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## On logarithmic derivatives

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## ON LOGARITHMIC DERIVATIVES

BY

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### 1. Introduction.

Let  $C$  be a ring, always commutative with identity and of prime characteristic  $p > 0$ . Let  $C^*$  denote the group of invertible elements of  $C$ . Given a derivation  $\partial$  on  $C$ , the mapping

$$\delta_0 : C^* \rightarrow C^+$$

defined by  $\delta_0(u) = (\partial u)/u$  is a group-homomorphism. Now assume  $\partial$  satisfies a polynomial

$$X = \alpha_0 t + \alpha_1 t^p + \dots + \alpha_i t^{p^i} + \dots + \alpha_n t^{p^n}$$

with coefficients in the ring  $A = \text{kernel } \partial$ . For any  $c$  in  $C$ , let  $Lc$  denote the map  $C \rightarrow C$  produced by multiplication by  $c$ . From the formula

$$(\partial + Lc)^p = \partial^p + L(\partial^{p-1}c + c^p) \quad ([3], \text{ p. } 201),$$

it is easily seen that

$$X(\partial + Lc) = L(\delta_1 c),$$

where

$$\delta_1(c) = \sum_{i=0}^n \alpha_i ([\partial^{p^i-1}c] + [\partial^{p^i-1}c]^p + \dots + [\partial^{p^i-j-1}c]^{p^j} + \dots + c^{p^i})$$

is an element in  $A$ . It is also immediately clear that

$$\delta_1 : C^+ \rightarrow A^+$$

is again a group-homomorphism. Let  $u$  be an element of  $C^*$ . Then

$$\partial + L(\delta_0 u) = (Lu)^{-1} \partial (Lu),$$

and so

$$X(\partial + L(\delta_0 u)) = (Lu)^{-1} X(\partial) (Lu) = 0.$$

This means given  $\partial$  and  $X$ , we have a complex :

$$0 \rightarrow A^* \xrightarrow{\varepsilon} C^* \xrightarrow{\partial_0} C^+ \xrightarrow{\partial_1} A^+ \rightarrow 0.$$

When  $C$  is a finite dimensional field extension over  $A$  and  $X$  is the characteristic polynomial for  $\partial$ , a theorem of N. JACOBSON ([7], theorem 15) states that the kernel of  $\partial_1$  coincides with the image of  $\partial_0$ .

The purpose of this paper is to describe, for a general commutative ring  $C$ , the group  $(\text{kernel } \partial_1)/(\text{image } \partial_0)$  in terms of classes of rank one projective  $A$ -modules which are split by  $C$ . If  $C$  is a noetherian integrally closed domain, a description is also given in terms of divisor classes of  $A$  which become principal in  $C$ . These are done in the next section. In the final section, some examples are given.

## 2. The rank one projective class group.

LEMMA 2.1. — *Let  $\mathfrak{g}$  be a set of derivations on a semi-local ring  $C$  of prime characteristic  $p > 0$ , and let  $A$  denote the kernel*

$$\{x \in C \mid \partial x = 0 \text{ for all } \partial \in \mathfrak{g}\}$$

*of  $\mathfrak{g}$ . Assume  $C$  is a finitely generated projective  $A$ -module and  $\text{Hom}_A(C, C) = C[\mathfrak{g}]$ . Then both  $C$  and  $A$  are finite ring direct sums of indecomposable semi-local rings*

$$C = C_1 + \dots + C_m, \quad A = A_1 + \dots + A_m;$$

*and for each  $i$ ,*

$$C_i \cong A_i[t_1, \dots, t_r]/(t_1^p - a_1, \dots, t_r^p - a_r),$$

*where  $a_1, \dots, a_r$  are in  $A_i$ ,  $t_1, \dots, t_r$  are indeterminates, and  $r$  depends on  $i$ .*

*Proof.* — Given a prime ideal  $q$  in  $A$ ,  $\mathfrak{Q} = \{x \in C \mid x^p \in q\}$  is a prime in  $C$ , and  $\mathfrak{Q} \cap A = q$ . If  $q$  is maximal, so is  $\mathfrak{Q}$ , hence  $A$  must be semi-local. Let  $e$  be any idempotent in  $C$ . We have  $\partial e = \partial e^p = p(\partial e) e^{p-1}$  is zero. This shows  $e$  is in  $A$ . The ring  $C$  being semi-local contains no more than finitely many indecomposable idempotents  $\{e_1, \dots, e_m\}$ . Put  $C_i = Ce_i$  and  $A_i = Ae_i$ . We have

$$C = C_1 + \dots + C_m, \quad A = A_1 + \dots + A_m.$$

Let  $N$  denote the radical of  $A_i$ , and put  $\bar{A} = A_i/N$ ,  $\bar{C} = C_i/NC_i$ . Of course  $\bar{A}$  is a finite direct sum  $\sum_j F_j$  of fields. Accordingly  $\bar{C}$  decomposes into a direct sum  $\sum_j R_j$ , where  $R_j$  is a finite dimensional

local  $F_j$ -algebra. Now  $C_i$  is a finitely generated projective module over a semi-local ring  $A_i$  with connected spectrum, so must be free ([1], p. 143). This shows the dimension of  $R_j$  over  $F_j$  is equal to the rank of  $C_i$  over  $A_i$  and hence is independent of  $j$ . If we denote by  $\bar{\partial}$  the derivation on  $R_j$  induced by  $\partial|_{C_i}$ , and by  $\bar{\mathfrak{g}}$  the set  $\{\bar{\partial} \mid \partial \in \mathfrak{g}\}$ , then  $\text{Hom}_{F_j}(R_j, R_j) = R_j[\bar{\mathfrak{g}}]$  because

$$\bar{A} \otimes_{A_i} \text{Hom}_{A_i}(C_i, C_i) = \text{Hom}_{\bar{A}}(\bar{C}, \bar{C}).$$

