C$ is a compact subset of Ω^2. There are numbers $r'_j > r_j$ $(1 \leqq j \leqq n)$ such that, for some constant $M(\lambda) > 0$ and all z in K',

$$|\sigma\, G^{(p,0)}(z, \lambda)| \leqq M(\lambda)\, p!\, r'^{-p}. \tag{6.6}$$

It is evident that we may take $M(\lambda)$ to depend continuously on λ in $\mathbf{C}^n$. On the other hand, the analog of estimate (6.3) holds for F, with C in the place of K. Therefore, for every $\varepsilon > 0$ there is $M'_\varepsilon > 0$ such that, for all λ in $\mathbf{C}^n$, all z in K',

$$|\sigma\, F(z, \lambda)| \leqq M'_\varepsilon \exp((r_1 + \varepsilon)|\lambda_1| + \ldots + (r_n + \varepsilon)|\lambda_n|).$$

If we combine this with Cauchy's inequalities, we obtain, for all p in $\mathbf{N}^n$, λ in $\mathbf{C}^n$, all n-tuples $\rho = (\rho_1, \ldots, \rho_n)$ of numbers $\geqq 0$, all z in K',

$$|\sigma\, F^{(0,p)}(z, \lambda)| \leqq M'_\varepsilon\, p!\, \rho^{-p} \exp\left(\sum_{j=1}^{n} (r_j + \varepsilon)(|\lambda_j| + \rho_j)\right),$$

agreeing that $0^0 = 1$. Choosing $\rho_j = p_j/(r_j + \varepsilon)$ for each $j = 1, \ldots, n$ and applying Stirling's formula, we get

$$|\sigma\, F^{(0,p)}(z, \lambda)| \leqq M''_\varepsilon (r + \varepsilon)^p \exp\left(\sum_{j=1}^{n} (r_j + \varepsilon)|\lambda_j|\right), \tag{6.7}$$

where $r+\varepsilon=(r_1+\varepsilon, \ldots, r_n+\varepsilon)$. We require that ε be so small as to have $r_j+\varepsilon<r'_j$ for every $j=1, \ldots, n$. Combining then (6.6) and (6.7) yields the desired result.

Even when C is not a polydisk, there are cases where (6.5) retains some kind of meaning, precisely, an asymptotic one, as follows. Suppose that Ω^2 is a Runge open set. Then $\sigma\, G(z, \lambda)$ is the limit, uniformly on the compact subsets of $\Omega^2 \times \mathbf{C}^n$, of functions $g_\nu(z, \lambda)$ which are entire with respect to both z and λ. Then, of course, the series

$$\sum_p (-1)^{|p|} \frac{1}{p!}\, g_\nu^{(0,p)}(z, \lambda)\, \sigma\, F^{(0,p)}(z, \lambda)$$

converges normally in $U \times \mathbf{C}^n$. When $\nu \to +\infty$, its sum converges, uniformly on the compact subsets of $U \times \mathbf{C}^n$, to $\sigma(F \circ G)(z, \lambda)$. To express this fact, we write

$$\sigma(F \circ G)(z, \lambda) \sim \sum_{p \in \mathbf{N}^n} \frac{(-1)^{|p|}}{p!}\, \sigma\, G^{(p,0)}(z, \lambda)\, \sigma\, F^{(0,p)}(z, \lambda). \tag{6.8}$$

The similarity between (6.8) and the formula giving the symbol of a product of pseudo-differential operators in the real space (*see* e. g. [3], form. (4.3)'') hardly needs stressing.

7. Expression of a hyper-differential operator in terms of its symbol.

We continue to deal with the open sets Ω^i $(i=1, 2)$ and the compact set K. We have, as before, $K+\Omega^1 \subset \Omega^2$. Let $G: H'(\Omega^1) \to H'(\Omega^2)$ be a hyper-differential operator with kernel $G_{z-w}(z)$, where

$$G_t(z) \in H'(K) \hat{\otimes} H(\Omega^2),$$

and let $\sigma\, G(z, \lambda)$ denote the symbol of G, i. e., the Fourier-Borel transform of $G_t(z)$ with respect to t. Let β be any analytic functional carried by a compact subset K' of Ω^1. We are now going to show that, under suitable restrictions upon the choice of Ω^1, Ω^2 and K, it is possible to give a formula expressing the Cauchy representation of $G\beta$ in terms of $\sigma\, G(z, \lambda)$ and of $\hat{\beta}(\lambda)$, the Fourier-Borel transform of β. This will be analoguous to the formula expressing a pseudo-differential operator in real space in terms of its symbol.

