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On the Picard number of divisors in Fano manifolds

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ON THE PICARD NUMBER OF DIVISORS IN FANO MANIFOLDS

BY CINZIA CASAGRANDE

ABSTRACT. – Let X be a complex Fano manifold of arbitrary dimension, and D a prime divisor in X . We consider the image $\mathcal{N}_1(D, X)$ of $\mathcal{N}_1(D)$ in $\mathcal{N}_1(X)$ under the natural push-forward of 1-cycles. We show that $\rho_X - \rho_D \leq \text{codim } \mathcal{N}_1(D, X) \leq 8$. Moreover if $\text{codim } \mathcal{N}_1(D, X) \geq 3$, then either $X \cong S \times T$ where S is a Del Pezzo surface, or $\text{codim } \mathcal{N}_1(D, X) = 3$ and X has a fibration in Del Pezzo surfaces onto a Fano manifold T such that $\rho_X - \rho_T = 4$.

RÉSUMÉ. – Soient X une variété de Fano lisse et complexe de dimension arbitraire, et D un diviseur premier dans X . Nous considérons l'image $\mathcal{N}_1(D, X)$ de $\mathcal{N}_1(D)$ dans $\mathcal{N}_1(X)$ par l'application naturelle de *push-forward* de 1-cycles. Nous démontrons que $\rho_X - \rho_D \leq \text{codim } \mathcal{N}_1(D, X) \leq 8$. De plus, si $\text{codim } \mathcal{N}_1(D, X) \geq 3$, alors soit $X \cong S \times T$ où S est une surface de Del Pezzo, soit $\text{codim } \mathcal{N}_1(D, X) = 3$ et X a une fibration en surfaces de Del Pezzo sur une variété de Fano lisse T , telle que $\rho_X - \rho_T = 4$.

1. Introduction

Let X be a complex Fano manifold of arbitrary dimension n , and consider a prime divisor $D \subset X$. We denote by $\mathcal{N}_1(X)$ the \mathbb{R} -vector space of one-cycles in X , with real coefficients, modulo numerical equivalence; its dimension is the *Picard number* ρ_X of X , and similarly for D . The inclusion $i: D \hookrightarrow X$ induces a push-forward of one-cycles $i_*: \mathcal{N}_1(D) \rightarrow \mathcal{N}_1(X)$, that does not need to be injective nor surjective. We are interested in the image

$$\mathcal{N}_1(D, X) := i_*(\mathcal{N}_1(D)) \subseteq \mathcal{N}_1(X),$$

which is the linear subspace of $\mathcal{N}_1(X)$ spanned by numerical classes of curves contained in D . The codimension of $\mathcal{N}_1(D, X)$ in $\mathcal{N}_1(X)$ is equal to the dimension of the kernel of the restriction $H^2(X, \mathbb{R}) \rightarrow H^2(D, \mathbb{R})$.

If X is a surface, then it follows from the classification of Del Pezzo surfaces that $\text{codim } \mathcal{N}_1(D, X) = \rho_X - 1 \leq 8$. Our main result is that the same holds in any dimension.

THEOREM 1.1. – *Let X be a Fano manifold of dimension n . For every prime divisor $D \subset X$, we have*

$$\rho_X - \rho_D \leq \text{codim } \mathcal{N}_1(D, X) \leq 8.$$

Moreover, suppose that there exists a prime divisor D with $\text{codim } \mathcal{N}_1(D, X) \geq 3$. Then one of the following holds:

- (i) $X \cong S \times T$, where S is a Del Pezzo surface with $\rho_S \geq \text{codim } \mathcal{N}_1(D, X) + 1$, and D dominates T under the projection;
- (ii) $\text{codim } \mathcal{N}_1(D, X) = 3$ and there exists a flat surjective morphism $\varphi: X \rightarrow T$, with connected fibers, where T is an $(n - 2)$ -dimensional Fano manifold, and $\rho_X - \rho_T = 4$.

When $n \geq 4$ and D is ample, one has $\mathcal{N}_1(D, X) = \mathcal{N}_1(X)$ and also $\dim \mathcal{N}_1(D, X) = \rho_D$ by Lefschetz Theorems on hyperplane sections, see [17, Example 3.1.25]. However in general $\dim \mathcal{N}_1(D, X)$ can be smaller than ρ_X : for instance, if $D \cong \mathbb{P}^{n-1}$ is the exceptional divisor of the blow-up X of any projective manifold at a point, we have $\rho_D = \dim \mathcal{N}_1(D, X) = 1 < \rho_X$.

In case (ii) of Theorem 1.1 the variety X does not need to be a product of lower dimensional varieties, see Example 3.4.

Theorem 1.1 generalizes an analogous result in [9] for toric Fano varieties, obtained in a completely different way, using combinatorial techniques.

We recall that the pseudo-index of a Fano manifold X is

$$\iota_X = \min\{-K_X \cdot C \mid C \text{ is a rational curve in } X\},$$

and is a multiple of the index of X ; one expects that Fano manifolds with large pseudo-index are simpler. When $\iota_X > 1$ (i.e., when X does not contain rational curves of anticanonical degree one), we show a stronger version of Theorem 1.1.

THEOREM 1.2. – *Let X be a Fano manifold with pseudo-index $\iota_X > 1$. For every prime divisor $D \subset X$, we have $\text{codim } \mathcal{N}_1(D, X) \leq 1$. More precisely, one of the following holds:*

- (i) $\iota_X = 2$ and there exists a smooth morphism $\varphi: X \rightarrow Y$ with fibers isomorphic to \mathbb{P}^1 , where Y is a Fano manifold with $\iota_Y > 1$;
- (ii) for every prime divisor $D \subset X$, we have $\mathcal{N}_1(D, X) = \mathcal{N}_1(X)$, $\rho_X \leq \rho_D$, and the restriction $H^2(X, \mathbb{R}) \rightarrow H^2(D, \mathbb{R})$ is injective. Moreover for every pair of prime divisors D_1, D_2 in X , we have $D_1 \cap D_2 \neq \emptyset$.

The author was led to this subject by the study of Fano manifolds with large Picard number (see [10] for an account of this problem). Let us mention two straightforward consequences of Theorem 1.1, which give bounds on ρ_X in some good situations. The first concerns the case $\dim X \leq 5$, while the second is about Fano manifolds having a morphism onto a curve.

COROLLARY 1.3. – *Let X be a Fano manifold, and suppose that there exists a prime divisor $D \subset X$ such that $\text{codim } \mathcal{N}_1(D, X) \geq 3$.*

If $\dim X = 4$ then either $\rho_X \leq 6$, or X is a product of Del Pezzo surfaces and $\rho_X \leq 18$.

If $\dim X = 5$ then either $\rho_X \leq 9$, or X is a product and $\rho_X \leq 19$.

COROLLARY 1.4. – *Let X be a Fano manifold, $\varphi: X \rightarrow \mathbb{P}^1$ a surjective morphism with connected fibers, and $F \subset X$ a general fiber. Then $\rho_X \leq \rho_F + 8$.*

Moreover if $\rho_X \geq \rho_F + 4$, then $X \cong S \times T$ where S is a Del Pezzo surface, φ factors through the projection $X \rightarrow S$, and $F \cong \mathbb{P}^1 \times T$.

Finally, we notice that some of the properties given by Theorem 1.1 are inherited by varieties dominated by a Fano manifold. We give two applications, and refer the reader to Lemma 4.1 for a more general statement.

COROLLARY 1.5. – *Let X be a Fano manifold and $\varphi: X \rightarrow Y$ a surjective morphism. Suppose that there exists a prime divisor $D \subset X$ such that $\dim \varphi(D) \leq 1$ (this always holds if $\dim Y = 2$). Then $\rho_Y \leq 9$.*

Moreover if $\rho_Y \geq 5$ then $\dim Y \leq 3$ and $X \cong S \times T$, where S is a Del Pezzo surface.

COROLLARY 1.6. – *Let X be a Fano manifold and $\varphi: X \rightarrow Y$ a surjective morphism with $\dim Y = 3$. Then $\rho_Y \leq 10$.*

Moreover if $\rho_Y \geq 6$ then $X \cong S \times T$ where S is a Del Pezzo surface, T has a contraction onto \mathbb{P}^1 , and φ factors through $X \rightarrow S \times \mathbb{P}^1$.

Outline of the paper

The idea that a special divisor should affect the geometry of X is classical. In [6] Fano manifolds containing a divisor $D \cong \mathbb{P}^{n-1}$ with normal bundle $\mathcal{N}_{D/X} \cong \mathcal{O}_{\mathbb{P}^{n-1}}(-1)$ are classified. This classification has been extended in [20] to the case $\mathcal{N}_{D/X} \cong \mathcal{O}_{\mathbb{P}^{n-1}}(-a)$ with $a > 0$; moreover [20, Proposition 5] shows that if X contains a divisor D with $\rho_D = 1$, then $\rho_X \leq 3$. More generally, divisors $D \subset X$ with $\dim \mathcal{N}_1(D, X) = 1$ or 2 play an important role in [10, 11].

In Section 2 we treat the main construction that will be used in the paper, based on the analysis of a Mori program for $-D$, where $D \subset X$ is a prime divisor; this is a development of a technique used in [11]. Let us give an idea of our approach, referring the reader to Section 2 for more details.

After [5, 13], we know that we can run a Mori program for any divisor in a Fano manifold X . In fact we need to consider *special Mori programs*, where all involved extremal rays have positive intersection with the anticanonical divisor (see Section 2.1).

Then, given a prime divisor $D \subset X$, we consider a special Mori program for $-D$, which roughly means that we contract or flip extremal rays having positive intersection with D , until we get a fiber type contraction such that (the transform of) D dominates the target.

If $c := \text{codim } \mathcal{N}_1(D, X) > 0$, by studying how the codimension of $\mathcal{N}_1(D, X)$ varies under the birational maps and the related properties of the extremal rays, we obtain $c - 1$ pairwise disjoint prime divisors $E_1, \dots, E_{c-1} \subset X$, all intersecting D , such that each E_i is a smooth \mathbb{P}^1 -bundle with $E_i \cdot f_i = -1$, where $f_i \subset E_i$ is a fiber (see Proposition 2.5 and Lemma 2.7). We call E_1, \dots, E_{c-1} the \mathbb{P}^1 -bundles determined by the special Mori program for $-D$ that we are considering; they play an essential role throughout the paper.

We conclude Section 2 proving Theorem 1.2 about the case with pseudo-index $\iota_X > 1$.

In Section 3 we consider the following invariant of X :

$$c_X := \max\{\text{codim } \mathcal{N}_1(D, X) \mid D \text{ is a prime divisor in } X\}.$$

In terms of this invariant, our main result is that $c_X \leq 8$, and if $c_X \geq 3$, then either X is a product, or $c_X = 3$ and X has a flat fibration onto an $(n - 2)$ -dimensional Fano manifold (see Theorem 3.3 for a precise statement). The proof of this result is quite long: it takes the whole Section 3, and is divided in several steps; see 3.5 for a plan. The strategy is to apply the construction of Section 2 to prime divisors of “minimal Picard number”, *i.e.*, with $\text{codim } \mathcal{N}_1(D, X) = c_X$. We show that there exists a prime divisor E_0 with $\text{codim } \mathcal{N}_1(E_0, X) = c_X$, such that E_0 is a smooth \mathbb{P}^1 -bundle with $E_0 \cdot f_0 = -1$, where $f_0 \subset E_0$ is a fiber. Applying the previous results to E_0 , we obtain a bunch of disjoint divisors with a \mathbb{P}^1 -bundle structure, and we use them to show that X is a product, or to construct a fibration in Del Pezzo surfaces.