Thus no non-trivial ideal of  $R_j$  can be stable under  $\bar{\mathfrak{g}}$ , the structure of  $R_j$  is therefore known ([9], corollary 2.8) :

$$R_j \cong F_j[t_1, \dots, t_r]/(t_1^p - f_1, \dots, t_r^p - f_r),$$

where  $f_1, \dots, f_r$  are elements of  $F_j$ ,  $t_1, \dots, t_r$  are indeterminates. But  $r$  is independent of  $j$ , so

$$\bar{C} = \sum R_j \cong \bar{A}[t_1, \dots, t_r]/(t_1^p - \bar{a}_1, \dots, t_r^p - \bar{a}_r) \quad (\bar{a}_i \in \bar{A}).$$

By [1], p. 105, this shows  $C_i$  is isomorphic to

$$A_i[t_1, \dots, t_r]/(t_1^p - a_1, \dots, t_r^p - a_r)$$

for some  $a_1, \dots, a_r$  in  $A_i$  as desired.

LEMMA 2.2. — *Let  $A$  be a commutative ring of prime characteristic  $p > 0$ , let*

$$C = A[t_0, \dots, t_n]/(t_0^p - a_0, \dots, t_n^p - a_n),$$

where  $a_0, \dots, a_n$  are elements of  $A$  and  $t_0, \dots, t_n$  are indeterminates. Assume  $\partial$  is an  $A$ -derivation on  $C$  such that  $\text{Hom}_A(C, C) = C[\partial]$ . Then the characteristic polynomial of  $\partial$  is of the form

$$\alpha_0 t + \alpha_1 t^p + \dots + \alpha_i t^{p^i} + \dots + \alpha_n t^{p^n} + t^{p^{n+1}} \quad (\alpha_i \in A).$$

*Proof.* — Let  $\partial_i = \frac{\partial}{\partial t_i}$  be the  $A$ -derivation on  $C$  given by  $\partial_i t_j = \delta_{ij}$  (the Kronecker delta function). So

$$\partial^{p^i} = b_{i0} \partial_0 + \dots + b_{in} \partial_n, \quad b_{ij} = \partial^{p^i}(t_j),$$

because  $\partial^{p^i}$  as a derivation is completely determined by its actions on the  $t_j$ 's. Now from  $\text{Hom}_A(C, C) = C[\partial]$ , we know  $\{\partial^i \mid 0 \leq i < p^{n+1}\}$  form a linearly independent  $C$ -basis for  $\text{Hom}_A(C, C)$ . (Notice that  $\partial$  as an  $A$ -endomorphism on the free  $A$ -module  $C$  of rank  $p^{n+1}$  has a characteristic polynomial of degree  $p^{n+1}$ . Therefore  $\text{Hom}_A(C, C) = C[\partial]$  implies that every  $A$ -endomorphism on  $C$  is a  $C$ -linear combination in

$\{\partial^i \mid 0 \leq i < p^{n+1}\}$ . But  $\text{Hom}_A(C, C)$  is a free  $C$ -module of rank  $p^{n+1}$ ,  $\{\partial^i \mid 0 \leq i < p^{n+1}\}$  must be  $C$ -linearly independent.) So

$$\partial_i = c_{i0} \partial + c_{i1} \partial^p + \dots + c_{in} \partial^{p^n} + \sum c'_{ij} \partial^j \quad (c_{ij}, c'_{ij} \in C),$$

where the summation runs through all  $j$ ,  $0 < j < p^{n+1}$  and  $j$  is not a power of  $p$ . So we have the matrix equation

$$\begin{pmatrix} \partial \\ \partial^p \\ \vdots \\ \partial^{p^n} \end{pmatrix} = \begin{pmatrix} b_{00} & \dots & b_{0n} \\ \vdots & & \vdots \\ b_{n0} & \dots & b_{nn} \end{pmatrix} \begin{pmatrix} c_{00} & \dots & c_{0n} \\ \vdots & & \vdots \\ c_{n0} & \dots & c_{nn} \end{pmatrix} \begin{pmatrix} c'_{ij} \end{pmatrix} \begin{pmatrix} \partial \\ \partial^p \\ \vdots \\ \partial^{p^n} \end{pmatrix}.$$

The linear independency of  $\{\partial^i \mid 0 \leq i < p^{n+1}\}$  therefore asserts that  $(b_{ij}) (c_{ij})$  is the identity  $n+1$  by  $n+1$  matrix and  $(b_{ij}) (c'_{ij})$  is a zero matrix. This shows  $(c'_{ij}) = (c_{ij}) (b_{ij}) (c'_{ij})$  is a zero matrix. In other words,

$$\partial_i = c_{i0} \partial + c_{i1} \partial^p + \dots + c_{in} \partial^{p^n} \quad \text{for all } i.$$

From  $\partial^{p^{n+1}} = b_{n+1,0} \partial_0 + \dots + b_{n+1,n} \partial_n$ , we see that  $\partial$  satisfies a polynomial

$$\alpha_0 t + \alpha_1 t^p + \dots + \alpha_i t^{p^i} + \dots + \alpha_n t^{p^n} + t^{p^{n+1}}.$$

That this polynomial must coincide with the characteristic polynomial of  $\partial$  follows from the fact that  $\{\partial^i \mid 0 \leq i < p^{n+1}\}$  are linearly independent over  $C$ . This completes the proof of the lemma.