We observe first that $G\beta$ is an analytic functional in Ω^2 carried by $K+K'$; this can be checked at once. Let then K'' be a compact subset of Ω^2. By combining (3.5) and (6.3), we see that, to every $\varepsilon>0$ there is a constant $C_\varepsilon>0$ such that, for all z in K'' and all λ in $\mathbf{C}^n$,

$$|\sigma\, G(z, \lambda)\, \hat{\beta}(\lambda)| \leq C_\varepsilon \exp(I_K(\lambda)+I_{K'}(\lambda)+\varepsilon|\lambda|). \tag{7.1}$$

Observe that

$$I_K(\lambda) + I_{K'}(\lambda) = I_{K+K'}(\lambda).$$

We make now the following assumption :

(7.2) $\Omega^i = \Omega^i_1 \times \ldots \times \Omega^i_n \quad (i = 1, 2), \qquad K = K_1 \times \ldots \times K_n,$

where the Ω^i_j and the K_j are convex sets.

Notice that (7.2) implies (5.1), (5.2), (5.3). Now we may and shall replace K' by its convex hull or, rather, assume that K' is convex.

Let $z = (z_1, \ldots, z_n)$ be an arbitrary point in $\mathbf{C}^n \ominus (K + K')$, that is, $z_j \notin K_j + K'_j$ whatever $j = 1, \ldots, n$. For each j, let z^0_j be the orthogonal projection of z_j on the convex compact set $K_j + K'_j$: z^0_j is the unique point in $K_j + K'_j$ such that $d(z_j, z^0_j) = d(z_j, K_j + K'_j)$. We denote by $L(z_j)$ the half straight line joining 0 to ∞ in the complex plane, and passing through $\bar{z}_j - \bar{z}^0_j$, and set

$$L(z) = L_1(z_1) \times \ldots \times L_n(z_n).$$

We shall need the following result :

Lemma 7.1. — *For each $j = 1, \ldots, n$, let K''_j be a compact subset of the complement of $K_j + K'_j$ with respect to Ω^2_j and let $K'' = K''_1 \times \ldots \times K''_n$. There exist constants $C, \delta_1, \ldots, \delta_n > 0$ such that, for all z in K'' and all λ in $L(z)$,*

(7.3) $| e^{-\langle \lambda, z \rangle} \sigma\, G(z, \lambda)\, \hat{\beta}(\lambda) | \leq C \exp(-\delta_1 |\lambda_1| - \ldots - \delta_n |\lambda_n|).$

Proof. — Fix arbitrarily λ on $L(z)$, which means that $\lambda_j = t_j(\bar{z}_j - \bar{z}^0_j)$ with $t_j \geq 0$ $(1 \leq j \leq n)$. Because of the compactness of $K + K'$ there is a point w in $K + K'$ such that

$$I_{K+K'}(\lambda) = \operatorname{Re}\langle \lambda, w \rangle.$$

In view of (7.1), the left hand side is $\leq C' \exp(-\operatorname{Re} < z - w, \lambda > + \varepsilon |\lambda|)$. But

$$\operatorname{Re}\langle z - w, \lambda \rangle = \sum_{j=1}^{n} t_j \operatorname{Re}(z_j - w_j)(\bar{z}_j - \bar{z}^0_j) \geq \sum_{j=1}^{n} t_j\, d(z_j, K_j + K'_j)^2.$$

Let m_j (resp. M_j) denote the minimum (resp. the maximum) of $d(z_j, K_j + K'_j)$ when z_j ranges over K''_j. We have

$$-\operatorname{Re}\langle z - w, \lambda \rangle + \varepsilon |\lambda| \leq -\sum_{j=1}^{n} t_j (m^2_j - \varepsilon M_j).$$

Choosing $\varepsilon < \inf_j (m^2_j / M_j)$ and observing that $|\lambda_j| \leq M_j t_j$ leads to the desired result.

An easy consequence of lemma 7.1 is that

$$\int_{L(z)} e^{-\langle z,\lambda\rangle}\sigma\, G(z, \lambda)\, \hat{\beta}(\lambda)\, d\lambda \tag{7.4}$$

is a holomorphic function of z in $\Omega^2 \ominus (K + K')$. Indeed, given any point Z of $\Omega^2 \ominus (K + K')$, for every z in a sufficiently small neighborhood of Z the integral (7.4) can be performed over $L(Z)$ instaed of $L(z)$: this follows at once from estimate (7.3). It suffices then to observe that the integrand in (7.4) is a holomorphic function of z in Ω^2.