Finally in Section 4 we use this result (Theorem 3.3) to prove the remaining results stated above: Theorem 1.1 and its Corollaries 1.3 to 1.6.

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Notation and terminology

We work over the field of complex numbers. A *manifold* is a smooth variety. A \mathbb{P}^1 -*bundle* is a projectivization of a rank 2 vector bundle.

Let X be a projective variety.

$\mathcal{N}_1(X)$ (respectively, $\mathcal{N}^1(X)$) is the \mathbb{R} -vector space of one-cycles (respectively, Cartier divisors) with real coefficients, modulo numerical equivalence.

$[C]$ is the numerical equivalence class in $\mathcal{N}_1(X)$ of a curve $C \subset X$; $[D]$ is the numerical equivalence class in $\mathcal{N}^1(X)$ of a \mathbb{Q} -Cartier divisor D in X .

If $E \subset X$ is an irreducible closed subset and $C \subset E$ is a curve, $[C]_E$ is the numerical equivalence class of C in $\mathcal{N}_1(E)$.

The symbol \equiv stands for numerical equivalence (for both 1-cycles and \mathbb{Q} -Cartier divisors).

For any \mathbb{Q} -Cartier divisor D in X , $D^\perp := \{\gamma \in \mathcal{N}_1(X) \mid D \cdot \gamma = 0\}$.

$\text{NE}(X) \subset \mathcal{N}_1(X)$ is the convex cone generated by classes of effective curves, and $\overline{\text{NE}}(X)$ is its closure.

An *extremal ray* R of X is a one-dimensional face of $\overline{\text{NE}}(X)$; $\text{Locus}(R) \subseteq X$ is the union of all curves whose class is in R .

If R is an extremal ray of X and D is a \mathbb{Q} -Cartier divisor in X , we say that $D \cdot R > 0$, respectively $D \cdot R = 0$, etc. if for $\gamma \in R \setminus \{0\}$ we have $D \cdot \gamma > 0$, respectively $D \cdot \gamma = 0$, etc.

Assume that X is normal.

A *contraction* of X is a surjective morphism with connected fibers $\varphi: X \rightarrow Y$, where Y is normal and projective; $\text{NE}(\varphi)$ is the face of $\overline{\text{NE}}(X)$ generated by classes of curves contracted by φ .

A contraction $\varphi: X \rightarrow Y$ is *elementary* if $\rho_X - \rho_Y = 1$.

We say that an elementary contraction $\varphi: X \rightarrow Y$ (or the extremal ray $\text{NE}(\varphi)$) is of type $(n - 1, n - 2)^{sm}$ if it is the blow-up of a smooth codimension 2 subvariety contained in the smooth locus of Y (here $n = \dim X$).

If $Z \subseteq X$ is a closed subset and $i: Z \hookrightarrow X$ is the inclusion, we set

$$\mathcal{N}_1(Z, X) := i_*(\mathcal{N}_1(Z)) \subseteq \mathcal{N}_1(X) \quad \text{and} \quad \text{NE}(Z, X) := i_*(\text{NE}(Z)) \subseteq \text{NE}(X) \subset \mathcal{N}_1(X).$$

2. Mori programs and prime divisors

2.1. Special Mori programs in Fano manifolds

In this section we recall what a Mori program is, and explain that by [13] and [5] we can run a Mori program for any divisor on a Fano manifold. We also introduce and show the existence of “special Mori programs”, where all involved extremal rays have positive intersection with the anticanonical divisor.

We begin by recalling the following fundamental result.

THEOREM 2.1 ([5], Corollary 1.3.2). – *Any Fano manifold is a Mori dream space.*

We refer the reader to [13] for the definition and properties of a Mori dream space; in particular, a Mori dream space is always a normal and \mathbb{Q} -factorial projective variety. We also need the following.

PROPOSITION 2.2 ([13], Proposition 1.11(1)). – *Let X be a Mori dream space and B a divisor in X . Then there exists a finite sequence*

$$(2.3) \quad X = X_0 \xrightarrow{\sigma_0} X_1 \dashrightarrow \cdots \dashrightarrow X_{k-1} \xrightarrow{\sigma_{k-1}} X_k$$

such that:

- every X_i is a normal and \mathbb{Q} -factorial projective variety;
- for every $i = 0, \dots, k - 1$ there exists an extremal ray Q_i of X_i such that $B_i \cdot Q_i < 0$, where $B_i \subset X_i$ is the transform⁽¹⁾ of B , $\text{Locus}(Q_i) \subsetneq X_i$, and σ_i is either the contraction of Q_i (if Q_i is divisorial), or its flip (if Q_i is small);
- either B_k is nef, or there exists an extremal ray Q_k in X_k , with a fiber type contraction $\varphi: X_k \rightarrow Y$, such that $B_k \cdot Q_k < 0$.

Moreover, the choice of the extremal rays Q_i is arbitrary among those that have negative intersection with B_i .

A sequence as above is called a *Mori program* for the divisor B . We refer the reader to [16, Definition 6.5] for the definition of flip.

An important remark is that when X is Fano, there is always a suitable choice of a Mori program where all involved extremal rays have positive intersection with the anticanonical divisor.

⁽¹⁾ More precisely, B_i is the transform of B_{i-1} if σ_{i-1} is a flip, and $B_i = (\sigma_{i-1})_*(B_{i-1})$ if σ_{i-1} is a divisorial contraction.

PROPOSITION 2.4. – *Let X be a Fano manifold and B a divisor on X . Then there exists a Mori program for B as (2.3), such that $-K_{X_i} \cdot Q_i > 0$ for every $i = 0, \dots, k$. We call such a sequence a special Mori program for B .*

This is a very special case of the MMP with scaling, see [5, Remark 3.10.9]. For the reader's convenience, we give a proof. The idea is to choose a facet of the cone of nef divisors $\text{Nef}(X) \subset \mathcal{N}^1(X)$ met by moving from $[B]$ to $[-K_X]$ along a line in $\mathcal{N}^1(X)$, and to repeat the same at each step.

Proof of Proposition 2.4. – By Theorem 2.1 X is a Mori dream space, therefore Proposition 2.2 applies to X , and there exists a Mori program for B . We have to prove that we can choose Q_0, \dots, Q_k with $B_i \cdot Q_i < 0$ and $-K_{X_i} \cdot Q_i > 0$ for all $i = 0, \dots, k$.

We can assume that B is not nef. Set

$$\lambda_0 := \sup\{\lambda \in \mathbb{R} \mid (1 - \lambda)(-K_X) + \lambda B \text{ is nef}\},$$

so that $\lambda_0 \in \mathbb{Q}$, $0 < \lambda_0 < 1$, and $H_0 := (1 - \lambda_0)(-K_X) + \lambda_0 B$ is nef but not ample. Then there exists an extremal ray Q_0 of $\text{NE}(X)$ such that $H_0 \cdot Q_0 = 0$ and $B \cdot Q_0 < 0$; in particular, $-K_X \cdot Q_0 > 0$.

If Q_0 is of fiber type, we are done. Otherwise, let $\sigma_0: X \dashrightarrow X_1$ be either the contraction of Q_0 (if divisorial), or its flip (if small), and let B_1 be the transform of B . Then $(1 - \lambda_0)(-K_{X_1}) + \lambda_0 B_1$ is nef in X_1 .

If B_1 is nef we are done. If not, we set

$$\lambda_1 := \sup\{\lambda \in \mathbb{R} \mid (1 - \lambda)(-K_{X_1}) + \lambda B_1 \text{ is nef}\},$$

so that $\lambda_1 \in \mathbb{Q}$, $\lambda_0 \leq \lambda_1 < 1$, and $H_1 := (1 - \lambda_1)(-K_{X_1}) + \lambda_1 B_1$ is nef but not ample. There exists an extremal ray Q_1 of $\text{NE}(X_1)$ such that $H_1 \cdot Q_1 = 0$ and $B_1 \cdot Q_1 < 0$, hence $-K_{X_1} \cdot Q_1 > 0$. Now we iterate the procedure. \square

2.2. Running a Mori program for $-D$

In this section we study in detail what happens when we run a Mori program for $-D$, where D is a prime divisor. This point of view has already been considered in [11], and is somehow opposite to the classical one: we consider extremal rays having *positive* intersection with D . In particular, we are interested in how the number $\text{codim } \mathcal{N}_1(D, X)$ varies under the Mori program.

We first describe the general situation for a prime divisor D in a Mori dream space (Lemma 2.6), and then consider the case of a *special* Mori program for $-D$ where D is a prime divisor in a Fano manifold (Lemma 2.7). In particular, we will show the following.

PROPOSITION 2.5. – *Let X be a Fano manifold and $D \subset X$ a prime divisor. Suppose that $\text{codim } \mathcal{N}_1(D, X) > 0$.*

Then there exist pairwise disjoint smooth prime divisors $E_1, \dots, E_s \subset X$, with $s = \text{codim } \mathcal{N}_1(D, X) - 1$ or $s = \text{codim } \mathcal{N}_1(D, X)$, such that every E_j is a \mathbb{P}^1 -bundle with $E_j \cdot f_j = -1$, where $f_j \subset E_j$ is a fiber; moreover $D \cdot f_j > 0$ and $[f_j] \notin \mathcal{N}_1(D, X)$. In particular $E_j \cap D \neq \emptyset$ and $E_j \neq D$.

It is important to point out that the \mathbb{P}^1 -bundles E_1, \dots, E_s are determined not only by D , but by the choice of a special Mori program for $-D$ (see Lemma 2.7). In fact the divisors E_j are the transforms of the loci of some of the extremal rays of the Mori program, the ones where $\text{codim } \mathcal{N}_1(D, X)$ drops.

Finally we study in more detail the case where $s = \text{codim } \mathcal{N}_1(D, X) - 1$ in the Proposition above; in this situation we show that there is an open subset of X which has a conic bundle structure (see Lemma 2.8).

We conclude the section with the proof of Theorem 1.2.

LEMMA 2.6. – *Let X be a Mori dream space and $D \subset X$ a prime divisor. Consider a Mori program for $-D$:*

$$X = X_0 \xrightarrow{\sigma_0} X_1 \dashrightarrow \dots \dashrightarrow X_{k-1} \xrightarrow{\sigma_{k-1}} X_k.$$

Let $D_i \subset X_i$ be the transform of D , for $i = 1, \dots, k$, and set $D_0 := D$, so that $D_i \cdot Q_i > 0$ for $i = 0, \dots, k$. We have the following.

