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A property of A-sequences


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A PROPERTY OF $A$-SEQUENCES

by

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Let $A$ be a noetherian local ring with maximal ideal $m$, containing a field $k$ (not necessarily its residue field). Recall ([1]; [7]) that an $A$-sequence is a finite set $x_1, \ldots, x_r$ of elements of $A$, contained in the maximal ideal $m$, such that $x_i$ is not a zero-divisor in $A$, and for each $i = 2, \ldots, r$, $x_i$ is not a zero-divisor in $A/(x_1, \ldots, x_{i-1})$. We will show that for many purposes, the elements of an $A$-sequence behave just like the variables in a polynomial ring over a field. In particular, the sum, product, intersection and quotient of ideals generated by monomials in a given $A$-sequence are just what one would expect (see Corollary 1 below for a precise statement).

PROPOSITION 1. — Let $A$ be a noetherian local ring containing a field $k$, and let $x_1, \ldots, x_r$ be an $A$-sequence. Then the natural map

$$\varphi: T = k[X_1, \ldots, X_r] \to A$$

of $k$-algebras, which sends $X_i$ into $x_i$ for each $i$, is injective, and $A$ is flat as a $T$-module.

Proof. — We show $\varphi$ is injective by induction on $r$, the case $r = 0$ being trivial. Let $r > 0$ be given. Then $x_2, \ldots, x_r$ is an $(A/x_1 A)$-sequence, so by the induction hypothesis, we may assume that

$$\bar{\varphi}: k[X_2, \ldots, X_r] \to A/x_1 A$$

is injective. Now let $t \in T$ be given and write

$$t = \sum_{n=0}^{\infty} X_1^n f_n(X_2, \ldots, X_r),$$

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where each \( f_n(x_1, \ldots, x_r) \in k[x_1, \ldots, x_r] \). Suppose that \( \varphi(t) = 0 \). If \( t \neq 0 \), let \( f_s \) be the first of the \( f_n \) which is non-zero. Then

\[
\varphi(t) = x_1^s \left( \sum_{n=s}^\infty x_1^{n-s} f_n(x_2, \ldots, x_r) \right).
\]

Since \( x_1 \) is a non-zero-divisor in \( A \), we have

\[
\sum_{n=s}^\infty x_1^{n-s} f_n(x_2, \ldots, x_r) = 0.
\]

Reducing modulo \( x_1 \), we find \( f_s(x_2, \ldots, x_r) = 0 \) in \( A/x_1 A \). Now since \( \bar{\varphi} \) is injective by the induction hypothesis, \( f_s(x_2, \ldots, x_r) = 0 \), which is a contradiction. Hence \( t = 0 \) and \( \varphi \) is injective.

Now to show \( A \) is flat over \( T \), we use the local criterion of flatness ([3], chap. III, § 5, n° 2, theorem 1, (iii)) applied to the ring \( T \), the ideal \( J = (x_1, \ldots, x_r) \), and the \( T \)-module \( A \). We must verify the four following statements:

(a) \( T \) is noetherian (well-known).
(b) \( A \) is separated for the \( J \)-adic topology, i.e. \( \bigcap J^n A = 0 \). This is true since \( JA \) is contained in the radical \( m \) of \( A \), and \( \bigcap m^n = 0 \) by Krull’s theorem ([3], chap. III, § 3, n° 2).
(c) \( A/JA \) is flat over \( k = T/J \). This is true since anything is flat over a field.
(d) \( \text{Tor}_i^T(T/J, A) = 0 \). To calculate this \( \text{Tor} \), we use the Koszul complex \( K.(X_1, \ldots, X_r; T) \) ([4], EGA, III, 1.1) which is a resolution of \( T/J \) since \( X_1, \ldots, X_r \) is a \( T \)-sequence. \( \text{Tor}_i(T/J, A) \) is the \( i \)th homology group of the complex \( K.(X_1, \ldots, X_r; T) \otimes_T A = K.(x_1, \ldots, x_r; A) \).

But since \( x_1, \ldots, x_r \) is an \( A \)-sequence, this homology is zero in degrees \( i > 0 \) ([4], EGA, III, 1.1.4). In particular \( \text{Tor}_i^T(T/J, A) = 0 \), which completes the proof of the proposition.

**Corollary 1.** — With the notations of the proposition, let \( a \) and \( b \) be any two ideals in \( T \). For any ideal \( c \) in \( T \), denote by \( cA \) its extension to \( A \). Then

(i) \( (a + b)A = aA + bA \);
(ii) \( (a \cdot b)A = (aA) \cdot (bA) \);
(iii) \( (a \cap b)A = (aA) \cap (bA) \);
(iv) \( (a : b)A = (aA) : (bA) \).

(Recall that for any two ideals \( a, b \) in a ring \( R \), \( a : b = \{ x \in R \mid x \cdot b \subseteq a \} \).
Proof. — (i) and (ii) are trivially true for any ring extension and are repeated here for convenience. (iii) and (iv) are true for any flat ring extension. (iii) is proved in ([3], chap. I, § 2, n° 6, Prop. 6).

To prove (iv), let \( y_1, \ldots, y_r \) be a set of generators for \( b \). Then \( a : b = \bigcap (a : (y_i)) \), and so using (iii) we are reduced to the case where \( b \) is generated by a single element \( y \). Now \( a : (y) \) is characterized by the exact sequence of \( T \)-modules

\[
0 \to a : (y) \to T \to T/a,
\]

where the last map is multiplication by \( y \). Tensoring with \( A \) we have an exact sequence of \( A \)-modules

\[
0 \to (a : (y)) A \to A \to A/a A
\]

from which we deduce that \( (a : (y)) A = a A : y A \) (Note that for any ideal \( b \) in \( T \), the natural map \( b \otimes T A \to b A \) is an isomorphism, since \( A \) is flat over \( T \), so we identify the two).