**REMARK 2.3.** — Derivations satisfying the hypothesis  $\text{Hom}_A(C, C) = C[\partial]$  always exist. For example, let  $\partial$  be given by  $\partial t_0 = 1$  and  $\partial t_i = (t_0 \dots t_{i-1})^{p-1}$  for all  $i > 0$ . It is easy to verify that the characteristic polynomial of this derivation is just  $t^{p^{n+1}}$ .

**THEOREM 2.4.** — Let  $\partial$  be a derivation on a ring  $C$  of prime characteristic  $p > 0$  with  $A$  as its kernel. Assume  $C$  is a finitely generated projective  $A$ -module of rank  $r$  and  $\text{Hom}_A(C, C) = C[\partial]$ . Then  $\partial$  satisfies a polynomial  $X = \alpha_0 t + \alpha_1 t^p + \dots + \alpha_{n-1} t^{p^{n-1}} + t^{p^n}$  with  $\alpha_i$  in  $A$  and  $r = p^n$ . Moreover  $XC[t] = \{f \in C[t] \mid f(\partial) = 0\}$ .

*Proof.* — Given a maximal ideal  $q$  in  $A$ , let  $Q$  denote the maximal ideal  $\{x \in C \mid x^p \in q\}$  in  $C$ . It is clear that  $C_Q = C \otimes_A A_q$ . So  $\text{Hom}_{A_q}(C_Q, C_Q) = A_q \otimes_A \text{Hom}_A(C, C) = C_Q[\partial]$ . Hence by lemma 2.1  $r = p^n$  for some  $n$ . Let  $M$  be the  $A$ -submodule of  $\text{Hom}_A(C, C)$  generated by  $\partial^{p^i}$ ,  $i = 0, 1, \dots, n$ , and denote by  $M'$  the  $A$ -submodule of  $M$  generated by  $\partial^{p^i}$ ,  $i = 0, \dots, n-1$ . In view of [1] (p. 112, cor. 1)

to show the inclusion map  $M' \rightarrow M$  is onto it suffices to show at each maximal ideal  $q$  in  $A$  the corresponding map  $M'_q \rightarrow M_q$  is onto which according to lemma 2.2 is indeed the case. So there is a polynomial  $X = \alpha_0 t + \alpha_1 t^p + \dots + \alpha_{n-1} t^{p^{n-1}} + t^{p^n}$ , with  $\alpha_i$  in  $A$  and  $X(\partial) = 0$ . Given  $f \in C[t]$ ,  $f(\partial) = 0$ , we may write  $f = gX + h$ , with  $g, h \in C[t]$  and degree  $h < p^n$ . So  $h(\partial) = 0$ . Since  $\{\partial^i \mid 0 \leq i < p^n\}$  is linearly independent over  $C_Q$  at every maximal ideal  $Q$  in  $C$ , all coefficients of  $h$  must vanish because they vanish locally. So  $f = gX$ . This completes the proof of the theorem.

**COROLLARY 2.5.** — *Let  $\partial$  be a derivation on a ring  $C$  of prime characteristic  $p > 0$  with  $A$  as its kernel. Assume  $C$  is a finitely generated projective  $A$ -module and  $\text{Hom}_A(C, C) = C[\partial]$ . Then*

$$\{f \in C[t] \mid f(\partial) = 0\} = XC[t]$$

for some  $X(t) = \alpha_0 t + \alpha_1 t^p + \dots + \alpha_i t^{p^i} + \dots + \alpha_n t^{p^n}$  with  $\alpha_i \in A$  and  $\alpha_n$  a non-zero idempotent.

*Proof.* — Since  $C$  is finitely generated and projective as  $A$ -module, the map  $\rho : q \rightarrow (\text{rank of } C_q \text{ over } A_q)$  is locally constant on  $\Omega = \text{Spec } A$ . For any positive integer  $r_i$  write  $\Omega_i = \{q \in \Omega \mid \rho(q) = r_i\}$ . So  $\Omega_i$  is both open and closed in  $\Omega$  and we have a finite disjoint union  $\Omega = \bigcup \Omega_i$  because  $\Omega$  is quasi-compact. If  $\tilde{A} = (\Omega, \tilde{A})$  is the sheaf of local rings associated to  $A$  and  $\tilde{A}_i = \tilde{A} \mid_{\Omega_i}$ , then  $A = \tilde{A}(\Omega)$  decomposes into a finite ring direct sum  $\bigoplus \tilde{A}_i(\Omega_i)$ . So  $A = \bigoplus Ae_i$  and  $C = \bigoplus Ce_i$  where  $e_i$  is the identity element of  $\tilde{A}_i(\Omega_i)$ . Since  $Ce_i$  is a finitely generated projective  $Ae_i$ -module of finite rank and  $\text{Hom}_{Ae_i}(Ce_i, Ce_i) = Ce_i[e_i\partial]$ . An application of the theorem completes the proof of the corollary.

