BULLETIN DE LA S. M. F.

TAKESHI KOTAKE
MUDUMBAI S. NARASIMHAN

Regularity theorems for fractional powers of a linear elliptic operator

Bulletin de la S. M. F., tome 90 (1962), p. 449-471
<http://www.numdam.org/item?id=BSMF_1962__90__449_0>

Bull. Soc. math. France,
90, 1962, p. 449 à 471.

REGULARITY THEOREMS FOR FRACTIONAL POWERS OF A LINEAR ELLIPTIC OPERATOR ;

BY

TAKESHI KOTAKE AND MUDUMBAI S. NARASIMHAN.

1. Introduction. — Let L be a linear elliptic operator with C^∞ coefficients in an open subset Ω of $\mathbf{R}^n (n \geq 2)$. We suppose that L admits a (strictly) positive self-adjoint realisation $\tilde{L}$ in $L^2(\Omega)$. Let $\{E_\lambda\}$ be the spectral resolution of $\tilde{L}$ so that

$$\tilde{L} = \int \lambda \, dE_\lambda.$$

We consider the family of operators $\tilde{L}^s$, depending on a complex parameter s, defined by

$$\tilde{L}^s = \int \lambda^s \, dE_\lambda.$$

The operators $\tilde{L}^s$ may be viewed as " fractional powers " of L. For $s = -1, -2, \ldots$, we obtain the Green's operator and its iterates.

We study in this paper the regularity properties of the operators $\tilde{L}^s$. For integral values of s, it is known that the operators $\tilde{L}^s$ define kernels which are " very regular " in the sense of SCHWARTZ ([17], chap. V, § 6) and that if further the coefficients of L are analytic the kernels of $\tilde{L}^s$ are analytically very regular. For positive integral values of s the results are trivial, for negative integral values of s these follow from well-known regularity theorems for elliptic operators [11]. The question arises whether these results are true for all values of s. We prove in this paper that this is in fact the case (Theorems 2 and 3). The case of elliptic operators with constant coefficients on a torus and on $\mathbf{R}^n$ has already been dealt with respectively by S. BOCHNER [3] and L. SCHWARTZ ([16], chap. VII, § 10, ex. 7).

That the operators $\tilde{L}^s$ possess kernels follows from regularity theorems for elliptic operators. In order to prove that the kernels are very regular, we represent the kernels, for $Rl(-s)$ sufficiently large, in terms of the Green's function $G(t, x, y)$ of the associated parabolic operator. By using some results of G. BERGENDAL [1] and S. D. EIDELMAN [6] and showing that $G(t, x, y)$ and its derivatives fall off exponentially as $t \to \infty$, we then prove that the kernel $\tilde{L}^s$ is very regular.

The proof of analytic regularity, when the coefficients are analytic, is more difficult. It involves in the first instance estimates for the norms $\| A^k u \|_{L^2}$, where u is a function that is to be proved to be analytic and A a linear elliptic operator with analytic coefficients. Next we need to prove a general theorem (Theorem 1) to the effect that if A is a linear elliptic operator of order m with analytic coefficients in an open set Ω' of $\mathbf{R}^n$, and u is a function satisfying the inequalities

$$\| A^k u \|_{L^2(\Omega')} \leqq (km)!\, c^{k+1}$$

for every integer $k \geqq 0$, with a positive constant c independent of k, then u is analytic in Ω'.

This theorem is a natural one in as much as the conditions

$$\| A^k u \| \leqq (km)!\, c^{k+1}$$

on every compact set are necessary for u to be analytic. We notice also that this theorem contains the well-known result : if A is linear elliptic operator and has analytic coefficients, and if $Au = f$ with f analytic, then u is analytic.

A weaker version of Theorem 1 has been proved by E. NELSON ([14], th. 7); he proves the analyticity of u under the stronger assumption

$$\| A^k u \| \leqq k!\, c^{k+1}.$$

Theorem 1 is proved by suitably estimating the L^2-norms of derivatives of order km of u in terms of L^2-norms of u, Au, ..., $A^k u$. The proof of this theorem uses some ideas of a paper of C. B. MORREY and L. NIRENBERG [13].

The use of the parabolic equation in the proofs of Theorems 2 and 3 was suggested by a paper of S. MINAKSHISUNDARAM [12].

For spaces of distributions we use the usual notation [17].

The results of this paper have been announced in [10].

2. Statement of the theorems. — Let Ω be an open subset of $\mathbf{R}^n$. Let $\mathcal{D}(\Omega)$ be the space of complex-valued C^∞ functions with compact support in Ω. $L^2(\Omega)$ is the Hilbert space of complex-valued square summable functions on Ω, with scalar product (φ, ψ) defined by

$$(\varphi, \psi) = \int_\Omega \varphi . \overline{\psi}\, dx$$

for $\varphi, \psi \in L^2(\Omega)$; $\| \varphi \|_{L^2(\Omega)}$ means $(\varphi, \varphi)^{1/2}$.

Let A be a linear differential operator of order m,

$$A = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$$

with sufficiently differentiable complex-valued coefficients $a_\alpha(x)$ defined in Ω, where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, α_i being integer ≥ 0 and we put :

$$|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n,$$
$$D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}.$$

We say now A is an elliptic operator in Ω, if the homogeneous form of order m

$$\sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha \neq 0$$

for every $x \in \Omega$ and for every non vanishing real vector $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$.

THEOREM 1. — *Let Ω be an open subset of $\mathbf{R}^n$. Let A be a linear elliptic operator of order m with analytic coefficients in Ω. Let A^k be the k^{th} iterate of A. Suppose that a function u (of class C^∞) satisfies the inequality*

$$\| A^k u \|_{L^2(\Omega)} \leq (km)!\, c^{k+1}$$

for every integer $k \geq 0$ with a positive constant c independent of k. Then the function u is analytic in Ω.

REMARK. — The above theorem is also valid for elliptic systems; the demonstration is the same as for the scalar case.

As for the following theorems, we consider a linear elliptic operator L defined on Ω such that

$$(L\varphi, \psi) = (\varphi, L\psi)$$

for every $\varphi, \psi \in \mathcal{D}(\Omega)$.

Suppose further that L when defined on $\mathcal{D}(\Omega)$ ($\subset L^2$), where it is symmetric, has a strictly positive self-adjoint tion extension $\tilde{L}$.

Remark that these conditions entail that the form

$$L(x, \xi) = \sum_{|\alpha| = m} b_\alpha(x) \xi^\alpha$$

is real and definite for every $x \in \Omega$ and ξ real vector, when $L = \sum_{|\alpha| \leq m} b_\alpha(x) D^\alpha$ has sufficiently smooth coefficients.

Let $\{E_\lambda\}$ be the spectral resolution of $\tilde{L}$. By the hypothesis on $\tilde{L}$, we have $\lambda > c_0 > 0$ on the spectrum.

We can now define a family of operators $\tilde{L}^s$ depending on the complex parameter s, by

$$\tilde{L}^s = \int \lambda^s \, dE_\lambda.$$

As we shall see in section 5, $\tilde{L}^s$ thus defined is a continuous linear map of $\mathcal{D}(\Omega)$ into the space of distributions $\mathcal{D}'(\Omega)$ for every s, so that $\tilde{L}^s$ defines a kernel $L^s(x, y)$ ([17], [19]); the theorems to be proved concern the regularity of the kernel $L^s(x, y)$.

THEOREM 2. — *Let L be a linear elliptic differential operator with C^∞ coefficients in an open set Ω of $\mathbf{R}^n$. We suppose further that L admits a strictly positive self-adjoint realisation*

$$\tilde{L} = \int \lambda \, dE_\lambda.$$

in $L^2(\Omega)$. Let s be a complex number. Then the operator

$$\tilde{L}^s = \int \lambda^s \, dE_\lambda.$$

defines a kernel which is very regular.

THEOREM 3. — *Let L be a linear elliptic differential operator with analytic coefficients in an open set Ω of $\mathbf{R}^n$, admitting a strictly positive self-adjoint realisation $\tilde{L}$ in $L^2(\Omega)$. Then, for every complex number s, the kernel of the operator*

$$\tilde{L}^s = \int \lambda^s \, dE_\lambda$$

is analytically very regular.

For the definition of very regular kernels and analytically very regular kernels *see* ([17], chap. V, § 6).