Corollary 2 (Theorem of Rees). — Let \( A \) be a noetherian local ring containing a field, and let \( J \) be an ideal generated by an \( A \)-sequence \( x_1, \ldots, x_r \). Then the images \( \bar{x}_1, \ldots, \bar{x}_r \) of the \( x_i \) in the graded ring

\[
\text{gr}_r(A) = \sum_{n=0}^{\infty} J^n/J^{n+1}
\]

are algebraically independent, so that \( \text{gr}_r(A) \) is isomorphic to the polynomial ring \( A/J[\bar{x}_1, \ldots, \bar{x}_r] \).

Proof (see also [7], Appendix 6, theorem 3). — It is sufficient to show that for each \( n \), \( J^n/J^{n+1} \) is a free \( A/J \)-module, with the images of the monomials in \( x_1, \ldots, x_r \) of degree \( n \) for basis. It is clear that these monomials generate \( J^n/J^{n+1} \). To show they are linearly independent, let \( z \) be a monomial of degree \( n \) in \( x_1, \ldots, x_r \), and let \( J' \) be the ideal generated by all the other monomials of degree \( n \) and by \( J^n/J^{n+1} \). Then we must show that \( J' : z = J \), which follows from Corollary 1.

Corollary 3. — Let \( A \) be a noetherian local ring containing a field \( k \), and let \( x_1, \ldots, x_r \) be an \( A \)-sequence. Then any ideal of \( A \) generated by polynomials in the \( x_i \), with coefficients in \( k \), is of finite homological dimension over \( A \).

Proof. — Using the notations of the proposition, any such ideal can be written as \( a A \), where \( a \) is an ideal in the polynomial ring \( T = k[X_1, \ldots, X_r] \). Over \( T \), \( a \) has a finite projective resolution ([7], chap. VII, § 13, theorem 43)

\[
0 \to L_0 \to \cdots \to L_1 \to L_0 \to a \to 0.
\]
Tensoring with $A$ gives an exact sequence
\[ 0 \to L_n \otimes A \to \cdots \to L_1 \otimes A \to L_0 \otimes A \to aA \to 0 \]
which is a finite projective resolution of $aA$.

**Remark.** — A refinement of the proof of proposition 1 due to D. Quillen allows one to dispense with the hypothesis that $A$ contains a field, provided that one is interested only in ideals of $A$ generated by monic monomials in the $x_i$. In particular this is sufficient for the result of Corollary 2, and of Proposition 2 below.

As an application we give the following:

**Proposition 2.** — Let $A$ be a noetherian local ring containing a field. Let $I$ be a radical ideal in $A$ (i.e. an ideal which is a finite intersection of prime ideals), and let $J$ be any ideal generated by an $A$-sequence whose radical is $I$. Then, to within isomorphism, the $A/I$-module
\[ M = \text{Hom}_A(A/I, A/J) \]
is independent of $J$.

**Example.** — An interesting case (already known [2]) is that of a local Cohen-Macaulay ring $A$, with $I = \mathfrak{m}$ the maximal ideal. Then there are ideals $J$ generated by an $A$-sequence with radical $\mathfrak{m}$, so that $M$ is defined. Its dimension as an $A/\mathfrak{m}$-vector space is an invariant of $A$, which is equal to 1 if and only if $A$ is a Gorenstein ring. (See [2], where if $n$ is the dimension of $M$, then $A$ is called a $MC_n$-ring. This number is also the "vordere Loewysche Invariante" of $A/I$ in [6], p. 28, and is the number $e$ of the exercises in [5], § 4, p. 67.)

**Proof of Proposition.** — Let $J$ be generated by the $A$-sequence $x_1, \ldots, x_r$. Then $r$ is the height of $J$, and so is independent of $J$. We consider the $r$th local cohomology group (see [5] for definition and methods of calculation)
\[ H = H^r_J(A) = \lim_{\rightarrow n} \text{Ext}^r(A/J^{(n)}, A), \]
where $J^{(n)} = (x_1^n, \ldots, x_r^n)$. Using the Koszul complex $K.(x_1^n, \ldots, x_r^n; A)$ to calculate the Ext, we find an isomorphism
\[ \varphi_n : \text{Ext}^r(A/J^{(n)}, A) \cong A/J^{(n)} \]
which transforms the maps of the direct system into the maps
\[ f_n : A/J^{(n)} \to A/J^{(n+1)} \]
which are defined by multiplication by $x_1 \cdots x_r$.

I claim that the maps $f_n$ are all injective. Indeed, it is sufficient to see that
\[ J^{(n+1)} : (x_1 \cdots x_r) = J^{(n)}, \]
This follows from Corollary 1 and the fact that the analogous relation holds in a polynomial ring. Therefore we can write $H$ as an increasing union

$$H = \bigcup_{n=1}^{\infty} E_n,$$

where $E_n$ is the isomorphic image of $A/J^{[n]}$ in $H$. Furthermore, I claim that for each $n$, $E_n$ is the set of elements of $H$ annihilated by $J^{[n]}$. Indeed, we have only to observe that for each $n$, $k > 0$,

$$J^{(n+k)} : J^{[n]} = (x_1 \cdots x_n)^{k}$$

which follows from Corollary 1 and the analogous formula in a polynomial ring. Now since $J \subseteq I$, anything in $H$ annihilated by $I$ is annihilated by $J$. Hence


But by definition, $H$ depends only on the radical of $J$ [5], so we are done.

**BIBLIOGRAPHY.**


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