Hereafter we shall always denote by  $X$  the polynomial given by corollary 2.5.

**THEOREM 2.6.** — *Let  $\partial$  be a derivation on a ring  $C$  of prime characteristic  $p > 0$  with  $A$  as its kernel. Assume  $C$  is a finitely generated projective module over  $A$  and  $\text{Hom}_A(C, C) = C[\partial]$ . Then the group  $P(C/A)$  of classes of rank one projective  $A$ -modules split by  $C$  is isomorphic to the homology group  $L(C/A) = (\text{kernel } \delta_1) / (\text{image } \delta_0)$  of the complex*

$$C^* \xrightarrow{\delta_0} C^+ \xrightarrow{\delta_1} A^+$$

defined by  $\partial$  and  $X$ .

*Proof* <sup>(1)</sup>. — Let  $M$  be a rank one projective  $A$ -module such that the  $C$ -module  $M \otimes C$  is free on one generator  $b$ . Let  $F$  be a finite subset

<sup>(\*)</sup> Henceforth all tensor-product signs without subscripts will denote tensor product over  $A$ .

of  $A$  such that the ideal in  $A$  generated by  $F$  is  $A$  and such that for any  $f \in F$ , the  $A_f$ -module  $M \otimes A_f$  is free on one generator  $b_f$  ([1], p. 138). Given  $f \in F$ ,  $b = b_f(\mathbf{1} \otimes u_f)$  for some invertible element  $u_f$  of  $A_f$ . Now let  $\mathfrak{Q}$  be a prime ideal of  $C$ , and let  $\mathfrak{q}$  denote the prime  $\mathfrak{Q} \cap A$  in  $A$ . To any generator  $b_{\mathfrak{Q}}$  for the free  $A_{\mathfrak{q}}$ -module  $M \otimes A_{\mathfrak{q}}$ , there is a unique invertible element  $u_{\mathfrak{Q}}$  in  $C_{\mathfrak{Q}}$  given by the equation  $b = b_{\mathfrak{Q}}(\mathbf{1} \otimes u_{\mathfrak{Q}})$ . It is easily seen that the correspondence  $\mathfrak{Q} \rightarrow (\partial u_{\mathfrak{Q}})/u_{\mathfrak{Q}}$  is independent of the choice of  $b_{\mathfrak{Q}}$ . In particular, if  $f \in F$  is not in  $\mathfrak{q}$ , then  $(\partial u_f)/u_f$  goes to  $(\partial u_{\mathfrak{Q}})/u_{\mathfrak{Q}}$  under the canonical homomorphism  $C_f \rightarrow C_{\mathfrak{Q}}$ . This shows  $\mathfrak{Q} \rightarrow (\partial u_{\mathfrak{Q}})/u_{\mathfrak{Q}}$  is a section for the structural sheaf of  $\text{Spec } C$ . By [4], p. 86, there is a unique element  $z \in C$  such that for all  $\mathfrak{Q} \in \text{Spec } C$ , the canonical image of  $z$  in  $C_{\mathfrak{Q}}$  is  $(\partial u_{\mathfrak{Q}})/u_{\mathfrak{Q}}$ . Now  $\partial_1 z$  must be trivial because at each  $\mathfrak{Q}$ ,

$$X(\partial + Lz) = (Lu_{\mathfrak{Q}})^{-1} X(\partial) (Lu_{\mathfrak{Q}}) = 0 \quad ([1], \text{p. 112}).$$

If  $b'$  is another generator for the free  $C$ -module  $M \otimes C$ , and  $z'$  is the element in  $C$  to correspond, then  $z' = z$  modulo image  $\partial_0$ . So we have a well-defined mapping  $\lambda : P(C/A) \rightarrow L(C/A)$ .

Obviously  $\lambda$  is a group-homomorphism. To show it is one-to-one, assume  $z = \partial u/u$  for some  $u \in C^*$ . Then for any  $\mathfrak{Q} \in \text{Spec } C$ ,  $u_{\mathfrak{Q}} = ua_{\mathfrak{Q}}$  for some  $a_{\mathfrak{Q}} \in A_{\mathfrak{q}}^*$ ,  $(\mathbf{1} \otimes \partial)(b[\mathbf{1} \otimes u^{-1}])$  must be zero in  $M \otimes C$  because at every  $\mathfrak{Q}$ ,

$$(\mathbf{1} \otimes \partial)(b[\mathbf{1} \otimes u^{-1}]) = (\mathbf{1} \otimes \partial)(b_{\mathfrak{Q}}[\mathbf{1} \otimes a_{\mathfrak{Q}}]) = 0.$$

But the sequence  $0 \rightarrow M \otimes A \rightarrow M \otimes C \xrightarrow{\mathbf{1} \otimes \partial} M \otimes C$  is exact,  $b(\mathbf{1} \otimes u^{-1})$  therefore is already contained in  $M$ . Let  $m$  be any element of  $M$ . Then  $m \otimes \mathbf{1} = b(\mathbf{1} \otimes u^{-1}c)$  for some  $c \in C$ . Therefore  $c$  must be an element of  $A$  because  $0 = (\mathbf{1} \otimes \partial)(m \otimes \mathbf{1}) = b(\mathbf{1} \otimes u^{-1}[\partial c])$ . This shows  $M$  is free over  $A$  and hence  $\lambda$  is one-to-one <sup>(2)</sup>.

