ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

Thomas J. HAINES

Base change for Bernstein centers of depth zero principal series blocks

BASE CHANGE FOR BERNSTEIN CENTERS
OF DEPTH ZERO PRINCIPAL SERIES BLOCKS

by Thomas J. HAINES

Abstract. – Let G be an unramified group over a p-adic field. This article introduces a base change homomorphism for Bernstein centers of depth-zero principal series blocks for G and proves the corresponding base change fundamental lemma. This result is used in the approach to Shimura varieties with $\Gamma_1(p)$-level structure initiated by M. Rapoport and the author in [14].

Résumé. – Soit G un groupe non-ramifié sur un corps p-adique. On définit un homomorphisme de changement de base pour les centres de Bernstein des blocs des séries principales de niveau zéro, et on démontre le lemme fondamental correspondant. Ce résultat est utilisé dans le calcul du facteur local en p des variétés de Shimura à structure de niveau $\Gamma_1(p)$ dans l'article avec M. Rapoport [14] publié en tandem avec cet article dans ce même journal.

1. Introduction

Let F denote a p-adic field, and let $F_r \supset F$ denote the unique degree r unramified extension of F contained in some algebraic closure $\bar{F}$ of F. Let θ denote a generator of $\mathrm{Gal}(F_r/F)$. Let G denote an unramified connected reductive group over F. The automorphism θ determines an automorphism of $G(F_r)$, which we also denote with the symbol θ.

Using θ, we have the notion of stable twisted orbital integral $\mathrm{SO}_{\delta\theta}(\phi)$ for any locally constant compactly-supported function ϕ on $G(F_r)$ and any element $\delta \in G(F_r)$ with semisimple norm. See [17] for the definition of the norm map $\mathcal{N}$ from stable θ-conjugacy classes in $G(F_r)$ to stable conjugacy classes in $G(F)$. For a precise definition of $\mathrm{SO}_{\delta\theta}$, see e.g. [18], or [11, 5.1.2].

This article is concerned with the matching of the (twisted) orbital integrals of certain functions on the groups $G(F_r)$ and $G(F)$, respectively. If $\phi \in \mathcal{H}(G(F_r))$ and $f \in \mathcal{H}(G(F))$ are functions in the corresponding Hecke algebras of locally constant compactly-supported functions, then we say ϕ, f are *associated* (or *have matching orbital integrals*), if the following

result holds for the stable (twisted) orbital integrals: for every semisimple element $\gamma \in G(F)$, we have

$$\mathrm{SO}_\gamma(f) = \sum_\delta \Delta(\gamma, \delta) \mathrm{SO}_{\delta\theta}(\phi)$$

where the sum is over stable θ-conjugacy classes $\delta \in G(F_r)$ with semisimple norm, and where $\Delta(\gamma, \delta) = 1$ if $\mathcal{N}\delta = \gamma$ and $\Delta(\gamma, \delta) = 0$ otherwise. See e.g. [18], [19], [7], or [11] for further discussion.

Of primary importance is the case of spherical Hecke algebras. Suppose $K_r \subset G(F_r)$ and $K \subset G(F)$ are hyperspecial maximal compact subgroups associated to a hyperspecial vertex in the Bruhat-Tits building $\mathcal{B}(G(F))$ for $G(F)$, and suppose $\phi \in \mathcal{H}(G(F_r))$ belongs to the corresponding spherical Hecke algebra $\mathcal{H}_{K_r}(G(F_r))$. The Satake isomorphism gives rise to a natural algebra homomorphism

$$b_r : \mathcal{H}_K(G(F_r)) \to \mathcal{H}_K(G(F)),$$

cf. [7]. The base change fundamental lemma for spherical functions asserts that ϕ and $b_r(\phi)$ are associated. This was proved by Clozel [7] and Labesse [24]. Even the earlier special cases of GL_2 [27] and GL_n [1] gave rise to important global and local applications, such as the existence of base-change lifts of certain automorphic representations for GL_n. The theorem of Clozel and Labesse played an important role in Kottwitz' work [20, 22, 21] on Shimura varieties with good reduction at p.

In [11] an analogous base change fundamental lemma is proved for centers of parahoric Hecke algebras. It plays a role in the study of Shimura varieties with parahoric level structure at p, see [10]. A very special case of [11] relates to the center of the Iwahori-Hecke algebra for $G(F)$. By Bernstein's theory [2], this can be viewed as the ring of regular functions on the variety of supercuspidal supports of the Iwahori block (the subcategory $\mathcal{R}_I(G)$ of the category $\mathcal{R}(G)$ of smooth representations of $G(F)$ whose objects are generated by their invariants under an Iwahori subgroup).

The purpose of this article is to generalize this result to certain other principal series blocks in the Bernstein decomposition, namely the *depth-zero* principal series blocks.

To state the theorem, we need some more notation. Denote the ring of integers of F by $\mathcal{O}$ and the residue field by k. Let A denote a maximal F-split torus, and set $T := \mathrm{Cent}_G(A)$, a maximal torus of G defined and unramified over F. Now choose an Iwahori subgroup $I \subset G(F)$ which is in good position relative to T, that is, the alcove $\mathbf{a}$ in $\mathcal{B}(G(F))$ corresponding to I is contained in the apartment $\mathcal{O}_T$ corresponding to A. Let I^+ denote the pro-unipotent radical of I.

Let $T(F)_1$ denote the maximal compact open subgroup of $T(F)$, and let $T(F)_1^+ = T(F)_1 \cap I^+$ denote its pro-unipotent radical. Throughout this article, χ will denote a depth-zero character on $T(F)_1$. This means that χ factors through a character $T(F)_1/T(F)_1^+ \to \mathbb{C}^\times$, which we also denote by χ. Via the canonical isomorphism

$$T(F)_1/T(F)_1^+ \xrightarrow{\sim} I/I^+,$$

we see that χ induces a smooth character $\rho := \rho_\chi$ on I which is trivial on I^+. Then we may consider the Hecke algebra $\mathcal{H}(G, \rho) := \mathcal{H}(G, I, \chi)$, which is defined as

$$\mathcal{H}(G, \rho) = \{f \in \mathcal{H}(G) \mid f(i_1 g i_2) = \rho^{-1}(i_1) f(g) \rho^{-1}(i_2) \ \ \forall i_1, i_2 \in I, \forall g \in G\}.$$

(Cf. [30].) Convolution is defined using the Haar measure which gives I volume 1. Write $\mathcal{Z}(G, \rho)$ for the center of $\mathcal{H}(G, \rho)$.

Let $\mathfrak{s} = \mathfrak{s}_\chi$ be the inertial equivalence class of the cuspidal pair $(T(F), \widetilde{\chi})_G$, for any extension of χ to a character $\widetilde{\chi} : T(F) \to \mathbb{C}^\times$. Then $\mathfrak{s}$ depends only on the relative Weyl group orbit of χ. We are concerned with the principal series Bernstein block

$$\mathcal{R}_{\mathfrak{s}}(G) = \mathcal{R}_\chi(G),$$

the category of smooth representations of $G(F)$ whose irreducible objects are constituents of some normalized principal series $i_B^G(\xi)$, where $\xi : T(F) \to \mathbb{C}^\times$ has $\xi|_{T(F)_1} = \chi$. It turns out (cf. Proposition 3.3.1) that (I, ρ_χ) is a Bushnell-Kutzko type for $\mathcal{R}_\chi(G)$, so that the Bernstein center of $\mathcal{R}_\chi(G)$ can be identified with the ring $\mathcal{Z}(G, \rho)$.

Let $N_r : T(F_r)_1 \to T(F)_1$ denote the norm homomorphism given by $t \mapsto t\,\theta(t) \cdots \theta^{r-1}(t)$. The character $\chi_r := \chi \circ N_r$ is a depth-zero character on $T(F_r)_1$, and gives rise to the Bernstein block $\mathcal{R}_{\chi_r}(G_r)$ for the p-adic group G_r, the character ρ_r on I_r, and the Hecke algebra $\mathcal{H}(G_r, \rho_r)$ with center $\mathcal{Z}(G_r, \rho_r)$. Here $G_r := G(F_r)$ and $I_r \subset G_r$ is the Iwahori subgroup corresponding to I.

In Definition 4.1.1 we define a *base change homomorphism*

(1.0.1) $$b_r : \mathcal{Z}(G_r, \rho_r) \to \mathcal{Z}(G, \rho).$$

This is analogous to the base change homomorphisms defined for spherical Hecke algebras or centers of parahoric Hecke algebra (cf. [11]). Our main theorem is the following.

THEOREM 1.1. – *If $\phi \in \mathcal{Z}(G_r, \rho_r)$, the functions ϕ and $b_r(\phi)$ are associated.*

Now write $\mathcal{H}(G, I^+)$ for $\mathcal{H}_{I^+}(G(F))$ and $\mathcal{Z}(G, I^+)$ for its center. In Section 10, we use (1.0.1) to define a natural algebra homomorphism

(1.0.2) $$b_r : \mathcal{Z}(G_r, I_r^+) \to \mathcal{Z}(G, I^+).$$

Moreover, we show how Theorem 1.1 immediately implies the following result.

COROLLARY 1.2. – *If $\phi \in \mathcal{Z}(G_r, I_r^+)$, the functions ϕ and $b_r(\phi)$ are associated.*

In [14], these results are used in the special case of $G = \mathrm{GL}_d$ to study Shimura varieties in the Drinfeld case with $\Gamma_1(p)$-level structure at p. In future works they will be applied to other Shimura varieties with $\Gamma_1(p)$-level structure. Here, by "$\Gamma_1(p)$-level" we mean that the compact open subgroup K_p coming from the Shimura data $(\mathbf{G}, X, K_p K^p)$ is the pro-unipotent radical of an Iwahori subgroup of $\mathbf{G}(\mathbb{Q}_p)$. Theorem 1.1 and Corollary 1.2 will play a role in the pseudostabilization of the counting points formula for the Shimura varieties just mentioned (see [14] for the Drinfeld case).

This article overlaps with [11] in the case where $\chi = \mathrm{triv}$, for then $\mathcal{H}(G, \rho)$ is just the Iwahori-Hecke algebra and the theorem proved here is a special case of [11]. Here we use Labesse's method of elementary functions [24], whereas in [11] we followed Clozel's method [7] more closely. When $\chi = \mathrm{triv}$, many arguments presented here become simpler. Thus in that special case, this article gives an alternative (somewhat easier) proof for the Iwahori-Hecke algebra special case of [11].

As in earlier papers, the proof of Theorem 1.1 is by induction on the semisimple rank of G, and so consists of two steps: (i) use descent formulas to reduce to the case of elliptic

elements, and (ii) use a global trace formula argument to prove the theorem for suitable elliptic elements. Labesse's elementary functions are used in step (ii) to give sufficiently many character identities that the required character identity for central elements is forced (see Section 9).

Here is an outline of the contents of the paper. In Section 2 we recall some standard notation which will be used throughout the article. In Section 3 we review the essential facts about the Bernstein decomposition related to depth-zero principal series blocks. In Section 4 we define the base change homomorphism and prove some of its basic properties. The constant term homomorphism (an essential ingredient for descent) is studied in Section 5, and in particular is proved to be compatible with the base change homomorphisms. The descent formulas themselves are the subject of Section 6. Section 7 reduces the general fundamental lemma to the case where G is adjoint and γ is a norm and is elliptic and strongly regular semisimple. (Unfortunately, this section is the most technical section of all.) In Section 8 we introduce and study suitable analogues of Labesse's elementary functions adapted to the Bernstein component $\mathscr{R}_\chi(G)$. In Section 9 we use all the preceding material to conclude the proof by establishing the character identity which is equivalent, by the existence of the local data, to the required identity of stable (twisted) orbital integrals. In Section 10 we define (1.0.2) and explain how Theorem 1.1 implies Corollary 1.2. Finally, in Section 11 we correct and clarify a few minor mistakes in [11].

2. Further notation

Let L denote $\widehat{F^{un}}$, the completion of the maximal unramified extension of F contained in $\bar{F}$. Let F_r/F be an unramified extension of degree r contained in L, with ring of integers $\mathcal{O}_r$ and residue field k_r. Let $\sigma \in \mathrm{Aut}(L/F)$ denote the Frobenius automorphism, and use the same symbol to denote the induced automorphism on groups of the form $G(L)$, etc. Fix an algebraic closure $\bar{L}$ for L and define the inertia subgroup[2] as $I = \mathrm{Gal}(\bar{L}/L)$.

Denote groups of F_r-points with a subscript r, e.g. $T_r := T(F_r)$ and $G_r := G(F_r)$. Fix a generator $\theta \in \mathrm{Gal}(F_r/F)$. We use the same symbol θ to denote the induced automorphisms of groups of F_r-points T_r, G_r, etc.

Let ϖ denote a uniformizer for the field F.

We recall the basic facts on the Kottwitz homomorphism [23]. In loc. cit. is defined a canonical surjective homomorphism for any connective reductive F-group H

$$(2.0.3) \qquad\qquad \kappa_H : H(L) \twoheadrightarrow X^*(Z(\widehat{H}))_I,$$

[2] Not to be confused with the Iwahori subgroup!

where $\widehat{H} = \widehat{H}(\mathbb{C})$ denotes the Langlands dual group of H. By loc. cit., it remains surjective on taking σ-fixed points:

$$\kappa_H : H(F) \twoheadrightarrow X^*(Z(\widehat{H}))_I^\sigma.$$

We define

$$H(L)_1 := \ker(\kappa_H)$$

$$H(F)_1 := \ker(\kappa_H) \cap H(F).$$

If H is any *unramified* F-torus, then $H(F)_1$ is indeed the unique maximal compact subgroup of $H(F)$, as the notation in the introduction indicated (cf. [11, Lemma 2.4.1]).

Let N_r denote the norm homomorphism $T(F_r)_1 \to T(F)_1$, given by

$$N_r(t) = t\,\theta(t)\,\cdots\,\theta^{r-1}(t).$$

We will use the same symbol to denote the norm homomorphisms $N_r : T(F_r) \to T(F)$ and $N_r : T(F_r)_1/T(F_r)_1^+ \to T(F)_1/T(F)_1^+$.

Let $\chi_r := \chi \circ N_r$. This will be thought of as a character on $T(F_r)_1/T(F_r)_1^+$ or as a depth-zero character on $T(F_r)_1$, depending on context.

The set of all smooth characters on $T(F)$ carries a natural (left) action under the relative Weyl group $W(F) = N_G T(F)/T(F)$; let W_χ denote the subgroup of $W(F)$ which fixes χ. Likewise we define W_{χ_r} in the relative Weyl group $W_r = W(F_r)$.

Fix an F-rational Borel subgroup B containing T and a Levi decomposition $B = TU$, where U is the unipotent radical of B. There is an Iwahori decomposition of I with respect to T and any such B:

(2.0.4)
$$I = I_U \cdot I_T \cdot I_{\bar{U}},$$

where $I_U := U \cap I$, etc.. Note that $I_T = T(F)_1$.

Let H denote any group. For $h \in H$ and any subset $S \subset H$, we set $^h S := hSh^{-1}$. For a function f on H, we define a new function $^h f$ by $^h f(x) := f(h^{-1}xh)$ for $x \in H$.

For two smooth $G(F)$-representations π and π', the notation $\pi \in \pi'$ will mean that π is isomorphic to a subquotient of π'.

3. Bernstein center for depth zero principal series

We assume G is any connected reductive group which is unramified over F.

3.1. Review of depth zero principal series blocks

When the base field F is understood, we will often abbreviate $G(F)$ by the symbol G and $W(F)$ by the symbol W.

Let χ denote a depth-zero character $\chi : T(F)_1 \to \mathbb{C}^\times$. Let $\widetilde{\chi}$ denote any extension of χ to a character $\widetilde{\chi} : T(F) \to \mathbb{C}^\times$. We consider the inertial class

$$\mathfrak{s} = \mathfrak{s}_\chi = [T(F), \widetilde{\chi}]_G.$$

The inertial class $\mathfrak{s}$ depends only on the W-orbit of χ.

Let $\mathscr{R}(G)$ denote the category of smooth representations of G. Let $\mathscr{R}_\mathfrak{s}(G)$ denote the Bernstein component indexed by $\mathfrak{s}$. Recall that this is the full subcategory of $\mathscr{R}(G)$ whose objects have the property that each of their subquotients is a subquotient of a principal

series representation $i_B^G(\widetilde{\chi}\eta)$, for some unramified character η of $T(F)$. We shall often denote $\mathscr{R}_{\mathfrak{s}}(G)$ by $\mathscr{R}_\chi(G)$.

3.2. Algebraic variety associated to a depth-zero block

Fix χ and $\mathfrak{s} = \mathfrak{s}_\chi$ as above, and the associated Bernstein component $\mathscr{R}_\chi(G)$. Recall (cf. e.g. [14, §9.2]) the set
$$\mathfrak{X}_\mathfrak{s} = \{(T,\xi)_G\}$$
of *supercuspidal supports* $(T,\xi)_G$ of irreducible representations π in the category $\mathscr{R}_\chi(G)$. Here $\xi : T(F) \to \mathbb{C}^\times$ is a smooth character extending some $W(F)$-conjugate of χ and $(T,\xi)_G$ denotes the G-conjugacy class of the pair (T,ξ). Recall that χ possesses at least one W_χ-invariant extension $\widetilde{\chi}$ (see [14, Remark 9.2.4][3]), and that having fixed such a $\widetilde{\chi}$, we have a bijection
$$\widehat{A}/W_\chi \xrightarrow{\sim} \mathfrak{X}_\mathfrak{s}$$
$$\eta \mapsto (T,\widetilde{\chi}\eta)_G,$$
where $\eta \in \widehat{A}$ is viewed as an unramified character on $T(F)$ (cf. e.g. [11, Lemma 2.4.2]). We use this bijection to endow $\mathfrak{X}_\mathfrak{s}$ with the structure of an affine algebraic variety. Up to isomorphism, this structure depends neither on the choice of χ in its $W(F)$-orbit, nor on the choice of the extension $\widetilde{\chi}$ of χ. We have
$$\mathfrak{X}_\mathfrak{s} = \mathrm{Spec}(\mathbb{C}[\mathfrak{X}_\mathfrak{s}]),$$
for a $\mathbb{C}$-algebra $\mathbb{C}[\mathfrak{X}_\mathfrak{s}]$ which is isomorphic to $\mathbb{C}[X_*(A)]^{W_\chi}$.