THEOREM 7.1. — *Let* ζ *be an arbitrary point in* $\mathbf{S}^n \ominus (K + K')$ *and for each* $j = 1, \ldots, n$, *let* c_j *be any simple closed rectifiable curve in* Ω_j^2 *encircling* $K_j + K'_j$ *and such that* ζ_j *lies in the exterior of* c_j. *Set* $c = c_1 \times \ldots \times c_n$. *Then, given any analytic functional* β *carried by* K',

$$\Phi(G\beta)(\zeta) = (2i\pi)^{-2n} \int_c \left(\int_{L(z)} e^{-\langle\lambda, z\rangle}\sigma\, G(z, \lambda)\, \hat{\beta}(\lambda)\, d\lambda \right) \times \prod_{j=1}^{n} (\zeta_j - z_j)^{-1}\, dz. \tag{7.5}$$

Proof. — We observe that (7.4) is equal to

$$\left\langle G_t(z), \int_{L(z)} e^{-\langle\lambda, z-t\rangle} \hat{\beta}(\lambda)\, d\lambda \right\rangle$$

where $\langle\ ,\ \rangle$ is the bracket of the duality between holomorphic functions and analytic functionals in the variable t (z plays the role of a parameter). We may apply Lemma 7.1 with $\sigma G(z, \lambda)$ replaced by $\exp\langle\lambda, t\rangle$. In view of condition (7.1), this is permitted provided that t remains in K. For these t's, we obtain the analog of (7.3). But by continuity, a similar inequality will be valid with $\delta_j/2$ substituted for δ_j ($1 \leq j \leq n$) for all t in a suitable neighborhood of K. For the t's in such a neighborhood, we see that $\hat{\beta}(\lambda)\, e^{-\langle\lambda, z-t\rangle}$ is absolutely integrable over $L(z)$ with respect to $|d\lambda|$. We have

$$(2i\pi)^{-n} \int_{L(z)} e^{-\langle\lambda, z-t\rangle} \hat{\beta}(\lambda)\, d\lambda = \Phi\, \beta(z - t) = (2i\pi)^{-n} \left\langle \beta_w, \prod_{j=1}^{n} (z_j - t_j - w_j)^{-1} \right\rangle.$$

Thus we see that (7.4) is equal to

$$(2i\pi)^n \langle \beta_w, \Phi\, G(z, z - w) \rangle$$

and therefore the right hand side of (7.5) is equal to

$$\int_c \langle \beta_w, \Phi\, G(z, z-w) \rangle h(z)\, dz, \tag{7.6}$$

where

$$h(z) = (2i\pi)^{-n}(\zeta_1 - z_1)^{-1} \ldots (\zeta_n - z_n)^{-1}.$$

Let then A_j be the inner region determined by c_j and $A = A_1 \times \ldots \times A_n$. It is clear that $h(z)$ is a holomorphic function of z in some open neighborhood of $\bar{A}$ which we might assume of the form $U = U_1 \times \ldots \times U_n$ with U_j open connected and simply connected, and contained in Ω_j^2 $(1 \leq j \leq n)$. We may then apply Lemma 5.1 with U substituted for Ω^2 (this is seen at once to be permitted). We conclude that (7.6) is equal to $\langle G\beta, h \rangle$; but the latter is nothing else but $\Phi(G\beta)(\zeta)$.

Q. E. D.

REMARK 7.1. — It is not true, in general, that we have

$$\Phi(G\beta)(z) = \int_{L(z)} e^{-\langle\lambda, z\rangle} \sigma\, G(z, \lambda)\, \hat{\beta}(\lambda)\, d\lambda,$$

as seen on the example $G_t(z) = \delta_t \otimes h(z)$, where $h \in H(\Omega^2)$.

We keep the same hypotheses as before, in particular (7.2). One of the implications of theorem 7.1 is that the hyper-differential operators are precisely those operators which can be expressed by formulas such as (7.5). Let us be a little more precise concerning this fact.

Let $g(z, \lambda)$ be a holomorphic function of (z, λ) in $\Omega^2 \times C^n$ and suppose that, for every compact subset K'' of Ω^2 and every $\varepsilon > 0$ there is a constant $C > 0$ such that, for all z in K'' and all λ in C^n,

$$|g(z, \lambda)| \leq C \exp(I_K(\lambda) + \varepsilon |\lambda|). \tag{7.7}$$

We may then apply Lemma 7.1 with $g(z, \lambda)$ instaed of $\sigma\, G(z, \lambda)$ and, consequently, form

$$\Gamma\beta(\zeta) = (2i\pi)^{-2n} \int_c \left(\int_{L(z)} e^{-\langle\lambda, z\rangle} g(z, \lambda)\, \hat{\beta}(\lambda)\, d\lambda \right) \times \prod_{j=1}^{n} (\zeta_j - z_j)^{-1}\, dz. \tag{7.8}$$