It remains to show  $\lambda$  is onto. So let  $C[t; \partial]$  be the non-commutative ring of differential polynomials with coefficients in  $C$  defined by  $tc = ct + \partial c$ . An inductive argument shows that

$$t^r c = ct^r + \binom{r}{1} (\partial c) t^{r-1} + \binom{r}{2} (\partial^2 c) t^{r-2} + \dots + (\partial^r c),$$

and so  $X$  is in the center of  $C[t; \partial]$  because  $t^r c = ct^r + \partial^r c$ .

Now to any  $z$  in the kernel of  $\partial_1 : C^+ \rightarrow A^+$ , we associate a ring-homomorphism

$$\rho_z : C[t; \partial] \rightarrow \text{Hom}_A(C, C) \quad \text{given by } \rho_z(g) = g(\partial + Lz).$$

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<sup>(2)</sup> Note that the hypotheses  $C$  over  $A$  being finitely generated projective and  $\text{Hom}_A(C, C) = C[\partial]$  are not needed for the existence and the injectivity of  $\lambda$ . Similar remark applies to theorem 2.9.

If  $g$  is in the kernel of  $\rho_0$ , then  $g(\partial + Lz)$  is the zero endomorphism on  $C$ . This shows the kernel of  $\rho_0$  is contained in the kernel of  $\rho_z$ . So we have a ring-homomorphism  $\rho_z \rho_0^{-1} : C[\partial] \rightarrow \text{Hom}_A(C, C)$ . In other words,  $X(\partial + Lz) = 0$  means that  $C$  is made into a  $C[\partial]$ -module with  $\partial$  acting on  $C$  as  $\partial + Lz$ . But if  $C[\partial] = \text{Hom}_A(C, C)$ , the modules over the latter are well-known. Write  $E = \text{Hom}_A(C, C)$ , then the formula is  $\text{Hom}_E(C, C) \otimes C \simeq C$  ([1], p. 181, exercise 18). Now each element of  $\text{Hom}_E(C, C)$  is determined by its action on  $1 \in C$  which must go to an element of  $C$  annihilated by the new operation of  $\partial$  since in the old operation of  $\partial$ ,  $\partial 1 = 0$ . Thus  $\text{Hom}_E(C, C) \cong \text{kernel}(\partial + Lz)$  and so  $C = C \cdot \text{kernel}(\partial + Lz)$ . But  $C$  over  $A$  is a faithfully flat module: given any prime ideal  $q$  in  $A$ ,  $Q = \{x \in C \mid x^p \in q\}$  is a prime in  $C$ , and  $Q \cap A = q$ ; if  $q$  is maximal, so is  $Q$  ([1], p. 51).  $\text{Hom}_E(C, C) \otimes C = C$  therefore implies that  $\text{Hom}_E(C, C)$  and hence  $\text{kernel}(\partial + Lz)$  is a rank one projective  $A$ -module ([1], p. 53, 142). Write  $\pi_z = \text{kernel}(\partial + Lz)$ , and let  $b$  be the element  $\sum m_i \otimes c_i$  in  $\pi_z \otimes C$  such that  $\sum m_i c_i = 1$  in  $C$ . For each  $Q \in \text{Spec } C$ , pick  $m_Q \in \pi_z$  such that  $b_Q = m_Q \otimes 1$  is a generator for the rank one free  $A_Q$ -module  $\pi_z \otimes A_Q$ . We have, for all  $i$ ,  $m_i \otimes 1 = m_Q \otimes a_i$  for some  $a_i \in A_Q$ . Now with the notations introduced earlier in this proof,  $u_Q = \sum a_i c_i$ . But in  $C_Q$

$$m_Q \sum a_i c_i = \sum m_i c_i = 1. \text{ So}$$

$$\begin{aligned} 0 &= (\partial m_Q) \left( \sum a_i c_i \right) + m_Q \sum a_i (\partial c_i) \\ &= -m_Q \left( z \sum a_i c \right) + m_Q \sum a_i (\partial c_i). \end{aligned}$$

This shows  $(\partial u_Q)/u_Q = \left( \partial \sum a_i c_i \right) / \left( \sum a_i c_i \right) = z$ , and hence  $\lambda$  is onto. This completes the proof of the theorem.

We list some special cases of theorem 2.6. When  $C$  is a field, the following is the well-known theorem of Jacobson ([7], theorem 15).

**COROLLARY 2.7.** — *Let  $C$  be a semi-local ring of prime characteristic  $p > 0$ . Let  $\partial$  be a derivation on  $C$  with  $A$  as its kernel such that  $C$  is a finitely generated projective module over  $A$  and  $\text{Hom}_A(C, C) = C[\partial]$  <sup>(3)</sup>. Then the sequence*

$$0 \rightarrow A^* \xrightarrow{\varepsilon} C^* \xrightarrow{\partial_0} C^+ \xrightarrow{\partial_1} A^+$$

is exact.

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<sup>(3)</sup> When  $C$  is a finite dimensional field extension of  $A$ , this is always satisfied.



*Proof.* — Since  $A$  is also semi-local, we have  $L(C/A) \cong P(C/A) = \mathfrak{o}$  ([1], p. 143) hence the corollary.