We will often denote $\mathfrak{X}_\mathfrak{s}$ by $\mathfrak{X}_\chi$.

3.3. The associated type for $\mathscr{R}_\chi(G)$

Let $\rho = \rho_\chi : I \to \mathbb{C}^\times$ be the smooth character which is induced by
$$\chi : I/I^+ = T(F)_1/T(F)_1^+ \to \mathbb{C}^\times.$$

PROPOSITION 3.3.1. – *The pair (I,ρ) is a Bushnell-Kutzko type for $\mathscr{R}_\chi(G)$.*

By definition, this means that an irreducible representation $\pi \in \mathscr{R}(G)$ belongs to $\mathscr{R}_\chi(G)$ if and only if its restriction to I contains the representation ρ. See [4].

Proof (Sketch). – For F-split groups G, Roche [30] produces a Bushnell-Kutzko type for $\mathscr{R}_\chi(G)$ for *any* smooth character χ, under certain restrictions on the residual characteristic of F (and in the case of depth-zero χ, his type is (I,ρ)). This relies on his theory of Hecke algebra isomorphisms and hence involves, in the case of general χ, some restrictions on the residual characteristic of F.

When χ is depth-zero, but now assuming only that G is unramified over F, one can prove more directly that (I,ρ) is a type for $\mathscr{R}_\chi(G)$. One proves that for any irreducible representation $\pi \in \mathscr{R}(G)$, the Jacquet functor $\pi \mapsto \pi_U$ induces a $T(F)_1$-equivariant isomorphism

(3.3.1) $$\pi^\rho \xrightarrow{\sim} \pi_U^\chi.$$

---

[3] This handles the split case; the unramified case is similar.

One then deduces (I, ρ) is a type for $\mathscr{R}_\chi(G)$ using Frobenius reciprocity. The proof of (3.3.1) relies on a consequence of certain Hecke algebra isomorphisms, namely: any non-zero element of $\mathscr{H}(G, \rho)$ supported on a single I-double coset is invertible. But for the depth-zero characters such isomorphisms were established by Goldstein [9] (in the split case) and by Morris [28, Theorem 7.12] (in general), and these are valid with no restrictions on the residual characteristic. Details for this approach in the split case are contained in [12], and the unramified case is handled in exactly the same way. $\qquad\square$

3.4. The action of the center $\mathcal{Z}(G, \rho)$ of $\mathscr{H}(G, \rho)$

Proposition 3.3.1 means that $\mathscr{R}_\chi(G)$ is naturally equivalent to the category of $\mathscr{H}(G, \rho)$-modules (cf. [30]). It also means that there is a canonical algebra isomorphism

$$(3.4.1) \qquad \beta : \mathbb{C}[\mathfrak{X}_\mathfrak{s}] \overset{\sim}{\to} \mathcal{Z}(G, \rho)$$

which is characterized as follows. For an extension ξ of some W-conjugate of χ, consider the space $i_B^G(\xi)^\rho$ of locally constant functions $f : G \to \mathbb{C}$ such that

$$f(tugi) = (\delta_B^{1/2}\xi)(t)f(g)\rho(i)$$

for all $t \in T(F)$, $u \in U(F)$, $g \in G$, and $i \in I$. Then

$$z \in \mathcal{Z}(G, \rho) \text{ acts on the left on } i_B^G(\xi)^\rho \text{ by the scalar } \beta^{-1}(z)(\xi).$$

Here $\beta^{-1}(z)$ is viewed as a regular function on the variety $\mathfrak{X}_\mathfrak{s}$ and the argument ξ is an abbreviation for the point $(T, \xi)_G \in \mathfrak{X}_\mathfrak{s}$.

REMARK 3.4.1. – Note that $\beta^{-1}(z)$ being well-defined as a function on the class $(T, \xi)_G$ means that $\beta^{-1}(z)(^w\xi) = \beta^{-1}(z)(\xi)$ for all $w \in W$; that is, $\beta^{-1}(z)$ is W-invariant as a function of ξ.

3.5. Right vs. left actions on principal series

For later use (e.g. in the proof of Proposition 5.4.1), we need to rephrase this in terms of the *right* action of $\mathcal{Z}(G, \rho)$ on principal series representations. Let $\iota : \mathscr{H}(G) \to \mathscr{H}(G)$ denote the involution of the Hecke algebra of locally constant compactly-supported functions $\mathscr{H}(G)$, defined by the identity

$$(\iota h)(g) = h(g^{-1}) \ , \ \ h \in \mathscr{H}(G), \ g \in G.$$

We use ι to convert left $\mathscr{H}(G)$-actions into right $\mathscr{H}(G)$-actions, and vice-versa. It is clear that $\mathscr{H}(G, \rho^{-1})$ (resp. $\mathscr{H}(G, \rho)$) acts on the left (resp. right) on the space $i_B^G(\xi^{-1})^{\rho^{-1}}$, and that for $\Psi \in i_B^G(\xi^{-1})^{\rho^{-1}}$, and $h \in \mathscr{H}(G, \rho)$, the left and right actions are related by the formula

$$(3.5.1) \qquad \iota h \cdot \Psi = \Psi \cdot h.$$

Duality gives a perfect pairing

$$(\cdot, \cdot) : i_B^G(\xi)^\rho \times i_B^G(\xi^{-1})^{\rho^{-1}} \to \mathbb{C}.$$

For $z \in \mathcal{Z}(G, \rho)$, $\phi_1 \in i_B^G(\xi)^\rho$, and $\phi_2 \in i_B^G(\xi^{-1})^{\rho^{-1}}$, we have the relation

$$(z(\phi_1), \phi_2) = (\phi, \iota z(\phi_2)).$$

Thus

(3.5.2) $$\beta^{-1}(z)(\xi) = \beta^{-1}(\iota z)(\xi^{-1}).$$

From (3.5.1) with $h = z \in \mathcal{Z}(G, \rho)$ and (3.5.2), we deduce the following result.

LEMMA 3.5.1. – *Let the Haar measure on $G(F)$ be normalized so that I has measure 1. Then $z \in \mathcal{Z}(G, \rho)$ acts via right convolutions on $i_B^G(\xi^{-1})^{\rho^{-1}}$ by the scalar*

$$ch_{\xi^{-1}}(z) := \beta^{-1}(z)(\xi).$$

4. Base change homomorphism

4.1. Construction

Again fix the depth-zero character χ on $T(F)_1$. Let (I, ρ) be the $\mathfrak{s}_\chi$-type described above. Let A^{F_r} denote the unique maximal F_r-split torus in G which contains A, and note that $T = \mathrm{Cent}_G(A^{F_r})$. Write T_r for $T(F_r)$. Consider $\chi_r := \chi \circ N_r$ as a depth-zero character on $T(F_r)_1$, and consider the corresponding inertial class $\mathfrak{s}_r := \mathfrak{s}_{\chi_r}$ for $G_r = G(F_r)$. Let (I_r, ρ_r) denote the $\mathfrak{s}_r$-type associated to the character χ_r. We denote the corresponding Hecke algebra (resp. its center) by $\mathcal{H}(G_r, \rho_r)$ (resp. $\mathcal{Z}(G_r, \rho_r)$).

There is a canonical morphism of algebraic varieties

$$N_r^* : \mathfrak{X}_{\mathfrak{s}} \to \mathfrak{X}_{\mathfrak{s}_r}$$
$$(T, \xi)_G \mapsto (T_r, \xi \circ N_r)_{G_r},$$

where here ξ denotes an extension of some $W(F)$-conjugate of χ. This induces an algebra homomorphism

(4.1.1) $$N_r : \mathbb{C}[\mathfrak{X}_{\mathfrak{s}_r}] \to \mathbb{C}[\mathfrak{X}_{\mathfrak{s}}].$$

DEFINITION 4.1.1. – Define the *base change homomorphism* $b_r : \mathcal{Z}(G_r, \rho_r) \to \mathcal{Z}(G, \rho)$ to be the unique morphism making the following diagram commute:

$$
\begin{array}{ccc}
\mathbb{C}[\mathfrak{X}_{\mathfrak{s}_r}] & \xrightarrow[\sim]{\beta} & \mathcal{Z}(G_r, \rho_r) \\
\downarrow{\scriptstyle N_r} & & \downarrow{\scriptstyle b_r} \\
\mathbb{C}[\mathfrak{X}_{\mathfrak{s}}] & \xrightarrow[\sim]{\beta} & \mathcal{Z}(G, \rho).
\end{array}
$$

4.2. Characterization by actions on principal series

LEMMA 4.2.1. – *For any character $\xi : T(F) \to \mathbb{C}^\times$ which extends some $W(F)$-conjugate of χ, define $\xi_r := \xi \circ N_r$, a character on $T(F_r)$ which extends some $W(F_r)$-conjugate of χ_r. Let $z \in \mathcal{H}(G_r, \rho_r)$. Then $b_r(z)$ is the unique element in $\mathcal{Z}(G, \rho)$ which acts on every module $i_B^G(\xi)^\rho$ by the same scalar by which z acts on $i_{B_r}^{G_r}(\xi_r)^{\rho_r}$. In other words,*

(4.2.1) $$\beta^{-1}(b_r(z))(\xi) = \beta^{-1}(z)(\xi_r).$$

In terms of right actions this means: for $z \in \mathcal{Z}(G_r, \rho_r)$, $b_r(z)$ acts on the right on $i_B^G(\xi^{-1})^{\rho^{-1}}$ by the scalar by which z acts on the right on $i_{B_r}^{G_r}(\xi_r^{-1})^{\rho_r^{-1}}$, that is,

$$(4.2.2) \qquad\qquad ch_{\xi^{-1}}(b_r(z)) = ch_{\xi_r^{-1}}(z)$$

(cf. Lemma 3.5.1).

This is reflective of the fact that b_r is compatible with the involutions ι_r (resp. ι) of $\mathcal{H}(G_r)$ (resp. $\mathcal{H}(G)$), in the sense that the following diagram commutes

$$
\begin{array}{ccc}
\mathcal{Z}(G_r, \rho_r) & \xrightarrow{b_r} & \mathcal{Z}(G, \rho) \\
{\scriptstyle \iota_r}\downarrow & & \downarrow{\scriptstyle \iota} \\
\mathcal{Z}(G_r, \rho_r^{-1}) & \xrightarrow{b_r} & \mathcal{Z}(G, \rho^{-1}).
\end{array}
$$

Indeed, this follows from (3.5.2) and the obvious compatibility of N_r and $\iota : \mathbb{C}[\mathfrak{X}_{\mathfrak{s}}] \to \mathbb{C}[\mathfrak{X}_{\mathfrak{s}}]$.

4.3. Compatibility with conjugation by $w \in W(F)$

Fix $w \in W(F)$, and use the same symbol to denote its lift in $N_G(T) \cap K$. The character ${}^w\chi$ is defined by ${}^w\chi(t) = \chi(w^{-1}tw)$. (Similarly define ${}^w\Phi(\cdot) = \Phi(w^{-1} \cdot w)$ for any suitable function Φ.) Also write ${}^wI := wIw^{-1}$ and ${}^wI^+ = wI^+w^{-1}$. The character ${}^w\rho : {}^wI/{}^wI^+ \to \mathbb{C}^\times$ is defined using ${}^w\chi$; in fact ${}^w\rho(wiw^{-1}) = \rho(i)$ for $i \in I$. There is an isomorphism $\mathcal{H}(G, I, \rho) \xrightarrow{\sim} \mathcal{H}(G, {}^wI, {}^w\rho)$, given by $h \mapsto {}^wh$.

Let ξ denote an extension of a $W(F)$-conjugate of χ. Write ${}^wB := wBw^{-1}$. Then $\Psi \mapsto {}^w\Psi$ gives an isomorphism

$$i_B^G(\xi^{-1})^{\rho^{-1}} \xrightarrow{\sim} i_{{}^wB}^G({}^w\xi^{-1})^{{}^w\rho^{-1}}.$$

This intertwines the right actions of $h \in \mathcal{H}(G, I, \rho)$ and ${}^wh \in \mathcal{H}(G, {}^wI, {}^w\rho)$, in the sense that

$$ {}^w(\Psi * h) = {}^w\Psi * {}^wh. $$

Applying this to $h = z \in \mathcal{Z}(G, I, \rho)$ and taking Remark 3.4.1 into account yields the next lemma.

LEMMA 4.3.1. – *The following diagram is commutative:*

$$
\begin{array}{ccc}
\mathbb{C}[\mathfrak{X}_\chi] & \xrightarrow[\sim]{\beta} & \mathcal{Z}(G, I, \rho) \\
{\scriptstyle =}\downarrow & & \downarrow{\scriptstyle z \mapsto {}^wz} \\
\mathbb{C}[\mathfrak{X}_\chi] & \xrightarrow[\sim]{\beta} & \mathcal{Z}(G, {}^wI, {}^w\rho).
\end{array}
$$

The compatibility of $z \mapsto {}^wz$ with base change follows.

LEMMA 4.3.2. – *For any $w \in W(F)$, the following diagram is commutative:*

$$
\begin{array}{ccc}
\mathcal{Z}(G_r, \rho_r) & \xrightarrow{z \mapsto {}^wz} & \mathcal{Z}(G_r, {}^w\rho_r) \\
{\scriptstyle b_r}\downarrow & & \downarrow{\scriptstyle b_r} \\
\mathcal{Z}(G, \rho) & \xrightarrow{z \mapsto {}^wz} & \mathcal{Z}(G, {}^w\rho).
\end{array}
$$

5. Constant term homomorphism

5.1. Abstract definition

Let M denote an F-Levi subgroup of G which contains the torus T. Let $\mathfrak{s}_\chi^M$ denote the inertial class associated to χ but for the group M rather than the group G. Let $\mathscr{R}_\chi(M)$ denote the corresponding principal series block of $\mathscr{R}(M)$, and let $\mathfrak{X}_\chi^M$ denote the corresponding variety of supercuspidal supports $(T, \xi)_M$.

There is a canonical surjective morphism of varieties

$$(5.1.1) \qquad\qquad c_M^{G*} : \mathfrak{X}_\chi^M \to \mathfrak{X}_\chi$$
$$(T, \xi)_M \mapsto (T, \xi)_G.$$

This induces an algebra homomorphism

$$(5.1.2) \qquad\qquad c_M^G : \mathbb{C}[\mathfrak{X}_\chi] \to \mathbb{C}[\mathfrak{X}_\chi^M].$$

We call c_M^G the *constant term homomorphism*. In the next few subsections we will give a concrete description which will also justify this terminology. We will also prove its compatibility with the base change homomorphism. As the first step, note that the following diagram is obviously commutative:

$$(5.1.3) \qquad\qquad
\begin{array}{ccc}
\mathbb{C}[\mathfrak{X}_{\chi_r}] & \xrightarrow{\ c_{M_r}^{G_r}\ } & \mathbb{C}[\mathfrak{X}_{\chi_r}^{M_r}] \\
{\scriptstyle N_r}\Big\downarrow & & \Big\downarrow{\scriptstyle N_r} \\
\mathbb{C}[\mathfrak{X}_\chi] & \xrightarrow{\ c_M^G\ } & \mathbb{C}[\mathfrak{X}_\chi^M].
\end{array}$$

5.2. Refined Iwasawa decomposition

Let us suppose M is a *standard*[4] F-Levi subgroup; this means that M is a Levi factor of a *standard* (i.e., containing B) F-parabolic subgroup P with unipotent radical N. Fix a Levi decomposition $P = MN$.

Consider the Kostant representatives $W^P \subset W(F)$, i.e. the set of minimal coset representatives for the elements of $W_M(F)\backslash W(F)$, with respect to the Bruhat order on $W(F)$ defined by the Borel subgroup B. We choose once and for all a lift $\dot{w} \in N_G(T)(F) \cap K$ for each $w \in W^P$; but from now on we denote $\dot{w}$ simply by w.

Let P_r denote $P(F_r)$, etc. Let I_r denote the Iwahori subgroup of $G_r := G(F_r)$ corresponding to I, so that $I_r \cap G(F) = I$. Abbreviate the relative Weyl group $W(F_r)$ by W_r and its set of Kostant representatives by W_r^P. The refined Iwasawa decomposition states that

$$(5.2.1) \qquad\qquad G_r = \coprod_{w \in W_r^P} P_r\, w\, I_r.$$

5.3. Levi factorization for $P \cap {}^w I$

LEMMA 5.3.1. – *For each $w \in W$, we have the factorization $P \cap {}^w I = (M \cap {}^w I)\,(N \cap {}^w I)$.*

Proof. – This follows from [3, 5.2.4] (cf. Section 11). $\qquad\qquad\square$

---

[4] This assumption is convenient but not necessary for this subsection and the next.

5.4. The concrete definition of the constant term

Fix an Iwahori subgroup J which is $W(F)$-conjugate to I.

For a compactly-supported locally constant function f on $G(F)$, we define the compactly-supported locally constant function $f^{(P)}$ on $M(F)$ by the formula

$$(5.4.1) \qquad f^{(P)}(m) = \delta_P^{1/2}(m) \int_{N(F)} f(mn)\, dn = \delta_P^{-1/2}(m) \int_{N(F)} f(nm)\, dn,$$

where the Haar measure dn on $N(F)$ is normalized to give $N(F) \cap J$ measure 1. Warning: $f^{(P)}$ depends on the choice of J, and in what follows we will allow J to vary.

Since $J/J^+ \cong T(F)_1/T(F)_1^+$, the character χ gives rise to a smooth character $\rho : J \to \mathbb{C}^\times$ as before, and (J, ρ) is also a type for $\mathscr{R}_\chi(G)$. Write the corresponding Hecke algebra (resp. its center) as $\mathscr{H}(G, J, \rho)$ (resp. $\mathcal{Z}(G, J, \rho)$) in order to emphasize the subgroup J.

Let $J_M = J \cap M$. Note that J_M is an Iwahori subgroup of M (cf. e.g., [11, Lemma 2.9.1]). We get the character $\rho : J_M \to \mathbb{C}^\times$ from χ and the isomorphism $J_M/J_M^+ \cong T(F)_1/T(F)_1^+$, and (J_M, ρ) is a type for $\mathscr{R}_\chi(M)$.

It is obvious that $f^{(P)} \in \mathscr{H}(M, J_M, \rho)$ if $f \in \mathscr{H}(G, J, \rho)$. The next proposition is the main goal of this section.