- (1) Every D_i is a prime divisor in X_i , and the program ends with an elementary contraction of fiber type $\varphi: X_k \rightarrow Y$ such that $\text{NE}(\varphi) = Q_k$ and $\varphi(D_k) = Y$.
- (2) $\#\{i \in \{0, \dots, k\} \mid Q_i \notin \mathcal{N}_1(D_i, X_i)\} = \text{codim } \mathcal{N}_1(D, X)$.
- (3) Set $c_i := \text{codim } \mathcal{N}_1(D_i, X_i)$ for $i = 0, \dots, k$. For every $i = 0, \dots, k - 1$ we have

$$c_{i+1} = \begin{cases} c_i & \text{if } Q_i \subset \mathcal{N}_1(D_i, X_i) \\ c_i - 1 & \text{if } Q_i \not\subset \mathcal{N}_1(D_i, X_i) \end{cases}, \text{ and } c_k = \begin{cases} 0 & \text{if } Q_k \subset \mathcal{N}_1(D_k, X_k) \\ 1 & \text{if } Q_k \not\subset \mathcal{N}_1(D_k, X_k). \end{cases}$$

- (4) Suppose that X is smooth. Let $A_1 \subset X_1$ be the indeterminacy locus of σ_0^{-1} , and for $i = 2, \dots, k$, if σ_{i-1} is a divisorial contraction (respectively, if σ_{i-1} is a flip), let $A_i \subset X_i$ be the union of $\sigma_{i-1}(A_{i-1})$ (respectively, the transform of A_{i-1}) and the indeterminacy locus of σ_{i-1}^{-1} .

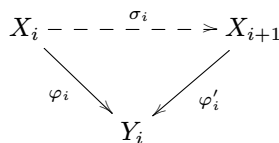
Then for all $i = 1, \dots, k$ we have $\text{Sing}(X_i) \subseteq A_i \subset D_i$, and the birational map $X_i \dashrightarrow X$ is an isomorphism over $X_i \setminus A_i$.

Proof. – Most of the statements are shown in [11] (see in particular Remarks 2.5 and 2.6, and Lemma 3.6); for the reader’s convenience we give a proof. We have $D_i \cdot Q_i > 0$ for every $i = 0, \dots, k$, just by the definition of Mori program for $-D$.

Let $i \in \{0, \dots, k - 1\}$ be such that σ_i is a divisorial contraction. Then $D_i \neq \text{Exc}(\sigma_i)$ (for otherwise $D_i \cdot Q_i < 0$), hence $D_{i+1} = \sigma_i(D_i) \subset X_{i+1}$ is a prime divisor. On the other hand D_i intersects every non-trivial fiber of σ_i (because $D_i \cdot Q_i > 0$), in particular $D_i \cap \text{Exc}(\sigma_i) \neq \emptyset$ and $D_{i+1} \supset \sigma_i(\text{Exc}(\sigma_i))$. Notice that $\sigma_i(\text{Exc}(\sigma_i))$ is the indeterminacy locus of σ_i^{-1} .

Consider the push-forward $(\sigma_i)_*: \mathcal{N}_1(X_i) \rightarrow \mathcal{N}_1(X_{i+1})$. We have $\ker(\sigma_i)_* = \mathbb{R}Q_i$ and $\mathcal{N}_1(D_{i+1}, X_{i+1}) = (\sigma_i)_*(\mathcal{N}_1(D_i, X_i))$, therefore $c_{i+1} = c_i$ if $Q_i \subset \mathcal{N}_1(D_i, X_i)$, and $c_{i+1} = c_i - 1$ otherwise.

Now let $i \in \{0, \dots, k - 1\}$ be such that σ_i is a flip, and consider the standard flip diagram:



where φ_i is the contraction of Q_i , and φ'_i is the corresponding small elementary contraction of X_{i+1} . We have $D_{i+1} \cdot \text{NE}(\varphi'_i) < 0$, in particular $\text{Exc}(\varphi'_i) \subset D_{i+1}$ and $\text{NE}(\varphi'_i) \subset \mathcal{N}_1(D_{i+1}, X_{i+1})$. Notice that $\text{Exc}(\varphi'_i)$ is the indeterminacy locus of σ_i^{-1} .

Moreover $\varphi_i(D_i) = \varphi'_i(D_{i+1})$, so that

$$(\varphi_i)_*(\mathcal{N}_1(D_i, X_i)) = \mathcal{N}_1(\varphi_i(D_i), Y_i) = (\varphi'_i)_*(\mathcal{N}_1(D_{i+1}, X_{i+1})).$$

Since $\ker(\varphi'_i)_* \subseteq \mathcal{N}_1(D_{i+1}, X_{i+1})$, we have $c_{i+1} = \text{codim } \mathcal{N}_1(\varphi_i(D_i), Y_i)$. We deduce again that $c_{i+1} = c_i$ if $Q_i \subset \mathcal{N}_1(D_i, X_i)$, and $c_{i+1} = c_i - 1$ otherwise.

In particular the preceding analysis shows that for every $i = 1, \dots, k$ the divisor D_i contains the indeterminacy locus of σ_i^{-1} , so that $A_i \subset D_i$. By definition, A_i contains the indeterminacy locus of the birational map $(\sigma_{i-1} \circ \dots \circ \sigma_0)^{-1}: X_i \dashrightarrow X$; in particular $X_i \setminus A_i$ is isomorphic to an open subset of X , thus it is smooth if X is smooth. This shows (4).

Consider now the prime divisor $D_k \subset X_k$. Clearly $-D_k$ cannot be nef, therefore the program ends with a fiber type contraction $\varphi: X_k \rightarrow Y$. Since $D_k \cdot Q_k > 0$, D_k intersects every fiber of φ , namely $\varphi(D_k) = Y$, and we have (1).

In particular $\varphi_*(\mathcal{N}_1(D_k, X_k)) = \mathcal{N}_1(Y)$, hence either $c_k = 0$ (i.e., $\mathcal{N}_1(D_k, X_k) = \mathcal{N}_1(X_k)$), or $c_k = 1$ and $Q_k \not\subset \mathcal{N}_1(D_k, X_k)$. Thus we have (3), which implies directly (2). \square

LEMMA 2.7. – *Let X be a Fano manifold and $D \subset X$ a prime divisor. Consider a special Mori program for $-D$:*

$$X = X_0 \xrightarrow{\sigma_0} X_1 \dashrightarrow \dots \dashrightarrow X_{k-1} \xrightarrow{\sigma_{k-1}} X_k.$$

Then we have the following (we keep the notation of Lemma 2.6).

- (1) Let $i \in \{0, \dots, k-1\}$ be such that $Q_i \not\subset \mathcal{N}_1(D_i, X_i)$.
Then Q_i is of type $(n-1, n-2)^{sm}$, i.e., $\sigma_i: X_i \rightarrow X_{i+1}$ is the blow-up of a smooth subvariety of codimension 2, contained in the smooth locus of X_{i+1} . Moreover $\text{Exc}(\sigma_i) \cap A_i = \emptyset$, hence $\text{Exc}(\sigma_i)$ does not intersect the loci of the birational maps σ_l for $l < i$.
- (2) Set $s := \#\{i \in \{0, \dots, k-1\} \mid Q_i \not\subset \mathcal{N}_1(D_i, X_i)\}$. We have two possibilities:
either $s = \text{codim } \mathcal{N}_1(D, X)$ and $\mathcal{N}_1(D_k, X_k) = \mathcal{N}_1(X_k)$,
or $s = \text{codim } \mathcal{N}_1(D, X) - 1$, $Q_k \not\subset \mathcal{N}_1(D_k, X_k)$, and $\text{codim } \mathcal{N}_1(D_k, X_k) = 1$.
- (3) Set $\{i_1, \dots, i_s\} := \{i \in \{0, \dots, k-1\} \mid Q_i \not\subset \mathcal{N}_1(D_i, X_i)\}$, and let $E_j \subset X$ be the transform of $\text{Exc}(\sigma_{i_j}) \subset X_{i_j}$ for every $j = 1, \dots, s$.
Then E_j is a smooth \mathbb{P}^1 -bundle, with fiber $f_j \subset E_j$, such that $E_j \cdot f_j = -1$, $D \cdot f_j > 0$, and $[f_j] \notin \mathcal{N}_1(D, X)$. In particular $E_j \cap D \neq \emptyset$ and $E_j \neq D$.
- (4) The prime divisors E_1, \dots, E_s are pairwise disjoint.

We call E_1, \dots, E_s the \mathbb{P}^1 -bundles determined by the special Mori program for $-D$ that we are considering. These divisors will play a key role throughout the paper.

Notice that Proposition 2.5 is a straightforward consequence of Proposition 2.4 and of Lemma 2.7, more precisely of 2.7(3) and 2.7(4).

Proof. – Statement (1) follows from [11, Lemma 3.9].

By 2.6(2) we have

$$s = \begin{cases} \text{codim } \mathcal{N}_1(D, X) & \text{if } Q_k \subset \mathcal{N}_1(D_k, X_k), \\ \text{codim } \mathcal{N}_1(D, X) - 1 & \text{if } Q_k \not\subset \mathcal{N}_1(D_k, X_k). \end{cases}$$

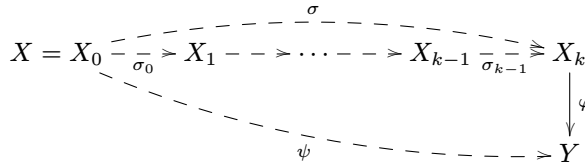
Together with 2.6(3) this yields (2).

Let $j \in \{1, \dots, s\}$. By (1) we have $E_j \cong \text{Exc}(\sigma_{i_j})$, thus E_j is a smooth \mathbb{P}^1 -bundle with $E_j \cdot f_j = -1$, where $f_j \subset E_j$ is a fiber, and $D \cdot f_j > 0$ because $D_{i_j} \cdot Q_{i_j} > 0$ in X_{i_j} . In particular $E_j \cap D \neq \emptyset$ and $E_j \neq D$. Moreover $[f_j] \subset \mathcal{N}_1(D, X)$ would yield $Q_{i_j} \subset \mathcal{N}_1(D_{i_j}, X_{i_j})$, which is excluded by definition. Therefore we have (3).

Finally E_1, \dots, E_s are pairwise disjoint, because for $j = 1, \dots, s$ the divisor $\text{Exc}(\sigma_{i_j})$ does not intersect the loci of the previous birational maps. \square

Here is a more detailed description of the case where $s = \text{codim } \mathcal{N}_1(D, X) - 1$ in Lemma 2.7.

LEMMA 2.8 (Conic bundle case). – *Let X be a Fano manifold and $D \subset X$ a prime divisor. Consider a special Mori program for $-D$; we keep the same notation as in Lemmas 2.6 and 2.7. Set $c := \text{codim } \mathcal{N}_1(D, X)$, $\sigma := \sigma_{k-1} \circ \dots \circ \sigma_0 : X \dashrightarrow X_k$, and $\psi := \varphi \circ \sigma : X \dashrightarrow Y$.*



We assume that $Q_k \not\subset \mathcal{N}_1(D_k, X_k)$, equivalently that $s = c - 1$ (see 2.7(2)). Then we have the following.