Of particular interest is the following corollary.

**COROLLARY 2.8.** — *Let  $C$  be either a noetherian ring or an integral domain of prime characteristic  $p > \mathfrak{o}$ . Let  $\partial$  be a derivation on  $C$  with  $A$  as its kernel such that  $C$  is a finitely generated projective  $A$ -module and  $\text{Hom}_A(C, C) = C[\partial]$ . Let  $L$  be the total ring of fractions of  $C$ , and denote by  $L(C/A)$  the group*

$$[\partial_0(L^*) \cap C^+] / \partial_0(C^*) = \{ \partial x/x \mid x \in L^*; \partial x/x \in C \} / \{ \partial x/x \mid x \in C^* \}.$$

Then there is an isomorphism

$$\pi : L(C/A) \rightarrow P(C/A)$$

which takes class  $z$  to class kernel  $(\partial + Lz)$ .

*Proof.* — Consider the commutative diagram given by  $\partial$  and  $X$ ,

$$\begin{array}{ccc} C^* & \xrightarrow{\partial_0} C^+ & \xrightarrow{\partial_1} A^+ \\ \cap & \cap & \cap \\ L^* & \xrightarrow{\partial_0} L^+ & \xrightarrow{\partial_1} K^+ \end{array} \quad (K = \text{the total ring of fractions of } A),$$

the lower sequence is exact by corollary 2.7. So  $z$  belongs to kernel  $\{C^+ \xrightarrow{\partial_1} A^+\}$  if and only if  $z = \partial x/x$  for some  $x \in L^*$ . By theorem 2.6, this shows  $\pi$  is an isomorphism as asserted.

In the above corollary, if  $C$  is a noetherian integrally closed domain, the hypothesis that  $C$  over  $A$  is finitely generated and projective can be relaxed to  $C$  over  $A$  is finitely presented, that is, there is an exact sequence of  $A$ -modules

$$F_2 \rightarrow F_1 \rightarrow C \rightarrow \mathfrak{o},$$

where  $F_1$  and  $F_2$  are finitely generated free  $A$ -modules. But instead of rank one projectives, we now have to describe  $L(C/A)$  in terms of divisor classes.

The definition of Krull domain can be found in [2]. Noetherian integrally closed domains form the main example of Krull domains. If  $g$  is a set of derivations on a field  $L$ , and  $z$  a non-zero element in  $L$ , we shall denote by  $\zeta_z : g \rightarrow L$  the map defined by  $\partial \rightarrow (\partial z/z)$ .

**THEOREM 2.9.** — *Let  $g$  be a finite set of derivations on a Krull domain  $C$  of characteristic  $p \neq \mathfrak{o}$ , and let  $A$  be the Krull domain*

$$\{x \in C \mid \partial x = \mathfrak{o} \text{ for all } \partial \in g\}.$$

Denote by  $L$  and  $K$  the fields of fractions of  $C$  and  $A$  respectively. Assume  $C$  is finitely presented as  $A$ -module and  $\text{Hom}_A(C, C) = C[g]$ . Then the group  $\Gamma(C/A)$  of divisor classes in  $A$  which become principal in  $C$  is isomorphic to

$$L(C/A) = \{ \zeta_z \mid z \in L^* \text{ and } \zeta_z(\partial) \in C \text{ for all } \partial \in g \} / \{ \zeta_z \mid z \in C^* \}.$$

*Proof.* — Let  $d$  be a divisor in  $A$  which becomes a principal divisor ( $z$ ) in  $C$ . Then for each prime ideal  $Q$  of height one in  $C$ , there is some  $z_Q$  in  $K$  such that  $|z|_Q = |z_Q|_Q$ , where  $| \cdot |_Q$  is the discrete valuation on  $C$  given by  $Q$ . So  $z = u_Q z_Q$  for some invertible element  $u_Q$  in  $C_Q$ . This shows for any  $\partial$  in  $g$ ,  $\partial z/z = \partial u_Q/u_Q$  is an element of  $C_Q$  for all prime  $Q$  of height one. So  $\partial z/z$  is an element of  $C$  because  $C$  is a Krull domain. Since  $\zeta_z = \zeta_u$  ( $z \in L^*$ ,  $u \in C^*$ ) is equivalent to  $\partial(z/u) = 0$  for all  $\partial$  in  $g$ , or in other words  $z/u \in K^*$ , the correspondence  $d \rightarrow \zeta_z$  gives rise to a one-to-one group-homomorphism  $\lambda : \Gamma(C/A) \rightarrow L(C/A)$ .