PROPOSITION 5.4.1. – *The operation* $f \mapsto f^{(P)}$ *sends* $\mathcal{Z}(G, J, \rho)$ *to* $\mathcal{Z}(M, J_M, \rho)$, *and the following diagram commutes:*

$$
\begin{array}{ccc}
\mathbb{C}[\mathfrak{X}_\chi] & \xrightarrow{\ \ \beta\ \ }_{\sim} & \mathcal{Z}(G, J, \rho) \\
{\scriptstyle c_M^G} \downarrow & & \downarrow {\scriptstyle f \mapsto f^{(P)}} \\
\mathbb{C}[\mathfrak{X}_\chi^M] & \xrightarrow{\ \ \beta\ \ }_{\sim} & \mathcal{Z}(M, J_M, \rho).
\end{array}
$$

In particular, when restricted to $\mathcal{Z}(G, J, \rho)$ *the operation* $f \mapsto f^{(P)}$ *is an algebra homomorphism and is independent of the choice of* P *having* M *as Levi factor.*

Because of this we will set $c_M^G(f) := f^{(P)}$ for $f \in \mathcal{Z}(G, J, \rho)$ and call it the *constant term of* f.

5.5. Compatibility of base change and constant term homomorphisms

Proposition 5.4.1 and Equation (5.1.3) have the following corollary.

COROLLARY 5.5.1. – *The following diagram commutes:*

$$(5.5.1) \qquad
\begin{array}{ccc}
\mathcal{Z}(G_r, J_r, \rho_r) & \xrightarrow{\ c_{M_r}^{G_r}\ } & \mathcal{Z}(M_r, J_{M_r}, \rho_r) \\
{\scriptstyle b_r} \downarrow & & \downarrow {\scriptstyle b_r} \\
\mathcal{Z}(G, J, \rho) & \xrightarrow{\ c_M^G\ } & \mathcal{Z}(M, J_M, \rho).
\end{array}
$$

5.6. Proof of Proposition 5.4.1

Fix an extension ξ of χ, and set $\xi_1 := \delta_P^{1/2}\xi$. Consider the subspace $i_B^G(\xi_1^{-1})^{\rho^{-1}} \subset i_B^G(\xi_1^{-1})$ whose elements transform under the left J-action according to the character ρ^{-1}. To prove Proposition 5.4.1, we need to prove the following equivalent statement.

PROPOSITION 5.6.1. – *Let* $z \in \mathcal{Z}(G, J, \rho)$. *Then for every* ξ, *the element* $z^{(P)}$ *acts on the right on* $i_{B_M}^M(\xi^{-1})^{\rho^{-1}}$ *by* $ch_{\xi^{-1}}(z)$, *the scalar by which z acts on the right on* $i_B^G(\xi^{-1})^{\rho^{-1}}$.

LEMMA 5.6.2. – *Given* $\psi \in i_{B_M}^M(\xi^{-1})^{\rho^{-1}}$, *there exists* $\Psi \in i_B^G(\xi_1^{-1})^{\rho^{-1}}$ *such that* $\Psi|_M = \psi$.

Proof. – We define Ψ to be the unique element in $i_B^G(\xi_1^{-1})^{\rho^{-1}}$ which is supported on PJ and which extends ψ. That is, for $mn \in MN$ and $j \in J$ we set

$$\Psi(mnj) := \psi(m)\rho^{-1}(j).$$

To see that Ψ belongs to the desired space, we use the fact that $\delta_B^{1/2}\xi_1^{-1} = \delta_{B_M}^{1/2}\xi^{-1}$. To see it is well-defined, we use the Iwahori-factorization of J with respect to $P = MN$, or more precisely a consequence of it: $J \cap P = J \cap M \cdot J \cap N$. See Lemma 5.3.1. $\square$

Proposition 5.6.1 will be proved using the following lemma.

LEMMA 5.6.3. – *Define the right convolution action* $*$ *of* $\mathcal{H}(M, J_M, \rho)$ *on* $i_{B_M}^M(\xi^{-1})^{\rho^{-1}}$ *using the measure dm which gives J_M measure 1. Let* $z \in \mathcal{Z}(G, J, \rho)$. *Let* $\psi \in i_{B_M}^M(\xi^{-1})^{\rho^{-1}}$. *Then for any* $y \in M$, *we have the identity*

$$(5.6.1) \qquad\qquad ch_{\xi_1^{-1}}(z) \cdot \psi(y) = \psi * [\delta_P^{1/2} z^{(P)}](y).$$

Proof. – In addition to dm and dn specified above, we let dg (resp. dj) denote the Haar measure on G (resp. J) giving J measure 1. Let $\Psi \in i_B^G(\xi_1^{-1})^{\rho^{-1}}$ denote the extension of ψ from Lemma 5.6.2. We have

$$\psi * [\delta_P^{1/2} z^{(P)}](y) = \int_M \psi(m)\,\delta_P^{1/2}(m^{-1}y)\,z^{(P)}(m^{-1}y)\,dm$$

$$= \int_M \int_N \Psi(mn)\,z((mn)^{-1}y)\,dn\,dm$$

$$= \int_M \int_N \int_J \Psi(mnj)\,z((mnj)^{-1}y)\,dj\,dn\,dm.$$

Now since Ψ is supported on MNJ, we may use the substitution $g = mnj$ (cf. [11, §4.3]) to write this as

$$\int_G \Psi(g)\,z(g^{-1}y)\,dg = (\Psi * z)(y)$$

$$= ch_{\xi_1^{-1}}(z) \cdot \Psi(y).$$

The lemma follows since $\Psi(y) = \psi(y)$. $\square$

Proof of Proposition 5.6.1: Recalling that $\xi^{-1} = \delta_P^{1/2}\xi_1^{-1}$, Lemma 5.6.3 shows that $\delta_P^{1/2} z^{(P)}$ acts on the right on $i_{B_M}^M (\delta_P^{1/2}\xi_1^{-1})^{\rho^{-1}}$ by $ch_{\xi_1^{-1}}(z)$, the scalar by which z acts on the right on $i_B^G(\xi_1^{-1})^{\rho^{-1}}$. It follows (cf. [11], 4.7.3]) that $z^{(P)}$ acts on the right on $i_{B_M}^M (\xi_1^{-1})^{\rho^{-1}}$ by the same scalar $ch_{\xi_1^{-1}}(z)$. Since ξ_1 ranges over all extensions of χ as ξ does, this proves the proposition. □

6. Descent formulas

6.1. Preliminaries

Fix once and for all Haar measures dg, di on G, I respectively, such that $\mathrm{vol}_{dg}(I) = \mathrm{vol}_{di}(I) = 1$. On $P = MN$ we fix a (left) Haar measure dp such that $\mathrm{vol}_{dp}(P \cap I) = 1$. For $w \in W(F)$, let dp_w (resp. dm_w, dn_w) denote the (left) Haar measure on P (resp. M, N) such that $\mathrm{vol}_{dp_w}(P \cap {}^wI)$ (resp. $\mathrm{vol}_{dm_w}(M \cap {}^wI)$, $\mathrm{vol}_{dn_w}(N \cap {}^wI)$) has the value 1.

Similar conventions will hold for measures on the groups G_r, P_r, I_r, etc.

Recall the Harish-Chandra function on $m \in M(F)$, defined by

$$D_{G(F)/M(F)}(m) = \det\big(1 - \mathrm{Ad}(m^{-1}); \mathrm{Lie}(G(F))/\mathrm{Lie}(M(F))\big).$$

6.2. Descent of (twisted) orbital integrals

Let us recall the set-up: $\gamma \in G(F)$ is a semisimple element, S denotes the F-split component of the center of G_γ°, and $M = \mathrm{Cent}_G(S)$, an F-Levi subgroup. We suppose γ is not elliptic in $G(F)$, so that M is a proper Levi subgroup of G. Choose an F-parabolic $P = MN$ with M as Levi factor. We may assume P and M are standard.

We have $\gamma \in M(F)$ and $G_\gamma^\circ = M_\gamma^\circ$. We assume $\gamma = \mathcal{N}\delta$ for an element $\delta \in G(F_r)$. By Lemma 4.2.1 of [11], we may assume $\delta \in M(F_r)$.

The twisted centralizer $G_{\delta\theta}$ of $\delta\theta$ is an inner form of G_γ whose group of F-points is

$$G_{\delta\theta}(F) = \{g \in G(F_r) \mid g^{-1}\delta\theta(g) = \delta\}.$$

We have $M_{\delta\theta}^\circ = G_{\delta\theta}^\circ$ (cf. [11], §4.4]). We choose compatible measures on the inner forms $G_{\delta\theta}^\circ$ and G_γ°, and use them to form the quotient measures $d\bar{g}$ in the (twisted) orbital integrals of $\phi \in C_c^\infty(G(F_r))$. By definition,

$$(6.2.1) \qquad \mathrm{TO}_{\delta\theta}^{G_r}(\phi) = \int_{G_{\delta\theta}^\circ \backslash G_r} \phi(g^{-1}\delta\theta(g)) \, d\bar{g}.$$

Here and in what follows we denote $G_{\delta\theta}^\circ(F)$ simply by $G_{\delta\theta}^\circ$.

Note that $\rho_r \circ \theta = \rho_r$. Using this and the refined Iwasawa decomposition (5.2.1), the argument in [11], §4.4] yields for $\phi \in \mathcal{H}(G_r, \rho_r)$ the descent formula

$$(6.2.2) \qquad \mathrm{TO}_{\delta\theta}^{G_r}(\phi) = |D_{G(F)/M(F)}(\gamma)|_F^{-1/2} \sum_{w \in W_r^P} \mathrm{TO}_{\delta\theta}^{M_r}\big(({}^{w,\theta}\phi)^{(P_r)}\big),$$

where ${}^{w,\theta}\phi(g) := \phi(w^{-1}g\theta(w))$. Here it is understood that TO^{M_r} is formed using dm_w and $(\cdot)^{(P_r)}$ is formed using dn_w.

Orbital integrals are special cases of twisted orbital integrals (take $r = 1$ and $\delta = \gamma$). For $\phi \in \mathcal{Z}(G_r, \rho_r)$, we have the following descent formula for $b_r(\phi) \in \mathcal{Z}(G, \rho)$:

$$(6.2.3) \qquad \mathrm{O}_\gamma^G(b_r(\phi)) = |D_{G(F)/M(F)}(\gamma)|_F^{-1/2} \sum_{w \in W^P} \mathrm{O}_\gamma^M[(\,^w b_r(\phi))^{(P)}].$$

Using the compatibility of b_r with conjugation by w (Lemma 4.3.2) and constant term (Corollary 5.5.1, taking $J = \,^w I$), we can write this as follows:

$$(6.2.4) \qquad \mathrm{O}_\gamma^G(b_r(\phi)) = |D_{G(F)/M(F)}(\gamma)|_F^{-1/2} \sum_{w \in W^P} \mathrm{O}_\gamma^M\big(b_r((\,^w \phi)^{(P_r)})\big).$$

6.3. Comparing descent formulas

We compare the formulas (6.2.2) and (6.2.4) following the method of [11, §4.5]. Note that $W^P = (W_r^P)^\theta$. Thus the key point is the following lemma.

LEMMA 6.3.1. – *Assume $\phi \in \mathcal{Z}(G_r, \rho_r)$. Then the summands in (6.2.2) indexed by elements $w \in W_r^P$ with $\theta(w) \neq w$ are zero.*

Proof. – This is the analogue of Lemma 4.5.3 in [11] which handled the case of parahoric Hecke algebras. The proof in pp. 602-608 of loc. cit. goes over to the present context nearly word-for-word; we omit the details. □

Comparing (6.2.2) and (6.2.4), it is now clear that the fundamental lemma for γ, δ, G follows from the fundamental lemma for γ, δ, M. Thus, by induction on the semisimple rank of G, we are reduced to considering elliptic semisimple elements γ.

REMARK 6.3.2. – Assume G_{ad} is split over F. Then $W^P = W_r^P$ and Lemma 6.3.1 is not needed to compare (6.2.2) with (6.2.4).

7. Reductions

7.1. Vanishing statement when γ is not a norm

LEMMA 7.1.1. – *Let $\phi \in \mathcal{Z}(G_r, \rho_r)$. If γ is not a norm from G_r, then $\mathrm{SO}_\gamma(b_r(\phi)) = 0$.*

We will prove the stronger fact that $\mathrm{O}_\gamma(b_r(\phi)) = 0$. The corresponding statement for spherical Hecke algebras was proved by Labesse [26, Lemme 3.7.1]. The proof here is similar, but we give some details for the sake of completeness.

First, our descent formula (6.2.4) reduces us to the case where γ is elliptic in G. Consider the canonical map $p : G_{\mathrm{sc}} \to G$ and the abelian group

$$\mathbf{H}_{ab}^0(F, G) = G(F)/p(G_{\mathrm{sc}}(F)).$$

Proposition 2.5.3 of loc. cit. shows that an elliptic element γ is a norm from G_r if and only if its image in $\mathbf{H}_{ab}^0(F, G)$ is a norm. Now the required vanishing follows from the following result (cf. [26, Lemme 3.7.1]).

LEMMA 7.1.2. – *Let $f = b_r(\phi)$ for $\phi \in \mathcal{Z}(G_r, \rho_r)$. Let $x \in G(F)$ be any element such that, for some character η on the group $\mathbf{H}_{ab}^0(F, G)$ which is trivial on the norms, we have $\eta(x) \neq 1$. Then $f(x) = 0$.*

Proof. – By pulling back along the projection $G(F) \to \mathbf{H}^0_{ab}(F, G)$, we view η as a character $\eta : G(F) \to \mathbb{C}^\times$; thus the condition $\eta(x) \neq 1$ makes sense.

It is not hard to show that η is necessarily an unramified character of $G(F)$, meaning it is trivial on $G(F)_1 = G(F) \cap G(L)_1$, where $G(L)_1$ is the kernel of the Kottwitz homomorphism $\kappa : G(L) \to X^*(Z(\widehat{G})^I)$ (cf. (2.0.3)). Equivalently, η is trivial on every parahoric subgroup of $G(F)$ (as $G(F)_1$ is the group generated by the parahoric subgroups of $G(F)$, cf. [13]). The restriction of η to $T(F)$ is also unramified (trivial on $T(F)_1$).

Thus, $f\eta \in \mathcal{H}(G, \rho)$. Now by examining the right convolution action of $f\eta$ on $i_B^G(\xi^{-1})^{\rho^{-1}}$, we see that $f\eta \in \mathcal{Z}(G, \rho)$ and in fact

$$ch_{\xi^{-1}}(f\eta) = ch_{(\eta\xi)^{-1}}(f).$$

But by (4.2.2), this is

$$ch_{(\eta_r \xi_r)^{-1}}(\phi) = ch_{\xi_r^{-1}}(\phi) = ch_{\xi^{-1}}(f),$$

the first equality holding since $\eta_r = \mathrm{triv}$. But this implies that $f\eta = f$, and in particular

$$f(x)(\eta(x) - 1) = 0.$$

Since $\eta(x) - 1 \neq 0$, the lemma follows. $\qquad\square$

7.2. Reductions when γ is a norm: three lemmas

Our strategy is as in [11, § 5] which handled the case of parahoric Hecke algebras. The first task is to state three key lemmas, which are analogous to Lemmas 5.3.1, 5.3.2, and 5.3.3 from [11]. The proofs go along the same lines; we shall explain the objects where they differ from loc. cit., but we shall omit detailed proofs. We also use freely the notation from loc. cit.

Choose a finite unramified extension $F' \supset F$ which contains F_r and splits G. Consider a z-extension of F-groups

$$1 \longrightarrow Z \longrightarrow H \overset{p}{\longrightarrow} G \longrightarrow 1,$$

where Z is a finite product of copies of $R_{F'/F}\mathbb{G}_m$. As usual, we are assuming $H_{\mathrm{der}} = H_{\mathrm{sc}}$. Recall that p is surjective on F_r-and F-points. Choose an extension of θ to an element, still denoted θ, in $\mathrm{Gal}(F'/F)$. As in loc. cit., the norm homomorphism $N : Z(F_r) \to Z(F)$ is surjective and induces a surjective map $N : Z(F_r)_1 \to Z(F)_1$.

Let $\lambda : Z(F) \to \mathbb{C}^\times$ denote a smooth character, and for $f \in C_c^\infty(H(F))$ set

$$f_\lambda(h) = \int_{Z(F)} f(hz)\lambda^{-1}(z)\, dz$$

where dz is the Haar measure on $Z(F)$ giving $Z(F)_1$ measure 1. Write $\lambda = 1$ for the trivial character, and $\overline{f}$ for the function f_1 when it is viewed as an element in $C_c^\infty(G(F))$. Write λN for the character $\lambda \circ N : Z(F_r) \to \mathbb{C}^\times$.

The depth-zero character χ on $T(F)_1$ determines a depth-zero character χ_H on $T_H(F)_1$, where $T_H := p^{-1}(T)$. Denote by I_H the Iwahori subgroup in H corresponding to the Iwahori subgroup I in G. Let $\rho_H : I_H \to \mathbb{C}^\times$ denote the character constructed from χ_H.

Lemma 7.2.1. – *Let* $\phi \in C_c^\infty(H(F_r))$ *and* $f \in C_c^\infty(H(F))$. *We have the following statements:*

(i) *The functions* ϕ, f *are associated if and only if* $\phi_{\lambda N}, f_\lambda$ *are associated for every* λ.

(ii) *Suppose that* $\phi \in \mathcal{H}(G_r, \widetilde{K}, \widetilde{\rho})$ *(resp.* $f \in \mathcal{H}(G, K, \rho)$*) for a compact open subgroup* $\widetilde{K} \subset G_r$ *and character* $\widetilde{\rho} : \widetilde{K} \to \mathbb{C}^\times$ *(resp.* $K \subset G$ *and* $\rho : K \to \mathbb{C}^\times$ *) such that*

- $N(\widetilde{K} \cap Z(F_r)) = K \cap Z(F)$,
- $\widetilde{K} \supset (1 - \theta)(Z(F_r))$, *and*
- $\widetilde{\rho}|_{\widetilde{K} \cap Z(F_r)} = \rho \circ N|_{\widetilde{K} \cap Z(F_r)}$.