Here the ingredients, ζ, c, β, ..., are the same as in formula (7.5). We see easily that $\beta \mapsto \Phi^{-1}\Gamma\beta$ is a continuous linear map of $H'(\Omega^1)$ into $H'(\Omega^2)$. Theorem 7.1 implies at once that this is a hyper-differential operator. Indeed, because of (7.7), we know that $g(z, \lambda)$ is the Fourier-Borel transform, with respect to the variable t, of an element $G_t(z)$ of $H'(K) \hat{\otimes} H(\Omega^2)$ [*cf.* the remark following (3.5) and (3.6)]. Now, $G_t(z)$ defines a hyper-differential operator $G : H'(\Omega^1) \to H'(\Omega^2)$ the symbol of G is of course $g(z, \lambda)$ and by Theorem 7.1, $\Phi(G\beta)$ is equal to $\Gamma\beta$ [defined in (7.8)].

8. The resolvents of systems of linear partial differential equations with analytic coefficients are hyper-differential operators.

As a justification for the introduction of hyper-differential operators in complex space, we are going to show its relevance to systems of linear partial differential equations with analytic coefficients. The systems under study will all be determined (i. e. contain as many equations as there are unknowns) and, for simplicity, the partial differential equations making up the systems will be of order one. It is convenient to deal with $(n+1)$ variables, denoted by $z = (z_1, \ldots, z_n)$ and t. The systems will be noncharacteristic in the t-direction. In fact, let us assume that they are of the form

$$L = \frac{\partial}{\partial t} - \sum_{j=1}^{n} \gamma_j(z, t) \frac{\partial}{\partial z_j} - \gamma_0(z, t),$$

where the $\gamma_j(z, t)$ $(0 \leq j \leq n)$ are matrix-valued holomorphic functions of (z, t) in some open neighborhood $\mathcal{O}$ of the origin in $\mathbf{C}^{n+1}$. That the system is determined means that the matrices $\gamma_j(z, t)$ are square, say with N rows and N columns $(N \geq 1)$, thus acting on N-vectors.

Let then Ω be a bounded open subset of $\mathbf{C}^n$, η a number > 0 such that the compact set $\{(z, t) \in \mathbf{C}^{n+1};\, z \in \overline{\Omega},\, |t| \leq \eta\}$ is contained in $\mathcal{O}$. Let then u_0 be an arbitrary analytic functional in Ω with values in $\mathbf{C}^N$, $\varphi(t)$ any holomorphic function of t, $|t| < \eta$, valued in $H'(\Omega; \mathbf{C}^N)$. We shall be interested in the Cauchy problem

$$(8.1) \qquad L\,u(t) = \varphi(t), \quad |t| < \delta < \eta; \qquad u(0) = u_0.$$

It has been proved in [6] (*see* Sections 12, 14), under the hypothesis that Ω is a Runge open set, that (8.1) admits one and only one solution $u(t)$: this is a holomorphic function of t, $|t| < \delta$, valued in the spaces of holomorphic mappings $\Omega_s \to \mathbf{C}^n$; here

$$\Omega_s = \{z \in \mathbf{C}^n;\, d(z, \Omega) < s\}$$

with s a number > 0, depending on δ, $0 < \delta < \eta$. As a matter of fact, s can be taken to be of the form $C\delta$ (C, a positive constant).

The solution $u(t)$ of (8.1) can be expressed as a function of the data φ and u_0 by means of the *resolvent* $\mathcal{R}(t, t')$ of the system L; $\mathcal{R}(t, t')$ is the (unique) solution of the problem

$$(8.2) \qquad L_t\,\mathcal{R}(t, t') = 0, \quad |t| < \delta, \quad |t - t'| < \delta; \qquad \mathcal{R}(t', t') = I,$$

where I is the identity mapping of the relevant spaces of analytic functionals (in the variables z; *see* [6]). The resolvent $\mathcal{R}(t, t')$ is a holomorphic function of (t, t') in a suitable neighborhood of (o, o) in $\mathbf{C}^2$, with values in the space of bounded linear operators $H'(\Omega_s; C^N) \to H'(\Omega_{s'}; C^N)$. Here s and s' are numbers > 0, depending on t and t' ([6], Section 8). We can view $\mathcal{R}(t, t')$ as an $N \times N$ matrix whose entries are bounded linear operators $H'(\Omega_s) \to H'(\Omega_{s'})$ depending holomorphically on (t, t') near (o, o).