To prove the map is onto, let  $z$  be an element of  $L^*$  such that  $\partial z/z \in C$  for all  $\partial$  in  $g$ . We claim that if  $|z|_Q \neq 0$  modulo  $p$ , then the ramification index  $e(Q)$  of  $Q$  over  $A$  must be one. Let  $t \in Q$  be a uniformizing variable for  $Q$ , that is,  $tC_Q = QC_Q$ . So  $z = ut^n$  for some invertible element  $u$  in  $C_Q$ , and

$$(\partial u/u) + n(\partial t/t) = \partial z/z \in C \text{ for all } \partial \text{ in } g.$$

This shows if  $n \neq 0 \pmod{p}$ , then  $tC_Q$  is stable under  $g$ . Now  $C$  is finitely presented as  $A$ -module, if  $q = Q \cap A$ , then

$$A_q \otimes_A \text{Hom}_A(C, C) \cong \text{Hom}_{A_q}(C_Q, C_Q) \quad ([1], \text{p. } 98).$$

But  $A_q$  is a discrete valuation ring,  $C_Q$  as a finitely generated torsion-free  $A_q$ -module must be free, so

$$\begin{aligned} \hat{C}_Q[g] &\cong \hat{A}_q \otimes_A \text{Hom}_A(C, C) \cong \hat{A}_q \otimes_{A_q} [A_q \otimes_A \text{Hom}_A(C, C)] \\ &\cong \hat{A}_q \otimes_{A_q} \text{Hom}_{A_q}(C_Q, C_Q) \cong \text{Hom}_{\hat{A}_q}(\hat{C}_Q, \hat{C}_Q), \end{aligned}$$

where  $\hat{\phantom{x}}$  means taking completion. Now the ramification index of  $t\hat{C}_Q$  is either 1 or  $p$ . If it is  $p$ , then there is an  $\hat{A}_q$ -derivation  $\Delta$  on  $\hat{C}_Q$  such that  $\Delta t = 1$ . From  $\hat{C}_Q[g] = \text{Hom}_{\hat{A}_q}(\hat{C}_Q, \hat{C}_Q)$ , we see that  $\partial t \notin t\hat{C}_Q$  for some  $\partial$  in  $g$ . This shows that if  $QC_Q = tC_Q$  is stable under  $g$ , then  $e(Q) = 1$ . Let  $d$  denote the divisor  $\sum_Q \frac{|z|_Q}{e(Q)} (Q \cap A)$ . Clearly  $\lambda$  maps class  $d$  to class  $\partial z/z$ . This completes the proof of the theorem.

REMARK 2.10. — When  $L$  is a field extension over  $K$  of dimension  $p$ ,  $g$  has only one element  $\partial$ , and  $\partial(C)$  contained in no prime ideal of

height one, theorem 2.9 is given by SAMUEL ([8], theorem 2). The monomorphism part of theorem 2.9 is also given by HALLIER ([6], p. 3924). That this monomorphism in general is by no means onto is clear from the following.

REMARK 2.11. — The hypothesis  $\text{Hom}_A(C, C) = C[\partial]$  cannot be dropped from theorems 2.6 and 2.9. Consider the polynomial ring  $C = E[x, y, z]$  where  $E$  is a field of characteristic 2. Let  $\partial'$  be the  $E$ -derivation on  $C$  given by

$$\partial'x = y^4, \quad \partial'y = x^2 \quad \text{and} \quad \partial'z = xyz.$$

Then  $C$  is a free module over  $A = \text{kernel } \partial' = E[x^2, y^2, z^2]$ . The latter is a unique factorization domain, so both  $P(C/A)$  and  $\Gamma(C/A)$  are trivial.  $L(C/A, \partial')$  however is not trivial:  $\partial'z/z = xy$  is an element of  $C$  while  $C^*$  is just  $E^*$ , the image of  $C^*$  in  $C^+$  is trivial.

If instead of  $\partial'$ , we consider the  $E$ -derivation  $\partial$  on  $C$  given by  $\partial x = 1$ ,  $\partial y = x$  and  $\partial z = xy$ , then  $\text{Hom}_A(C, C) = C[\partial]$ . The sequence  $C^* \rightarrow C^+ \rightarrow A^+$  given by  $\partial$  and its characteristic polynomial  $t^8$  is exact, and

$$L(C/A) = L(C/A, \partial) = 0.$$

### 3. Examples.

3.1. *Counter-example for a conjecture of Samuel.* — Let  $C$  be the polynomial ring  $E[x, y]$  where  $E$  is a field of characteristic 2. Let  $\partial$  be the  $E$ -derivation on  $C$  given by  $\partial x = 1$  and  $\partial y = y^2$ . Then  $C$  is a free module over  $A = \text{kernel } \partial = E[x^2, y^2, xy^2 + y]$  and  $\text{Hom}_A(C, C) = C[\partial]$ . The characteristic polynomial for  $\partial$  is  $t^2$ , and the map  $\partial_1: C^+ \rightarrow A^+$  given by  $\partial$  and  $t^2$  is  $c \rightarrow \partial c + c^2$ . Now  $C^*$  is just  $E^*$ , so  $\partial_0(C^*)$  is trivial. The kernel of  $\partial_1$  is  $\{0, y\}$ . So  $P(C/A) = \Gamma(C/A) [= P(A) = \Gamma(A)$  because  $C$  is a unique factorization domain] is cyclic of order 2. The non-trivial rank one projective  $A$ -module is the ideal  $y^2A + (xy^2 + y)A$ . Since  $\partial y/y = y$  is an element of  $C = (\partial C)C$ , we get a counter-example for the following conjecture of Pierre SAMUEL ([8], p. 88):

Let  $\partial$  be a derivation on an integral domain of characteristic  $p > 0$ . If  $Q$  is the ideal in  $C$  generated by the image of  $\partial$ , then  $\partial c/c \in Q$  ( $c \in C$ ) implies  $\partial u/u = \partial c/c$  for some  $u \in C^*$ .

Some special cases of this statement have been verified by HALLIER [5] and also by SAMUEL [8], and was used by SAMUEL to compute the divisor class group of the following example when the characteristic of  $C$  is 2, 3 and 5.