- (1) Every fiber of φ has dimension 1, $\dim Y = n - 1$, and φ is finite on D_k .
- (2) Let $j \in \{1, \dots, c-1\}$ and consider $\sigma_{i_j}(\text{Exc}(\sigma_{i_j})) \subset X_{i_j+1}$. For every $m = i_j+1, \dots, k-1$ the set $\text{Locus}(Q_m) \subset X_m$ is disjoint from the image of $\sigma_{i_j}(\text{Exc}(\sigma_{i_j}))$ in X_m , so that the birational map $X_{i_j+1} \dashrightarrow X_k$ is an isomorphism on $\sigma_{i_j}(\text{Exc}(\sigma_{i_j}))$, and σ is regular on $E_j \subset X$.
- (3) There exist open subsets $U \subseteq X$ and $V \subseteq Y$, with $E_1, \dots, E_{c-1} \subset U$, such that V and $\varphi^{-1}(V)$ are smooth, $\varphi|_{\varphi^{-1}(V)} : \varphi^{-1}(V) \rightarrow V$ and $\psi : U \rightarrow V$ are conic bundles, and $\sigma|_U$ is the blow-up of pairwise disjoint smooth subvarieties $T_1, \dots, T_{c-1} \subset \varphi^{-1}(V)$, of dimension $n - 2$, with exceptional divisors E_1, \dots, E_{c-1} .

$$U \begin{array}{c} \xrightarrow{\psi} \\ \xrightarrow{\sigma|_U} \varphi^{-1}(V) \xrightarrow{\varphi} V \end{array}$$

In particular we have $\text{Locus}(Q_m) \subseteq X_m \setminus (\sigma_{m-1} \circ \dots \circ \sigma_0)(U)$ for every $m \in \{0, \dots, k-1\} \setminus \{i_1, \dots, i_{c-1}\}$.

(4) Set $Z_j := \psi(E_j) \subset V$ for every $j \in \{1, \dots, c-1\}$. Then $Z_1, \dots, Z_{c-1} \subset Y$ are pairwise disjoint smooth prime divisors, and $\psi^*(Z_j) = E_j + \widehat{E}_j$, where $\widehat{E}_j \subset U$ is a smooth \mathbb{P}^1 -bundle with fiber $\widehat{f}_j \subset \widehat{E}_j$, $f_j + \widehat{f}_j$ is numerically equivalent to a general fiber of ψ , and

$$\widehat{E}_j \cdot \widehat{f}_j = -1, \quad E_j \cdot \widehat{f}_j = \widehat{E}_j \cdot f_j = 1, \quad \text{and} \quad [\widehat{f}_j] \notin \mathcal{N}_1(E_j, X),$$

for every $j \in \{1, \dots, c-1\}$. In particular the divisors $D, E_1, \dots, E_{c-1}, \widehat{E}_1, \dots, \widehat{E}_{c-1}$ are all distinct, and $E_1 \cup \widehat{E}_1, \dots, E_{c-1} \cup \widehat{E}_{c-1}$ are pairwise disjoint.

We refer the reader to [9, p. 1478-1479] for an explicit description of the rational conic bundle ψ in the toric case.

Proof of Lemma 2.8. – Let $F \subset X_k$ be a fiber of φ . Then $F \cap D_k \neq \emptyset$ because $D_k \cdot Q_k > 0$; on the other hand $\dim(F \cap D_k) = 0$, because if there exists a curve $C \subset F \cap D_k$, then $[C] \in Q_k$ and $[C] \in \mathcal{N}_1(D_k, X_k)$, thus $Q_k \subset \mathcal{N}_1(D_k, X_k)$ against our assumptions. Hence every fiber of φ has dimension 1, $\dim Y = n - 1$, and we have (1).

Recall from 2.6(4) that $\text{Sing}(X_k) \subseteq A_k$, and notice that $\text{codim } A_k \geq 2$, therefore A_k cannot dominate Y . Restricting φ we get a contraction $X_k \setminus \varphi^{-1}(\varphi(A_k)) \rightarrow Y \setminus \varphi(A_k)$ of a smooth variety, with $-K_{X_k}$ relatively ample (because $-K_{X_k} \cdot Q_k > 0$), and one-dimensional fibers. We conclude that $Y \setminus \varphi(A_k)$ is smooth and that $\varphi|_{X_k \setminus \varphi^{-1}(\varphi(A_k))}$ is a conic bundle (see [3, Theorem 4.1(2)]).

By 2.6(4), $\sigma: X \dashrightarrow X_k$ is an isomorphism over $X_k \setminus A_k$. If $U_1 := \sigma^{-1}(X_k \setminus \varphi^{-1}(\varphi(A_k)))$, then $\psi: U_1 \rightarrow Y \setminus \varphi(A_k)$ is again a conic bundle; in particular it is flat, and induces an injective morphism $\iota: Y \setminus \varphi(A_k) \rightarrow \text{Hilb}(X)$. Let $H \subset \text{Hilb}(X)$ be the closure of the image of ι , and $\mathcal{C} \subset H \times X$ the restriction of the universal family over $\text{Hilb}(X)$. We get a diagram:

$$\begin{array}{ccccc} \mathcal{C} & \xrightarrow{e} & X & \xrightarrow{\sigma} & X_k \\ \pi \downarrow & & & \searrow \psi & \downarrow \varphi \\ H & \xleftarrow{\iota} & & & Y \end{array}$$

where $\pi: \mathcal{C} \rightarrow H$ and $e: \mathcal{C} \rightarrow X$ are the projections, and ι is birational. We want to compare the degenerations in X and in X_k of the general fibers the conic bundle $\psi|_{U_1}$.

Fix $j \in \{1, \dots, c-1\}$, and recall from 2.7(1) that $\text{Exc}(\sigma_{i_j}) \cap A_{i_j} = \emptyset$, so that the birational map $X \dashrightarrow X_{i_j}$ is an isomorphism over $\text{Exc}(\sigma_{i_j})$. In X_{i_j+1} we have

$$A_{i_j+1} = \sigma_{i_j}(\text{Exc}(\sigma_{i_j}) \cup A_{i_j}),$$

hence $\sigma_{i_j}(\text{Exc}(\sigma_{i_j}))$ is a connected component of A_{i_j+1} .

Let $x \in \sigma_{i_j}(\text{Exc}(\sigma_{i_j})) \subset X_{i_j+1}$ and let $l \subseteq E_j \subset X$ be the transform of the fiber of σ_{i_j} over x .

Let $B_0 \subseteq H$ be a general irreducible curve which intersects $\pi(e^{-1}(l))$. Since π is equidimensional and the general fiber of π over B_0 is \mathbb{P}^1 , the inverse image $\pi^{-1}(B_0) \subseteq \mathcal{C}$ is irreducible. Set $S := e(\pi^{-1}(B_0)) \subseteq X$, then $S \cap l \neq \emptyset$ by construction.

Consider the normalizations $B \rightarrow B_0$ and $\mathcal{C}_B \rightarrow \pi^{-1}(B_0)$ of B_0 and $\pi^{-1}(B_0)$ respectively; we have induced morphisms $e_B: \mathcal{C}_B \rightarrow S$ and $\pi_B: \mathcal{C}_B \rightarrow B$.

$$\begin{array}{ccc} \mathcal{C}_B & \longrightarrow & \pi^{-1}(B_0) \subseteq \mathcal{C} \xrightarrow{e} X \supseteq S := e(\pi^{-1}(B_0)) \\ \downarrow \pi_B & & \downarrow \pi \\ B & \longrightarrow & B_0 \subseteq H \end{array}$$

Because B_0 is general, $B_0 \cap \text{dom}(\iota^{-1}) \neq \emptyset$, and ι^{-1} induces a morphism $\eta: B \rightarrow Y$. Set $B_1 := \eta(B) \subset Y$.

Again, since φ is equidimensional and the general fiber of φ over B_1 is \mathbb{P}^1 , the inverse image $\varphi^{-1}(B_1) \subset X_k$ is irreducible; call S_k this surface, which is just the transform of $S \subset X$ under σ .

Recall that φ is finite on D_k by (1), and $A_k \subset D_k$ by 2.6(4), hence no component of a fiber of φ can be contained in A_k . On the other hand, by the generality of B_0 , the general fiber of $\varphi|_{S_k}$ does not intersect A_k . Therefore S_k can intersect A_k at most in a finite number of points.

Consider now $\sigma_S := \sigma|_S: S \dashrightarrow S_k$. Then σ_S is an isomorphism over $S_k \setminus (S_k \cap A_k)$ and $\dim(S_k \cap A_k) \leq 0$, hence by Zariski's main theorem $\xi := \sigma_S \circ e_B: \mathcal{C}_B \rightarrow S_k$ is a morphism.

$$\begin{array}{ccc} \mathcal{C}_B & \xrightarrow{e_B} S \subset X & \xrightarrow{\sigma_S} S_k \subset X_k \\ \downarrow \pi_B & & \downarrow \varphi \\ B & \xrightarrow{\eta} & B_1 \subset Y \end{array}$$

ξ (curved arrow from \mathcal{C}_B to S_k)

Let $y \in B$ be such that $C := e_B(\pi_B^{-1}(y)) \subset S$ intersects l ; in particular $C \cap E_j \neq \emptyset$, because $l \subseteq E_j$. Since C is numerically equivalent in X to a general fiber of ψ , we have $-K_X \cdot C = 2$ and $E_j \cdot C = 0$; in particular C has at most two irreducible components, because $-K_X$ is ample.

Set $r := \varphi^{-1}(\eta(y))$. Since r is numerically equivalent in X_k to a general fiber of φ , we have $-K_{X_k} \cdot r = 2$. Recall that no irreducible component of r can be contained in A_k ; on the other hand, r must intersect A_k , otherwise σ_S would be an isomorphism over r , $C = \sigma_S^{-1}(r)$, and $C \cap E_j = \emptyset$, a contradiction.

Let us show that r is an integral fiber of φ . Indeed let C_1 be an irreducible component of r . If $C_1 \cap A_k = \emptyset$, then C_1 is contained in the smooth locus of X_k and $-K_{X_k} \cdot C_1 \geq 1$. If instead $C_1 \cap A_k \neq \emptyset$, then [11, Lemma 3.8] gives $-K_{X_k} \cdot C_1 > 1$. Since $-K_{X_k} \cdot r = 2$ and r must intersect A_k , it must be irreducible and reduced.

For every $i \in \{0, \dots, k-1\}$ let $\tilde{r}_i \subset X_i$ be the transform of $r \subset X_k$ (where $X_0 = X$). Again by [11, Lemma 3.8] we get $-K_X \cdot \tilde{r}_0 < -K_{X_k} \cdot r = 2$, hence $-K_X \cdot \tilde{r}_0 = 1$.

Notice that $\xi(\pi_B^{-1}(y)) \subset S_k$ is contained in r ; on the other hand ξ cannot contract to a point a fiber of π_B , hence $\xi(\pi_B^{-1}(y)) = r$. Then $\tilde{r}_0 \subseteq C$, because $C = e_B(\pi_B^{-1}(y))$, and we get $C = \tilde{r}_0 \cup C'$, where $C' \subset X$ is an irreducible curve (and possibly $C' = \tilde{r}_0$ if C is non-reduced).

Since $r \not\subset A_k$, we have $\tilde{r}_0 \not\subset E_j$; in particular $E_j \cdot \tilde{r}_0 \geq 0$. If $E_j \cdot \tilde{r}_0 = 0$, then also $E_j \cdot C' = 0$ and $C \subset E_j$, which is impossible. Hence $E_j \cdot \tilde{r}_0 > 0$, and since $E_j \cdot C = 0$, we have $E_j \cdot C' < 0$ and $C' \neq \tilde{r}_0$.