It does not require much acumen to realize that information about the resolvent $\mathcal{R}(t, t')$ can be extremely useful in the study of problem (8.1). This is particularly true if one wants to find out whether this problem admits calssical solutions whenever the data $\varphi(t)$ and u_0 are themselves „ classical " (meaning by this that their values are carried by *real* compact sets and defined by functions or distributions with support in these sets). Such a study will be greatly facilitated if we can show that the resolvents $\mathcal{R}(t, t')$ are hyper-differential operators. For then we may rely on the symbolic calculus for such operators, and therefore deal with holomorphic functions instaed of having to deal with „ operators ".

In this section, we show that the resolvent $\mathcal{R}(t, t')$ of the system L is a hyper-differential operator. But we shall have to impose certain conditions on the sets carrying the analytic functionals : namely those conditions that enabled us to define the symbolic calculus of the previous sections — essentially, the open sets will have to be *convex polydomains*. For the sake of simplicity, we restrict ourselves to the case $t' = 0$; from there a translation $t \mapsto t - t'$, affecting all intervening functions of t (including the coefficients γ_j of the system L), enables us easily to settle the general case.

Like in (7.2), consider two convex open polydomains

$$\Omega^i = \Omega^i_1 \times \ldots \times \Omega^i_n \qquad (i = 1, 2)$$

and a convex polycompact set in $\mathbf{C}^n$. We assume that $K + \Omega^1 \subset \Omega^2$. Let then $G : H'(\Omega^1) \to H'(\Omega^2)$ be a hyper-differential operator, whose translations are carried by K, and let $\sigma\, G(z, \lambda)$ denote its symbol. We shall use the notation

$$\mathcal{E}(G\beta)(z) = (2\, i\, \pi)^{-n} \int_{L(z)} e^{-\langle \lambda, z \rangle} \sigma\, G(z, \lambda)\, \hat{\beta}(\lambda)\, d\lambda,$$

where β is an analytic functional carried by a convex compact subset of Ω^1 of the form $K' = K'_1 \times \ldots \times K'_n$, and z is a point in $\Omega^2 \cap (\mathbf{S}^n \ominus (K + K))$. Then formula (7.5) reads

$$(8.3) \qquad \Phi(G\beta)(\zeta) = (2\, i\, \pi)^{-n} \int_c \mathcal{E}(G\beta)(z) \prod_{j=1}^{n} (\zeta_j - z_j)^{-1}\, dz.$$

Here c is an n-cycle in Ω^2 appropriately positioned with respect to $K + K'$.

All this remains meaningful, and (7.5) remains valid, if β is an analytic functional valued in $\mathbf{C}^N$, i. e., an N-vector whose components are analytic functionals; then G will be a hyper-differential operator $H'(\Omega^1; \mathbf{C}^N) \to H'(\Omega^2; \mathbf{C}^N)$ or, equivalently, an $N \times N$ matrix whose entries are hyper-differential operators $H'(\Omega^1) \to H'(\Omega^2)$ (the translations of all these hyper-differential operators are carried by the same compact set, K); the symbol $\sigma\, G(z, \lambda)$ is a matrix-valued holomorphic function of (z, λ) in $\Omega^2 \times \mathbf{C}^n$, satisfying an estimate of the kind of (6.3).

Suppose now that G depends holomorphically on the complex variable t, for $|t| < \eta$; we write then $G(t)$, and $\sigma\, G(z, t, \lambda)$ for the symbol of $G(t)$. Let us assume, on the other hand, that the coefficients $\gamma_j(z, t)$ of L are holomorphic functions of (z, t) in some open neighborhood of the closure of the product set $\Omega^2 \times]-\eta, \eta[$, for instance by requiring that this closure be contained in our initial set $\mathcal{O}$. In view of this, we may make the differential operator L act on the vector-valued functions $\mathcal{C}(G(t)\beta)(z)$. It is easily seen, for instance on the formula giving the symbol of the compose of two hyper-differential operators, that

$$(8.4)\quad \mathcal{C}(L(G(t)\beta))(z) = L\{\mathcal{C}(G(t)\beta)(z)\} = (2i\pi)^{-n}\int_{L(z)} e^{-\langle\lambda, z\rangle}\left\{\frac{\partial}{\partial t} - \sum_{j=1}^{n}\gamma_j(z, t)\left(\frac{\partial}{\partial z_j} - \lambda_j\right) - \gamma_0(z, t)\right\}\sigma\, G(z, t, \lambda)\,\hat{\beta}(\lambda)\, d\lambda.$$

Keeping this in mind, we consider the following Cauchy problem :

$$(8.5)\qquad \left\{\frac{\partial}{\partial t} - \sum_{j=1}^{n}\gamma_j(z, t)\left(\frac{\partial}{\partial z_j} - \lambda_j\right) - \gamma_0(z, t)\right\} r(z, t, \lambda) = 0;$$

$$(8.6)\qquad r(z, 0, \lambda) = 1_N, \text{ the } N\times N \text{ identity matrix.}$$

Here the unknown $r(z, t, \lambda)$ is $N \times N$-matrix valued function; the coefficients $\gamma_j(z, t)$ act on it by matrix multiplication.