Then in (i) *we only need to consider characters* λ *such that*

$$\lambda|_{K \cap Z(F)} = \rho^{-1}|_{K \cap Z(F)}.$$

(iii) *If* $\phi \in \mathcal{H}(H_r, \rho_{Hr})$ *and* $f \in \mathcal{H}(H, \rho_H)$, *then in (i) we need consider only the characters* λ *with* $\lambda|_{Z(F)_1} = \chi^{-1}|_{Z(F)_1}$.

(iv) *The pair* ϕ_1, f_1 *are associated if and only if* $\overline{\phi}, \overline{f}$ *are associated.*

Proof. – Parts (i) and (iv) are proved in [11]. Part (ii) is proved that same way as loc. cit. Lemma 5.3.1(ii). Part (iii) follows from part (ii), taking $K = I_H$, $\widetilde{K} = I_{Hr}$ and $\widetilde{\rho} = \rho_{Hr}$. □

The following lemma is proved similarly to [11, Lemma 5.3.2].

Lemma 7.2.2. – *The map* $\phi \mapsto \overline{\phi}$ *determines a surjective homomorphism* $\mathcal{Z}(H_r, \rho_{Hr}) \to \mathcal{Z}(G_r, \rho_r)$. *It is compatible with the base change homomorphisms in the sense that*

(7.2.1) $$b_r(\overline{\phi}) = \overline{b_r(\phi)}.$$

Now for any character $\lambda : Z(F) \to \mathbb{C}^\times$ such that $\lambda|_{Z(F)_1} = \chi^{-1}|_{Z(F)_1}$, consider the algebra $\mathcal{H}_{\rho_{Hr}, \lambda N}$ which consists of the locally constant functions ϕ on $H(F_r)$ which are compactly supported modulo $Z(F_r)$ and which transform under right and left multiplications by $i \in I_{Hr}$ (resp. $z \in Z(F_r)$) by $\rho_{Hr}^{-1}(i)$ (resp. $\lambda N(z)$). One proves the following lemma by imitating the argument of [11, Lemma 5.3.3], using this algebra in place of the algebra $\mathcal{H}_{J, \chi N}$ used there. Along the way, the argument relies on our Lemma 7.2.1(iii-iv) and Lemma 7.2.2.

Lemma 7.2.3. – *Assume* $Z = Z(H)$, *so that* $G = H_{\mathrm{ad}}$. *Suppose that* $\overline{\phi}, b_r(\overline{\phi})$ *are associated for every* $\phi \in \mathcal{Z}(H_r, \rho_{Hr})$. *Then* $\phi, b_r(\phi)$ *are associated for every* $\phi \in \mathcal{Z}(H_r, \rho_{Hr})$.

7.3. The reduction steps

We may assume $\gamma = \mathcal{N}(\delta)$. We follow [11, § 5.4].

(1) *We may assume $G_{\mathrm{der}} = G_{\mathrm{sc}}$.* Using Lemmas 7.2.1 and 7.2.2, we see that the fundamental lemma for H implies the fundamental lemma for G.

(2) *We may assume γ is elliptic.* We use our descent formulas (6.2.2) and (6.2.4), along with Lemma 4.2.1 of [11]. (See [11, § 5.4(2)].)

(3) *We may assume γ is regular.* This is proved by Clozel [7, Prop. 7.2].

(4) *We may assume G is such that $G_{\mathrm{der}} = G_{\mathrm{sc}}$ and $Z(G)$ is an induced torus (of the same form as Z above).* We work under assumptions (1-3), and follow Clozel's argument in [7, Section 6.1(b)], with a few modifications. Where Clozel uses the Satake isomorphism, we use the inverse β^{-1} of the Bernstein isomorphism (3.4.1). The homomorphism j used by Clozel is replaced by the natural homomorphism

$$j : \mathcal{Z}(G, \rho) \hookrightarrow \mathcal{Z}(G', \rho')$$

coming from the exact sequence

$$1 \to G \to G' \to Q \to 1$$

of loc. cit. in the obvious way. Here G and G' are unramified groups with the properties that $G_{\mathrm{der}} = G_{\mathrm{sc}}$, $G'_{\mathrm{der}} = G'_{\mathrm{sc}}$, and $Z(G')$ is an induced torus; we assume the fundamental lemma holds for the group G' and our present goal is to deduce that it must hold for G. The main point is to justify the analogues of Clozel's equalities (which we write here in slightly different notation)

$$(7.3.1) \qquad \mathrm{SO}_\gamma^G(f, dt, dg) = c_T \, \mathrm{SO}_\gamma^{G'}(jf, dt', dg')$$

$$\mathrm{SO}_{\delta\theta}^{G_r}(\phi, dt, dg_{F_r}) = C_T \, \mathrm{SO}_{\delta\theta}^{G'_r}(j\phi, dt', dg'_{F_r}).$$

We need to prove the equalities (7.3.1) for all functions $f \in \mathcal{Z}(G, \rho)$ and $\phi \in \mathcal{Z}(G_r, \rho_r)$ and for all $(\theta\text{-})$regular $(\theta\text{-})$elliptic elements γ and δ in G whose $(\theta\text{-})$centralizer is the elliptic F-torus T[5]. Crucially, the constants c_T and C_T making (7.3.1) hold must depend only on the torus T and various choices of measures $(dt, dt', \mathrm{etc.})$, and we must have $c_T = C_T$.

The proof of the existence of the constant C_T is unfortunately rather laborious. The details we present apply as well to the spherical case considered by Clozel and to other settings, so we hope their potential use to the reader justifies their inclusion here.

To prove the existence of the constants c_T and C_T, one first examines those functions f, ϕ which correspond under Morris's Hecke algebra isomorphisms [28] to Bernstein functions in suitable Iwahori-Hecke algebras, and uses the fact that the homomorphisms j are compatible with the Hecke algebra isomorphisms. Let us make this more precise, using some notation from [14, § 9], and [30]. The Hecke algebra isomorphisms take the form

$$(7.3.2) \qquad \mathcal{H}(G, \rho) \cong H_{\mathrm{aff}}(\widetilde{W}_H) \widetilde{\otimes} \mathbb{C}[W_\chi / W_\chi^\circ],$$

where H is a certain connected reductive F-group with finite Weyl group W_χ°, with extended affine Weyl group $\widetilde{W}_H = X_*(A) \rtimes W_\chi^\circ$, and with associated affine Hecke algebra

---

[5] Warning: in this subsection and the next, T will denote an arbitrary elliptic torus, not the standard Cartan we denote with this symbol elsewhere!

$H_{\text{aff}}(\widetilde{W}_H)$. Each W_χ°-orbit representative $\lambda \in X_*(A)$ gives rise to a Bernstein function $z_\lambda \in Z(H_{\text{aff}}(\widetilde{W}_H))$ and for each W_χ-orbit representative μ we have the Bernstein function

$$Z_\mu := \sum_\lambda z_\lambda$$

where λ ranges over W_χ°-orbit representatives in $W_\chi \cdot \mu$; the Z_μ form a basis for the center of the right hand side of (7.3.2). If $f \in \mathcal{Z}(G, \rho)$ corresponds to Z_μ, then $jf \in \mathcal{Z}(G', \rho')$ corresponds to Z'_μ defined in the same way as Z_μ, and we may write

$$f = \sum_{w \in \mathscr{J}} a_w \, [In_w I]_{\check{\chi}}$$

$$jf = \sum_{w \in \mathscr{J}'} a'_w \, [I'n_w I']_{\check{\chi}'},$$

for constants $a_w, a'_w \in \mathbb{C}$ indexed by certain finite subsets $\mathscr{J}, \mathscr{J}' \subset \widetilde{W}_H$ determined by μ. (We are writing the basis elements $[In_w I]_{\check{\chi}} \in \mathcal{H}(G, \rho)$, etc., following the notation of [14, §9].) In fact $\mathscr{J} = \mathscr{J}'$ and the corresponding coefficients are equal, namely $a_w = a'_w$. This follows, in view of the precise form of the Hecke algebra isomorphisms (see [14, Theorem 9.3.1] for the case where G is split and $W_\chi^\circ = W_\chi$), from the corresponding equality of Iwahori-Matsumoto coefficients of Z_μ and Z'_μ (which is easy to prove from their definitions).

In light of this, the existence of the constants c_T, C_T follows via the next lemma. We state this only in the twisted case (the untwisted case being the special case $r = 1$ of the twisted case). Recall that $\widetilde{W}_r$ denotes the extended affine Weyl group for G_r.

LEMMA 7.3.1. – *There exists a constant C_T such that, for every $\delta \in G_r$ with elliptic regular norm in T, and for every $w \in \widetilde{W}_r$, we have*

$$(7.3.3) \qquad \mathrm{SO}_{\delta\theta}^{G_r}([I_r n_w I_r]_{\check{\chi}_r}) = C_T \, \mathrm{SO}_{\delta\theta}^{G'_r}([I'_r n_w I'_r]_{\check{\chi}'_r}).$$

We postpone the proof of this lemma to the following subsection.

The matching of $jf, j\phi$ will force that of f, ϕ provided that $c_T = C_T$ for all maximal tori T in G. Since the homomorphisms j are compatible with the base change homomorphisms, this will show that the fundamental lemma for G' indeed implies the fundamental lemma for G. In his situation, Clozel justifies the equality $c_T = C_T$ by invoking the matching of unit elements 1_K in spherical Hecke algebras and using the fact that orbital integrals of 1_K are never identically zero on any torus T in G.

In our case, the construction of c_T, C_T given in the proof of Lemma 7.3.1 shows that they do not depend on χ and they may be understood by taking $\chi = \text{triv}$ and $w = 1$. Thus, our equality $c_T = C_T$ follows from the fact that 1_{I_r} and 1_I are associated [18], and from the fact that the stable orbital integrals of 1_I do not vanish identically on any torus T in G. This last fact is evident for elliptic tori by the description of $O_\gamma(1_I)$ in terms of the cardinality of the (non-empty) fixed-point set for the action of an elliptic $\gamma \in G$ on the building for G. (Although we do not need this, the case of non-elliptic tori follows, by using our descent formula (6.2.3), with $b(\phi_r)$ replaced by 1_I, to reduce to the elliptic case.)

(5) *We may assume G is adjoint.* We know by (4) that the fundamental lemma holds provided it holds for all groups having the form of H. By Lemma 7.2.3, the fundamental lemma for $G := H_{\text{ad}}$ implies the fundamental lemma for H.

(6) *We may assume γ is strongly regular elliptic.* This is explained by Clozel [7, p. 292] (see [11, §5.4(6)].

Conclusion: We may assume that $G = G_{\mathrm{ad}}$ and γ is a strongly regular elliptic semisimple element.

7.4. Proof of Lemma 7.3.1

I am very grateful to Robert Kottwitz for his invaluable help with this lemma. Recall that the F-unramified groups G, G' we are concerned with satisfy $G_{\mathrm{sc}} = G_{\mathrm{der}} = G'_{\mathrm{der}} = G'_{\mathrm{sc}}$ and $G/Z = G'/Z'$ where Z resp. Z' denotes the center of G resp. G'[6]. By construction, the group G' sits in a commutative diagram

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & G & \longrightarrow & G' & \overset{p}{\longrightarrow} & Q & \longrightarrow & 1 \\
& & \uparrow & & \uparrow & & \| & & \\
1 & \longrightarrow & S & \longrightarrow & S' & \longrightarrow & Q & \longrightarrow & 1
\end{array}
$$

and the (maximal Cartan) tori S, S' and Q are all unramified over F.

Let A_G denote the maximal F-split torus in the center of G. Set $\Lambda = X_*(S)/Q^\vee$ where $Q^\vee$ denotes the coroot lattice. Write $\Lambda(F)$ for the $\mathrm{Gal}(\bar{F}/F)$-fixed points on Λ. Recall from [23] that there is a *surjective* homomorphism

$$(7.4.1) \qquad \kappa : G(F) \twoheadrightarrow \Lambda(F)$$

(cf. (2.0.3)). Let $G(F)_1$ denote the kernel of κ. By Lemma 7.4.1(iii) below, the canonical map $S(F)_1 \to (G/G_{\mathrm{der}})(F)_1$ is surjective and hence $G(F)_1 = G_{\mathrm{sc}}(F)\, S(F)_1$. Since $S(F)_1 \subset I$ (i.e. we may assume S and I are in good relative position), we have

$$(7.4.2) \qquad G(F)_1 = G_{\mathrm{sc}}(F)\, I.$$

Let Λ_{ad} denote the analogue of Λ for the group G_{ad}. There is a short exact sequence

$$0 \to X_*(A_G) \to \Lambda(F) \to \Lambda_{\mathrm{ad}}(F),$$

and hence $\Lambda(F)/X_*(A_G)$ is a finite abelian group.

We can apply the constructions above to get corresponding objects $A_{G'}$, Λ', etc., for G'. There is a canonical map $\Lambda(F)/X_*(A_G) \to \Lambda'(F)/X_*(A_{G'})$. It turns out that this is injective (one ingredient is the fact that $Q = Z'/Z$, which comes from the construction of G' as in [7]). Also, let I'_r denote the Iwahori subgroup of $G'(F_r)$ corresponding to $I_r \subset G(F_r)$.

We will need the following lemmas. For the rest of this subsection, we assume all elements δ_0, δ, etc. belonging to $G(F_r)$ or $G'(F_r)$ have strongly regular semisimple norms. Let $D := G/G_{\mathrm{der}}$ resp. $D' = G'/G'_{\mathrm{der}}$ and denote by $c : G' \to D'$ the projection homomorphism.

LEMMA 7.4.1. – *The following statements hold:*
(i) $p : S'(F)_1 \to Q(F)_1$ *is surjective.*
(ii) $p(G'(F)_1) = Q(F)_1$.
(iii) *If $H \subset G$ is any maximal F-torus, then $c : H(F)_1 \to D(F)_1$ is surjective.*

---

[6] Much of the following discussion does not require G_{der} to be simply-connected.

(iv) *If $\delta \in G'(F_r)$, we have $G'(F)_1 \subseteq G'_{\delta\theta}(F)_1 \, G_{\mathrm{sc}}(F_r)$.*

Proof. – Part (i). Since S, S', and Q are all unramified tori, their Kottwitz homomorphisms yield a commutative diagram with exact rows and surjective vertical arrows

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & S(L) & \longrightarrow & S'(L) & \xrightarrow{\;p\;} & Q(L) & \longrightarrow & 1 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X_*(S) & \longrightarrow & X_*(S') & \xrightarrow{\;p\;} & X_*(Q) & \longrightarrow & 0.
\end{array}
$$

The snake lemma gives an exact sequence

$$
1 \longrightarrow S(L)_1 \longrightarrow S'(L)_1 \xrightarrow{\;p\;} Q(L)_1 \longrightarrow 1.
$$

Now (i) follows since $H^1(\langle\sigma\rangle, S(L)_1) = 0$ ([23], 7.6.1]). For part (iii), set $H_{\mathrm{sc}} := H \cap G_{\mathrm{sc}}$, and argue the same way; we get the exact sequence

$$
1 \longrightarrow H_{\mathrm{sc}}(L)_1 \longrightarrow H(L)_1 \xrightarrow{\;c\;} D(L)_1 \longrightarrow 1
$$

because $X_*(H_{\mathrm{sc}})_I$ is torsion-free.

Now, part (ii) follows from part (i), since (i) gives the non-trivial inclusion in the chain

$$
Q(F)_1 \subseteq p(S'(F)_1) \subseteq p(G'(F)_1) \subseteq Q(F)_1.
$$

Finally, we deduce (iv) from (iii) applied to the group G' with $H := G'_\gamma$, where $\gamma \in G'(F)$ is the norm of $\delta \in G'(F_r)$. The groups $G'_{\delta\theta}$ and G'_γ are inner forms over F hence are F-isomorphic since they are tori. There is an explicit isomorphism over F_r ([17]), yielding a commutative diagram

$$
\begin{array}{ccc}
G'_{\delta\theta}(F_r)_1 & \xrightarrow{\;c\;} & D'_{c(\delta)\theta}(F_r)_1 \\
\wr \downarrow & & \wr \downarrow \\
G'_\gamma(F_r)_1 & \xrightarrow{\;c\;} & D'(F_r)_1.
\end{array}
$$

Because of (iii), we see that the horizontal arrows are surjective and remain so upon taking $\mathrm{Gal}(F_r/F)$-invariants. Doing so to the top arrow just gives the natural map

$$
c : G'_{\delta\theta}(F)_1 \to D'(F)_1,
$$

hence this is surjective. Since $\ker(c) = G'_{\mathrm{sc}} = G_{\mathrm{sc}}$, (iv) follows easily. $\qquad\square$

We use the symbol w to denote any lift $n_w \in N_G(S)(F_r)$ for $w \in \widetilde{W}_r$, which we identify with $N_G(S)(F_r)/S(F_r)_1$.

LEMMA 7.4.2. – *Let $\delta_0 \in G(F_r)$. Suppose $\delta' \in G'(F_r)$ is stably θ-conjugate to δ_0 and belongs to $I'_r w I'_r$. Then δ' is I'_r-θ-conjugate to an element in $G(F_r) \cap I'_r w I'_r$.*

Proof. – First note that since $p(I'_r) \subseteq Q(F_r)_1$ and $p(w) = 1$, we have $p(\delta') \in Q(F_r)_1$. Writing $g'^{-1}\delta_0\theta(g') = \delta'$ for some $g' \in G'(\bar{F})$ and applying p, one checks that $N_r(p(\delta')) = 1$. Now since $H^1(F_r/F, Q(F_r)_1) = 1$ (comp. (8.3.4)), it follows that $p(\delta') \in (1 - \theta)(Q(F_r)_1)$. By Lemma 7.4.1 (i), $p : S'(F_r)_1 \to Q(F_r)_1$ is surjective. Hence there exists $s' \in S'(F_r)_1$ such that $p(s'\theta(s')^{-1}) = p(\delta')$. But then $s'^{-1}\delta'\theta(s') \in G(F_r)$. $\qquad\square$

Now we fix an element $\delta_0 \in G(F_r)$ which we assume is θ-elliptic and θ-regular semisimple. Also fix $w \in \widetilde{W}_r$ as above, and abbreviate the functions $[I_r n_w I_r]_{\check{\chi}_r}$ resp. $[I'_r n_w I'_r]_{\check{\chi}'_r}$ by 1_w resp. $1'_w$. We seek to compare $\mathrm{SO}^{G_r}_{\delta_0\theta}(1_w)$ with $\mathrm{SO}^{G'_r}_{\delta_0\theta}(1'_w)$.