As a consequence of the above theorems, $\tilde{L}^s(T)$ can be defined for T, a distribution with compact support and when L has the C^∞ (analytic) coefficients, $\tilde{L}^s(T)$ is an infinitely differentiable (resp. analytic) function in an open set of Ω where T is an infinitely differentiable (resp. analytic) function.

3. **Preliminary lemmas.** — We consider in this section some lemmas which are required in the proof of Theorem 1.

Let Ω' be any open subset of Ω. Let u be of class C^∞ on the closure $\overline{\Omega'}$ of Ω'. Let k be an integer ≥ 0. We define the k-norm of $u \in C^\infty(\overline{\Omega'})$ by

$$\| u \|_{k,\Omega'} = \sum_{|\alpha| = k} \frac{k\,!}{\alpha\,!} \| D^\alpha u \|_{L^2(\Omega')},$$

where we put $\alpha\,! = \alpha_1\,!\,\alpha_2\,!\ldots\alpha_n\,!$ for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$.

LEMMA 3.1. — *Let k, k', be given integers $\geqq 0$. Then we have*

$$\| u \|_{k+k',\Omega'} = \sum_{|\alpha|=k} \frac{k\,!}{\alpha\,!} \| D^\alpha u \|_{k',\Omega'},$$

PROOF. — We have

$$\frac{(k+k')\,!}{\gamma\,!} = \sum_{\substack{|\alpha|=k \\ |\beta|=k' \\ \alpha+\beta=\gamma}} \frac{k\,!}{\alpha\,!} \frac{k'\,!}{\beta\,!}.$$

The lemma follows immediately from this equality.

The next lemma is a refined version of Friedrichs' inequality [7]. The proof is a modification of Friedrichs' proof as in [13].

We denote by Ω_r the ball $|x| < r$ of radius r in $\mathbf{R}^n$.

LEMMA 3.2. — *Let A be a linear elliptic operator of order m with C^∞ coefficients in Ω. Let r, δ be positive numbers such that $\delta < r$ and $\Omega_{r+\delta} \subset \Omega$. Then there exists a constant $c > 0$ independent of δ such that for every $u \in C^\infty(\Omega)$ we have*

$$\| u \|_{m,\Omega_r} \leqq c \{ \| A u \|_{o,\Omega_{r+\delta}} + \delta^{-m} \| u \|_{o,\Omega_{r+\delta}} \}.$$

PROOF. — Let $\zeta \in \mathcal{D}(\Omega)$ have its support in $\Omega_{r+\delta}$ and be such that $\zeta \equiv 1$ on Ω_r and satisfies

$$\sup_{\Omega_{r+\delta}} | D^\alpha \zeta(x) | \leqq c_\alpha \delta^{-|\alpha|} \qquad (\delta < r) \tag{3.1}$$

with $c_\alpha > 0$ depending only on α.

For any $u \in C^\infty(\Omega)$, we shall consider $\zeta^m u$, which is of class C^∞ having its support in $\Omega_{r+\delta}$. Since A is an elliptic operator with C^∞ coefficients, we have the well-known inequality [10]

$$\| \zeta^m u \|_{m,\Omega_{r+\delta}} \leqq c \{ \| A(\zeta^m u) \|_{o,\Omega_{r+\delta}} + \| \zeta^m u \|_{o,\Omega_{r+\delta}} \} \tag{3.2}$$

with a constant $c > 0$ depending only on A and $\Omega_{r+\delta}$.

By using the estimate (3.1), we obtain

$$\| A(\zeta^m u) \|_{o,\Omega_{r+\delta}} \leqq c' \left\{ \| \zeta^m A u \|_{o,\Omega_{r+\delta}} + \sum_{k=0}^{m-1} \delta^{-m+k} \| \zeta^k u \|_{k,\Omega_{r+\delta}} \right\},$$

$$\sum_{|\alpha|=m} \| \zeta^m D^\alpha u \|_{o,\Omega_{r+\delta}} \leqq c'' \left\{ \| \zeta^m u \|_{m,\Omega_{r+\delta}} + \sum_{k=0}^{m-1} \delta^{-m+k} \| \zeta^k u \|_{k,\Omega_{r+\delta}} \right\}.$$

It follows then from (3.2),

$$(3.3) \qquad \sum_{k=0}^{m} \delta^{-m+k} \| \zeta^k u \|_{k,\Omega_{r+\delta}} \leq c \left\{ \| \zeta^m A u \|_{o,\Omega_{r+\delta}} + \sum_{k=0}^{m-1} \delta^{-m+k} \| \zeta^k u \|_{k,\Omega_{r+\delta}} \right\}$$

with $c > 0$ independent of k.

To complete the proof of the lemma, we need the following fact : for every ε, $\delta > 0$, there exists a constant c independent of ε, δ and u such that

$$(3.4) \qquad \sum_{|\alpha|=k} \| \zeta^k D^\alpha u \|_{o,\Omega} \leq \varepsilon \sum_{|\alpha|=k+1} \| \zeta^{k+1} D^\alpha u \|_{o,\Omega} + c(\varepsilon^{-1} + \delta^{-1}) \sum_{|\alpha|=k-1} \| \zeta^{k-1} D^\alpha u \|_{o,\Omega}$$

where $k \geq 1$.

In fact we have the equality

$$-(\zeta^k D^\alpha u, \zeta^k D^\alpha u) = (\zeta^{k-1} D^{\alpha'} u, \zeta^{k+1} D_1 D^\alpha u) + 2k((D_1 \zeta) \zeta^{k-1} D^{\alpha'} u, \zeta^k D^\alpha u),$$

where $\alpha' = (\alpha_1 - 1, \alpha_2, \ldots, \alpha_n)$ (we suppose $\alpha_1 \neq 0$) and $D_1 = \partial/\partial x_1$.

Now we can obtain the inequality (3.4) by Schwarz's inequality and by taking into account the estimate (3.1) for ζ.

In (3.4) we take $k = m - 1$ and choose ε as $\varepsilon = \delta/2c$. Bringing the inequality thus obtained in the right side of (3.3), we have

$$(3.5) \qquad \sum_{k=0}^{m} \delta^{-m+k} \| \zeta^k u \|_{k,\Omega_{r+\delta}} \leq c \left\{ \| \zeta^m A u \|_{o,\Omega_{r+\delta}} + \sum_{k=0}^{m-2} \delta^{-m+k} \| \zeta^k u \|_{k,\Omega_{r+\delta}} \right\}$$

with $c > 0$ independent of k. Thus in the right side of (3.3), the terms corresponding to $k = m - 1$ can be absorbed in the left side. Repeating this procedure by using (3.4) with appropriate ε, we arrive finally at the desired inequality stated in the lemma.

LEMMA 3.3. — *Let q be positive integer such that $q < m$. Let $r < r_0$, r_0 being fixed. Then there exists a constant $c_m > 0$ depending only on m and r_0 such that for every $\varepsilon > 0$ and $u \in C^\infty(\Omega)$ one has*

$$\| u \|_{q,\Omega_r} \leq \varepsilon \| u \|_{m,\Omega_r} + c_m \varepsilon^{-q/(m-q)} \| u \|_{o,\Omega_r}.$$

A proof of this lemma can be given by using Fourier transforms after extending the functions suitably to $\mathbf{R}^n$. Another proof can be found in [15] (Appendix).

REMARK. — Let p be any integer $\geqq 0$. By applying the above inequality to $D^\alpha u$ and by summing up the inequality thus obtained with respect to α such that $|\alpha| = mp$, we obtain from Lemma 3.1,

$$\| u \|_{pm+q,\Omega_r} \leqq \varepsilon \| u \|_{(p+1)m,\Omega_r} + c_m \varepsilon^{-q/(m-q)} \| u \|_{pm,\Omega_r}$$

with the same constant c_m as in the above lemma.

4. Proof of theorem 1. — In this section we shall prove Theorem 1. The proof is preceeded by several lemmas which permit one to estimate suitably $\| u \|_{km}$ in terms of zero-norms of u, Au, ..., $A^k u$.

We suppose throughout this section that A has analytic coefficients. In this section, $c(c_1, c_2, \ldots$, etc.) will denote a positive constant, always independent of k, which may vary from place to place.

The first lemma gives an estimate for the commutator of the operator D^α and the operator of multiplication by an analytic function.