Consider now the blow-up $\sigma_{i_j} : X_{i_j} \rightarrow X_{i_j+1}$. We have $\text{Exc}(\sigma_{i_j}) \cdot \tilde{r}_{i_j} = E_j \cdot \tilde{r}_0 \geq 1$, hence using the projection formula we get $-K_{X_{i_j+1}} \cdot \tilde{r}_{i_j+1} \geq -K_{X_{i_j}} \cdot \tilde{r}_{i_j} + 1$. On the other hand [11, Lemma 3.8] gives

$$1 = -K_X \cdot \tilde{r}_0 \leq -K_{X_{i_j}} \cdot \tilde{r}_{i_j} \quad \text{and} \quad -K_{X_{i_j+1}} \cdot \tilde{r}_{i_j+1} \leq -K_{X_k} \cdot r = 2.$$

We conclude that $\text{Exc}(\sigma_{i_j}) \cdot \tilde{r}_{i_j} = 1$, $-K_X \cdot \tilde{r}_0 = -K_{X_{i_j}} \cdot \tilde{r}_{i_j}$, and $-K_{X_{i_j+1}} \cdot \tilde{r}_{i_j+1} = -K_{X_k} \cdot r$, and again by [11, Lemma 3.8] this implies that:

$$(2.9) \quad \text{for every } m \in \{0, \dots, k-1\}, m \neq i_j, \text{Locus}(Q_m) \text{ is disjoint from } \tilde{r}_m.$$

We show that $C' = l$ (recall that $l \subset X$ is the transform of $\sigma_{i_j}^{-1}(x) \subset X_{i_j}$). Since C' intersects \tilde{r}_0 (because $C = \tilde{r}_0 \cup C'$ is connected), and $\tilde{r}_0 \cap \text{Locus}(Q_0) = \emptyset$ by (2.9), we see that C' is not contained in $\text{Locus}(Q_0)$. Iterating this reasoning for every σ_m with $m \in \{0, \dots, i_j-1\}$, we see that C' intersects the open subset where the birational map $X \dashrightarrow X_{i_j}$ is an isomorphism; let $\tilde{C}' \subset X_{i_j}$ be its transform.

If $\sigma_{i_j}(\tilde{C}')$ were a curve, then by the same reasoning it could not be contained in $\text{Locus}(Q_m)$ for any $m = i_j+1, \dots, k-1$, and in the end we would get a curve $\tilde{C}'_k \subset X_k$, distinct from r , which should belong to $\xi(\pi_B^{-1}(y))$, which is impossible. Thus \tilde{C}' must be a fiber of σ_{i_j} . On the other hand $\text{Exc}(\sigma_{i_j}) \cdot \tilde{r}_{i_j} = 1$, thus \tilde{r}_{i_j} intersects a unique fiber of σ_{i_j} , and $C' = l$.

In particular this yields that $x \in \tilde{r}_{i_j+1} \cap \sigma_{i_j}(\text{Exc}(\sigma_{i_j}))$. Since $x \in \sigma_{i_j}(\text{Exc}(\sigma_{i_j}))$ was arbitrary, (2.9) implies statement (2).

Let $T_j \subset X_k$ be the image of $\sigma_{i_j}(\text{Exc}(\sigma_{i_j})) \subset X_{i_j+1}$. By (2) the birational map $X_{i_j+1} \dashrightarrow X_k$ yields an isomorphism between $\sigma_{i_j}(\text{Exc}(\sigma_{i_j}))$ and T_j , hence T_j is smooth of dimension $n-2$, and is contained in the smooth locus of X_k . Since $\sigma_{i_j}(\text{Exc}(\sigma_{i_j}))$ is a connected component of A_{i_j+1} , we deduce that T_j is a connected component of A_k , and $A_k \setminus T_j$ is closed in X_k .

By (2.9) the birational map $X_{i_j+1} \dashrightarrow X_k$ yields also an isomorphism between \tilde{r}_{i_j+1} and r , and $r \cap (A_k \setminus T_j) = \emptyset$.

Consider the point $x' \in T_j$ corresponding to $x \in \sigma_{i_j}(\text{Exc}(\sigma_{i_j}))$. Then $x' \in r \cap T_j$ because $x \in \tilde{r}_{i_j+1}$, i.e., r is the fiber of φ through $x' \in T_j$. Again since x was arbitrary in $\sigma_{i_j}(\text{Exc}(\sigma_{i_j}))$, from $r \cap (A_k \setminus T_j) = \emptyset$ we deduce that $\varphi^{-1}(\varphi(T_j)) \cap (A_k \setminus T_j) = \emptyset$, and hence that $\varphi(T_j) \cap \varphi(A_k \setminus T_j) = \emptyset$ in Y .

Summing up, we have shown that T_1, \dots, T_{c-1} are connected components of A_k (so that $A_k \setminus (T_1 \cup \dots \cup T_{c-1})$ is closed in X_k , and the images $\varphi(T_1), \dots, \varphi(T_{c-1})$, $\varphi(A_k \setminus (T_1 \cup \dots \cup T_{c-1}))$ are pairwise disjoint in Y).

Now set

$$(2.10) \quad V := Y \setminus \varphi(A_k \setminus (T_1 \cup \dots \cup T_{c-1})).$$

Then V is open in Y , $\varphi^{-1}(V) \subseteq \sigma(\text{dom}(\sigma))$, and $T_1 \cup \dots \cup T_{c-1} \subset \varphi^{-1}(V)$. Set $U := \sigma^{-1}(\varphi^{-1}(V)) \subseteq X$. By definition, $\varphi^{-1}(V) \cap (A_k \setminus (T_1 \cup \dots \cup T_{c-1})) = \emptyset$; this means that for every $m \in \{0, \dots, k-1\} \setminus \{i_1, \dots, i_{c-1}\}$, $\text{Locus}(Q_m)$ is disjoint from the image of U in X_m .

We have $E_1, \dots, E_{c-1} \subset U$, because $E_j = \sigma^{-1}(T_j)$, and $\psi: U \rightarrow V$ is regular and proper. More precisely, every fiber of ψ over V is one-dimensional, and as before [3, Theorem 4.1(2)] shows that this is a conic bundle and that V is smooth. We have a factorization

$$U \begin{array}{c} \xrightarrow{\psi} \\ \xrightarrow{\sigma|_U} \varphi^{-1}(V) \xrightarrow{\varphi} V \end{array}$$

and $\sigma|_U$ is just the blow-up of $T_1 \cup \dots \cup T_{c-1}$, so we get (3). For every $j \in \{1, \dots, c-1\}$ we have $Z_j = \psi(E_j) = \varphi(T_j)$, so Z_1, \dots, Z_{c-1} are pairwise disjoint. Now let $\widehat{E}_j \subset U$ be the transform of $\varphi^{-1}(Z_j)$. Then $\psi^{-1}(Z_j) = E_j \cup \widehat{E}_j$, and the rest of statement (4) follows from standard arguments on conic bundles. Just notice that if for some $j \in \{1, \dots, c-1\}$ we have $[\widehat{f}_j] \in \mathcal{N}_1(E_j, X)$, then $[\sigma(\widehat{f}_j)] \in \mathcal{N}_1(T_j, X_k) \subseteq \mathcal{N}_1(A_k, X_k) \subseteq \mathcal{N}_1(D_k, X_k)$, which is impossible because $\sigma(\widehat{f}_j)$ is a fiber of φ and $\text{NE}(\varphi) \not\subset \mathcal{N}_1(D_k, X_k)$ by assumption. \square

Proof of Theorem 1.2. – If $\mathcal{N}_1(D, X) = \mathcal{N}_1(X)$ for every prime divisor $D \subset X$, then we have (ii) (just notice that if $D_1, D_2 \subset X$ are two disjoint divisors, then $\mathcal{N}_1(D_1, X) \subseteq D_2^\perp \subsetneq \mathcal{N}_1(X)$, see Remark 3.1.2).

Suppose now that there exists a prime divisor $D \subset X$ with $\text{codim } \mathcal{N}_1(D, X) > 0$, and consider a special Mori program for $-D$ (which exists by Proposition 2.4). Let $E_1, \dots, E_s \subset X$ be the \mathbb{P}^1 -bundles determined by the Mori program.

If $s \geq 1$, by 2.7(3) we have $-K_X \cdot f_1 = 1$, where $f_1 \subset E_1$ is a fiber of the \mathbb{P}^1 -bundle; this is impossible because $\iota_X > 1$.

Therefore $s = 0$, and 2.7(2) yields that $\text{codim } \mathcal{N}_1(D, X) = 1$ and $Q_k \not\subset \mathcal{N}_1(D_k, X_k)$, so that Lemma 2.8 applies.

We show that $k = 0$ and $X = X_k$. Indeed if not, we have $A_k \neq \emptyset$ in X_k (see 2.6(4)). Take r a fiber of φ intersecting A_k . Then, using [11, Lemma 3.8] as in the proof of Lemma 2.8, we see that r is integral, and that the transform $\tilde{r} \subset X$ of r has anticanonical degree 1 in X , a contradiction.

Thus $X = X_k$ and we get a conic bundle $\varphi: X \rightarrow Y$, which is finite on D . Since X contains no curves of anticanonical degree 1, φ must be a smooth fibration in \mathbb{P}^1 . Then Y is Fano by [21, Proposition 4.3], and finally we have $\iota_Y \geq \iota_X = 2$ by [7, Lemme 2.5]. \square

3. Divisors with minimal Picard number

Let X be a Fano manifold, and consider

$$c_X := \max\{\text{codim } \mathcal{N}_1(D, X) \mid D \text{ is a prime divisor in } X\}.$$

We always have $0 \leq c_X \leq \rho_X - 1$. If S is a Del Pezzo surface, then $c_S = \rho_S - 1 \in \{0, \dots, 8\}$.

EXAMPLE 3.1. – Consider a Fano manifold $X = S \times T$, where S is a Del Pezzo surface. Then $c_X = \max\{\rho_S - 1, c_T\}$. More precisely, for any prime divisor $D \subset X$, we have three possibilities:

- $D = C \times T$ where $C \subset S$ is a curve, and $\text{codim } \mathcal{N}_1(D, X) = \rho_S - 1$;
- $D = S \times D_T$ where $D_T \subset T$ is a divisor, and $\text{codim } \mathcal{N}_1(D, X) = \text{codim } \mathcal{N}_1(D_T, T) \leq c_T$;
- D dominates both S and T under the projections, and $\text{codim } \mathcal{N}_1(D, X) \leq \rho_S - 1$.

Indeed suppose that $D \subset X$ is a prime divisor with $\text{codim } \mathcal{N}_1(D, X) > \rho_S - 1$. Then $\dim \mathcal{N}_1(D, X) < \rho_T + 1$, so that D cannot dominate T under the projection, and $D = S \times D_T$.

EXAMPLE 3.2. – If X is a Fano manifold with pseudo-index $\iota_X \geq 3$ (for instance $X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ with $n_i \geq 2$ for all $i = 1, \dots, r$), then $c_X = 0$ by Theorem 1.2.

We are going to use the results of Section 2.2 to prove the following.