Lemma 7.4.2 implies that if a $G'(F_r)$-θ-conjugacy class in the stable G'-θ-conjugacy class of δ_0 contributes to $\mathrm{SO}^{G'_r}_{\delta_0\theta}(1'_w)$, then that class meets $G(F_r)$. Hence we may write $\mathrm{SO}^{G'_r}_{\delta_0\theta}(1'_w)$ as a sum of terms $\mathrm{TO}^{G'_r}_{\delta\theta}(1'_w)$ where $\delta \in G(F_r)$ is stably G'-θ-conjugate to δ_0.

Now we fix such a δ. We will express $\mathrm{TO}^{G'_r}_{\delta\theta}(1'_w)$ in terms of certain integrals $\mathrm{TO}^{G_r}_{\delta_\lambda\theta}(1_w)$. However, we start by studying $\mathrm{TO}^{G_r}_{\delta\theta}(1_w)$. Since δ is θ-elliptic, we may use the following version of twisted orbital integrals

$$\mathrm{TO}^{G_r}_{\delta\theta}(1_w) = \int_{A_G(F)\backslash G(F_r)} 1_w(g^{-1}\delta\theta(g)) \frac{dg}{dz},$$

where $\mathrm{vol}_{dg}(I_r) = 1$ and $\mathrm{vol}_{dz}(A_G(\mathcal{O})) = 1$. This integral vanishes unless some θ-conjugate δ_1 of δ satisfies $\kappa(\delta_1) = \kappa(w)$. So from now on we may assume that $\kappa(\delta) = \kappa(w)$. We embed $X_*(A_G)$ in $A_G(F) \subset G(F)$ by $\nu \mapsto \varpi^\nu$ for any uniformizer ϖ. The integral can be written as

$$\sum_{g \in X_*(A_G)\backslash G(F_r)/I_r} 1_w(g^{-1}\delta\theta(g)).$$

Clearly the sum may be taken only over elements such that $g^{-1}\delta\theta(g) \in I_r n_w I_r$. For these, $\kappa(g) \in \Lambda(F_r)/X_*(A_G)$ is fixed by θ, i.e., $\kappa(g) \in \Lambda(F)/X_*(A_G)$. We can then write the above sum as

$$\sum_{\substack{\lambda \in \Lambda(F)/X_*(A_G)}} \sum_{\substack{g \in X_*(A_G)\backslash G_r/I_r \\ \kappa(g)=\lambda}} 1_w(g^{-1}\delta\theta(g)).$$

For each $\lambda \in \Lambda(F)$, use (7.4.1) to *choose* a $g_\lambda \in G(F)$ with $\kappa(g_\lambda) = \lambda$. Using (7.4.2), we see that

$$\{g \in G(F_r)/I_r \mid \kappa(g) = \lambda\} = g_\lambda G_{\mathrm{sc}}(F_r)I_r/I_r = g_\lambda G_{\mathrm{sc}}(F_r)/I_{\mathrm{sc},r},$$

where $I_{\mathrm{sc},r} := I_r \cap G_{\mathrm{sc}}(F_r)$. Set $\delta_\lambda := g_\lambda^{-1}\delta g_\lambda$. We get the expression

$$(7.4.3) \qquad \mathrm{TO}^{G_r}_{\delta\theta}(1_w) = \sum_{\lambda \in \Lambda(F)/X_*(A_G)} \sum_{g \in G_{\mathrm{sc}}(F_r)/I_{\mathrm{sc},r}} 1_w(g^{-1}\delta_\lambda\theta(g)).$$

Now we apply the same reasoning to G'. Since $G'_{\mathrm{sc}} = G_{\mathrm{sc}}$, we find

$$(7.4.4) \qquad \mathrm{TO}^{G'_r}_{\delta\theta}(1'_w) = \sum_{\lambda' \in \Lambda'(F)/X_*(A_{G'})} \sum_{g \in G_{\mathrm{sc}}(F_r)/I_{\mathrm{sc},r}} 1_w(g^{-1}\delta_{\lambda'}\theta(g)).$$

(We used $1'_w(g^{-1}\delta_{\lambda'}\theta(g)) = 1_w(g^{-1}\delta_{\lambda'}\theta(g))$ here.) Now it follows that

$$(7.4.5) \qquad \mathrm{TO}^{G'_r}_{\delta\theta}(1'_w) = \sum_{\lambda' \in \frac{\Lambda'(F)/X_*(A_{G'})}{\Lambda(F)/X_*(A_G)}} \mathrm{TO}^{G_r}_{\delta_{\lambda'}\theta}(1_w).$$

By Lemma 7.4.2, all relevant $G'(F_r)$-θ-conjugacy classes in the stable θ-conjugacy class of δ_0 can be represented by elements δ in $G(F_r)$, so in forming $\mathrm{SO}^{G'_r}_{\delta_0\theta}(1'_w)$ we only sum $\mathrm{TO}^{G'_r}_{\delta\theta}(1'_w)$ over such elements $\delta \in G(F_r)$, for which the relation (7.4.5) holds. Each such δ gives rise to elements $\delta_{\lambda'} = g_{\lambda'}^{-1}\delta g_{\lambda'}$ for various $g_{\lambda'} \in G'(F)$. It follows that $\delta_{\lambda'} \in G(F_r)$ and is stably θ-conjugate to δ, hence to δ_0.

The following helps us understand the right hand side of (7.4.5).

LEMMA 7.4.3. – *We have the following statements.*

(1) *Two elements of the form $\delta_{\lambda'}$ and $\delta_{\lambda''}$ are in the same $G(F_r)$-θ-conjugacy class if and only if λ' and λ'' belong to the same coset in $\frac{\Lambda'(F)/X_*(A_{G'})}{\Lambda(F)/X_*(A_G)}$.*

(2) *Any element $\delta_1 \in G(F_r) \cap I'_r w I'_r$ belonging to the $G'(F_r)$-θ-conjugacy class of δ is $G(F_r)$-θ-conjugate to some element of the form $\delta_{\lambda'}$.*

Proof. – It is clearly enough to prove (1) in the special case $\lambda'' = 0$. First assume there exists $g \in G(F_r)$ such that $\delta = (g_{\lambda'}g)^{-1}\delta\theta(g_{\lambda'}g)$. Then $g_{\lambda'}g \in G'_{\delta\theta}(F)$. Write T' for the F-torus $G'_{\delta\theta}$ and $\kappa_{T'} : T'(F) \to X_*(T')^{\mathrm{Gal}(\bar{F}/F)}$ for the corresponding Kottwitz homomorphism. We see that $\kappa_{T'}(g_{\lambda'}g) \in X_*(T')^{\mathrm{Gal}(\bar{F}/F)} = X_*(A_{G'})$, the equality holding since T' is elliptic over F. Recall that $\kappa(g_{\lambda'}g)$ is the image of $\kappa_{T'}(g_{\lambda'}g)$ in $\Lambda'(F)$. Thus $\lambda' \in X_*(A_{G'})\Lambda(F_r) \cap \Lambda'(F) = X_*(A_{G'})\Lambda(F)$.

For the converse, suppose that $\lambda' \in X_*(A_{G'})\,\Lambda(F)$. Note that $X_*(A_{G'})$ can be embedded into $A_{G'}(F) \subset G'_{\delta\theta}(F)$ by $\nu \mapsto \varpi^\nu$, and that $\kappa_{G'}(\varpi^\nu)$ is the image of ν in $\Lambda'(F)$. Then using Lemma 7.4.1(iv), we have

$$g_{\lambda'} \in A_{G'}(F)\,G(F)\,G'(F)_1$$
$$\subseteq G'_{\delta\theta}(F)\,G(F_r).$$

This implies that $\delta_{\lambda'} = g_{\lambda'}^{-1}\delta\theta(g_{\lambda'})$ is $G(F_r)$-θ-conjugate to δ.

For (2), assume $\delta_1 = g'^{-1}\delta\theta(g')$ for some $g' \in G'(F_r)$. Since $\kappa(\delta_1) = \kappa(w) = \kappa(\delta) \in \Lambda(F_r)$, we see $\kappa(g') \in \Lambda'(F)$; suppose $\kappa(g') = \lambda'$. Then $g' \in g_{\lambda'}G'(F_r)_1 \subseteq g_{\lambda'}G_{\mathrm{sc}}(F_r)\,I'_r$ (by (7.4.2)). We can write

(7.4.6) $$i'^{-1}\delta_1\theta(i') = h^{-1}\delta_{\lambda'}\theta(h)$$

for some $i' \in I'_r$ and $h \in G_{\mathrm{sc}}(F_r)$.

Now $\delta_1 \in G(F_r)$ means that $p(\delta_1) = 1$. Also, the right hand side of (7.4.6) is trivial under p. It follows that $p(i') \in Q(F_r)_1^\theta = Q(F)_1$. Now we apply Lemma 7.4.1(ii,iv) to δ_1. We see that $p(i') \in Q(F)_1 \subseteq p(G'(F)_1) \subseteq p(G'_{\delta_1\theta}(F)_1)$ (since $p(G_{\mathrm{sc}}(F_r)) = 1$). This means that

$$i' \in G'_{\delta_1\theta}(F)_1\,G(F_r).$$

Thus the left hand side of (7.4.6) is $G(F_r)$-θ-conjugate to δ_1, and it follows that δ_1 is $G(F_r)$-θ-conjugate to $\delta_{\lambda'}$. $\square$

Finally, Lemma 7.4.2, (7.4.5), and Lemma 7.4.3 together imply that

$$\mathrm{SO}^{G'_r}_{\delta_0\theta}(1'_w) = \mathrm{SO}^{G_r}_{\delta_0\theta}(1_w)$$

with the normalizations for twisted orbital integrals we fixed above. This completes the proof of Lemma 7.3.1. $\square$

8. Labesse's elementary functions adapted to $\mathscr{R}_{\chi_r}(G_r)$

8.1. Definition

We will now construct the analogues of Labesse's (twisted) elementary functions [24] which are adapted to the Bernstein component $\mathscr{R}_{\chi_r}(G_r)$.

Recall that A^{F_r} denotes a maximal F_r-split torus in G, with centralizer T, and $B = TU$ is the F-rational Borel subgroup defining the dominant Weyl chamber in $X_*(A^{F_r})_{\mathbb{R}}$ or $X_*(A)_{\mathbb{R}}$. Let ρ denote the half-sum of the B-positive roots of G.

Fix any uniformizer ϖ for the field F. Consider a regular dominant cocharacter $\nu \in X_*(A^{F_r})$ and set $u = \nu(\varpi) \in T(F_r)$. Also, let $\tau = N_r(\nu)$, a regular dominant cocharacter in $X_*(A)$, and set $t = \tau(\varpi) \in T(F)$. Thus $t = N_r(u)$.

A construction of Casselman [5] and Deligne [8] associates to any element $g \in G(F)$ an F-rational parabolic $P_g = M_g N_g$ with unipotent radical N_g and F-rational Levi factor M_g. Following Labesse [24], we may construct the analogous objects associated to $u\theta \in G_r \rtimes \langle\theta\rangle$. The resulting subgroups $M_{u\theta}$ (resp. $P_{u\theta}$) of G can be characterized as the set of elements $g \in G$ such that $(u\theta)^n g (u\theta)^{-n}$ remains bounded as n ranges over all integers (resp. all positive integers). In our situation $M_{u\theta} = M_t = T$, and $P_{u\theta} = P_t = B$.

Now fix any F-Levi subgroup M. Write G_r (resp. M_r) for the set of F_r-points $G(F_r)$ (resp. $M(F_r)$). Then M_r acts on $G_r \times M_r$ on the left by $m_1(g, m) = (m_1 g, m_1 m \theta(m_1)^{-1})$. Let $[g, m]$ denote the equivalence class of (g, m) in the quotient space $M_r \backslash G_r \times M_r$. There is a natural morphism of p-adic analytic manifolds

$$(8.1.1) \qquad M_r \backslash G_r \times M_r \to G_r$$
$$[g, m] \mapsto g^{-1} m \theta(g).$$

It is well-known (cf. [7, 24]) that this morphism is generically one-to-one and that the normalized absolute value of its Jacobian at the point $[g, m]$ is $|D_{G(F)/M(F)}(\mathscr{N}(m))|_F$.

Let us apply this to our Levi subgroup $M_{u\theta} = T$; we are now concerned only with the quotient by $T(F_r)_1$ of the compact open subset $I_r \times T(F_r)_1 u \subset G_r \times T_r$. The action of $T(F_r)_1$ on $I_r \times T(F_r)_1 u$ is defined by restricting the above-defined action of T_r on $G_r \times T_r$. The map

$$(8.1.2) \qquad T(F_r)_1 \backslash I_r \times T(F_r)_1 u \to G_r$$
$$[k, mu] \mapsto k^{-1} m u \theta(k)$$

is injective and has the absolute value of its Jacobian everywhere equal to the non-zero number

$$(8.1.3) \qquad |\mathrm{Jac}_{[k,mu]}|_F = \delta_{B_r}^{-1}(u) = q^{r\langle 2\rho, \nu\rangle}.$$

Thus its image is a compact *open* subset, which we will denote $\mathfrak{L}_u$.

DEFINITION 8.1.1. – We define the elementary function ϕ_{u,χ_r} on G_r to vanish off of $\mathfrak{L}_u$, and to have value on $\mathfrak{L}_u$ given by

$$(8.1.4) \qquad \phi_{u,\chi_r}(k^{-1} m u \theta(k)) = \chi_r^{-1}(m)$$

for $k \in I_r$ and $m \in T(F_r)_1$.

We claim that ϕ_{u,χ_r} is well-defined. Indeed, suppose

$$k_1^{-1} m_1 u \theta(k_1) = k_2^{-1} m_2 u \theta(k_2),$$

for $k_1, k_2 \in I_r$ and $m_1, m_2 \in T(F_r)_1$. By Lemma 8.1.2(i) below, the element $k = k_2 k_1^{-1}$ belongs to T_r. Thus it belongs to $T(F_r)_1$ and commutes with u, and we have

$$m_1 k \theta(k^{-1}) = m_2$$

which implies $\chi_r(m_1) = \chi_r(m_2)$. Thus we need only the following result due to Labesse [24].

LEMMA 8.1.2. – *Suppose $g_1, g_2 \in G(\bar{F})$ and $m_1, m_2 \in T(F_r)_1$. Extend θ to an element of $\mathrm{Gal}(\bar{F}/F)$ and use the same symbol θ to denote the induced automorphism of $G(\bar{F})$. We have the following statements.*

(i) *If $g_1^{-1} m_1 u \theta(g_1) = g_2^{-1} m_2 u \theta(g_2)$, then $g_1 \in T(\bar{F}) g_2$.*

(ii) *The θ-centralizer of $m_1 u$ is T; hence $m_1 u$ is strongly θ-regular.*

(iii) *If $g_1 \in G_r$ and $g_1^{-1} m_1 u \theta(g_1)$ lies in the support of ϕ_{u,χ_r}, then $g_1 \in T_r I_r$.*

Proof. – Part (i). Let $g = g_1 g_2^{-1}$. As in Lemma 1 of [24] one proves, using the fact that u commutes with m_1 and m_2, that $(u\theta)^n g (u\theta)^{-n}$ remains bounded as n ranges over $\mathbb{Z}$, and thus $g \in M_{u\theta} = T$. It is enough to let n range only over multiples of r.

Let n be any positive integer. We have

$$g_1^{-1}(m_1 u \theta)^{rn} g_1 = g_2^{-1}(m_2 u \theta)^{rn} g_2,$$

and thus

$$[N_r(m_1) N_r(u) \theta^r]^n = g [N_r(m_2) N_r(u) \theta^r]^n g^{-1}.$$

The terms in each set of brackets $[\cdots]$ pairwise commute, so that we deduce

$$[N_r(u) \theta^r]^n g [N_r(u) \theta^r]^{-n} = N_r(m_1)^{-n} g N_r(m_2)^n.$$

The right hand side remains bounded as $n \in \mathbb{Z}$. The left hand side is $(u\theta)^{rn} g (u\theta)^{-rn}$. This shows $g \in M_{u\theta} = T$ and proves part (i).

Parts (ii) and (iii) follow from part (i). $\qquad\square$

Thus, ϕ_{u,χ_r} is a smooth compactly-supported function on G_r, which is invariant under θ-conjugation by elements in I_r. For $r = 1$ (hence $\theta = \mathrm{id}$ and $u = t$) we get the analogous smooth compactly supported function $f_{t,\chi}$ on G, and it is invariant under conjugation by I.

For later use in Subsection 9.2, we record the following lemma.

LEMMA 8.1.3. – *The functions ϕ_{u,χ_r} belong to $\mathcal{H}(G_r, \rho_r)$.*

Proof. – Labesse shows in [25, Proposition IV.1.1], that $\mathfrak{L}_u = I_r u I_r$; thus the support of ϕ_{u,χ_r} is stable under left or right multiplication by I_r. The first step is to check left I_r^+-invariance of ϕ_{u,χ_r} (the right invariance is similar).

Using the normality of I_r^+ in I_r, it is sufficient to show that an identity of the form

$$(8.1.5) \qquad k_1^{-1} i^+ m_1 u \theta(k_1) = k_2^{-1} m_2 u \theta(k_2),$$

with $i^+ \in I_r^+$, $k_1, k_2 \in I_r$, and $m_1, m_2 \in T(F_r)_1$, implies that

$$(8.1.6) \qquad \chi_r(m_1) = \chi_r(m_2).$$

Let $k := k_2 k_1^{-1}$. Then (8.1.5) gives

(8.1.7) $$m_2^{-1} k i^+ m_1 = u\theta(k) u^{-1}.$$

Now write $k \in I_r$ according to the Iwahori decomposition (2.0.4) as

(8.1.8) $$k = k_- \cdot k_0 \cdot k_+ \in I_{r\bar{U}} \cdot T(F_r)_1 \cdot I_{rU}.$$

The element $k i^+$ has an Iwahori decomposition of the form

(8.1.9) $$k i^+ = k_- i_-^+ \cdot k_0 i_0^+ \cdot k_+ i_+^+$$

for suitable elements $i_-^+ \in I_r^+ \cap \bar{U}$, $i_0^+ \in I_r^+ \cap T$ and $i_+^+ \in I_r^+ \cap U$. So Equation (8.1.7) has the form

(8.1.10) $m_2^{-1}(k_- i_-^+) m_2 \cdot m_2^{-1}(k_0 i_0^+) m_1 \cdot m_1^{-1}(k_+ i_+^+) m_1 = u\theta(k_-) u^{-1} \cdot \theta(k_0) \cdot u\theta(k_+) u^{-1}.$

Using the uniqueness of the decompositions in the "big cell" $\bar{U} \cdot T \cdot U$, we deduce that

$$m_2^{-1}(k_0 i_0^+) m_1 = \theta(k_0).$$

This implies (8.1.6) and proves the I_r^+-bi-invariance.