LEMMA 4.1. — *Let a be an analytic function in $\overline{\Omega'}$. We define the commutator $[a, D^\alpha]$ by $[a, D^\alpha]u = a.D^\alpha u - D^\alpha(au)$, then we have for every integer $k > 0$.*

$$(4.1) \qquad \sum_{|\alpha|=k} \frac{k!}{\alpha!} \| [a, D^\alpha] u \|_{o,\Omega'} \leqq k!\, c^k \sum_{p=0}^{k-1} (p!)^{-1} c^{-p} \| u \|_{p,\Omega'}$$

with $c > 0$ independent of k.

PROOF. — Since a is analytic in $\overline{\Omega'}$, we have

$$(4.2) \qquad \sup_{\Omega'} | D^\alpha a | \leqq \alpha!\, c^{|\alpha|+1}.$$

The Leibniz formula gives

$$D^\alpha(au) = \sum_{\beta \leqq \alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} (D^\beta a)(D^{\alpha-\beta} u)$$

where $\alpha - \beta = (\alpha_1 - \beta_1, \ldots, \alpha_n - \beta_n)$ and $\beta \leqq \alpha$ means $\beta_i \leqq \alpha_i$ for each $i (i = 1, 2, \ldots, n)$.

From (4.2) and the definition of $[a, D^\alpha]$, it follows immediately

$$\sum_{|\alpha|=k} \frac{k!}{\alpha!} \| [a, D^\alpha] u \|_{o,\Omega'} \leqq \sum_{|\alpha|=k} \sum_{\substack{\gamma \leqq \alpha \\ \gamma \neq \alpha}} \frac{k!}{\gamma!} c^{k-|\gamma|} \| D^\gamma u \|_{o,\Omega_r}.$$

Now the number of α's such that $\alpha > \gamma$ for fixed γ is at most of order $n^{k-|\gamma|}$, so that the right side is majorised by

$$\sum_{p=0}^{k} \frac{k!}{p!} (nc)^{k-p} \sum_{|\gamma|=p} \frac{p!}{\gamma!} \| D^\gamma u \|_{o,\Omega'};$$

this proves the lemma.

Lemma 4.2. — *Let r, δ be as in the lemma 3.2. Let ε be any positive number. Then there exist constants c, c_1 (c depending only on A and c_1 depending on A and ε) such that one has for every k and $u \in C^\infty(\Omega)$,*

$$(4.3)\quad \| u \|_{(k+1)m,\Omega_r} \leq c \Big\{ \| Au \|_{km,\Omega_{r+\delta}} + \delta^{-m} \| u \|_{km,\Omega_{r+\delta}} + \varepsilon \| u \|_{(k+1)m,\Omega_{r+\delta}} + ((k+1)m)!\, c_1^{k+1} \sum_{p=0}^{k} ((pm)!)^{-1} c_1^{-p} \| u \|_{pm,\Omega_{r+\delta}} \Big\}.$$

Proof. — From Lemma 3.1 and the Friedrichs' inequality (Lemma 3.2), we have

$$(4.4)\quad \| u \|_{(k+1)m,\Omega_r} = \sum_{|\alpha|=km} \frac{(km)!}{\alpha!} \| D^\alpha u \|_{m,\Omega_r}$$
$$\leq c \Big\{ \| Au \|_{km,\Omega_{r+\delta}} + \delta^{-m} \| u \|_{km,\Omega_{r+\delta}} + \sum_{|\alpha|=km} \frac{(km)!}{\alpha!} \| [A, D^\alpha] u \|_{o,\Omega_{r+\delta}} \Big\}.$$

Now, writting A explicitly as $A = \sum_{|\beta| \leq m} a_\beta D^\beta$ with analytic coefficients a_β and applying the Lemma 4.1 for $[A, D^\alpha] u = \sum_{|\beta| \leq m} [a, D^\alpha] D^\beta u$, we obtain

$$(4.5)\quad \sum_{|\alpha|=km} \frac{(km)!}{\alpha!} \| [A, D^\alpha] u \|_{o,\Omega_{r+\delta}} \leq \sum_{p=0}^{km-1} \sum_{q=0}^{m} \frac{(km)!}{p!} c_1^{km-p} \| u \|_{p+q,\Omega_{r+\delta}}.$$

Since we may suppose $c_1 > 1$ in (4.5), it follows immediately that there exists a constant $c_2 > 0$ independent of k such that

$$(4.6)\quad \sum_{|\alpha|=km} \frac{(km)!}{\alpha!} \| [A, D^\alpha] u \|_{o,\Omega_{r+\delta}} \leq \sum_{s=0}^{(k+1)m-1} \frac{((k+1)m)!}{s!} c_2^{(k+1)m-s} \| u \|_{s,\Omega_{r+\delta}}.$$

We wish now to majorize the right side of (4.6), containing terms $\| u \|_s$, for $s = 0, 1, \ldots, (k+1)m - 1$, by an expression which contains only $\| u \|_{pm}$, for $p = 0, 1, \ldots, (k+1)$.

For this purpose, we write s as $s = pm + q$ with $0 \leq p \leq k$, and $0 \leq q < m$. Then the remark of Lemma 3.3 gives

$$(4.7)\quad \| u \|_{pm+q,\Omega_{r+\delta}} \leq \varepsilon' \| u \|_{(p+1)m,\Omega_{r+\delta}} + c_m \varepsilon'^{-q/(m-q)} \| u \|_{pm,\Omega_{r+\delta}}$$

with c_m independent of ε' and δ.

In (4.7), we choose ε' as

$$\varepsilon' = \varepsilon \frac{(pm+q)!}{((p+1)m)!} c_2^{-(m-q)}$$

where $\varepsilon\,(0<\varepsilon<1)$ is given.

Then we have

$$\varepsilon'^{-q/(m-q)} \leqq \left(\frac{m}{\varepsilon}\right)^m \frac{(pm+q)!}{(pm)!} c_2^q$$

so that we obtain for $s = pm + q$,

$$\frac{c^{-s}}{s!} \| u \|_{s,\Omega_{r+\delta}} \leqq \varepsilon \frac{c_2^{-(p+1)m}}{((p+1)m)!} \| u \|_{(p+1)m,\Omega_{r+\delta}} + c_m \left(\frac{m}{\varepsilon}\right)^m \frac{c_2^{-pm}}{(pm)!} \| u \|_{pm,\Omega_{r+\delta}}. \tag{4.8}$$

Bringing this in the expression (4.6), we have

$$\sum_{|\alpha|=km} \frac{(km)!}{\alpha!} \| [A, D^\alpha] u \|_{0,\Omega_{r+\delta}} \leqq m\varepsilon \| u \|_{(k+1)m,\Omega_{r+\delta}} + c'(\varepsilon) \sum_{p=0}^{k} \frac{((k+1)m)!}{(pm)!} c_2^{(k+1)m-pm} \| u \|_{pm,\Omega_{r+\delta}} \tag{4.9}$$

where we put $c'(\varepsilon) = 1 + m\varepsilon + \left(\frac{m}{\varepsilon}\right)^m c_m$. We take now in (4.9) the constant c_2 large enough to absorb the constant $c'(\varepsilon)$ which is independent of k. Then, from (4.4), the desired inequality follows.

