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On $\Sigma$-groups


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ON Σ-GROUPS

BY

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Recently, IRWIN and WALKER [2] introduced the notion of Σ-groups. One question left open is whether or not the property of being a Σ-group is hereditary, (i.e.) whether every subgroup of a Σ-group is again a Σ-group. We will show that this is not always the case by constructing an easy example. Further, it is asked in [3] whether every high subgroup of a Σ-group $G$ is an endomorphic image of $G$. We will give an affirmative answer to this question. Lastly, we investigate when a cotorsion group is a Σ-group.

All groups considered in this note are abelian. If $G$ is any group, then $G^1$ denotes the subgroup of elements of infinite height in $G$, that is, $G^1 = \bigcap_{n=1}^{\infty} nG$. We will sometimes refer to it as the Radical of $G$.

A subgroup maximally disjoint with $G^1$ is known as a high subgroup of $G$. A group is called a Σ-group, if all its high subgroups are direct sums of cyclic groups. It is known [3] that if $G$ is a Σ-group, then all its high subgroups are isomorphic. A group $G$ is called cotorsion if it is reduced and $\text{Ext}(Q, G) = 0$, where $Q$ is the additive group of rational numbers. For general properties of high subgroups we refer to [2]. If $G$ is any group, then $G_t$ denotes the torsion part of $G$.

**Lemma 1.**

(i) Every torsion group $G$ contains a Σ-group $R$ such that $G^1 = R^1$, $R$ is pure in $G$ and $G/R$ divisible.

(ii) Every torsion-free group $G$ contains a Σ-group $R$ such that $G^1 = R^1$ and $R$ is pure in $G$. 

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Proof. — Assertion (i) is the content of theorem 10 of [2], while (ii) follows on noting that for the torsion free group $G$, the radical $G^1$ is divisible and so is itself a $\Sigma$-group.

Lemma 2. — Every reduced torsion group $G$ is isomorphic to the radical of some reduced torsion $\Sigma$-group $R$.

Proof. — Appealing to the assertion (i) of lemma 1, we see that it is enough if we show that $G$ is a subgroup of a torsion group $H$, with $H' = G$. To each $a \in G$, associate a sequence of elements $(a_1, a_2, \ldots, a_n, \ldots)$ with the conditions that $na_n = a$. Let $H$ be the group obtained by adjoining to $G$ all such sequences of elements with the stated conditions. It is then immediately checked up that $H' = G$ and that $H$ is torsion.

Theorem 1 — Not every subgroup of a $\Sigma$-group need again be a $\Sigma$-group (1).

Proof. — Since every divisible group is a $\Sigma$-group, the assertion follows trivially, when we consider a divisible group containing a group $G$ which is not a direct sum of cyclic groups and for which $G^1 = 0$.

We could even give an example of a reduced group which is a $\Sigma$-group and not all subgroups of which are $\Sigma$-groups.

Let $G$ be an unbounded closed $p$-group. Then it follows from [1] (p. 116) that $G$ is not a $\Sigma$-group. But, by lemma 2, there is a reduced $\Sigma$-group $R$ which contains $G$ (even as its radical). This $R$ is the required counter example.

The following theorem answers a question raised in [3].

Theorem 2. — Let $G$ be a $\Sigma$-group. Then every high subgroup of $G$ is an endomorphic image of $G$.

Proof. — We assume, without loss in generality, that $G$ is reduced. If $G$ is torsion, then every high subgroup of $G$ is a basic subgroup of $G$ and hence is an endomorphic image of $G$. If $G$ is torsion free, since $G$ is reduced, $G^1 = 0$ and so, $G$ is its own high subgroup, so that it clearly has the required property. Let now $G$ be a mixed group. Then we distinguish two cases:

Case 1. — Let $G/G_1$ be reduced. Now every high subgroup $H$ of $G$ is a direct sum of cyclic groups and hence $H$ splits. Then, by theorem 2 of [3], $G$ also splits; $G = G_1 + F$. Since $G_1$ and $F$ are $\Sigma$-groups, all their high subgroups are endomorphic images. Then we readily check up that every high subgroup of $G$ is an endomorphic image of $G$.

(1) The author thanks Prof. E. A. Walker for offering comments towards the simplification of the example. He also thanks him for having given the benefit of papers [2] and [3] long before their publication.
Case 2. — Let \( G/G_i \) be not reduced. Let \( M/G_i \) be the maximal divisible subgroup of \( G/G_i \). Let \( H = H_i + H_f \) be a high subgroup of \( G \). Then by theorem 1 of [3], \( G = M + H_f \).

Now \( H_i \) is high in \( G_i \) and \( G_i \subseteq M \). We show that \( H_i \) is high in \( M \). Clearly, \( H_i \cap M^i = \emptyset \). If \( H_i \) is not high in \( M \), let \( K \subseteq H \) be high in \( M \). Then, \( K + H_f \) includes \( H_i + H_f = H \) and is high in \( G \), which contradicts the maximality of \( H \). Hence \( K = H_i \) so that \( H_i \) is high in \( M \).