To finish the proof, we need to consider an equality of the form

$$m_0 k_1^{-1} m_1 u\theta(k_1) = k_2^{-1} m_2 u\theta(k_2)$$

for k_1, k_2, m_1, m_2 as above and $m_0 \in T(F_r)_1$, and show that $\chi_r(m_2) = \chi_r(m_0 m_1)$. We can write this in the form

$$k_1^{-1} i^+ (m_0 m_1 u)\theta(k_1) = k_2^{-1} m_2 u\theta(k_2)$$

for a suitable element $i^+ \in I_r^+$. Now the argument above applied to this expression yields $\chi_r(m_0 m_1) = \chi_r(m_2)$, as desired. $\square$

The utility of the functions ϕ_{u,χ_r} stems from two principles:

– The twisted orbital integrals of ϕ_{u,χ_r} are easy to compute; we can thus prove ϕ_{u,χ_r} and $f_{t,\chi}$ are associated functions for all u.
– The character of ϕ_{u,χ_r} on any θ-stable representation Π of G_r is easy to compute, and is non-vanishing only on representations Π belonging to $\mathscr{R}_{\chi_r}(G_r)$; thus we will get explicit character identities in the parameter u coming from the matching of ϕ_{u,χ_r} and $f_{t,\chi}$.

We will establish these facts in the next few subsections. The next subsection contains a key lemma about the norm homomorphism.

8.2. A lemma on the norm homomorphism

LEMMA 8.2.1. – *Let T be any F-torus. Then the norm homomorphism $N_r : T(F_r)_1 \to T(F)_1$ is surjective.*

This is well-known (and easy) when T is F-split; in the general case it can be proved indirectly from the fact that the characteristic functions $1_{T(F_r)_1}$ and $1_{T(F)_1}$ are associated ([18]). Our purpose here is only to give a more elementary and direct proof. In this article we will use this lemma only in the case where T is split over L.

Proof. – Following [13], let $\mathscr{T}^\circ$ denote the neutral component of the lft Neron model $\mathscr{T}$ associated to T. Then $\mathscr{T}^\circ$ is a smooth $\mathcal{O}$-group scheme with connected geometric fibers. By loc. cit. we have $T(F)_1 = \mathscr{T}^\circ(\mathcal{O})$ and $T(F_r)_1 = \mathscr{T}^\circ(\mathcal{O}_r)$.

Therefore, we need to prove that the group

$$\frac{\mathscr{T}^\circ(\mathcal{O}_r)^\theta}{N_r(\mathscr{T}^\circ(\mathcal{O}_r))} = \widehat{H}^0(F_r/F, \mathscr{T}^\circ(\mathcal{O}_r))$$

vanishes.

First note that for each positive integer n, we have

(8.2.1) $$H^1(F_r/F, \mathscr{T}^\circ(\mathcal{O}_r/\varpi^n \mathcal{O}_r)) = 0.$$

Indeed, since inflation is always injective on H^1, it is enough to prove the vanishing of $H^1(\langle\theta\rangle, \mathscr{T}^\circ(\mathcal{O}_L/\varpi^n \mathcal{O}_L))$. But this is a quotient of the group $H^1(\langle\theta\rangle, \mathscr{T}^\circ(\mathcal{O}_L))$ which is well-known to be trivial (cf. e.g. [23, 7.6.1]).

Further, the $\mathrm{Gal}(F_r/F)$-module $\mathscr{T}^\circ(\mathcal{O}_r/\varpi^n \mathcal{O}_r)$ is finite and thus has trivial Herbrand quotient (cf. [32, VIII, Prop. 8]). Hence from (8.2.1) we deduce

$$\widehat{H}^0(F_r/F, \mathscr{T}^\circ(\mathcal{O}_r/\varpi^n \mathcal{O}_r)) = 0.$$

Thus to prove the lemma it would be enough to prove that the natural map

(8.2.2) $$\frac{\mathscr{T}^\circ(\mathcal{O}_r)^\theta}{N_r(\mathscr{T}^\circ(\mathcal{O}_r))} \longrightarrow \varprojlim_n \frac{\mathscr{T}^\circ(\mathcal{O}_r/\varpi^n \mathcal{O}_r)^\theta}{N_r(\mathscr{T}^\circ(\mathcal{O}_r/\varpi^n \mathcal{O}_r))}$$

is injective. But this is a straightforward exercise which goes as follows. Let t_n be the reduction modulo ϖ^n of $t \in \mathscr{T}^\circ(\mathcal{O}_r)^\theta$, and suppose for every n we have $t_n = N_r(s_n)$ for some $s_n \in \mathscr{T}^\circ(\mathcal{O}_r/\varpi^n \mathcal{O}_r)$. Using (8.2.1) and the surjectivity of the transition maps

(8.2.3) $$\mathscr{T}^\circ(\mathcal{O}_r/\varpi^{n+1} \mathcal{O}_r) \to \mathscr{T}^\circ(\mathcal{O}_r/\varpi^n \mathcal{O}_r)$$

we may inductively alter the s_n's so that they are compatible under the maps (8.2.3), yet still satisfy $N_r(s_n) = t_n$ for all n. Then $s := \varprojlim_n s_n$ is an element in $\mathscr{T}^\circ(\mathcal{O}_r)$ with $N_r(s) = t$. This proves the lemma. □

8.3. Orbital integrals

Normalize the Haar measure dg on G_r (resp. ds on T_r) so that I_r (resp. $T(F_r)_1$) has measure 1. Use this to define the quotient measures $d\bar{g}$ on $T(F)\backslash G_r$ (resp. $d\bar{s}$ on $T(F)\backslash T_r$).

PROPOSITION 8.3.1. – *The following statements hold.*

(1) *For $\delta \in G_r$, $\mathrm{TO}_{\delta\theta}(\phi_{u,\chi_r})$ is non-zero if and only if δ is θ-conjugate in G_r to an element of the form mu, $m \in T(F_r)_1$. For such m we have*

(8.3.1) $$\mathrm{TO}_{mu\theta}(\phi_{u,\chi_r}) = \chi_r^{-1}(m).$$

(2) *For $\gamma \in G$, $\mathrm{O}_\gamma(f_{t,\chi})$ is non-zero if and only if γ is conjugate in G to an element of the form $m_0 t$, $m_0 \in T(F)_1$. For such m_0 we have*

(8.3.2) $$\mathrm{O}_{m_0 t}(f_{t,\chi}) = \chi^{-1}(m_0).$$

Proof. – It is enough to prove (1). Let $\delta \in G_r$. If $\mathrm{TO}_{\delta\theta}(\phi_{u,\chi_r}) \neq 0$, then δ is θ-conjugate under G_r to an element of the form mu, where $m \in T(F_r)_1$. Hence we might as well assume $\delta = mu$; in that case $G_{\delta\theta}^\circ(F) = G_{mu\theta}^\circ(F) = T(F)$ (Lemma 8.1.2(ii)). Now if $g \in G_r$ is such that $g^{-1}mu\theta(g) \in \mathrm{supp}(\phi_{u,\chi_r})$, then by Lemma 8.1.2(ii) we have $g \in T_r I_r$. Then using our choice of measures we get

$$\int_{T(F)\backslash G_r} \phi_{u,\chi_r}(g^{-1}mu\theta(g))\, d\bar{g} = \int_{T(F)\backslash T_r} \phi_{u,\chi_r}(s^{-1}mu\theta(s))\, d\bar{s}$$
$$= \int_{T(F)\backslash T_r} \phi_{1,\chi_r}^{T_r}(s^{-1}m\theta(s))\, d\bar{s}.$$

Here $\phi_{1,\chi_r}^{T_r}$ denotes the elementary function on T_r associated to $u = 1$. This is precisely the characteristic function $1_{T(F_r)_1}$ of $T(F_r)_1$ times the character χ_r^{-1}. Thus we have proved

(8.3.3) $$\mathrm{TO}_{mu\theta}^{G_r}(\phi_{u,\chi_r}) = \chi_r^{-1}(m)\, \mathrm{TO}_{m\theta}^{T_r}(1_{T(F_r)_1}).$$

Using

(8.3.4) $$H^1(F_r/F, T(F_r)_1) = 0$$

(see the proof of (8.2.1)), one easily checks that $\mathrm{TO}_{m\theta}^{T_r}(1_{T(F_r)_1}) = \mathrm{vol}_{d\bar{s}}(T(F)\backslash T(F)T(F_r)_1)$. By our choice of measures, this volume equals 1.   □

LEMMA 8.3.2. – *We have the following statements.*

(i) *Let $m_0, m_0' \in T(F)_1$. Then $m_0 t$ and $m_0' t$ are stably-conjugate in G if and only if $m_0 = m_0'$.*

(ii) *Let $m, m' \in T(F)_1$. Then mu and $m'u$ are stably θ-conjugate in G_r if and only if they are θ-conjugate by an element of $T(F_r)_1$.*

Proof. – Part (i) follows from Lemma 8.1.2(i), taking $r = 1$ (hence $\theta = \mathrm{id}$ and $u = t$). Next, mu is stably θ-conjugate to $m'u$ if and only if $N_r(m)t$ is stably-conjugate to $N_r(m')t$ (by [17, Prop. 5.7]). By part (i), this holds if and only if $N_r(m) = N_r(m')$. Because of (8.3.4), this holds if and only if m and m' are θ-conjugate in $T(F_r)_1$.   □

COROLLARY 8.3.3. – *For every $\delta \in G_r$, $\mathrm{SO}_{\delta\theta}(\phi_{u,\chi_r}) = \mathrm{TO}_{mu\theta}(\phi_{u,\chi_r})$, where mu, $m \in T(F_r)_1$, is any element of this form which is stably θ-conjugate to δ.*

Proof. – If δ is not stably-conjugate to an element of the form mu, then both sides are zero. If $\delta = mu$, then the non-zero summands in $\mathrm{SO}_{\delta\theta}(\phi_{u,\chi_r})$ are of the form $\mathrm{TO}_{m'u}(\phi_{u,\chi_r})$ (they come with coefficient 1 since $m'u$ is strongly θ-regular). By Lemma 8.3.2(ii), only one term appears in this sum.   □

PROPOSITION 8.3.4. – *The functions $f_{t,\chi}$ and ϕ_{u,χ_r} are associated.*

Proof. – First we assume that $\mathrm{SO}_\gamma(f_{t,\chi}) \neq 0$ and prove this implies that γ is a norm from G_r. By Proposition 8.3.1(2), we see that γ is stably-conjugate in G to an element of the form $m_0 t$, $m_0 \in T(F)_1$. By Lemma 8.2.1, $m_0 = N_r(m)$ for some $m \in T(F_r)_1$. Since $t = N_r(u)$, this proves that γ is a norm from G_r.

Next we assume γ is the norm of an element $\delta \in G_r$. If $\mathrm{SO}_{\delta\theta}(\phi_{u,\chi_r}) \neq 0$, then δ is stably θ-conjugate to an element of the form mu, and then by Corollary 8.3.3 and Proposition 8.3.1 we have

(8.3.5) $$\mathrm{SO}_{\delta\theta}(\phi_{u,\chi_r}) = \mathrm{TO}_{mu\theta}(\phi_{r,\chi_r}) = \chi_r^{-1}(m).$$

But letting $m_0 := N_r(m)$, [17, Prop. 5.7] shows that γ is stably-conjugate to the element $m_0 t$ and the same reasoning then gives us

$$\mathrm{SO}_\gamma(f_{t,\chi}) = \mathrm{O}_{m_0 t}(f_{t,\chi}) = \chi^{-1}(m_0),$$

which coincides with (8.3.5).

If $\mathrm{SO}_{\delta\theta}(\phi_{u,\chi_r}) = 0$, then δ is not stably θ-conjugate to an element of the form mu. But then γ is not stably conjugate to an element of the form $m_0 t$, and thus $\mathrm{SO}_\gamma(f_{t,\chi}) = 0$ as well. Indeed, if γ were stably conjugate to $m_0 t$, then writing $m_0 = N_r(m)$ for some m (Lemma 8.2.1), we would have that $N_r(\delta)$ is stably conjugate to $N_r(mu)$. But then by [17, Prop. 5.7], it would be the case that δ is stably θ-conjugate to mu, contrary to what we saw above. $\square$

8.4. Traces of elementary functions

Let Π denote a θ-stable admissible representation of $G(F_r)$, and fix an intertwiner $I_\theta : \Pi \xrightarrow{\sim} \Pi^\theta$. Let $\Theta_{\Pi\theta}$ denote the locally integrable function of Harish-Chandra representing the functional $\phi \mapsto \langle \mathrm{trace}\,\Pi I_\theta, \phi \rangle$, for $\phi \in C_c^\infty(G(F_r))$. Thus,

(8.4.1) $$\langle \mathrm{trace}\,\Pi I_\theta, \phi \rangle = \int_{G_r} \Theta_{\Pi\theta}(g)\phi(g)\, dg.$$

Following [24, §3], we calculate this trace for $\phi = \phi_{u,\chi_r}$ using the map (8.1.2). Let $d\bar{k}$ denote the quotient measure on $T(F_r)_1 \backslash I_r$ denoted $d\bar{g}$ earlier. The change of variable formula yields

$$\langle \mathrm{trace}\,\Pi I_\theta, \phi_{u,\chi_r} \rangle = \int_{T(F_r)_1 \backslash I_r} \int_{T(F_r)_1} |\mathrm{Jac}_{[k,mu]}|_F \cdot \Theta_{\Pi\theta}(k^{-1}mu\theta(k)) \cdot \phi_{u,\chi_r}(k^{-1}mu\theta(k))\, dm\, d\bar{k}$$

$$= q^{r\langle 2\rho,\nu \rangle} \int_{T(F_r)_1} \Theta_{\Pi\theta}(mu)\, \chi_r^{-1}(m)\, dm$$

(8.4.2) $$= q^{r\langle 2\rho,\nu \rangle} \int_{T(F_r)_1} \Theta_{\Pi_U\theta}(mu)\, \chi_r^{-1}(m)\, dm.$$

Here Π_U is the Jacquet module of Π corresponding to the Borel subgroup $B_r = T_r U_r$, and the equality $\Theta_{\Pi\theta}(mu) = \Theta_{\Pi_U\theta}(mu)$ we used is the twisted version due to Rogawski [31] of a theorem of Casselman [5].

LEMMA 8.4.1. – *Suppose Π is an irreducible θ-stable object in $\mathcal{R}(G_r)$. If $\langle \mathrm{trace}\,\Pi I_\theta, \phi_{u,\chi_r} \rangle \neq 0$, then Π belongs to the subcategory $\mathcal{R}_{\chi_r}(G_r)$.*

Proof. – The non-vanishing of (8.4.2) shows that $\Pi_U^{\chi_r} \neq 0$. Then by (3.3.1), we see $\Pi^{\rho_r} \neq 0$. Now the result follows by Proposition 3.3.1. $\square$

8.5. Computation of traces of elementary functions

In this subsection, we write W (resp. W_r) for the relative Weyl group associated to the F-split (resp. F_r-split) torus A (resp. A^{F_r}) in G.

For any irreducible θ-stable representation Π of G_r with intertwiner I_θ, we will compute here the trace

$$\langle \text{trace } \Pi I_\theta, \phi_{u,\chi_r} \rangle.$$

The computation is similar[7] to that in [24]. By Lemma 8.4.1, we may assume Π belongs to the category $\mathscr{R}_{\chi_r}(G_r)$ (otherwise the trace is zero). Suppose the supercuspidal support of Π is $(T_r, \xi')_{G_r}$ for some extension ξ' of a W_r-conjugate of χ_r.

Let Ξ denote the set of characters on $T(F_r)$ which extend some W_r-conjugate of χ_r. Thus, Ξ contains $W_r \xi'$. Let $\Xi(\chi_r) \subset \Xi$ consist of those whose restriction to $T(F_r)_1$ is precisely χ_r. Let Ξ^θ resp. $\Xi(\chi_r)^\theta$ denote the subset of θ-fixed elements in Ξ resp. $\Xi(\chi_r)$. It will become clear below that the trace above is zero unless $\Xi^\theta(\chi_r) \neq \varnothing$. From now on, ξ' will denote an arbitrary element of Ξ, not just the one we started with.

Now we fix further notation related to Π. Recall that Π_U denotes the Jacquet module of Π relative to U_r. Because $(\Pi^\theta)_U = (\Pi_U)^\theta$, the intertwiner $I_\theta : \Pi \xrightarrow{\sim} \Pi^\theta$ induces an intertwiner $I_\theta : \Pi_U \xrightarrow{\sim} \Pi_U^\theta$. Recall that Π_U is a subquotient of

$$(i_{B_r}^{G_r}(\xi'))_U = \delta_{B_r}^{1/2} \bigoplus_{w \in W_r} \mathbb{C}_{w\xi'}$$

(cf. [6]), where $\mathbb{C}_{w\xi'}$ is the 1-dimensional representation of T_r corresponding to the character ${}^w\xi'$. Thus there is a well-defined subset $\Xi(\Pi) \subset \Xi$ and positive multiplicities $a_{\xi',\Pi} =: a_{\xi'}$ for $\xi' \in \Xi(\Pi)$ such that, as T_r-representations,

$$(8.5.1) \qquad \Pi_U = \delta_{B_r}^{1/2} \bigoplus_{\xi' \in \Xi(\Pi)} \mathbb{C}_{\xi'}^{a_{\xi'}}.$$

Since Π_U^θ is isomorphic to Π_U, the set $\Xi(\Pi)$ is stabilized by θ.