DEFINITION (*see* [13]). — Let λ be a positive number. For each integer $k \geqq 0$, we define

$$\sigma^k(u, \lambda, R) = ((km)!)^{-1} \lambda^{-k} (R-r)^{km} \sup_{R/2 \leqq r < R} \| u \|_{km,\Omega_r}.$$

LEMMA 4.3. — *Let $R<1$. There exists a constant λ depending only on A and R such that for every k and $u \in C^\infty(\Omega)$ we have*

$$\sigma^{k+1}(u, \lambda, R) \leqq [(km+1)\ldots((k+1)m)]\, \sigma^k(Au, \lambda, R) + \sum_{p=0}^{k} \sigma^p(u, \lambda, R). \tag{4.10}$$

PROOF. — Multipliying by $[((k+1)m)!]^{-1} \lambda^{-(k+1)} (R-r)^{(k+1)m}$ on both sides of the inequality of Lemma 4.2 and taking the supremum for $R/2 \leqq r < R$, we obtain

$$\sigma^{k+1}(u, \lambda, R) \leqq \sup_{R/2 \leqq r < R} (I_1 + \varepsilon I_2 + I_3 + I_4), \tag{4.11}$$

where

$$(4.12)\quad \begin{cases} I_1 = c[((k+1)m)!]^{-1}\lambda^{-(k+1)}(R-r)^{(k+1)m}\|Au\|_{km,\Omega_{r+\delta}}, \\ I_2 = c[((k+1)m)!]^{-1}\lambda^{-(k+1)}(R-r)^{(k+1)m}\|u\|_{(k+1)m,\Omega_{r+\delta}}, \\ I_3 = c[((k+1)m)!]^{-1}\lambda^{-(k+1)}(R-r)^{(k+1)m}\delta^{-m}\|u\|_{km,\Omega_{r+\delta}}, \\ I_4 = c\lambda^{-(k+1)}(R-r)^{(k+1)m}\sum_{p=0}^{k}\frac{c_1^{k+1-p}}{(pm)!}\|u\|_{pm,\Omega_{r+\delta}}. \end{cases}$$

We choose in what follows $\delta = \dfrac{R-r}{k+1}$; then we have

$$\left(\frac{R-r}{R-r-\delta}\right)^{km} = \left(1-\frac{1}{k+1}\right)^{-km} < c_2$$

with c_2 independent of k. It follows now from the definition of $\sigma^k(u, \lambda, R)$,

$$(4.13)\quad I_1 \leqq [(km+1)\ldots((k+1)m)]^{-1}\left(\frac{cc_2}{\lambda}\right)\sigma^k(Au, \lambda, R).$$

Similarly

$$(4.14)\quad I_2 \leqq (cc_2)\,\sigma^{k+1}(u, \lambda, R).$$

For I_3, we have

$$(4.15)\quad I_3 \leqq \frac{c}{\lambda}\left(\frac{R-r}{R-r-\delta}\right)^{km}\left(\frac{R-r}{\delta}\right)^{m}\frac{(km)!}{((k+1)m)!}\sigma^k(u, \lambda, R).$$

Since we have from the definition of δ.

$$\left(\frac{R-r}{\delta}\right)^{m} = (k+1)^m$$

it follows from (4.15)

$$(4.16)\quad I_3 \leqq \left(\frac{cc_2}{\lambda}\right)\sigma^k(u, \lambda, R).$$

Finally we obtain for I_4,

$$(4.17)\quad I_4 \leqq \left(\frac{cc_1c_2}{\lambda}\right)\sum_{p=0}^{k}\left(\frac{c_1}{\lambda}\right)^{k-p}\sigma^p(u, \lambda, R) \qquad (\lambda \geqq 1).$$

It follows now for every $k \geqq 0$,

$$(4.18)\quad (1-\varepsilon c)\,\sigma^{k+1}(u, \lambda, R) \leqq [(km+1)\ldots((k+1)m)]^{-1}\left(\frac{c_1}{\lambda}\right)\sigma^k(Au, \lambda, R) + \left(\frac{c_1}{\lambda}\right)\sum_{p=0}^{k}\left(\frac{c_1}{\lambda}\right)^{k-p}\sigma^p(u, \lambda, R)$$

for sufficiently large constants c, $c_1 > 0$, c being independent of ε, while c_1 depends on ε. After we have chosen $\varepsilon = 1/2c$ in (4.18) c_1 is a constant dependent only on A and R so that it is possible to find λ independent of k such that $\lambda > 2c_1$; thus we obtain the inequality (4.10).

LEMMA 4.4. — *Let λ be the same constant as in lemma 4.3; we have then*

$$(4.19)\quad \sigma^{k+1}(u, \lambda, R) \leq \sum_{p=0}^{k+1} 2^{k-p+1} \binom{k+1}{p} ((mp)!)^{-1} \sigma^0(A^p u, \lambda, R).$$

PROOF. — The proof is by induction on k. For $k = 0$, the Lemma is valid (*see* Lemma 4.3). Suppose that the lemma is valid upto $k-1$. Applying the induction hypothesis to the function Au, we have

$$(4.20)\quad \sigma^k(Au, \lambda, R) \leq \sum_{p=0}^{k} 2^{k-p} \binom{k}{p} ((pm)!)^{-1} \sigma^0(A^{p+1}u, \lambda, R).$$

Also, we have for $q \leq k$,

$$(4.21)\quad \sigma^q(u, \lambda, R) \leq \sum_{p=0}^{q} 2^{q-p} \binom{q}{p} ((pm)!)^{-1} \sigma^0(A^p u, \lambda, R).$$

From Lemma 4.3, we get

$$(4.22)\quad \begin{aligned} \sigma^{k+1}(u, \lambda, R) \leq & [(km+1)\ldots((k+1)m)]^{-1} \\ & \times \sum_{p=0}^{k} 2^{k-p} \binom{k}{p} ((pm)!)^{-1} \sigma^0(A^{p+1}u, \lambda, R) \\ & + \sum_{q=0}^{k} \sum_{p=0}^{q} 2^{q-p} \binom{q}{p} ((pm)!)^{-1} \sigma^0(A^p u, \lambda, R). \end{aligned}$$

Now, let c_p be the coefficient of $\sigma^0(A^p u, \lambda, R)$. Then for $0 \leq p \leq k$

$$c_p = [(km+1)\ldots((k+1)m)]^{-1} 2^{k-p+1} \binom{k}{p-1} [((p-1)m)!]^{-1} + \sum_{q=p}^{k} 2^{q-p} \binom{q}{p} [(mp)!]^{-1}.$$

Since

$$\sum_{q=p}^{k} 2^{q-p} \binom{q}{p} \leq 2^{k-p+1} \binom{k}{p}$$

we get

$$c_p \leq 2^{k-p+1} \binom{k+1}{p} ((pm)!)^{-1}.$$

On the other hand, for $p = k + 1$, we have evidently,

$$c_{k+1} = [((k+1)m)\,!]^{-1}.$$

Hence, it follows

$$(4.23) \qquad \sigma^{k+1}(u, \lambda, R) \leq \sum_{p=0}^{k+1} 2^{k-p+1} \binom{k+1}{p} [((pm)\,!)]^{-1} \sigma^0(A^p u, \lambda, R);$$

this is the inequality which we wanted to prove; thus the induction is completed.

Proof of theorem 1. — Let $u \in C^\infty(\Omega)$ such that

$$(4.24) \qquad \| A^k u \|_{L^2(\Omega')} \leq (km)\,!\; c^{k+1}$$

for Ω' an open set of Ω and for all $k \geq 0$ with a constant c independent of k.

Since the analyticity is a local property, we may suppose that the origin of $\mathbf{R}^n$ belongs to Ω' and it is sufficient to prove analyticity at the origin. Take $R < 1$ with $\Omega_R \subset \Omega'$, then

$$(4.25) \qquad \sigma^0(A^k u, \lambda, R) = \| A^k u \|_{L^2(\Omega_R)} \leq km\,!\; c^{k+1}.$$

Now from Lemma 4.4, we have

$$(4.26) \qquad \sigma^{k+1}(u, \lambda, R) \leq \sum_{p=0}^{k+1} 2^{k-p+1} \binom{k+1}{p} [((pm)\,!)]^{-1} \sigma^0(A^p u, \lambda, R)$$

$$\leq \sum_{p=0}^{k+1} 2^{k-p+1} c^{p+1} \binom{k+1}{p} = c\,(c+2)^{k+1}.$$

From the definition of $\sigma^{k+1}(u, \lambda, R)$ we obtain

$$\| u \|_{(k+1)m, \Omega_{R/2}} \leq ((k+1)m)\,!\,.\, c^{k+1}$$

with a certain constant c independent of k.

Then, Lemma 3.3 permits us to estimate $\| u \|_p$ for $p = 0, 1, \ldots$ by $\| u \|_{(k+1)m}$ for $k = 0, 1, \ldots$ and we have

$$(4.27) \qquad \| u \|_{p, \Omega_{R/2}} \leq p\,!\; c^{p+1}$$

for all $p\,(= 0, 1, \ldots)$, where c is a constant depending only on A and Ω_R. Now, by Sobolev's lemma [13], we see that u is analytic at the origin. Hence, the proof of Theorem 1 is completed.

5. **Regurarity of the kernel of $\tilde{L}^s$.** — We denote by $D(\tilde{L}^s)$ the domain of $\tilde{L}^s$, that is the set of elements $f \in L^2(\Omega)$ such that $\int |\lambda^s|^2 d\|E_\lambda f\| < \infty$.