Suppose for a moment that we show that (8.5)-(8.6) has a (unique) solution $r(z, t, \lambda)$ which is a holomorphic function of (z, t, λ) for z in Ω^2, $|t| < \delta$ and all λ in $\mathbf{C}^n$. Furthermore, suppose that $r(z, t, \lambda)$ satisfies an estimate of the kind of (6.3). The considerations at the end of Section 7 tell us, then, that $r(z, t, \lambda)$ is the symbol of a hyper-differential operator $\mathcal{R}(t) : H'(\Omega^1; \mathbf{C}^N) \to H'(\Omega^2; \mathbf{C}^N)$, with translations carried by K. Now, obviously $\mathcal{R}(t)$ will satisfy (8.2) with $t' = 0$. The uniqueness of the solution to the latter problem demands therefore that $\mathcal{R}(t) = \mathcal{R}(t, 0)$.

Via the " change of unknown "

$$r(z, t, \lambda) = e^{\langle \lambda, z \rangle} \rho(z, t, \lambda),$$

Problem (8.5)-(8.6) gets transformed into :

$$(8.7) \quad \left(\frac{\partial}{\partial t} - \sum_{j=1}^{n} \gamma_j(z, t) \frac{\partial}{\partial z_j} - \gamma_0(z, t) \right) \rho(z, t, \lambda) = 0, \qquad |t| < \delta;$$

$$(8.8) \qquad \rho(z, 0, \lambda) = e^{-\langle \lambda, z \rangle} 1_N.$$

Let there be given n convex open subsets Ω_j of the complex plane. For any j and any $\delta > 0$, set

$$(\Omega_j)_\delta = \{ t \in \mathbf{C}^1 ; d(t, \Omega_j) < \delta \}.$$

Set also $(\Omega_j)_0 = \Omega_j$ and

$$\Omega(\delta) = (\Omega_1)_\delta \times \ldots \times (\Omega_n)_\delta, \qquad \delta \geqq 0.$$

We shall assume that the $\Omega(\delta)$, or equivalently, the Ω_j, are bounded. We select the number $\eta > 0$ in such a way that that, for some $\delta_0 > 0$, the closure of the set

$$(8.9) \qquad \{ (z, t) \in \mathbf{C}^{n+1} ; z \in \Omega(\delta_0), |t| < \eta \}$$

is contained in our initial open set $\mathcal{O}$. This has, as a consequence, that the differential operator L is defined, and that its coefficients are holomorphic in an open neighborhood of the closure of the set (8.9). For any number s, $0 \leqq s \leqq 1$, we call X_s the space of continuous mappings $\overline{\Omega(s\,\delta_0)} \to L(\mathbf{C}^N; \mathbf{C}^N)$ which are holomorphic in $\Omega(s\,\delta_0)$; X_s is turned into a Banach space by the maximum norm [over the compact set $\overline{\Omega(s\,\delta_0)}$]. In the framework of these spaces X_s, we may apply Ovciannikov theorem to problem (8.7)-(8.8) [6]. We reach the following conclusion :

(8.10) *For each λ in $\mathbf{C}^n$, problem* (8.7)-(8.8) *has a unique solution* $\rho(z, t, \lambda)$. *For every s, $0 \leqq s < 1$, $\rho(z, t, \lambda)$ is a holomorphic function of t in an open disk $|t| < \delta(s)$, valued in X_s. We may take $\delta(s) = \alpha(1 - s)$ with α a sufficiently small constant* > 0.