For the following statement and proof, we set $\Xi(\Pi)^\theta = \Xi^\theta \cap \Xi(\Pi)$, and $\Xi(\Pi, \chi_r)^\theta := \Xi(\chi_r)^\theta \cap \Xi(\Pi)$. Further, for $\xi' \in \Xi^\theta$, we set

$$\text{tr}(I_\theta, \Pi, \xi') := \langle \text{trace } I_\theta ; \delta_{B_r}^{1/2} \mathbb{C}_{\xi'}^{a_{\xi'}} \rangle.$$

Finally, for $\xi' \in \Xi(\chi_r)$, we may write

$$\xi' = \widetilde{\chi}_r^{\varpi} \, \eta'$$

for a unique unramified character η' on T_r.

PROPOSITION 8.5.1. – *In the notation above, we have*

$$(8.5.2) \qquad \langle \text{trace } \Pi I_\theta, \phi_{u,\chi_r} \rangle = q^{\langle \rho, \tau \rangle} \sum_{\xi' \in \Xi(\Pi, \chi_r)^\theta} \eta'(u) \, \text{tr}(I_\theta, \Pi, \xi').$$

---

[7] In the final displayed equation on p. 526 of [24], the calculation of $\text{trace}(I_{\tilde{\lambda}})_{N_0}(u\theta)$ is done assuming a particularly simple form for the intertwiner on the induced module $I_{\tilde{\lambda}}$ (more precisely, on its Jacquet module $(I_{\tilde{\lambda}})_{N_0}$). Unless $\tilde{\lambda}$ is regular, there will be other possible intertwiners on $(I_{\tilde{\lambda}})_{N_0}$. The formula in Proposition 7' of loc. cit. implicitly assumes that the intertwiner on Π is inherited from such a simple intertwiner on a suitable induced module $I_{\tilde{\lambda}}$. The statement of our Proposition 8.5.1 is more complicated because we do not make any assumptions about the intertwiner on Π.

Proof. – We will rewrite the integral in (8.4.2). Using (8.5.1) it takes the form

$$\int_{T(F_r)_1} \chi_r^{-1}(m) \, \langle \operatorname{trace} \Pi_U(mu)I_\theta \, ; \, \delta_{B_r}^{1/2} \bigoplus_{\xi' \in \Xi(\Pi)} \mathbb{C}_{\xi'}^{a_{\xi'}} \rangle \, dm.$$

The isotypical component $\delta_{B_r}^{1/2} \mathbb{C}_{\xi'}^{a_{\xi'}}$ can contribute to the trace of $\Pi_U I_\theta$ only if $\xi' \in \Xi(\Pi)^\theta$, and in that case $\Pi_U I_\theta$ preserves that component. Thus the integral can be expressed as

$$(8.5.3) \qquad \sum_{\xi' \in \Xi(\Pi)^\theta} \int_{T(F_r)_1} \chi_r^{-1}(m) \, \langle \operatorname{trace} \Pi_U(mu)I_\theta \, ; \, \delta_{B_r}^{1/2} \mathbb{C}_{\xi'}^{a_{\xi'}} \rangle \, dm.$$

On $\delta_{B_r}^{1/2} \mathbb{C}_{\xi'}^{a_{\xi'}}$ appearing here, $\Pi_U(mu)$ acts by the scalar

$$\delta_{B_r}^{1/2}(u) \, \xi'(mu) = q^{-\langle \rho, \tau \rangle} \, \xi'(m) \, \eta'(u).$$

(We used that $\widetilde{\chi}_r^\varpi(u) = 1$.) Thus (8.5.3) becomes

$$q^{-\langle \rho, \tau \rangle} \sum_{\xi' \in \Xi(\Pi)^\theta} \eta'(u) \, \operatorname{tr}(I_\theta, \Pi, \xi') \int_{T(F_r)_1} \chi_r^{-1}(m) \, \xi'(m) \, dm.$$

The integral on the right hand side is non-vanishing (and equal to 1) if and only if ξ' equals χ_r on $T(F_r)_1$, that is, if and only if $\xi' \in \Xi(\Pi, \chi_r)^\theta$. The result now follows. □

We shall also need the following result. Recall the idempotent $e_\rho \in \mathcal{Z}(G, \rho)$ defined to have support I and to take value $\rho(k)^{-1}$ at $k \in I$.

LEMMA 8.5.2. – *In the notation above, we have*

$$(8.5.4) \qquad \sum_{\xi' \in \Xi(\Pi, \chi_r)^\theta} \operatorname{tr}(I_\theta, \Pi, \xi') = \langle \operatorname{trace} \Pi I_\theta \, , \, e_{\rho_r} \rangle.$$

Proof. – The right hand side can be written as

$$\int_{I_r} \langle \operatorname{trace} \Pi(g)I_\theta \, , \, \Pi^{\rho_r} \rangle \, e_{\rho_r}(g) \, dg = \langle \operatorname{trace} I_\theta \, , \, \Pi^{\rho_r} \rangle.$$

By (3.3.1) and (8.5.1), this can be written as

$$\langle \operatorname{trace} I_\theta \, , \, \delta_{B_r}^{1/2} \bigoplus_{\xi' \in \Xi(\Pi, \chi_r)^\theta} \mathbb{C}_{\xi'}^{a_{\xi'}} \rangle,$$

which is the left hand side in (8.5.4). □

COROLLARY 8.5.3. – *We have*

$$(8.5.5) \qquad \langle \operatorname{trace} \Pi I_\theta, \phi_{u, \chi_r} \rangle|_{u=1} = \langle \operatorname{trace} \Pi I_\theta, e_{\rho_r} \rangle.$$

In particular, if $r = 1$, we have

$$(8.5.6) \qquad \sum_{\xi \in \Xi(\pi, \chi)} \dim(\mathbb{C}_\xi^{a_{\xi, \pi}}) = \langle \operatorname{trace} \pi, e_\rho \rangle.$$

9. Proof in the strongly regular elliptic case

9.1. Local data adapted to $\mathscr{R}_\chi(G)$

In this subsection, we assume G is any unramified group over F with $G = G_{\mathrm{ad}}$. Let $\mathrm{Irr}_\chi(G)$ (resp. $\mathrm{Irr}^\theta_{\chi_r}(G_r)$) denote the set of irreducible (resp. irreducible θ-stable) admissible representations in $\mathscr{R}_\chi(G)$ (resp. $\mathscr{R}_{\chi_r}(G_r)$).

Inspired by Hales [15], we define *local data adapted to $\mathscr{R}_\chi(G)$* to consist of the data (a),(b), and (c), subject to conditions (1) and (2) below:

(a) An indexing set $\mathscr{I}$ (possibly infinite);
(b) A collection of complex numbers $a_i(\pi)$ for $i \in \mathscr{I}$ and $\pi \in \mathrm{Irr}_\chi(G)$;
(c) A collection of complex numbers $b_i(\Pi)$ for $i \in \mathscr{I}$ and $\Pi \in \mathrm{Irr}^\theta_{\chi_r}(G_r)$.

(1) For i fixed, the constants $a_i(\pi)$ and $b_i(\Pi)$ are zero for all but finitely many π and Π.
(2) For $\phi \in \mathscr{H}(G_r, \rho_r)$ and $f \in \mathscr{H}(G, \rho)$ the following are equivalent:

(A) For all i, we have $\sum_\pi a_i(\pi)\langle \mathrm{trace}\,\pi, f\rangle = \sum_\Pi b_i(\Pi)\langle \mathrm{trace}\,\Pi I_\theta, \phi\rangle$;
(B) For all strongly regular elliptic semisimple norms $\gamma = \mathscr{N}(\delta)$, we have

$$\mathrm{SO}_\gamma(f) = \mathrm{SO}_{\delta\theta}(\phi).$$

One can prove that such local data exist using a result of Clozel (cf. [11, Proposition 8.3.1]), in exactly the same way as for parahoric Hecke algebras (see loc. cit. §8.5).

A word about the intertwiners I_θ: these are local intertwiners coming from canonically defined intertwiners at the global (adelic) level (cf. [11, Proposition 8.3.1]). But the global-to-local process involves some choices and hence the I_θ here are not canonically defined. However, since Π is irreducible, I_θ is defined up to scalar, and such scalars can be absorbed into the coefficients $b(\Pi)$. Hence we are free to normalize the I_θ however we like.

9.2. End of proof

Assume $G = G_{\mathrm{ad}}$. Fix $\phi \in \mathscr{Z}(G_r, \rho_r)$ and $f = b_r(\phi) \in \mathscr{Z}(G, \rho)$.

We now complete the proof of Theorem 1.1. By Lemma 7.1.1, we may assume γ is a norm. By our reduction section, we may assume γ is a strongly regular elliptic semisimple norm, say $\gamma = \mathscr{N}(\delta)$. To check the identity $\mathrm{SO}_{\delta\theta}(\phi) = \mathrm{SO}_\gamma(f)$, we follow the method of Labesse [24]: we use the associated elementary functions $\phi_{u,\chi_r}, f_{t,\chi}$ to give, as u ranges, sufficiently many character identities to establish (2)(A) in the local data for the pair ϕ, f. (Recall $u = \varpi^\nu$ and $t = N_r(u) = \varpi^\tau$, and that ν ranges over the semigroup of regular dominant cocharacters $X_*(A^{F_r})^{++}$ in the group $X_*(A^{F_r})$).

We need to show that for each i, the identity in (2)(A) above holds. Fixing i and dropping it from our notation, we need to prove that

$$\text{(9.2.1)} \qquad \sum_\pi a(\pi)\,\langle \mathrm{trace}\,\pi, b_r(\phi)\rangle = \sum_\Pi b(\Pi)\,\langle \mathrm{trace}\,\Pi I_\theta, \phi\rangle.$$

By Lemma 8.1.3 and Proposition 8.3.4, the equivalent statements (2)(A) and (2)(B) hold for each pair $\phi_u := \phi_{u,\chi_r}$, $f_t := f_{t,\chi}$. Therefore

$$\text{(9.2.2)} \qquad \sum_\pi a(\pi)\,\langle \mathrm{trace}\,\pi, f_t\rangle = \sum_\Pi b(\Pi)\,\langle \mathrm{trace}\,\Pi I_\theta, \phi_u\rangle.$$

Let us rewrite the left hand side using (8.5.2) in the case where $r = 1$. We get

$$(9.2.3) \quad \sum_\pi a(\pi) \langle \operatorname{trace} \pi, f_t \rangle = q^{\langle \rho, \tau \rangle} \sum_{\xi_0 \in \Xi(\chi)/W_\chi} \sum_{\pi \in i_B^G(\xi_0)} a(\pi) \sum_{\xi \in \Xi(\pi, \chi)} \eta(t) \, (\dim \mathbb{C}_\xi^{a_{\xi, \pi}}).$$

Here ξ_0 ranges over any set of representatives for the W_χ-orbits on the set $\Xi(\chi)$ of characters on $T(F)$ extending χ. Since $\eta(t) = \eta_r(u)$, (9.2.3) is a linear combination of characters which are norms evaluated on $u = \varpi^\nu$.

We would like to write down a similar expression for the right hand side of (9.2.2) and then compare it to (9.2.3). An apparent difficulty is that given two different elements in $\Xi(\chi_r)^\theta$ which are W_{χ_r} conjugate, it could happen that one of them is a norm and the other is not. To remedy this, we will group the elements according to W_χ-conjugacy rather than W_{χ_r}-conjugacy (note that $W_\chi \subseteq W_{\chi_r}^\theta$). The following lemma shows that this avoids the problem just mentioned. We leave the proof to the reader.

LEMMA 9.2.1. – *If ξ_1', ξ_2' belong to $\Xi(\chi_r)^\theta$ and are W_χ-conjugate, then one is a norm if and only if the other one is.*

Now again using (8.5.2), we can write the right hand side of (9.2.2) in the following way:

$$\sum_\Pi b(\Pi) \langle \operatorname{trace} \Pi I_\theta, \phi_u \rangle =$$

$$(9.2.4) \qquad q^{\langle \rho, \tau \rangle} \sum_{\xi_0' \in \Xi(\chi_r)^\theta / W_\chi} \sum_{\substack{\Pi \text{ s.t.} \\ W_\chi \xi_0' \cap \Xi(\Pi, \chi_r)^\theta \neq \varnothing}} b(\Pi) \sum_{\substack{\xi' \in \\ W_\chi \xi_0' \cap \Xi(\Pi, \chi_r)^\theta}} \eta'(u) \operatorname{tr}(I_\theta, \Pi, \xi').$$

Let $(9.2.4)_n$ (resp. $(9.2.4)_{nn}$) denote the contribution to (9.2.4) coming from those ξ_0' which are norms (resp. not norms). By Lemma 9.2.1, all (resp. none) of the terms η' appearing in $(9.2.4)_n$ (resp. $(9.2.4)_{nn}$) are norms.

Now we regard each side of the identity

$$(9.2.5) \qquad\qquad (9.2.3) = (9.2.4)_n + (9.2.4)_{nn}$$

as a linear combination of characters on $\nu \in X_*(A^{F_r})^{++}$. The independence of characters asserts that any non-empty set of distinct characters on $X_*(A^{F_r})$ is linearly independent over $\mathbb{C}$. The proof shows that the set of their restrictions to $X_*(A^{F_r})^{++}$ remains independent. The equation (9.2.5) holds for all regular dominant ν, thus by this linear independence statement, it must hold for $\nu = 0$, that is, for $u = 1$. In the self-evident notation, this means that we have two identities

$$(9.2.6) \qquad\qquad (9.2.3)|_{u=1} = (9.2.4)_n|_{u=1}$$

$$(9.2.7) \qquad\qquad 0 = (9.2.4)_{nn}|_{u=1}.$$

In fact the linear independence gives something stronger: the contributions of each $\xi_0' \in \Xi(\chi_r)^\theta / W_\chi$, to the above equations satisfy corresponding identities. Making use of Corollary 8.5.3, this may be stated as follows.

LEMMA 9.2.2. – *For each $\xi_0' \in \Xi(\chi_r)^\theta / W_\chi$, we have*

$$(9.2.8) \qquad \sum_{\substack{\xi_0 \in \Xi(\chi)/W_\chi \\ s.t. \ \xi_{0r} = \xi_0'}} \sum_{\pi \in i_B^G(\xi_0)} a(\pi) \langle \operatorname{trace} \pi \, , e_\rho \rangle \ =$$

$$\sum_{\substack{\Pi \ s.t. \\ W_\chi \xi_0' \cap \Xi(\Pi,\chi_r)^\theta \neq \varnothing}} b(\Pi) \sum_{\substack{\xi' \in \\ W_\chi \xi_0' \cap \Xi(\Pi,\chi_r)^\theta}} \operatorname{tr}(I_\theta, \Pi, \xi').$$

Note that if ξ_0' is not a norm, the left hand side of (9.2.8) vanishes, and the right hand does as well by our linear independence argument.

Next we fix any $\xi_0' \in \Xi(\chi_r)^\theta$ and sum this equation over all W_χ-orbits in $W_{\chi_r}\xi_0' \cap \Xi(\chi_r)^\theta$. In that sum, the right hand side will simplify, because of the identity (8.5.4). Indeed, if $W_{\chi_r}\xi_0'$ contains a norm, we may assume ξ_0' itself is a norm, and the W_χ-orbits in $W_{\chi_r}\xi_0'$ which are not norms contribute nothing to the right hand side of (9.2.8), and hence we get an equality

$$(9.2.9) \qquad \sum_{\substack{\xi_0 \in \Xi(\chi)/W_\chi \\ s.t. \ \xi_{0r} \in W_{\chi_r}\xi_0'}} \sum_{\pi \in i_B^G(\xi_0)} a(\pi) \langle \operatorname{trace} \pi \, , e_\rho \rangle$$

$$= \sum_{\Pi \in i_{B_r}^{G_r}(\xi_0')} b(\Pi) \langle \operatorname{trace} \Pi I_\theta \, , e_{\rho_r} \rangle.$$

On the other hand, if $W_{\chi_r}\xi_0'$ contains no norms, then (9.2.9) remains true: the left hand side obviously vanishes, and the right hand side vanishes as a result of the linear independence discussion above.

Recall that if $\xi_0' = \xi_{0r}$, then $b(\phi)$ acts on $i_B^G(\xi_0)^\rho$ by the same scalar by which ϕ acts on $i_{B_r}^{G_r}(\xi_0')^{\rho_r}$ (Lemma 4.2.1). Thus multiplying (9.2.9) by $ch_{\xi_0}(b(\phi)) = ch_{\xi_{0r}}(\phi)$ if $\xi_0' = \xi_{0r}$, and by $ch_{\xi_0'}(\phi)$ in the case where ξ_0' is not a norm, we get (9.2.9) but with e_{ρ_r} resp. e_ρ replaced by ϕ resp. $b(\phi)$. Then summing the resulting formula over all $\xi_0' \in \Xi(\chi_r)/W_{\chi_r}$ yields the desired formula (9.2.1). This completes the proof of Theorem 1.1. $\qquad\square$

REMARK 9.2.3. – The results of this article are closely related to at least four earlier works on the fundamental lemma, namely [7], [24], [15], [11]. We point out that all of these rely crucially on a result of Keys [16]. It is interesting to note that the present article does not need Keys' result at all. The reason is that we are using Labesse's method of elementary functions: in [24], Keys' result is used only to separate, in the character identities appearing in the local data, the spherical representation from other representations in the same unramified principal series representation. There is no need for any such separation argument in this article.

10. Base change for I^+-level

In this section we will construct a base change homomorphism

$$(10.0.10) \qquad b_r : \mathcal{Z}(G_r, I_r^+) \to \mathcal{Z}(G, I^+)$$

and show that the fundamental lemma also holds for it, as claimed in Corollary 1.2.

First we abuse notation by writing $T(k_r)$ in place of $\mathcal{T}^\circ(k_r)$, where $\mathcal{T}^\circ$ is the neutral component of the lft Néron model $\mathcal{T}$ for the F-torus T. Thus there is a canonical isomorphism

$$T(k_r) = T(F_r)_1/T(F_r)_1^+$$

and the set of depth-zero characters on $T(F_r)_1$ can be canonically identified with $T(k_r)^\vee$, the set of characters of the finite group $T(k_r)$. For $\chi' \in T(k_r)^\vee$, we consider the corresponding Hecke algebra $\mathcal{H}(G_r, I_r, \chi')$ and its center $\mathcal{Z}(G_r, I_r, \chi')$.