Then, under our hypothesis on $\tilde{L}$, it is easy to see that

(5.1) $$D(\tilde{L}^s) \subseteq D(\tilde{L}^{s'}) \quad \text{if} \quad Rl\, s \geqq Rl\, s',$$

(5.2) for every complex number s,

$$\tilde{L}^s f \in \bigcap_{k=0}^{\infty} D(\tilde{L}^k) \quad \text{if} \quad f \in \bigcap_{k=0}^{\infty} D(\tilde{L}^k).$$

Let $f \in \bigcap_{k=0}^{\infty} D(\tilde{L}^k)$. It follows from (5.1), (5.2) that $f \in D(\tilde{L}^{s+k})$ and $\tilde{L}^s f \in D(\tilde{L}^k)$ for every complex number s and integer $k \geqq 0$. We have then

(5.3) $$\tilde{L}^k \tilde{L}^s f = \tilde{L}^s \tilde{L}^k f = \tilde{L}^{s+k} f$$

(for these properties, *see* [16], § 228; [18], p. 222).

PROPOSITION 5.1. — *For any complex number s, $\tilde{L}^s$ defines a kernel $L^s(x, y)$, that is, a distribution in the product space $\Omega \times \Omega$.*

PROOF. — We first consider the case $Rl\, s < 0$. In this case, $\tilde{L}^s$ is a continuous map of $L^2(\Omega)$ into itself. For, by hypothesis on $\tilde{L} = \int \lambda \, dE_\lambda$, we have a positive constant c_0 such that $\lambda > c_0$ on the spectrum, hence $\lambda^{Rl\, s} \leqq c_0^{Rl\, s}$ for $\lambda \leqq 1$ and $\lambda^{Rl\, s} \leqq 1$ for $\lambda > 1$, since $Rl\, s < 0$.

Thus, λ^s is bounded on the spectrum of $\tilde{L}$. Hence $\tilde{L}^s$ is a continuous linear map of $L^2(\Omega)$ into itself. *A fortiori*, $\tilde{L}^s$ is a continuous linear map of $\mathcal{D}(\Omega)$ into $\mathcal{D}'(\Omega)$. By the kernel theorem of L. SCHWARTZ [19], $\tilde{L}^s$ defines a kernel.

For general s, we take a positive integer m such that $Rl(s - m) < 0$. Then, as seen above, $\tilde{L}^{s-m}$ is a continuous map of $\mathcal{D}(\Omega)$ into $\mathcal{D}'(\Omega)$ while $\tilde{L}^m$, m^{th} iterate of L with C^∞ coefficients, is evidently a continuous map of $\mathcal{D}(\Omega)$ into itself.

Now, the proposition follows from (5.3), by remarking that

$$\tilde{L}^s \varphi = \tilde{L}^{s-m} \tilde{L}^m \varphi \quad \text{for} \quad \varphi \in \mathcal{D}(\Omega) \quad \text{since} \quad \mathcal{D}(\Omega) \subseteq \bigcap_{k=0}^{\infty} D(\tilde{L}^k).$$

From now on, we denote by $L^s(x, y)$ the kernel of $\tilde{L}^s$.

PROPOSITION 5.2. — *For every complex number s, the kernel $L^s(x, y)$ is regular.*

PROOF. — We have to prove that $\tilde{L}^s$ maps continuously $\mathcal{D}(\Omega)$ into $\mathcal{E}(\Omega)$ and can be extended to a continuous linear map of $\mathcal{E}'(\Omega)$ into $\mathcal{D}'(\Omega)$.

Suppose that for every s, $\tilde{L}^s$ maps continuously $\mathcal{D}(\Omega)$ into $\mathcal{E}(\Omega)$. Let φ, ψ be in $\mathcal{D}(\Omega)$. We have then $(\tilde{L}^s\varphi, \psi) = (\varphi, \tilde{L}^{\bar{s}}\psi)$, $\bar{s}$ denoting the conjugate complex of s, this implies that $\tilde{L}^s$ can be identified on the dense subspace $\mathcal{D}(\Omega)$ of $\mathcal{E}'(\Omega)$ with the transpose of $\tilde{L}^{\bar{s}}$, while the transpose of $\tilde{L}^{\bar{s}}$ is a continuous map of $\mathcal{E}'(\Omega)$ into $\mathcal{D}'(\Omega)$ when $\tilde{L}^{\bar{s}}$ is a continuous map of $\mathcal{D}(\Omega)$ into $\mathcal{E}(\Omega)$. Hence, $\tilde{L}^s$ can be extended to a continuous map of $\mathcal{E}'(\Omega)$ into $\mathcal{D}'(\Omega)$.

It remains now to prove that $\tilde{L}^s$ maps continuously $\mathcal{D}(\Omega)$ into $\mathcal{E}(\Omega)$.

Remark first that the image of $\mathcal{D}(\Omega)$ by $\tilde{L}^s$ is contained in $\mathcal{E}(\Omega)$. For, if $\varphi \in \mathcal{D}(\Omega)$, then $\varphi \in \bigcap_{k=0}^{\infty} D(\tilde{L}^k)$, so that by (5.2) we have $\tilde{L}^s\varphi \in \bigcap_{k=0}^{\infty} D(\tilde{L}^k)$. From the regularity theorem for a linear elliptic operator with C^∞ coefficients ([7], [15]), it follows that $\tilde{L}^s\varphi$ is of class C^∞.

As for the continuity of the mapping $\tilde{L}^s$, it is sufficient [17] to verify that the image of every bounded set in $\mathcal{D}(\Omega)$ by $\tilde{L}^s$ is also a bounded set in $\mathcal{E}(\Omega)$.

Let s be such that $Rl\, s < 0$. Let B be a bounded set in $\mathcal{D}(\Omega)$. Then, by definition [17], the image $\tilde{L}^k(B)$ of B by $\tilde{L}^k$ is bounded in $\mathcal{D}(\Omega)$, *a fortiori*, bounded in $L^2(\Omega)$. Now $\tilde{L}^s$ is a continuous map of $L^2(\Omega)$ into itself, so that $\tilde{L}^s\tilde{L}^k(B)$ is bounded in $L^2(\Omega)$. On the other hand, $\tilde{L}^s(B)$ is a family of C^∞ functions belonging to the domain of $\tilde{L}^k$; hence it follows from (5.3) that $\tilde{L}^k\tilde{L}^s(B)$ is bounded in $L^2(\Omega)$, from this, we see, according to Lemma 3.2 and Sobolev's lemma [13], that $\tilde{L}^s(B)$ is a family of C^∞ functions whose derivatives of orders $mk - \left[\frac{n}{2}\right] - 1$ are uniformly bounded on every compact of Ω. Since k is arbitrary, this proves that $\tilde{L}^s(B)$ is bounded in $\mathcal{E}(\Omega)$.

For general s, as in the proof of Proposition 5.1, choose m so large that $Rl(s - m) < 0$ and remark that $\tilde{L}^s\varphi = \tilde{L}^{s-m}\tilde{L}^m\varphi$ for $\varphi \in \mathcal{D}(\Omega)$, then $\tilde{L}^m$ and $\tilde{L}^{s-m}$ map respectively $\mathcal{D}(\Omega)$ into $\mathcal{D}(\Omega)$ and $\mathcal{E}(\Omega)$ continuously. This completes the proof.

6. Estimates for the Green's function of the associated parabolic operator. — Consider the family of operators $G_t = \int e^{-\lambda t}\, dE_\lambda$ for $t > 0$.

G_t is a bounded and Hermitian operator in $L^2(\Omega)$. Associated with these operators we have a C^∞ function in $\mathbf{R}\times\Omega\times\Omega$,

$$G(t, x, y) = \int e^{-\lambda t}\, de(\lambda, x, y),$$

where $e(\lambda, x, y)$ denotes the spectral function of $\tilde{L}$ [8].

We have then

$$\left(\frac{\partial}{\partial t} + L_x\right) G(t, x, y) = 0 \qquad \text{and} \qquad \left(\frac{\partial}{\partial t} + L_y\right) \overline{G(t, x, y)} = 0$$

for $t > 0$.

The next lemma shows that the function $G(t, x, y)$ and its derivatives fall off exponentially as $t \to \infty$.