3.2. — Let  $C = E[[x, y]]$  be the formal power series ring over a field  $E$  of characteristic  $p > 0$ . Let  $\partial$  be the  $E$ -derivation on  $C$  given by  $\partial x = x$  and  $\partial y = -y$ . So  $A = \text{kernel } \partial = E[[x^p, y^p, xy]]$ . Both  $A$  and  $C$  are noetherian integrally closed. Since  $C$  is finitely generated

as  $A$ -module,  $C$  is finitely presented also [1], p. 36. The rank one projective class group  $P(A)$  is trivial because  $A$  is a local ring. We propose to verify the following statements :

- (i)  $C[\partial] = \text{Hom}_A(C, C)$ ;
- (ii)  $\Gamma(A) = \Gamma(C/A)$  is cyclic of order  $p$ ;
- (iii) the  $A$ -module  $C$  is not flat, and hence not projective.

Given  $f$  in  $\text{Hom}_A(C, C)$ , we have  $f = x_0 + x_1\partial + \dots + x_{p-1}\partial^{p-1}$  with  $x_i \in L$  because  $\text{Hom}_K(L, L) = L[\partial]$  and  $[L : K] = p$ . Now  $x_0 = f(1) \in C$ , so we may assume  $x_0 = 0$  and

$$f = x_1\partial + \dots + x_{p-1}\partial^{p-1}.$$

From  $\partial^i(x^j) = j^i x^j$ ,  $\partial^i(y^j) = (-j)^i y^j$ , we get two systems of linear equations in  $x_i$  :

- (I)  $i x_1 + i^2 x_2 + \dots + i^{p-1} x_{p-1} = f(x^i)/x^i \quad (0 < i < p)$ ;
- (II)  $(-i) x_1 + (-i)^2 x_2 + \dots + (-i)^{p-1} x_{p-1} = f(y^i)/y^i \quad (0 < i < p)$ .

The first system of equations shows  $x_i$  is a polynomial in  $1/x$  with coefficients in  $C$ , while the second system shows  $x_i$  is a polynomial in  $1/y$  also with coefficients in  $C$ . So  $x_i \in C$  and  $f \in C[\partial]$ .

The divisor class group  $\Gamma(A)$  is just  $\Gamma(C/A)$  because  $C$  is a unique factorization domain. So  $\Gamma(A) = [\partial_0(L^+) \cap C^+] / \partial_0(C^+)$ . Now the minimal polynomial for  $\partial$  is  $t^p - t$ . The mapping  $\delta_1 : C^+ \rightarrow A^+$  with respect to  $\partial$  and  $t^p - t$  is given by  $\delta_1(s) = \partial^{p-1} s - s + s^p (s \in C)$ .

Assume  $z$  is an element of kernel  $\delta_1$ , and write

$$z = \alpha + \beta + \sum_{i=1}^{p-1} (u_i x^i + v_i y^i),$$

where  $\alpha \in E$ ,  $\beta, u_i, v_i \in A$ , and  $\beta$  has no constant term. We have

$$(\alpha^p - \alpha) + (\beta^p - \beta) + \sum_{i=1}^{p-1} (u_i x^i + v_i y^i)^p = 0.$$

So  $\alpha = \alpha^p$ , which implies  $\alpha$  is an element of  $\{0, 1, \dots, p-1\}$ , and

$$\begin{aligned} \beta &= \sum_{i=1}^{p-1} (u_i x^i + v_i y^i)^p + \beta^p \\ &= \sum_{i=1}^{p-1} (u_i x^i + v_i y^i)^p + \sum_{i=1}^{p-1} (u_i x^i + v_i y^i)^{p^2} + \beta^{p^2} \\ &= \sum_{n=1}^{\infty} \sum_{i=1}^{p-1} (u_i x^i + v_i y^i)^{p^n}. \end{aligned}$$

This shows  $z$  is an element of kernel  $\delta_1$  if and only if

$$z = \alpha + \sum_{i=0}^{\infty} \sum_{n=1}^{p-1} (u_i x^i + v_i y^i)^{pn},$$

with  $\alpha \in \{0, 1, \dots, p-1\}$ ,  $u_i, v_i \in A$ . But given  $u \in A$ ,  $0 < i < p$ , the element  $ux^i + (ux^i)^p + (ux^i)^{p^2} + \dots$  always lies in the image of  $\delta_0 : C^* \rightarrow C^+$  because the equation

$$d\left(\sum_{j=0}^{p-1} s_j x^j\right) = \left(\sum_{j=0}^{p-1} s_j x^j\right) \sum_{n=0}^{\infty} (ux^i)^{pn} \quad (s_j \in A)$$

is solvable in  $s_j$ . This proves  $\Gamma(A)$  is cyclic of order  $p$  since elements in the image of  $\delta_0 : C^* \rightarrow C^+$  has no constant terms.

Finally,  $C$  is finitely presented as  $A$ -module, if  $C$  is flat over  $A$ ,  $C$  would be projective over  $A$  ([1], p. 140); according to corollary 2.8, that would imply  $P(C/A) = L(C/A) = \Gamma(C/A)$  is cyclic of order  $p$ . But  $A$  is a local ring,  $P(C/A)$  must be trivial, therefore the  $A$ -module  $C$  is not flat.

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