Now \( M \) is a mixed group such that it has a high subgroup which is torsion. On the other hand, the group \( M \), being a direct summand of \( G \), is a \( \Sigma \)-group. Thus all high subgroups of \( M \) are isomorphic and hence torsion. Now if \( y \in M \) and \( o(y) = \infty \), then \( \{ y \} \cap M^i = \emptyset \) since otherwise, \( \{ y \} \) can be expanded to a high subgroup of \( M \) which will no longer be torsion. Hence there exists an integer \( k \) such that \( ky \in M^i \). This implies, in particular, that \( M^* = M/M^i \) is a torsion group. Let \( H^* \) be the image of \( H \) in \( M^* \). Then \( H^* \cong H_i \) and hence is a direct sum of cyclic groups.

We show that \( H^* \) is pure in \( M^* \). Let \( nx^* = a^* \in H^* \), where \( x^* \in M^* \). Then \( nx = a + b \), where \( x \in M \), \( a \in H \), \( b \in M^i \). Since \( b \) is in \( M^i \), \( b = ny \), \( y \in M \). Therefore,

\[
n(x - y) = a \in H,
\]
\[
= nz, \quad z \in H_i, \quad \text{since } H_i \text{ is pure.}
\]

Hence \( nx = nz + b \)

(i.e.) \( nx^* = nz^* \), \( z^* \in H^* \).

This proves that \( H^* \) is pure in \( M^* \).

Thus \( H^* \) is a direct sum of cyclic groups and is pure in the torsion group \( M^* \). Then \( H^* \) is a direct summand of a basic subgroup \( B^* \) of \( M^* \) and so \( H^* \) is an endomorphic image of \( M^* \). Since \( M^* \) is an epimorphic image of \( M \), it follows that \( H^* \) is an epimorphic image of \( M \). But \( H^* \) is isomorphic to \( H \), so that we are assured of an epimorphism of \( M \) on \( H_i \). That is, \( H_i \) is an endomorphic image of \( M \). Let this mapping be \( \delta \).

Now \( G = M + H_f \). Let \( \pi \) and \( \pi' \) be the corresponding projections, \( \pi(G) = M \), \( \pi'(G) = H_f \). Then define a mapping \( \hat{\delta} \) of \( G \) in to itself as \( \hat{\delta} = 0 \pi + \pi' \). Now we can readily check up that \( \hat{\delta} \) is an endomorphism and \( \hat{\delta}(G) = H \). Thus \( H \) is an endomorphic image of \( G \) and this completes the proof.

Remark. — It is worth noting that the subgroup \( M \) is fully invariant in \( G \). This follows from the fact that \( M \) is the unique complementary summand of \( H_f \) in \( G \) which together with theorem 22.3 of [1] implies that \( M \) is fully invariant.

We now investigate when a cotorsion group is a \( \Sigma \)-group. Before that we consider two lemmas which are of independent interest.
Lemma 3. — A high subgroup \( H \) of a cotorsion group \( G \) is an endomorphic image of \( G \) if and only if \( G = H \).

Proof. — We need only to prove the necessary part. If \( H \) is an endomorphic image of \( G \), then clearly it should be cotorsion. Since \( H \) is high in \( G \), \( G/H \) is divisible and so, the exact sequence

\[ 0 \rightarrow H \rightarrow G \rightarrow G/H \rightarrow 0 \]

gives the exact sequence,

\[ \text{Hom}(Q, G) = 0 \rightarrow \text{Hom}(Q, G/H) \rightarrow \text{Ext}(Q, H) \rightarrow \text{Ext}(Q, G) = 0, \]

where the first and the last terms are zero since \( G \) is cotorsion. Since \( G/H \) is divisible, \( \text{Hom}(Q, G/H) = 0 \). This means that \( \text{Ext}(Q, H) = 0 \) so that \( H \) is not cotorsion, which is a contradiction unless \( G/H = 0 \), (i. e.) \( G = H \).

Lemma 4. — A cotorsion group is a direct sum of cyclic groups if and only if it is bounded.

The proof follows from the observations:
(i) A direct summand of a cotorsion group is again cotorsion.
(ii) An infinite cyclic group \( S \) is cotorsion if and only if \( S = 0 \).
(iii) A torsion cotorsion group is bounded.

Theorem 3. — A cotorsion group is a \( \Sigma \)-group if and only if it is bounded.

Proof. — If \( G \) is a \( \Sigma \)-group, by theorem 2, \( H \) (high in \( G \)) is an endomorphic image of \( G \). But then, by lemma 3, \( G \) coincides with \( H \). Now \( H \) is a direct sum of cyclic groups and so lemma 4 settles that \( G \) is bounded.

On the other hand, if \( G \) is bounded, then clearly it is a \( \Sigma \)-group.

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References.


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