LEMMA 6.1. — *Let H be a compact in $\Omega\times\Omega$. Under our assumption that $\tilde{L}$ is strictly positive operator ($\lambda > c_0 > 0$ on the spectrum), we have*

$$\left|\left(\frac{\partial}{\partial t}\right)^p D_x^\alpha D_y^\beta G(t, x, y)\right| \leq c\, e^{-c_0 t/2}$$

for $t > 1$ and uniformly for $(x, y) \in H$, where c depends on p, α, β and H.

PROOF. — Denote by $\bar{L}$ the elliptic operator with conjugate complex coefficients of L.

Consider the operator :

$$L_x + \bar{L}_y = L\left(x, \frac{\partial}{\partial x}\right) + \bar{L}\left(y, \frac{\partial}{\partial y}\right)$$

which is evidently elliptic with C^∞ coefficients in the product space $\Omega\times\Omega$.

Now, by Lemma 3.2 and Sobolev's lemma [13] applied to $(L_x + \bar{L}_y)$, it is easy to see that the desired estimate is a simple consequence of the following : let U be a relatively compact open subset in Ω such that $H \subset U\times U$. Then for every positive integers k', k'', we have

$$\left|(L_x + \bar{L}_y)^{k'}\left(\frac{\partial}{\partial t}\right)^{k''} G(t, x, y)\right| \leq c\, e^{-c_0 t/2}$$

for $t > 1$ and for $(x, y) \in U\times U$. Since

$$L_x G(t, x, y) = \bar{L}_y G(t, x, y) = -\frac{\partial}{\partial t} G(t, x, y) \qquad \text{for} \quad t > 0,$$

it is sufficient to estimate $\left(\frac{\partial}{\partial t}\right)^k G(t, x, y)$ for every positive integer k.

Let m be a sufficiently large positive integer such that $\tilde{L}^{-m}$ has a kernel $K(x, y)$ of the Carleman type ([4], [5], [8]). For $x \in \Omega$, let $K_x \in L^2$ denote the function $K(x, \hat{\ })$.

Now

$$\left|\left(\frac{\partial}{\partial t}\right)^k G(t, x, y)\right| = \left|\left(\frac{\partial}{\partial t}\right)^k \int e^{-\lambda t}\, de(\lambda, x, y)\right|$$
$$= \left|\int e^{-\lambda t}(-\lambda)^k\, de(\lambda, x, y)\right|$$
$$= \left|\int e^{-\lambda t}(-\lambda)^k \lambda^{2m}\, de\,(E_\lambda K_x, K_y)\right|$$
$$\leqq e^{-c_0 t/2} \int e^{-\lambda/2} \lambda^{2m+k} |\, d(E_\lambda K_x, K_y)\,|$$

since $\lambda > c_0$ and $t \geqq 1$. Now the variation of $(E_\lambda K_x, K_y)$ in $\mathbf{R}$ is majorised by $\| K_x \|_{L^2} \| K_y \|_{L^2}$ ([16], § 126) and $\| K_x \|_{L^2} \| K_y \|_{L^2} \leqq c(U)$ for $(x, y) \in U \times U$, where $c(U)$ is a constant depending only on U and $\tilde{L}$.

It follows that

$$\left|\left(\frac{\partial}{\partial t}\right)^k G(t, x, y)\right| \leqq c\, e^{-c_0 t/2}$$

for $t > 1$ and $(x, y) \in H$ with a constant c depending on k, H and $\tilde{L}$. Thus Lemma 5.1 is proved.

We next consider the behaviour of $G(t, x, y)$ and its derivatives as $t \to 0$. The required information is given by the results of G. Bergendal [1] and S. D. Eidelman [6].

Let K be a relatively compact open subset of Ω. Consider now the parabolic operator $\left(\frac{\partial}{\partial t} + L\right)$ on $\mathbf{R} \times K$ assiociated with L. According to S. D. Eidelman, we have a fundamental solution $E(t, x, y)$ of $\left(\frac{\partial}{\partial t} + L_x\right)$. It is of class C^∞ in (t, x, y) when $t > 0$ and satisfies near $t = 0$ the following estimate.

Lemma 6.2 (S. D. Eidelman). — *For* $0 < t < 1$ *and* $(x, y) \in K \times K$, *we have*

$$\left|\left(\frac{\partial}{\partial t}\right)^p D_x^\alpha D_y^\beta E(t, x, y)\right| \leqq c t^{-(pm+|\alpha|+|\beta|+n)/m} e^{-c_1 |x-y|^{1+\mu} t^{-\mu}},$$

where $\mu = 1/(m-1)$ *and* c_1 *depends only on* L, K, *while* c *depends also on* p, α, β.

As for the behaviour of $G(t, x, y)$ we have

Lemma (6.3) (G. Bergendal). — *Let* H *be a compact subset of* $\Omega \times \Omega$ *such that* $H \subset K \times K$. *Let* $E(t, x, y)$ *be the same as in lemma* 6.2. *Then there exist positive constants* c, c_1 *such that*

$$\left|\left(\frac{\partial}{\partial t}\right)^p D_x^\alpha D_y^\beta [\, G(t, x, y) - E(t, x, y)\,]\right| \leqq c\, e^{-c_1 t^{-\mu}}$$

for $0 < t < 1$ *and for* $(x, y) \in H$, *where* c_1 *depends only on* L *and* H, *while* c *depends also on* p, α, β.

For $p + |\alpha| + |\beta| = 0$, this is proved in [1]. The general case can be proved in a similar fashion (*see* [2], § 2.3).

7. A representation for the kernel $L^s(x, y)$ in terms of the Green's function $G(t, x, y)$.

PROPOSITION 7.1. — *Let* s *be a complex number such that* $Rl\, s < -n/m$. *Then we have*

$$L^s(x, y) = \frac{1}{\Gamma(-s)} \int_0^\infty t^{-s-1} G(t, x, y)\, dt. \tag{7.1}$$

The integral on the right converges uniformly on every compact subset of $\Omega \times \Omega$ *and represents a continuous function of* (x, y) *in* $\Omega \times \Omega$, *where we denote by* $\Gamma(-s)$ *the Gamma function.*

PROOF. — From Lemma 6.1 we have for $t \geq 1$ and for $(x, y) \in H$,

$$|G(t, x, y)| \leq c\, e^{-c_0 t/2} \tag{7.2}$$

while for $0 < t < 1$ and for $(x, y) \in H$, it follows from Lemma 6.2 and Lemma 6.3,

$$\begin{aligned} |G(t, x, y)| &\leq |E(t, x, y)| \\ &+ |(E - G)(t, x, y)| \leq c\, t^{-n/m} + c\, e^{-c_1 t^{-\mu}} \end{aligned} \tag{7.3}$$

with positive constants c, c_1 depending on H.

From these estimates, it is easy to see that the integral converges uniformly for $(x, y) \in H$ when $Rl\, s < -n/m$ and represents a continuous function of (x, y) since $G(t, x, y)$ is of class C^∞ for $t > 0$.

We shall prove now the equality stated in proposition 7.1. For φ, $\psi \in \mathcal{D}(\Omega)$, consider

$$P = \frac{1}{\Gamma(-s)} \left\langle \int_0^\infty t^{-s-1} G(t, x, y)\, dt,\ \varphi(x)\, \overline{\psi}(y) \right\rangle,$$

where $\langle\ ,\ \rangle$ denote the scalar product between $\mathcal{D}'(\Omega \times \Omega)$ and $\mathcal{D}(\Omega \times \Omega)$. By what has been seen,

$$\begin{aligned} P &= \frac{1}{\Gamma(-s)} \int_0^\infty t^{-s-1}\, dt \int_{\Omega \times \Omega} G(t, x, y)\, \varphi(x)\, \overline{\psi}(y)\, dx\, dy \\ &= \frac{1}{\Gamma(-s)} \int_0^\infty t^{-s-1}\, dt \int_{c_0}^\infty e^{-\lambda t}\, d(E_\lambda \varphi, \psi), \end{aligned}$$

where the integration $\int e^{-\lambda t}\, d(E_\lambda \varphi, \psi)$ is taken in the sense of the Radon-Stieltjes integral with respect to the complex-valued function of bounded variation $(E_\lambda \varphi, \psi)$ in $-\infty < \lambda < \infty$.