We use now the resolvent $\mathcal{R}^\natural(t, t')$ of Problem (8.7)-(8.8) relatively to our choice of the spaces X_s— or rather, we use $\mathcal{R}^\natural(t, 0)$: for $s < 1$ and $|t| < \delta(s)$, $\mathcal{R}^\natural(t, 0)$ is a bounded linear operator $X_1 \to X_s$ (*see* [6]). We have (*see* [6]).

$$\rho(z, t, \lambda) = \mathcal{R}^\natural(t, 0) (e^{-\langle \lambda, z \rangle} 1_N).$$

Since the restriction to $\overline{\Omega(\delta_0)}$ of $1_N \exp(-\langle \lambda, z\rangle)$ is an entire function of λ in $\mathbf{C}^n$ with values in X_1, we derive :

(8.11) *For each s, $0 \leqq s < 1$, and for $|t| < \delta(s)$, the solution $\rho(z, t, \lambda)$ is an entire function of λ in $\mathbf{C}^n$, valued in X_s.*

By Hartog's theorem, we see that, for $0 \leqq s < 1$, $\rho(z, t, \lambda)$ is a holomorphic function of (z, t, λ) in the region

$$(8.12) \qquad z \text{ in } \Omega(s\,\delta_0), \quad |t| < \delta(s), \quad \lambda \text{ in } \mathbf{C}^n.$$

All these properties also hold for the solution

$$r(z, t, \lambda) = e^{\langle \lambda, z\rangle}\rho(z, t, \lambda)$$

of (8.5)-(8.6). But $r(z, t, \lambda)$ satisfies an inequality which is very important in the present context, and which does not hold for $\rho(z, t, \lambda)$.

Let us introduce the following notation :

$$A(t) = \sum_{j=1}^{n} \gamma_j(z, t)\frac{\partial}{\partial z_j} + \gamma_0(z, t),$$

$$B(t, \lambda) = \sum_{j=1}^{n} \gamma_j(z, t)\,\lambda_j.$$

We view $r(z, t, \lambda)$ as a function of t, valued in the spaces X_s (with respect to the variable z) and depending on the parameter λ. For the sake of brevity, we denote by $r(t, \lambda)$ this function. We regard problem (8.5)-(8.6) as a Cauchy problem for an evolution equation (which it is!) and rewrite it

$$(8.13) \qquad (d/dt)\, r(t, \lambda) = A(t)\, r(t, \lambda) - B(t, \lambda)\, r(t, \lambda), \qquad |t| < \delta;$$

$$(8.14) \qquad r(0, \lambda) = 1_N.$$

We observe that $B(t, \lambda)$ is a holomorphic function of t (and a linear one of λ) valued in the Banach space of bounded linear operators $X_s \to X_s$ for any s, $0 \leqq s \leqq 1$; the norm of $B(t, \lambda)$ in this Banach space is $\leqq B|\lambda|$, with B a constant > 0, independent of t, $|t| < \eta$, and of s, $0 \leqq s \leqq 1$. As for $A(t)$ it is a bounded linear operator (depending holomorphically on t) from X_s into $X_{s'}$ whatever s, s' such that $0 \leqq s' < s \leqq 1$; its norm in $L(X_s; X_{s'})$ is $\leqq C(s - s')^{-1}$, where C is a constant > 0 independent of t and of s, s'. The latter property is an easy consequence of our choice of the spaces X_s and of Cauchy's inequalities.

Fix λ arbitrarily in $\mathbf{C}^n$. We define recursively a sequence $\{\varphi_k\}$ ($k = 0, 1, \ldots$) of holomorphic functions of t, $|t| < \eta$, valued in X_s for any s, $0 \leq s < 1$, in the following manner :

$$(8.15) \qquad \varphi_0 = 1_N;$$

$$(8.16) \qquad \varphi'_{k+1} = A(t)\varphi_k - B(t, \lambda)\varphi_k, \quad |t| < \eta; \qquad \varphi_{k+1}(0) = 0$$

for $k = 0, 1, \ldots$. Here φ' stands for the derivative of φ with respect to t. Let us write $\|\ \|_s$ for the norm of X_s. The properties of $A(t)$ and $B(t, \lambda)$ are such that

$$(8.17) \quad \|\varphi_{k+1}(t)\|_s \leq \frac{C}{\varepsilon}\int_0^t \|\varphi_k(t')\|_{s+\varepsilon}|dt'| + B|\lambda|\int_0^t \|\varphi_k(t')\|_s |dt'|,$$

where ε is any number such that $0 < \varepsilon < 1 - s$, and where the integrations are performed over the straight line segment joining 0 to t. Let us set

$$M_k(s) = \sup_{|t|<\eta} (\|\varphi_k(t)\|_s |t|^{-k}).$$

Our contention will be that, for all s, $0 \leq s < 1$, and all $k = 0, 1, \ldots$,

$$(8.18) \qquad M_k(s) \leq \left(\frac{Ce}{1-s} + \frac{e}{k}B|\lambda|\right)^k.$$