THEOREM 3.3. – For any Fano manifold X we have $c_X \leq 8$. Moreover:

- if $c_X \geq 4$ then $X \cong S \times T$ where S is a Del Pezzo surface, $\rho_S = c_X + 1$, and $c_T \leq c_X$;
- if $c_X = 3$ then there exists a flat, quasi-elementary contraction $X \rightarrow T$ where T is an $(n - 2)$ -dimensional Fano manifold, $\rho_X - \rho_T = 4$, and $c_T \leq 3$.

A contraction φ is *quasi-elementary* if $\ker \varphi_*$ is generated by the numerical classes of the curves contained in a general fiber of φ ; we refer the reader to [10] for properties of quasi-elementary contractions. In particular, in the case where $c_X = 3$ in Theorem 3.3, the general fiber of the contraction $X \rightarrow T$ is a Del Pezzo surface S with $\rho_S \geq 4$.

EXAMPLE 3.4 (Codimension 3). – Let $n \geq 3$ and $Z = \mathbb{P}_{\mathbb{P}^{n-2}}(\mathcal{O}^{\oplus 2} \oplus \mathcal{O}(1))$. Then Z is a toric Fano manifold with $\rho_Z = 2$, and the \mathbb{P}^2 -bundle $Z \rightarrow \mathbb{P}^{n-2}$ has three pairwise disjoint sections $T_1, T_2, T_3 \subset Z$ which are closed under the torus action. Let $X \rightarrow Z$ be the blow-up of T_1, T_2, T_3 . Then X is Fano with $\rho_X = 5$, and it has a smooth morphism $X \rightarrow \mathbb{P}^{n-2}$ such that every fiber is the Del Pezzo surface S with $\rho_S = 4$. If $E \subset X$ is one of the exceptional divisors of the blow-up, one easily checks that $\rho_X - \rho_E = \text{codim } \mathcal{N}_1(E, X) = 3$, hence $c_X \geq 3$. However X is not a product, thus $c_X = 3$ by Theorem 3.3.

3.5. – The proof of Theorem 3.3 will take all the rest of Section 3; we will proceed in several steps. Section 3.1 gathers some preliminary remarks and lemmas. In Section 3.2 we treat the case $c_X \geq 4$, and we show that $X \cong S \times T$, where S is a Del Pezzo surface with $\rho_S = c_X + 1$, and T a Fano manifold with $c_T \leq c_X$ (see Proposition 3.2.1, and 3.2.3 for an outline of its proof). In particular this implies that $c_X \leq 8$, because $\rho_S \leq 9$.

The case $c_X = 3$ is more delicate, as we have to treat separately the two following cases:

- (3.6.a) for every prime divisor $D \subset X$ with $\text{codim } \mathcal{N}_1(D, X) = 3$, and for every special Mori program for $-D$, we have $\mathcal{N}_1(D_k, X_k) = \mathcal{N}_1(X_k)$ (notation as in Lemma 2.6);
- (3.6.b) there exist a prime divisor $D \subset X$ with $\text{codim } \mathcal{N}_1(D, X) = 3$, and a special Mori program for $-D$, such that $\mathcal{N}_1(D_k, X_k) \subsetneq \mathcal{N}_1(X_k)$.

The first case (3.6.a) is treated together with the case $c_X \geq 4$, in Section 3.2. In the end we reach a contradiction, hence *a posteriori* we conclude that (3.6.a) never happens (see Corollary 3.2.2). The second case (3.6.b) is treated in Section 3.3, where we show the existence of a flat, quasi-elementary contraction $X \rightarrow T$, where T is an $(n - 2)$ -dimensional Fano manifold, $\rho_X - \rho_T = 4$, and $c_T \leq 3$ (see Proposition 3.3.1, and 3.3.3 for an outline of its proof).

3.1. Preliminary results

In this section we collect some remarks and lemmas which will be used in the proof of Theorem 3.3.

REMARK 3.1.1. – Let X be a projective manifold, $\varphi: X \rightarrow Y$ a contraction such that $-K_X$ is φ -ample and $\dim Y > 0$, and D a divisor in X such that $\ker \varphi_* \subseteq D^\perp$. Then we have the following:

- (1) $\dim Y = 1 + \dim \varphi(\text{Supp } D)$ and $D = \varphi^*(D_Y)$, D_Y a Cartier divisor in Y ;
- (2) if D is a prime divisor, then $\varphi(D)$ is a prime Cartier divisor, and $D = \varphi^*(\varphi(D))$;
- (3) if D is a smooth prime divisor, let $\varphi(D)^\nu \rightarrow \varphi(D)$ be the normalization. Then the morphism $\varphi_D: D \rightarrow \varphi(D)^\nu$ induced by $\varphi|_D$ is a contraction, and $-K_D$ is φ_D -ample;
- (4) if D is a smooth prime divisor and Y is smooth, then $\varphi(D)$ is a smooth prime divisor.

Proof. – By [16, Theorem 3.7(4)] there exists a Cartier divisor D_Y on Y such that $D = \varphi^*(D_Y)$. Then $\text{Supp } D_Y = \varphi(\text{Supp } D)$, so we have (1).

If D is a prime divisor, then D_Y is a prime divisor supported on $\varphi(D)$, namely $D_Y = \varphi(D)$, and we have (2).

For (3), φ_D is surjective with connected fibers onto a normal projective variety, hence a contraction. Let $i: D \hookrightarrow X$ be the inclusion and take $\gamma \in \overline{\text{NE}}(D) \cap \ker(\varphi_D)_*$ with $\gamma \neq 0$. The restriction $(-K_X)|_D$ is φ_D -ample, hence $(-K_X)|_D \cdot \gamma > 0$. Moreover $i_*(\gamma) \in \ker \varphi_*$, so that

$$-K_D \cdot \gamma = -(K_X + D) \cdot i_*(\gamma) = -K_X \cdot i_*(\gamma) > 0,$$

and $-K_D$ is φ_D -ample.

For (4), let $y \in \varphi(D)$ and let $f \in \mathcal{O}_{Y,y}$ be a local equation for $\varphi(D)$. Then $\varphi^*(f)$ is a local equation for D near the fiber over y . Since D is smooth, the differential $d_x(\varphi^*(f))$ is non-zero, where $x \in \varphi^{-1}(y)$. Then $d_y f$ is non-zero, hence $\varphi(D)$ is smooth at y . \square

REMARK 3.1.2. – Let X be a projective manifold, $Z \subset X$ a closed subset, and $D \subset X$ a prime divisor. If $Z \cap D = \emptyset$, then $D \cdot C = 0$ for every curve $C \subset Z$, hence $\mathcal{N}_1(Z, X) \subseteq D^\perp$.

REMARK 3.1.3. – Let X be a projective manifold, $E \subset X$ a smooth prime divisor which is a \mathbb{P}^1 -bundle with fiber $f \subset E$, and $D \subset X$ a prime divisor with $D \cdot f > 0$. Then the following holds:

- (1) $\dim \mathcal{N}_1(D \cap E, X) \geq \dim \mathcal{N}_1(E, X) - 1$ and $\mathcal{N}_1(E, X) = \mathbb{R}[f] + \mathcal{N}_1(D \cap E, X)$;
- (2) either $[f] \in \mathcal{N}_1(D \cap E, X)$ and $\mathcal{N}_1(D \cap E, X) = \mathcal{N}_1(E, X)$, or $[f] \notin \mathcal{N}_1(D \cap E, X)$ and $\mathcal{N}_1(D \cap E, X)$ has codimension 1 in $\mathcal{N}_1(E, X)$;
- (3) for every irreducible curve $C \subset E$ we have $C \equiv \lambda f + \mu C'$, where C' is an irreducible curve contained in $D \cap E$, $\lambda, \mu \in \mathbb{R}$, and $\mu \geq 0$.

Proof. – Let $\pi: E \rightarrow F$ be the \mathbb{P}^1 -bundle structure on E , and consider the push-forward $\pi_*: \mathcal{N}_1(E) \rightarrow \mathcal{N}_1(F)$. This is a surjective linear map with kernel $\mathbb{R}[f]_E$.

Since $D \cdot f > 0$, we have $\pi(D \cap E) = F$, thus $\pi_*(\mathcal{N}_1(D \cap E, E)) = \mathcal{N}_1(F)$. Therefore $\mathcal{N}_1(E) = \mathbb{R}[f]_E + \mathcal{N}_1(D \cap E, E)$, and applying i_* (where $i: E \hookrightarrow X$ is the inclusion) we get (1) and (2). Statement (3) follows from [18, Lemma 3.2 and Remark 3.3]. \square

REMARK 3.1.4. – Let X be a Fano manifold and $D, E \subset X$ prime divisors with

$$\mathcal{N}_1(D \cap E, X) \subseteq E^\perp.$$

Suppose that E is a smooth \mathbb{P}^1 -bundle with fiber $f \subset E$, such that $E \cdot f = -1$ and $D \cdot f > 0$.

Then the half-line $\mathbb{R}_{\geq 0}[f] \subset \text{NE}(X)$ is an extremal ray of type $(n-1, n-2)^{sm}$, with contraction $\varphi: X \rightarrow Y$ where $E = \text{Exc}(\varphi)$ and Y is Fano.

Proof. – Notice first of all that $(-K_X + E) \cdot f = 0$.

Let $C \subset X$ be an irreducible curve. If $C \not\subseteq E$, then $(-K_X + E) \cdot C > 0$. If $C \subseteq D \cap E$, then $E \cdot C = 0$, and again $(-K_X + E) \cdot C > 0$.

Assume now that $C \subseteq E$. By 3.1.3(3) we have $C \equiv \lambda f + \mu C'$, where C' is a curve contained in $D \cap E$, $\lambda, \mu \in \mathbb{R}$, and $\mu \geq 0$. Thus

$$(-K_X + E) \cdot C = \mu(-K_X + E) \cdot C' \geq 0,$$

and $(-K_X + E) \cdot C = 0$ if and only if $\mu = 0$, if and only if $[C] \in \mathbb{R}_{\geq 0}[f]$. Therefore $-K_X + E$ is nef, and $(-K_X + E)^\perp \cap \text{NE}(X) = \mathbb{R}_{\geq 0}[f]$ is an extremal ray.

Let $\varphi: X \rightarrow Y$ be the contraction of $\mathbb{R}_{\geq 0}[f]$; clearly $\text{Exc}(\varphi) = E$. Since $(-K_X + E) \cdot C > 0$ for every curve $C \subset D \cap E$, φ is finite on $D \cap E$. Thus if $F \subset E$ is a fiber of φ , then $F \cap D \neq \emptyset$ (because $D \cdot \text{NE}(\varphi) > 0$), and $\dim(F \cap D) = 0$. This yields that $\dim F = 1$, and by [1, Theorem 2.3] $\mathbb{R}_{\geq 0}[f]$ is of type $(n-1, n-2)^{sm}$ and Y is smooth.

Finally $-K_X + E = \varphi^*(-K_Y)$, thus $-K_Y$ is ample and Y is Fano (notice that $\text{NE}(Y)$ is closed, because $\text{NE}(Y) = \varphi_*(\text{NE}(X))$). \square

LEMMA 3.1.5. – Let X be a Fano manifold and $D, E \subset X$ prime divisors with

$$\mathcal{N}_1(D \cap E, X) = \mathcal{N}_1(E, X) \cap D^\perp \subseteq E^\perp.$$

Suppose that E is a smooth \mathbb{P}^1 -bundle with fiber $f \subset E$, such that $E \cdot f = -1$ and $D \cdot f > 0$.