Let

$$(E_\lambda \varphi, \psi) = [\rho_1(\lambda) - \rho_2(\lambda)] + i[\rho_3(\lambda) - \rho_4(\lambda)]$$

be the canonical resolution of $(E_\lambda \varphi, \psi)$ with the real valued monotone increasing functions of bounded variation $\rho_k(\lambda)$, $k = 1, 2, 3, 4$ ([20], p. 202).

Then we have

$$\int_{c_0}^{\infty} e^{-\lambda t}\, d(E_\lambda \varphi, \psi) = \sum_{k=1}^{4} \varepsilon_k \int_{c_0}^{\infty} e^{-\lambda t}\, d\rho_k(\lambda)$$

where $\varepsilon_1 = -\varepsilon_2 = -i\varepsilon_3 = i\varepsilon_4 = 1$.

Consider now

$$\int_{c}^{\infty} t^{-s-1}\, dt \int_{c_0}^{\infty} e^{-\lambda t}\, d\rho_k(\lambda).$$

Since $t^{-s-1} e^{-\lambda t}$ is a continuous function of (t, λ) in the integration domain: $0 < t < \infty$, $c_0 < \lambda < \infty$ and the ovbious estimate $|t^{-s-1} e^{-\lambda t}| \leq t^{-Rl\, s-1} e^{-c_0 t}$ implies that it is integrable there with respect to the product measure $dt\, d\rho_k(\lambda)$ when $Rl\, s < 0$.

By *Fubini's theorem*, we have,

$$\int_{0}^{\infty} t^{-s-1}\, dt \int_{c_0}^{\infty} e^{-\lambda t}\, d\rho_k(\lambda) = \int_{c_0}^{\infty} d\rho_k \int_{0}^{\infty} t^{-s-1} e^{-\lambda t}\, dt.$$

Noting that $\int_{0}^{\infty} t^{-s-1} e^{-\lambda t}\, dt = \Gamma(-s)\lambda^s$ and summing up the above integral with respect to k, we have

$$P = \sum_{k=1}^{4} \varepsilon_k \int \lambda^s\, d\rho_k(\lambda)$$

which is equal to

$$\int \lambda^s\, d(E_\lambda \varphi, \psi) = (\tilde{L}^s \varphi, \psi).$$

This completes the proof.

8. Proof of theorem 2. — As in paragraph 5, we see that it is sufficient to prove Theorem 2 for $Rl\, s < -\frac{n}{m}$. Since we have already proved that $L^s(x, y)$ is regular, it is sufficient to prove that $L^s(x, y)$ is of class C^∞ outside the diagonal [17].

For $Rl\, s < -\frac{n}{m}$, we have by Proposition 7.1,

$$L^s(x, y) = \frac{1}{\Gamma(-s)} \int_0^\infty t^{-s-1} G(t, x, y)\, dt. \tag{8.1}$$

If (x, y) belongs to a compact set H in the complement of the diagonal we see from Lemmas 6.1, 6.2 and 6.3 that

$$\left| \left(\frac{\partial}{\partial t} \right)^k D_x^\alpha D_y^\beta G(t, x, y) \right| \leq c\, e^{-c_1(t + t^{-\mu})} \qquad (0 < t < \infty) \tag{8.2}$$

with positive constants c, c_1, where c_1 is independent of k, α, β.

It now follows from (8.1) and (8.2) that $L^s(x, y)$ is of class C^∞ outside the diagonal, since we may differentiate under the integral sign any number of times.

9. **Proof of theorem 3.** — In this section c, $c_i (i = 1, 2, \ldots)$ will denote positive constants independent of k. We suppose that L has analytic coefficients.

To prove Theorem 3, it is sufficient to prove the following two statements :

(i) $L^s(x, y)$ is an analytic function in the complement of the diagonal in $\Omega \times \Omega$.

(ii) For each $\varphi \in \mathcal{D}(\Omega)$, $\tilde{L}^s \varphi$ is an analytic function in every open set where φ is analytic.

Proof of (i). — $(L_x + \bar{L}_y)^k$ is a linear elliptic operator of order m with analytic coefficients in $\Omega \times \Omega$. Applying Theorem 1, we see that to prove (i) it is sufficient to prove the following : for each compact set H in the complet ment of the diagonal, there exists a constant c independent of k such tha-

$$\sup_{(x, y) \in H} |(L_x + \bar{L}_y)^k L^s(x, y)| \leq (mk)!\, c^{k+1}. \tag{9.1}$$

It is sufficient to consider the case $Rl\, s < -\frac{n}{m}$.

As in paragraph 8, we start from the integral representation of $L^s(x, y)$:

$$L^s(x, y) = \frac{1}{\Gamma(-s)} \int_0^\infty t^{-s-1} G(t, x, y)\, dt. \tag{9.2}$$

If $(x, y) \in H$, we have the estimate (8.2) which permits us to differentiate under the integral sign, so that we have

$$(L_x + \bar{L}_y)^k L^s(x, y) = \frac{(-1)^k 2^k}{\Gamma(-s)} \int_0^\infty t^{-s-1} \left(\frac{\partial}{\partial t} \right)^k G(t, x, y)\, dt. \tag{9.3}$$

For we have

$$\left(\frac{\partial}{\partial t}+L_x\right)G(t,\, x,\, y)=\left(\frac{\partial}{\partial t}+\bar{L}_y\right)G(t,\, x,\, y)=0$$

for $t>0$. Let us first suppose that s is not a negative integer. By integration by parts in (9.3) [which is permitted by (8.2)] we obtain

$$(9.4)\quad (L_x+\bar{L}_y)^k L^s(x,\, y) \\ =\frac{2^k}{\Gamma(-s)}(-s-1)(-s-2)\dots(-s-k)\int_0^\infty t^{-s-k-1}\, G(t,\, x,\, y)\, dt.$$

Now as a special case of (8.2) we have

$$|\, G(t,\, x,\, y)\, | \leqq c\, e^{-c_1(t+t^{-\mu})}$$

uniformly for $(x,\, y)\in H$ with positive constants c, c_1 depending on H.

Remembering that $\mu=(m-1)^{-1}$, it follows from a simple calculation that

$$(9.5)\qquad \sup_{(x,y)\in H}\left|\int_0^\infty t^{-s-k-1}\, G(t,\, x,\, y)\, dt\right| \leqq ((m-1)k)!\, c^{k+1},$$

c being independent of k, which gives evidently, from (9.4),

$$\sup_{(x,y)\in H} |\, (L_x+\bar{L}_y)^k L^s(x,\, y)\, | \\ \leqq \frac{2^k}{|\,\Gamma(-s)\,|}\,|\,(-s-1)(-s-2)\dots(-s-k)\,|\,((m-1)k)!\, c^{k+1} \leqq (mk)!\, c_1^k.$$

If s is a negative integer, we see that the integral

$$\int_0^\infty t^{-s-1}\left(\frac{\partial}{\partial t}\right)^k G(t,\, x,\, y)\, dt \qquad (x,\, y)\in H$$

vanishes for all large k and (9.1) is trivially valid. So (i) is proved.

Proof of (ii). — Let $\varphi\in\mathcal{D}(\Omega)$. We suppose φ is analytic in an open subset Ω_0 of Ω. We shall show that $\tilde{L}^s\varphi$ is analytic in Ω_0.

Let Ω_1, Ω_2 be any relatively compact open subsets of Ω_0 such that

$$\bar{\Omega}_1\subset\Omega_2\subset\bar{\Omega}_2\subset\Omega_0.$$

Let $\alpha\in\mathcal{D}(\Omega_0)$ and $\alpha\equiv 1$ on Ω_2. One has then

$$\tilde{L}^s(\varphi)=\tilde{L}^s(\alpha\varphi)+\tilde{L}^s((1-\alpha)\varphi).$$

Now, $(1-\alpha)\varphi\in\mathcal{D}(\Omega)$ and its support does not inersect Ω_1; by what has been seen in (i), $L^s(x,\, y)$ is an analytic function of $(x,\, y)$ outside the diagonal in $\Omega\times\Omega$, so that it follows immediately from the integral representation of $L^s(x,\, y)$ that $\tilde{L}^s((1-\alpha)\varphi)$ is analytic in Ω_1.