Recall the idempotents $e_{\chi'} \in \mathcal{Z}(G_r, I_r, \chi')$: the function $e_{\chi'}$ is supported on I_r and takes value $\mathrm{vol}(I_r)^{-1}\rho_{\chi'}(x)^{-1}$ at $x \in I_r$. These form a complete set of orthogonal idempotents in $\mathcal{H}(G_r, I_r^+)$ (cf. e.g. [14, proof of Prop. 12.1.1]). Hence there is an algebra *monomorphism*

$$(10.0.11) \qquad \mathcal{Z}(G_r, I_r^+) \hookrightarrow \prod_{\chi' \in T(k_r)^\vee} \mathcal{Z}(G_r, I_r, \chi')$$

$$z \mapsto (z e_{\chi'})_{\chi'}.$$

We need to identify the image of (10.0.11). As it clearly suffices, we discuss this in the case $r = 1$ and make self-evident adjustments to our notation.

LEMMA 10.0.1. – *The image of* (10.0.11) *consists of the elements* $(z_\chi)_\chi$ *with the following property: for any relative Weyl group element* $w \in W$, *any* $\chi \in T(k)^\vee$, *and any extension* $\tilde{\chi} : T(F) \to \mathbb{C}^\times$ *of* χ, *the scalar by which* z_χ *acts on* $i_B^G(\tilde{\chi})^{\rho_\chi}$ *coincides with the scalar by which* z_{w_χ} *acts on* $i_B^G(\tilde{\chi})^{\rho_{w_\chi}}$.

Proof. – It is clear that elements in the image of (10.0.11) satisfy the property in question. We assume $(z_\chi)_\chi$ satisfies the property and show that it comes from a $z \in \mathcal{Z}(G, I^+)$.

Fix χ_0 and consider its W-orbit $[\chi_0]$. Recall ([14, §12]) the *central* idempotent $e_{[\chi_0]} := \sum_{\chi \in [\chi_0]} e_\chi$ in $\mathcal{H}(G, I^+)$. This is the projector onto the Bernstein component $\mathcal{R}_{\chi_0}(G)$, in the sense that for every smooth representation V of $G(F)$, the $G(F)$-module spanned by $e_{[\chi_0]}V$ is the component of V belonging to $\mathcal{R}_{\chi_0}(G)$. Using Proposition 3.3.1, we deduce

$$(10.0.12) \qquad \bigoplus_{\chi \in [\chi_0]} i_B^G(\widetilde{\chi_0})^{\rho_\chi} = i_B^G(\widetilde{\chi_0})^{I^+} = e_{[\chi_0]} i_B^G(\widetilde{\chi_0}),$$

and the G-module generated by the latter is just $i_B^G(\widetilde{\chi_0})$. By our hypothesis, the element $\sum_{\chi \in [\chi_0]} z_\chi$ operates by a scalar $ch_{\widetilde{\chi_0}}$ on (10.0.12). As $\widetilde{\chi_0}$ varies these scalars give a regular function on the Bernstein variety $\mathfrak{X}_{\chi_0}$. Let $\mathcal{Z}(G)$ denote the Bernstein center of $G(F)$, viewed as certain distributions on $C_c^\infty(G(F))$ or equally as regular functions on the union of all Bernstein varieties. We see there is an element $z_{[\chi_0]} \in \mathcal{Z}(G) * e_{[\chi_0]} \subset \mathcal{Z}(G, I^+)$ such that $z_{[\chi_0]}e_\chi = z_\chi$ for each $\chi \in [\chi_0]$. Now let $z := \sum_{\chi_0 \in W \backslash T(k)^\vee} z_{[\chi_0]} \in \mathcal{Z}(G, I^+)$. Then z goes to $(z_\chi)_\chi$ under (10.0.11). $\qquad\square$

If χ' is a norm, then $\chi' = \chi_r := \chi \circ N_r$ for some $\chi \in T(k)^\vee$, and since N_r is surjective, the character χ is uniquely determined by χ'. Now consider the composition of the projection onto the factors indexed by norms

$$\prod_{\chi' \in T(k_r)^\vee} \mathcal{Z}(G_r, I_r, \chi') \to \prod_{\chi \in T(k)^\vee} \mathcal{Z}(G_r, I_r, \chi_r)$$

followed by the map

$$\prod_{\chi \in T(k)^\vee} \mathcal{Z}(G_r, I_r, \chi_r) \to \prod_{\chi \in T(k)^\vee} \mathcal{Z}(G, I, \chi)$$

which is given in each factor by the homomorphism b_r of Definition 1.0.1.

LEMMA 10.0.2. – *This composition $\prod_{\chi'} \mathcal{Z}(G_r, I_r, \chi') \to \prod_\chi \mathcal{Z}(G, I, \chi)$ above takes the image of (10.0.11) into the corresponding image for $r = 1$.*

Proof. – Let $z \in \mathcal{Z}(G_r, I_r^+)$. We need to check that the tuple $(b_r(ze_{\chi_r}))_\chi$ satisfies the property of Lemma 10.0.1. By Lemma 4.2.1, the scalar by which $b_r(ze_{\chi_r})$ acts on $i_B^G(\chi)^{\rho_\chi}$ is the scalar by which ze_{χ_r} acts on $i_{B_r}^{G_r}(\chi_r)^{\rho_{\chi_r}}$. Apply this to both χ and ${}^w\chi$, for w any element of W, the relative Weyl group over F. $\qquad\square$

This lemma allows us to define the base change homomorphism for I^+-level.

DEFINITION 10.0.3. – The homomorphism $b_r : \mathcal{Z}(G_r, I_r^+) \to \mathcal{Z}(G, I^+)$ is the unique homomorphism making the following diagram commute:

$$
\begin{array}{ccc}
\mathcal{Z}(G_r, I_r^+) & \longrightarrow & \prod_{\chi'} \mathcal{Z}(G_r, I_r, \chi') \\
{\scriptstyle b_r}\downarrow & & \downarrow \\
\mathcal{Z}(G, I^+) & \longrightarrow & \prod_\chi \mathcal{Z}(G, I, \chi).
\end{array}
$$

The following lemma explains why it is permissible to forget all the factors corresponding to χ' which are not norms.

LEMMA 10.0.4. – *Suppose $\chi' \in T(k_r)^\vee$ is not a norm. Then all twisted orbital integrals at θ-semisimple elements vanish on $\mathcal{H}(G_r, I_r, \chi')$.*

Proof. – Fix $\phi \in \mathcal{H}(G_r, I_r, \chi')$ and consider any $i \in I_r$. The substitution $g \mapsto gi$ in $\mathrm{TO}_{\delta\theta}(\phi) = \int_{G_{\delta\theta}^\circ \backslash G_r} \phi(g^{-1}\delta\theta(g)) \, d\bar{g}$ yields

$$\mathrm{TO}_{\delta\theta}(\phi) = (\chi'\theta(\chi')^{-1})(i) \, \mathrm{TO}_{\delta\theta}(\phi).$$

Thus if the twisted orbital integral does not vanish, we have $\chi' = \theta(\chi')$. But then χ' is a norm: the θ-invariant elements in $T(k_r)^\vee$ are precisely the norms. $\qquad\square$

With these remarks in hand, it is easy to see how Corollary 1.2 follows from Theorem 1.1. $\qquad\square$

11. Corrigenda to [11]

All notation will be that of [11], but note the correction in notation discussed below.

11.1. [11, Section 2.2]

In Section 2.2 of [11], there is a minor misstatement. It has no effect on the main results of [11], but nevertheless this corrigendum seems necessary in order to avoid potential confusion. I am very grateful to Brian Smithling and Tasho Kaletha, who brought this misstatement to my attention.

11.1.1. *Correction*. – Here is the precise misstatement. In [11, Section 2.2], the "ambient" group scheme $\mathcal{G}_{\mathbf{a}_J}$ was incorrectly identified with the group scheme whose group of $\mathcal{O}_L$-points is the full fixer of the facet $\mathbf{a}_J$. In the notation of Bruhat-Tits [3], which I intended to follow in [11], the group scheme whose group of $\mathcal{O}_L$-points is the full fixer of $\mathbf{a}_J$ is denoted $\widehat{\mathcal{G}}_{\mathbf{a}_J}$. The group scheme $\widehat{\mathcal{G}}_{\mathbf{a}_J}$ is defined and characterized in this way in [3, 4.6.26-28].

The group scheme denoted $\mathcal{G}_{\mathbf{a}_J}$ is defined in loc. cit. 4.6.26 (cf. also 4.6.3-6). In general, it can be a bit smaller than $\widehat{\mathcal{G}}_{\mathbf{a}_J}$ (see below).

The correction is the following: in [11], the symbol $\mathcal{G}_{\mathbf{a}_J}$ should now be interpreted as the group denoted by this symbol in [3], a potentially proper subgroup of the full fixer $\widehat{\mathcal{G}}_{\mathbf{a}_J}$.

We still have, as stated in [11, 2.3.2 and 2.3.3], the equalities

$$(11.1.1) \qquad\qquad J(L) = \mathcal{G}_{\mathbf{a}_J}^{\circ}(\mathcal{O}_L) = T(L)_1 \cdot \mathfrak{U}_{\mathbf{a}_J}(\mathcal{O}_L)$$

$$(11.1.2) \qquad\qquad \mathcal{G}_{\mathbf{a}_J}(\mathcal{O}_L) = T(L)_b \cdot \mathfrak{U}_{\mathbf{a}_J}(\mathcal{O}_L).$$

In general,

$$\mathcal{G}_{\mathbf{a}_J}^{\circ}(\mathcal{O}_L) = \widehat{\mathcal{G}}_{\mathbf{a}_J}^{\circ}(\mathcal{O}_L) \subset \mathcal{G}_{\mathbf{a}_J}(\mathcal{O}_L) \subset \widehat{\mathcal{G}}_{\mathbf{a}_J}(\mathcal{O}_L),$$

and both inclusions can be strict.

11.1.2. *Clarifications*. – We discuss the effect of the above correction on subsequent statements in [11].

1. Theorem 2.3.1 of [11] remains valid as stated, but can be slightly augmented: Equation (2.3.1) can be replaced by

$$(11.1.3) \qquad J(L) = \mathrm{Fix}(\mathbf{a}_J^{\mathrm{ss}}) \cap G(L)_1 = \mathcal{G}_{\mathbf{a}_J}(\mathcal{O}_L) \cap G(L)_1 = \widehat{\mathcal{G}}_{\mathbf{a}_J}(\mathcal{O}_L) \cap G(L)_1.$$

Cf. [13, Remark 11].

2. Contrary to [11, line above Equation (2.3.2)], our $\mathcal{G}_{\mathbf{a}_J}$ should not now be identified with the scheme $\widehat{\mathcal{G}}_{\mathbf{a}_J^{\mathrm{ss}}}$ of [3].

3. Corollary 2.3.2 of [11] remains valid, with the same proof. Indeed, when G_L is split we have $T(L)_b = T(\mathcal{O}_L) = T(L)_1$ and then from (11.1.1) and (11.1.2) above we see that $\mathcal{G}_{\mathbf{a}_J}^{\circ}(\mathcal{O}_L) = \mathcal{G}_{\mathbf{a}_J}(\mathcal{O}_L)$.

4. Lemma 2.9.1 of [11] remains valid as stated, but in the proof (especially in Equations (2.9.1) and (2.9.2)) the symbols $\mathcal{G}_{\mathbf{a}_J}(\mathcal{O}_L)$ and $\mathcal{G}_{\mathbf{a}_J^M}(\mathcal{O}_L)$ should be replaced by $\widehat{\mathcal{G}}_{\mathbf{a}_J}(\mathcal{O}_L)$ and $\widehat{\mathcal{G}}_{\mathbf{a}_J^M}(\mathcal{O}_L)$, respectively.

11.1.3. *Example*. – It is sometimes but usually not the case that $\mathcal{G}_{\mathbf{a}_J}(\mathcal{O}_L) = \widehat{\mathcal{G}}_{\mathbf{a}_J}(\mathcal{O}_L)$. The following is perhaps the simplest example where this equality fails[8]. Take G to be the split group $\mathrm{PSp}(4)$, and let $\mathbf{a}_J$ denote the non-special vertex in a base alcove. Then let τ denote the element in the stabilizer $\Omega \subset \widetilde{W}(L)$ of the base alcove, which interchanges the two special vertices and fixes $\mathbf{a}_J$. The element τ does not belong to the group $\mathcal{G}_{\mathbf{a}_J}^{\circ}(\mathcal{O}_L) = \mathcal{G}_{\mathbf{a}_J}(\mathcal{O}_L)$ (cf. **3** above), since τ does not belong to $G(L)_1$. On the other hand $\tau \in \widehat{\mathcal{G}}_{\mathbf{a}_J}(\mathcal{O}_L)$ since it fixes $\mathbf{a}_J$ and $G(L)^1 = G(L)$ (cf. [3, 4.6.28]).

---

[8] Brian Smithling and Tasho Kaletha provided me with another example for the split group $\mathrm{SO}(2n)$.

11.2. [11, Lemma 2.9.1(b)]

The proof of the equality $J \cap P = (J \cap M)(J \cap N)$ given in Lemma 2.9.1(b) is flawed. Instead, one can deduce this from the following equality

$$J = (J \cap B)(J \cap \overline{N})(J \cap N)(J \cap M),$$

which is a consequence of [3, 5.2.4].

REFERENCES

[1] J. ARTHUR, L. CLOZEL, *Simple algebras, base change, and the advanced theory of the trace formula*, Annals of Math. Studies **120**, Princeton Univ. Press, 1989.

[2] J. N. BERNSTEIN, Le "centre" de Bernstein, in *Représentations des groupes réductifs sur un corps local* (P. Deligne, ed.), Travaux en Cours, Hermann, 1984, 1–32.

[3] F. BRUHAT, J. TITS, Groupes réductifs sur un corps local. II, *Publ. Math. I.H.É.S.* **60** (1984), 5–184.

[4] C. J. BUSHNELL, P. C. KUTZKO, Smooth representations of reductive p-adic groups: structure theory via types, *Proc. London Math. Soc.* **77** (1998), 582–634.

[5] W. CASSELMAN, Characters and Jacquet modules, *Math. Ann.* **230** (1977), 101–105.

[6] W. CASSELMAN, Introduction to the theory of admissible representations of p-adic reductive groups, unpublished notes, 1995.

[7] L. CLOZEL, The fundamental lemma for stable base change, *Duke Math. J.* **61** (1990), 255–302.

[8] P. DELIGNE, Le support du caractère d'une représentation supercuspidale, *C. R. Acad. Sci. Paris Sér. A-B* **283** (1976), A155–A157.

[9] D. J. GOLDSTEIN, Hecke algebra isomorphisms for tamely ramified characters, Ph.D. Thesis, The University of Chicago, 1990.

[10] T. J. HAINES, Introduction to Shimura varieties with bad reduction of parahoric type, in *Harmonic analysis, the trace formula, and Shimura varieties*, Clay Math. Proc. **4**, Amer. Math. Soc., 2005, 583–642.

[11] T. J. HAINES, The base change fundamental lemma for central elements in parahoric Hecke algebras, *Duke Math. J.* **149** (2009), 569–643.

[12] T. J. HAINES, On Hecke algebra isomorphisms and types for depth-zero principal series, expository note available at http://www.math.umd.edu/~tjh, 2009.

[13] T. J. HAINES, M. RAPOPORT, On parahoric subgroups, *Adv. Math.* **219** (2008), 188–198, appendix to [29].

[14] T. J. HAINES, M. RAPOPORT, Shimura varieties with $\Gamma_1(p)$-level via Hecke algebra isomorphisms: the Drinfeld case, *Ann. Sci. École Norm. Sup.* **45** (2012), 719–785.

[15] T. C. HALES, On the fundamental lemma for standard endoscopy: reduction to unit elements, *Canad. J. Math.* **47** (1995), 974–994.

[16] D. KEYS, Reducibility of unramified unitary principal series representations of p-adic groups and class-1 representations, *Math. Ann.* **260** (1982), 397–402.

[17] R. E. KOTTWITZ, Rational conjugacy classes in reductive groups, *Duke Math. J.* **49** (1982), 785–806.

[18] R. E. KOTTWITZ, Base change for unit elements of Hecke algebras, *Compositio Math.* **60** (1986), 237–250.

[19] R. E. KOTTWITZ, Tamagawa numbers, *Ann. of Math.* **127** (1988), 629–646.

[20] R. E. KOTTWITZ, Shimura varieties and λ-adic representations, in *Automorphic forms, Shimura varieties, and L-functions, Vol. I (Ann Arbor, MI, 1988)*, Perspect. Math. **10**, Academic Press, 1990, 161–209.

[21] R. E. KOTTWITZ, On the λ-adic representations associated to some simple Shimura varieties, *Invent. Math.* **108** (1992), 653–665.

[22] R. E. KOTTWITZ, Points on some Shimura varieties over finite fields, *J. Amer. Math. Soc.* **5** (1992), 373–444.

[23] R. E. KOTTWITZ, Isocrystals with additional structure. II, *Compositio Math.* **109** (1997), 255–339.

[24] J.-P. LABESSE, Fonctions élémentaires et lemme fondamental pour le changement de base stable, *Duke Math. J.* **61** (1990), 519–530.

[25] J.-P. LABESSE, Noninvariant base change identities, *Mém. Soc. Math. France (N.S.)* **61** (1995).

[26] J.-P. LABESSE, Cohomologie, stabilisation et changement de base, *Astérisque* **257** (1999).

[27] R. P. LANGLANDS, *Base change for* GL(2), Annals of Math. Studies **96**, Princeton Univ. Press, 1980.

[28] L. MORRIS, Tamely ramified intertwining algebras, *Invent. Math.* **114** (1993), 1–54.

[29] G. PAPPAS, M. RAPOPORT, Twisted loop groups and their affine flag varieties, *Adv. Math.* **219** (2008), 118–198.

[30] A. ROCHE, Types and Hecke algebras for principal series representations of split reductive p-adic groups, *Ann. Sci. École Norm. Sup.* **31** (1998), 361–413.

[31] J. D. ROGAWSKI, Trace Paley-Wiener theorem in the twisted case, *Trans. Amer. Math. Soc.* **309** (1988), 215–229.

[32] J-P. SERRE, *Local fields*, Graduate Texts in Math. **67**, Springer, 1979.

Thomas J. HAINES
University of Maryland
Department of Mathematics
College Park, MD 20742-4015, USA
E-mail: tjh@math.umd.edu