Then $E \cong \mathbb{P}^1 \times F$ where F is a Fano manifold, and $D \cap E = \{\text{pts}\} \times F$. Moreover the half-line $\mathbb{R}_{\geq 0}[f]$ is an extremal ray of type $(n-1, n-2)^{sm}$, it is the unique extremal ray having negative intersection with E , and the target of its contraction is Fano.

Proof. – Consider the divisor $D|_E$ in E . We have $\text{Supp}(D|_E) = D \cap E$, and if $C \subseteq D \cap E$ is an irreducible curve, then $[C] \in \mathcal{N}_1(D \cap E, X) \subseteq D^\perp$, so that $D|_E \cdot C = D \cdot C = 0$. Therefore $D|_E$ is nef.

Let $i: E \hookrightarrow X$ be the inclusion and take $\gamma \in \overline{\text{NE}}(E) \cap (D|_E)^\perp$ with $\gamma \neq 0$. Then $i_*(\gamma) \in \mathcal{N}_1(E, X) \cap D^\perp \subseteq E^\perp$, hence:

$$-K_E \cdot \gamma = -(K_X + E) \cdot i_*(\gamma) = -K_X \cdot i_*(\gamma) = (-K_X)|_E \cdot \gamma > 0.$$

By the contraction theorem, there exists a contraction $g: E \rightarrow Z$ such that $-K_E$ is g -ample and $\text{NE}(g) = \overline{\text{NE}}(E) \cap (D|_E)^\perp$ (see [16, Theorem 3.7(3)]). Notice that $D|_E \cdot f = D \cdot f > 0$, hence g does not contract the fibers of the \mathbb{P}^1 -bundle on E , and $\dim Z \geq 1$. On the other hand g sends $D \cap E$ to a union of points, so that $\dim Z = 1$ by 3.1.1(1). More precisely, since $g(f) = Z$, we get $Z \cong \mathbb{P}^1$. The general fiber F of g is a Fano manifold of dimension $n-2$, because $-K_E$ is g -ample.

By [11, Lemma 4.9] we conclude that $E \cong \mathbb{P}^1 \times F$ and g is the projection onto \mathbb{P}^1 . Since $D \cdot f > 0$, $D \cap E$ dominates F under the projection, and is sent by g to a union of points; therefore $D \cap E = \{\text{pts}\} \times F$.

Using Remark 3.1.4, we see that $\mathbb{R}_{\geq 0}[f]$ is an extremal ray of type $(n - 1, n - 2)^{sm}$, and the target of its contraction is Fano.

Finally let R be an extremal ray of X with $E \cdot R < 0$. Then $R \subseteq \text{NE}(E, X) \subseteq \text{NE}(X)$, thus R must be a one-dimensional face of $\text{NE}(E, X)$.⁽²⁾ Since $E \cong \mathbb{P}^1 \times F$, we have $\text{NE}(E) = \mathbb{R}_{\geq 0}[f]_E + \text{NE}(\{pt\} \times F, E)$ and $\text{NE}(E, X) = \mathbb{R}_{\geq 0}[f] + \text{NE}(\{pt\} \times F, X)$. On the other hand $\text{NE}(\{pt\} \times F, X) \subset \mathcal{N}_1(\{pt\} \times F, X) = \mathcal{N}_1(D \cap E, X) \subseteq E^\perp$, therefore $R = \mathbb{R}_{\geq 0}[f]$. \square

REMARK 3.1.6. – Let X be a projective manifold and $E_0 \subset X$ a smooth prime divisor which is a \mathbb{P}^1 -bundle with fiber $f_0 \subset E_0$. Let $E_1, \dots, E_s \subset X$ be pairwise disjoint prime divisors such that $E_0 \neq E_i$ and $E_0 \cap E_i \neq \emptyset$ for every $i = 1, \dots, s$. Then either $E_1 \cdot f_0 = \dots = E_s \cdot f_0 = 0$, or $E_i \cdot f_0 > 0$ for $i = 1, \dots, s$.

Proof. – For every $i = 1, \dots, s$ we have $E_i \cdot f_0 \geq 0$, because $E_0 \neq E_i$.

Suppose that there exists $j \in \{1, \dots, s\}$ such that $E_j \cdot f_0 = 0$. Since $E_0 \cap E_j \neq \emptyset$, this implies that E_j contains a fiber \bar{f}_0 of the \mathbb{P}^1 -bundle structure on E_0 . If $i \in \{1, \dots, s\}, i \neq j$, we have $E_i \cap E_j = \emptyset$, in particular $E_i \cap \bar{f}_0 = \emptyset$ and hence $E_i \cdot f_0 = 0$. \square

LEMMA 3.1.7. – Let X be a Fano manifold and $D \subset X$ a prime divisor with $\text{codim } \mathcal{N}_1(D, X) = c_X$. Let $E_1, \dots, E_s \subset X$ be pairwise disjoint prime divisors such that:

$D \cap E_i \neq \emptyset, D \neq E_i,$ and $\text{codim } \mathcal{N}_1(D \cap E_i, X) \leq c_X + 1,$ for every $i = 1, \dots, s$.

If $s \geq 2,$ then $\text{codim } \mathcal{N}_1(D \cap E_i, X) = c_X + 1$ for every $i = 1, \dots, s,$ and

$$\mathcal{N}_1(D \cap E_i, X) = \mathcal{N}_1(D, X) \cap E_j^\perp \text{ for every } i \neq j.$$

If $s \geq 3,$ then there exists a linear subspace $L \subset \mathcal{N}_1(X),$ of codimension $c_X + 1,$ such that $L = \mathcal{N}_1(D \cap E_i, X) = \mathcal{N}_1(D, X) \cap E_i^\perp$ for every $i = 1, \dots, s.$

Proof. – Assume that $s \geq 2,$ and let $i, j \in \{1, \dots, s\}$ with $i \neq j$. Since $E_i \cap E_j = \emptyset,$ we have $\mathcal{N}_1(D \cap E_i, X) \subseteq E_j^\perp$ by Remark 3.1.2. On the other hand, since $D \cap E_j \neq \emptyset$ and $D \neq E_j,$ there exists some curve $C \subset D$ with $E_j \cdot C > 0,$ so that $\mathcal{N}_1(D, X) \not\subseteq E_j^\perp$. Therefore we get:

$$\mathcal{N}_1(D \cap E_i, X) \subseteq \mathcal{N}_1(D, X) \cap E_j^\perp \subsetneq \mathcal{N}_1(D, X),$$

hence $\rho_X - c_X - 1 \leq \dim \mathcal{N}_1(D \cap E_i, X) \leq \dim \mathcal{N}_1(D, X) \cap E_j^\perp = \dim \mathcal{N}_1(D, X) - 1 = \rho_X - c_X - 1,$ and this yields the statement.

Assume now that $s \geq 3,$ and set $L := \mathcal{N}_1(D \cap E_1, X);$ the first part already gives that $\text{codim } L = c_X + 1$ and that $L = \mathcal{N}_1(D, X) \cap E_i^\perp$ for every $i = 2, \dots, s.$ If $i, j \in \{2, \dots, s\}$ are distinct, again by the first part we get

$$L = \mathcal{N}_1(D, X) \cap E_i^\perp = \mathcal{N}_1(D \cap E_j, X) = \mathcal{N}_1(D, X) \cap E_1^\perp. \quad \square$$

⁽²⁾ Since F and E are Fano, the cones $\text{NE}(F), \text{NE}(E), \text{NE}(E, X),$ etc. are closed and polyhedral.

LEMMA 3.1.8. – Let X be a Fano manifold and $D \subset X$ a prime divisor with $\text{codim } \mathcal{N}_1(D, X) = c_X$. Let $E_1, \dots, E_s \subset X$ be pairwise disjoint smooth prime divisors, and suppose that E_i is a \mathbb{P}^1 -bundle with fiber $f_i \subset E_i$, such that $E_i \cdot f_i = -1$ and $D \cdot f_i > 0$, for every $i = 1, \dots, s$.

Assume that $s \geq 2$. Then $\text{codim } \mathcal{N}_1(E_i, X) = c_X$ and $\text{codim } \mathcal{N}_1(D \cap E_i, X) = c_X + 1$ for every $i = 1, \dots, s$; moreover $\mathcal{N}_1(D \cap E_i, X) = \mathcal{N}_1(D, X) \cap E_j^\perp$ for every $i \neq j$.

Proof. – Let $i \in \{1, \dots, s\}$. We have $D \cap E_i \neq \emptyset$ and $D \neq E_i$ because $D \cdot f_i > 0$ and $E_i \cdot f_i = -1$. Since $D \cdot f_i > 0$, by 3.1.3(1) and by the definition of c_X we have

$$(3.1.9) \quad \text{codim } \mathcal{N}_1(D \cap E_i, X) \leq \text{codim } \mathcal{N}_1(E_i, X) + 1 \leq c_X + 1.$$

Therefore Lemma 3.1.7 yields that $\mathcal{N}_1(D \cap E_i, X) = \mathcal{N}_1(D, X) \cap E_j^\perp$ if $i \neq j$, and $\text{codim } \mathcal{N}_1(D \cap E_i, X) = c_X + 1$. By (3.1.9) we get $\text{codim } \mathcal{N}_1(E_i, X) = c_X$. \square

LEMMA 3.1.10. – Let X be a Fano manifold and $D \subset X$ a prime divisor with $\text{codim } \mathcal{N}_1(D, X) = c_X$. Let $E_1, \dots, E_s, \widehat{E}_1, \dots, \widehat{E}_s \subset X$ be prime divisors such that E_i and \widehat{E}_i are smooth \mathbb{P}^1 -bundles, with fibers respectively $f_i \subset E_i$ and $\widehat{f}_i \subset \widehat{E}_i$, and moreover:

$$E_i \cdot f_i = \widehat{E}_i \cdot \widehat{f}_i = -1, \quad D \cdot f_i > 0, \quad E_i \cdot \widehat{f}_i > 0, \quad \widehat{E}_i \cdot f_i > 0, \quad [\widehat{f}_i] \notin \mathcal{N}_1(E_i, X),$$

and no fiber \widehat{f}_i is contained in D , for every $i = 1, \dots, s$. We assume also that $E_1 \cup \widehat{E}_1, \dots, E_s \cup \widehat{E}_s$ are pairwise disjoint, and that $s \geq 2$.

Then $\text{codim } \mathcal{N}_1(E_i, X) = \text{codim } \mathcal{N}_1(\widehat{E}_i, X) = c_X$ and $[\widehat{f}_i] \notin \mathcal{N}_1(\widehat{E}_i, X)$ for every $i = 1, \dots, s$.

Proof. – Lemma 3.1.8 (applied to D and E_1, \dots, E_s) shows that $\text{codim } \mathcal{N}_1(E_i, X) = c_X$ for every $i = 1, \dots, s$.