We reason by induction on k. Because of (8.15), (8.18) is true when $k = 0$. Next we apply (8.17) and obtain

$$M_{k+1}(s) \leq \frac{C}{\varepsilon}\frac{1}{k+1}M_k(s+\varepsilon) + \frac{B}{k+1}|\lambda| M_k(s).$$

We take $\varepsilon = (1-s)/(k+1)$ and apply the induction hypothesis

$$M_{k+1}(s) \leq \frac{C}{1-s}\left(\left(1+\frac{1}{k}\right)\frac{Ce}{1-s} + \frac{e}{k}B|\lambda|\right)^k + \frac{B|\lambda|}{k+1}\left(\frac{Ce}{1-s} + \frac{e}{k}B|\lambda|\right)^k$$

$$= \frac{C}{1-s}\left(1+\frac{1}{k}\right)^k\left(\frac{Ce}{1-s} + \frac{e}{k+1}B|\lambda|\right)^k$$

$$+ \frac{B|\lambda|}{k+1}\left(1+\frac{1}{k}\right)^k\left(\frac{Ce}{1-s}\left(1+\frac{1}{k}\right)^{-1} + \frac{Be|\lambda|}{k+1}\right).$$

Observing that $\left(1+\frac{1}{k}\right)^k \leq e$, we obtain at once (8.18).

Using the fact that $(a+b)^k \leq 2^k(a^k + b^k)$ ($a, b \geq 0$), we derive from (8.18), that, for all t, $|t| < \eta$,

$$\|\varphi_k(t)\|_s \leq \left(\frac{2Ce}{1-s}|t|\right)^k + \left(\frac{e}{k}\right)^k (2B|t|.|\lambda|)^k.$$

By Stirling's formula, we know that for a suitable constant M_0,

$$\left(\frac{e}{k}\right)^k \leq M_0 (k!)^{-1}.$$

Finally, we see that, for $|t| < (2Ce)^{-1}(1-s)$,

$$(8.19) \qquad \sum_{k=0}^{+\infty} \| \varphi_k(t) \|_s \leq \left\{ 1 - \frac{2Ce}{1-s} |t| \right\}^{-1} + M_0 \exp(2B|t||\lambda|).$$

But from (8.15), (8.16), we derive that the series

$$(8.20) \qquad \sum_{k=0}^{+\infty} \varphi_k(t),$$

which converges absolutely for $|t| < (2Ce)^{-1}(1-s)$, must be, for these values of t, a solution of problem (8.13)-(8.14). The uniqueness of the solution to this problem demands that the sum of (8.20) be equal to $r(t, \lambda)$. Therefore, for the same values of t, we reach the conclusion [*cf.* (8.19)] that

$$(8.21) \qquad \| r(t, \lambda) \|_s \leq \left\{ 1 - \frac{2Ce}{1-s} |t| \right\}^{-1} + M_0 \, e^{2B|t|.|\lambda|}.$$

This estimate is essentially what we wanted. Take

$$\Omega^1 = \Omega(0) = \Omega_1 \times \ldots \times \Omega_n, \qquad \Omega^2 = \Omega(s\,\delta_0)$$

for any s, $0 < s < 1$. In view of (8.11), we know that $r(z, t, \lambda)$ is a holomorphic function of (z, t, λ) for z in Ω^2, $|t| < \delta(s)$ and λ in $\mathbf{C}^n$, valued in the space of $N \times N$ matrices. Select then $\varepsilon > 0$ small enough so that, if

$$(8.22) \qquad K = \{ z \in \mathbf{C}^n;\ |z_j| \leq \varepsilon,\ j = 1, \ldots, n \},$$

then (2.1) holds, i. e., $K + \Omega^1 \subset \Omega^2$ (it is clear that it suffices to require $\varepsilon < s\,\delta_0$). Imposing then simultaneously the conditions

$$(8.23) \qquad |t| < (2Ce)^{-1}(1-s), \qquad |t| < \delta(s), \qquad 2B|t| < \varepsilon,$$

we derive from (8.21) that, for all z in Ω^2 and all λ in $\mathbf{C}^n$,

$$|r(z, t, \lambda)| \leq C(t) \exp(I_K(\lambda)).$$

Then the considerations at the end of Section 7 show that $r(t, z, \lambda)$ is the symbol of a hyper-differential operator $H'(\Omega^1; \mathbf{C}^N) \to H'(\Omega^2; \mathbf{C}^N)$ whose translations are carried by the compact polydisk K defined in (8.22), and which depends holomorphically on t in the open disk (8.23). In view of what, we have said earlier, this hyper-differential operator must be equal to the resolvent $\mathcal{R}(t, 0)$.

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