It remains to show that $\tilde{L}^s(\alpha\varphi)$ is analytic in Ω_1. It is sufficient to consider the case $Rl\, s < -\frac{n}{m}\cdot$ Then we have for each integer $k \geqq 0$

$$(9.6) \qquad L^k \tilde{L}^s(\alpha\varphi)(x) = \frac{1}{\Gamma(-s)} \int_0^\infty t^{-s-1}\, dt \int_\Omega G(t, x, y)\, (\alpha L^k \varphi)_y\, dy + \frac{1}{\Gamma(-s)} \sum_{p=0}^{k=1} L_x^p \int_0^\infty t^{-s-1}\, dt \times \int_\Omega G(t, x, y)\, ([L, \alpha] L^{k-p-1}\varphi)_y\, dy,$$

where $[L, \alpha]$ is the commutator of L and α.

Consider the second term in the above expression, which we write as

$$(9.7) \qquad \frac{1}{\Gamma(-s)} \sum_{p=0}^{k-1} F_p(x),$$

where

$$F_p(x) = L_x^p \int_0^\infty t^{-s-1}\, dt \int_\Omega G(t, x, y)\, ([L, \alpha] L^{k-p-1}\varphi)_y\, dy.$$

Now $[L, \alpha]$ is a differential operator of order $(m-1)$ whose coefficients have their supports in $(\Omega_0 - \Omega_2)$, so that if we consider x in Ω_1 we may perform the differentiation L_x^p under the integral sign as in paragraph 8 and we obtain,

$$(9.8) \qquad F_p(x) = (-s-1)(-s-2)\dots(-s-p) \int_0^\infty t^{-s-p-1}\, dt \times \int_\Omega G(t, x, y)\, ([L, \alpha] L^{k-p-1}\varphi)_y\, dy, \quad \text{for } s \text{ non-integral}$$

$$= 0 \text{ for all large } p \text{ if } s \text{ is a negative integer.}$$

Since the coefficients of $[L, \alpha]$ have their supports in $(\Omega_0 - \Omega_2)$ and φ is analytic in Ω_0 by hypothesis, we have

$$(9.9) \qquad \sup_{x \in \Omega_0} |[L, \alpha] L^{k-p-1}\varphi| \leqq ((k-p)m)!\, c^{k-p+1}$$

with c independent of k and p. Further we have (*see* § 8)

$$(9.10) \qquad \sup_{(x, y) \in \Omega_1 \times (\Omega_0 - \Omega_2)} |G(t, x, y)| \leqq c\, e^{-c_1(t + t^{-\mu})};$$

we obtain from (9.8), (9.9) and (9.10)

$$\sup_{x \in \Omega} |F_p(x)| \leqq (pm)!\, ((k-p)m)!\, c^{k+1}$$

with a constant c independent of k, p.

Consequently, we have

$$(9.11)\qquad \sup_{x\in\Omega_1}\left|\frac{1}{\Gamma(-s)}\sum_{p=0}^{k-1}F_p(x)\right|\leq (km)!\,c^{k+1}.$$

On the other hand, since φ is analytic in Ω_0, we have

$$\sup_{x\in\Omega_1}|\alpha L^k\varphi|\leq (km)!\,c^{k+1}$$

and from the results of paragraph 6 [*see* (7.2), (7.3)], we have

$$\sup_{(x,y)\in\Omega_1\times\Omega_0}|G(t,x,y)|\leq c\,t^{-n/m}\,e^{-c_1 t}$$

so that it follows for $Rl\,s < -n/m$,

$$(9.12)\qquad \sup_{x\in\Omega_1}\left|\frac{1}{\Gamma(-s)}\int_0^\infty t^{-s-1}\,dt\int_{\Omega_0}G(t,x,y)\,(\alpha L^k\varphi)_y\,dy\right|\leq (km)!\,c^{k+1}.$$

From (9.6), (9.11) and (9.12) we obtain finally

$$\sup_{x\in\Omega_1}|L^k\tilde{L}^s(\alpha\varphi)|\leq (km)!\,c^{k+1}.$$

with c independent of k; now from Theorem 1 we see that $\tilde{L}^s(\alpha\varphi)$ is analytic in Ω_1, which was an arbitrary open subset of Ω_0. This proves (ii) and the proof of Theorem 3 is thus completed.

BIBLIOGRAPHY.

[1] BERGENDAL (G.). — On spectral functions belonging to an elliptic differential operator with variable coefficients, *Math. Scand.*, t. 5, 1957, p. 241-254.

[2] BERGENDAL (G.). — Convergence and summability of eigenfunction expansions connected with elliptic differential operators, *Medd. Lunds Univ. Mat. Sem.*, t. 15, 1959.

[3] BOCHNER (S.). — Zeta functions and Green's functions for linear partial differential operators of elliptic type with constant coefficients, *Annals of Math.*, Series 2, t. 57, 1953, p. 32-56.

[4] BROWDER (Felix E.). — The eigenfunction expansion theorem for the general self-adjoint singular elliptic partial differential operator, I : The analytic foundation, *Proc. Nat. Acad. Sc. U. S. A.*, t. 40, 1954, p. 454-459.

[5] BROWDER (Felix E.). — Eigenfunction expansions for singular elliptic operators, II : The Hilbert space argument, parabolic equations on open manifolds, *Proc. Nat. Acad. Sc. U. S. A.*, t. 40, 1954, p. 459-463.

[6] EIDELMAN (S. D.). — On fundamental solutions of parabolic systems [in russian], *Mat. Sbornik*, t. 38, (80), 1956, p. 51-92.

[7] FRIEDRICHS (K. O.). — On the differentiability of the solutions of the linear elliptic differential equations, *Comm. pure and appl. Math.*, t. 6, 1953, p. 299-326.

[8] GÅRDING (L.). — On the asymptotic properties of the spectral function belonging to a self-adjoint semi-bounded extension of an elliptic differential operator, *Kungl. Fysiogr. Sallsk. Lund Forh.*, t. 24, 1954, n° 21, p. 1-18

[9] GÅRDING (L.). — Dirichlet's problem for linear elliptic partial differential equations, *Math. Scand.*, t. 1, 1953, p. 55-72.

[10] KOTAKE (T.) and NARASIMHAN (M. S.). — Sur la régularité de certains noyaux associés à un opérateur elliptique, *C. R. Acad. Sc. Paris*, t. 252, 1961, p. 1549-1550.

[11] LIONS (J.-L.). — *Lectures on elliptic partial differential equations.* — Bombay, Tata Institute of fundamental Research, 1957.

[12] MINAKSHISUNDARAM (S.). — A generalization of Epstein zeta functions, *Canadian J. Math.*, t. 1, 1947, p. 320-327.

[13] MORREY Jr (C. B.) and NIRENBERG (L.). — On the analyticity of the solutions of linear elliptic systems of partial differential equations, *Comm. pure and appl. Math.*, t. 10, 1957, p. 271-290.

[14] NELSON (Edward). — Analytic vectors, *Annals of Math.*, t. 70, 1959, p. 572-615.

[15] NIRENBERG (Louis). — Remarks on strongly elliptic partial differential equations, *Comm. pure and appl. Math.*, t. 8, 1955, p. 648-674.

[16] RIESZ (F.) and NAGY (Béla, Sz). — *Leçons d'analyse fonctionnelle*, 3e éd. — Paris, Gauthier-Villars; Budapest, Akademiai Kiodo, 1955.

[17] SCHWARTZ (Laurent). — *Théorie des distributions*, Tome 1, 2e éd. — Paris, Hermann, 1957 (*Act. scient. et ind.*, 1091-1245; *Publ. Inst. Math. Univ. Strasbourg*, 9).

[18] SCHWARTZ (Laurent). — *Théorie des distributions*, Tome 2. — Paris, Hermann, 1951 (*Act. scient. et ind.*, 1122; *Publ. Inst. Math. Univ. Strasbourg*, 10).

[19] SCHWARTZ (Laurent). — Théorie des noyaux, *Proc. Intern. Congr. Math.* [11, 1950. Cambridge (Mass.)]; p. 220-230. — Providence, American mathematical Society, 1952.

[20] STONE (Marshall H.). — *Linear transformations in Hilbert space and their applications to analysis.* — New York, American mathematical Society, 1932 (American mathematical Society, *Colloquim Publicalions*, 15).

(Manuscrit reçu le 30 octobre 1961.)

Takeshi KOTAKE,
Attaché de Recherches C. N. R. S., Paris,
Yale University,
New Haven (Conn.) U. S. A.

Mudumbai S. NARASIMHAN,
Tata Institute of fundamental Research,
Bombay (Inde).