Fix $i \in \{1, \dots, s\}$. Since $\mathcal{N}_1(E_i \cap \widehat{E}_i, X) \subseteq \mathcal{N}_1(E_i, X)$, we have $[\widehat{f}_i] \notin \mathcal{N}_1(E_i \cap \widehat{E}_i, X)$. Because $E_i \cdot \widehat{f}_i > 0$, 3.1.3(2) yields that $\mathcal{N}_1(E_i \cap \widehat{E}_i, X)$ has codimension 1 in $\mathcal{N}_1(\widehat{E}_i, X)$. Recall that by the definition of c_X we have $\text{codim } \mathcal{N}_1(\widehat{E}_i, X) \leq c_X$, so that $\text{codim } \mathcal{N}_1(E_i \cap \widehat{E}_i, X) \leq c_X + 1$.

Let us show that

$$(3.1.11) \quad \text{codim } \mathcal{N}_1(E_i \cap \widehat{E}_i, X) = c_X + 1 \quad \text{and} \quad \text{codim } \mathcal{N}_1(\widehat{E}_i, X) = c_X.$$

If $D \cap \widehat{E}_i = \emptyset$, then $\mathcal{N}_1(E_i \cap \widehat{E}_i, X) \subseteq \mathcal{N}_1(E_i, X) \cap D^\perp$ (see Remark 3.1.2); on the other hand $\mathcal{N}_1(E_i, X) \cap D^\perp \subsetneq \mathcal{N}_1(E_i, X)$, because $D \cdot f_i > 0$. This yields $\text{codim } \mathcal{N}_1(E_i \cap \widehat{E}_i, X) = c_X + 1$.

If instead $D \cap \widehat{E}_i \neq \emptyset$, then $D \cdot \widehat{f}_i > 0$, because D cannot contain any curve \widehat{f}_i . Thus we can apply Lemma 3.1.8 to the divisors D and $E_1, \dots, E_{i-1}, \widehat{E}_i, E_{i+1}, \dots, E_s$, and we deduce that $\text{codim } \mathcal{N}_1(\widehat{E}_i, X) = c_X$. Hence we have (3.1.11).

Since $\widehat{E}_i \cdot f_i > 0$ and $\text{codim } \mathcal{N}_1(E_i, X) = c_X = \text{codim } \mathcal{N}_1(E_i \cap \widehat{E}_i, X) - 1$, again by 3.1.3(2) we get $[\widehat{f}_i] \notin \mathcal{N}_1(E_i \cap \widehat{E}_i, X)$. For dimensional reasons $\mathcal{N}_1(E_i \cap \widehat{E}_i, X) = \mathcal{N}_1(E_i, X) \cap \mathcal{N}_1(\widehat{E}_i, X)$, and we conclude that $[\widehat{f}_i] \notin \mathcal{N}_1(\widehat{E}_i, X)$. \square

3.2. The case where X is a product

The main results of this section are the following.

PROPOSITION 3.2.1. – *Let X be a Fano manifold such that either $c_X \geq 4$, or $c_X = 3$ and X satisfies (3.6.a).*

Then $X \cong S \times T$, where S is a Del Pezzo surface with $\rho_S = c_X + 1$, and $c_T \leq c_X$. In particular, $c_X \leq 8$.

COROLLARY 3.2.2. – *Let X be a Fano manifold with $c_X = 3$. Then X satisfies (3.6.b).*

Proof of Corollary 3.2.2. – By contradiction, suppose that X satisfies (3.6.a). Then by Proposition 3.2.1 we have $X \cong S \times T$ and $\rho_S = 4$, i.e., S is the blow-up of \mathbb{P}^2 in three non-collinear points. Consider the sequence:

$$X \cong S \times T \longrightarrow S_1 \times T \longrightarrow \mathbb{F}_1 \times T \longrightarrow \mathbb{P}^1 \times T,$$

where S_1 is the blow-up of \mathbb{P}^2 in two distinct points. Let $C \subset \mathbb{F}_1$ be the section of the \mathbb{P}^1 -bundle containing the two points blown up under $S \rightarrow \mathbb{F}_1$. Let moreover $\tilde{C} \subset S$ be its transform, and $D := \tilde{C} \times T \subset X$. Then $\text{codim } \mathcal{N}_1(D, X) = 3$, and the sequence above is a special Mori program for $-D$. The image of D in $\mathbb{F}_1 \times T$ is $C \times T$, and $\mathcal{N}_1(C \times T, \mathbb{F}_1 \times T) \subsetneq \mathcal{N}_1(\mathbb{F}_1 \times T)$. Thus we have a contradiction with (3.6.a). \square

3.2.3. – *Outline of the proof of Proposition 3.2.1.* There are three preparatory steps, and then the actual proof.

The first step is to apply the construction of Section 2.2 to a prime divisor $D \subset X$ with $\text{codim } \mathcal{N}_1(D, X) = c_X$. We consider a special Mori program for $-D$, and this determines pairwise disjoint \mathbb{P}^1 -bundles $E_1, \dots, E_s \subset X$ as in Lemma 2.7; we denote by $f_i \subset E_i$ a fiber. The crucial property here is that $s \geq 3$: indeed $s \geq \text{codim } \mathcal{N}_1(D, X) - 1 = c_X - 1$, so that $s \geq 3$ if $c_X \geq 4$. On the other hand if $c_X = 3$ we have $s = 3$ by (3.6.a). Then for $i = 1, \dots, s$ we show that $\text{codim } \mathcal{N}_1(E_i, X) = c_X$ and that $\mathbb{R}_{\geq 0}[f_i]$ is an extremal ray of type $(n-1, n-2)^{sm}$, such that the target of its contraction is again Fano. This is Lemma 3.2.4.

In particular, this shows that X has at least one extremal ray R_0 of type $(n-1, n-2)^{sm}$ such that if $E_0 := \text{Locus}(R_0)$, then $\text{codim } \mathcal{N}_1(E_0, X) = c_X$, and the target of the contraction of R_0 is Fano.

Now we replace D by E_0 , and apply again the same construction. Let $p: E_0 \rightarrow F$ be the \mathbb{P}^1 -bundle structure. Since E_1, \dots, E_s are pairwise disjoint, either $E_0 \cap E_i$ is a union of fibers of p for every $i = 1, \dots, s$, or $p(E_0 \cap E_i) = F$ for every $i = 1, \dots, s$. The second preparatory step is to show that if E_1, \dots, E_s intersect E_0 horizontally with respect to the \mathbb{P}^1 -bundle (i.e., $p(E_0 \cap E_i) = F$), the divisors E_0, \dots, E_s have very special properties; in particular, for every $i = 0, \dots, s$, $E_i \cong \mathbb{P}^1 \times F$ where F is an $(n-2)$ -dimensional Fano manifold. This is Lemma 3.2.7.

The third preparatory step is show that we can always choose the extremal ray R_0 , and the special Mori program for $-E_0$, in such a way that E_1, \dots, E_s actually intersect E_0 horizontally with respect to the \mathbb{P}^1 -bundle, so that the previous result applies. This is Lemma 3.2.10.

Then we are ready for the proof of Proposition 3.2.1. We use the properties given by Lemma 3.2.7 to show that E_1, \dots, E_s are the exceptional divisors of the blow-up

$\sigma: X \rightarrow X_s$ of a Fano manifold X_s in s smooth codimension 2 subvarieties. Moreover there is an elementary contraction of fiber type $\varphi: X_s \rightarrow Y$ such that if $\psi := \varphi \circ \sigma: X \rightarrow Y$, then $\psi(E_0) = Y$, and ψ is finite on $\{pt\} \times F \subset E_0$ (recall that $E_0 \cong \mathbb{P}^1 \times F$). We have then two possibilities: either ψ is not finite on E_0 and $\dim Y = n - 2$, or ψ is finite on E_0 and $\dim Y = n - 1$.

We first consider the case where ψ is not finite on E_0 , in 3.2.21. We use the divisors E_0, \dots, E_s to define a contraction $X \rightarrow S$ onto a surface, such that the induced morphism $\pi: X \rightarrow S \times Y$ is finite. Finally we show that in fact π is an isomorphism; here the key property is that E_0, \dots, E_s are products.

Then we consider in 3.2.24 the case where ψ is finite on E_0 . In this situation Y is smooth, and both ψ and φ are conic bundles. If $T_1, \dots, T_s \subset X_s$ are the subvarieties blown up by σ , the transforms $\widehat{E}_1, \dots, \widehat{E}_s \subset X$ of $\varphi^{-1}(\varphi(T_i))$ are smooth \mathbb{P}^1 -bundles.

Similarly to what previously done for E_0, \dots, E_s , we show that $\widehat{E}_i \cong \mathbb{P}^1 \times F$ for every $i = 1, \dots, s$.

Since $\psi(E_0) = Y$, Y is covered by the family of rational curves $\psi(\mathbb{P}^1 \times \{pt\})$. We use a result from [8] to show that in fact these rational curves are the fibers of a smooth morphism $Y \rightarrow Y'$, where $\dim Y' = n - 2$.

In this way we get a contraction $X \rightarrow Y'$, and we proceed similarly to the previous case: we use the divisors $E_0, E_1, \dots, E_s, \widehat{E}_1, \dots, \widehat{E}_s$ to define a contraction $X \rightarrow S$ onto a surface, and show that the induced morphism $X \rightarrow S \times Y'$ is an isomorphism.

Let us start with the first preparatory result.

LEMMA 3.2.4. – *Let X be a Fano manifold such that either $c_X \geq 4$, or $c_X = 3$ and X satisfies (3.6.a).*

Let $D \subset X$ be a prime divisor with $\text{codim } \mathcal{N}_1(D, X) = c_X$, consider a special Mori program for $-D$, and let $E_1, \dots, E_s \subset X$ be the \mathbb{P}^1 -bundles determined by the Mori program. For $i = 1, \dots, s$ let $f_i \subset E_i$ be a fiber of the \mathbb{P}^1 -bundle, and set $R_i := \mathbb{R}_{\geq 0}[f_i]$. Then we have the following:

- (1) $s \in \{c_X - 1, c_X\}$ and $s \geq 3$;
- (2) R_i is an extremal ray of type $(n - 1, n - 2)^{sm}$, the target of the contraction of R_i is Fano, and $\text{codim } \mathcal{N}_1(E_i, X) = c_X$, for every $i = 1, \dots, s$;
- (3) there exists a linear subspace $L \subset \mathcal{N}_1(X)$, of codimension $c_X + 1$, such that

$$L = \mathcal{N}_1(D \cap E_i, X) = \mathcal{N}_1(D, X) \cap E_i^\perp = \mathcal{N}_1(E_i, X) \cap E_i^\perp \text{ for every } i = 1, \dots, s.$$

We will call R_1, \dots, R_s the extremal rays determined by the special Mori program for $-D$ that we are considering. Notice that differently from the case of the \mathbb{P}^1 -bundles E_1, \dots, E_s , the extremal rays R_1, \dots, R_s are defined only when X satisfies the assumptions of Lemma 3.2.4, and $D \subset X$ is a prime divisor with $\text{codim } \mathcal{N}_1(D, X) = c_X$.

Proof. – We know by Lemma 2.7 that: $E_i \cdot f_i = -1$ and $D \cdot f_i > 0$ for $i = 1, \dots, s$, E_1, \dots, E_s are pairwise disjoint, and $s \in \{c_X - 1, c_X\}$ because $\text{codim } \mathcal{N}_1(D, X) = c_X$. Moreover, if $c_X = 3$, then $s = 3$ by (3.6.a), so that in any case $s \geq 3$, and we get (1).

Therefore, by Lemma 3.1.8, we have $\text{codim } \mathcal{N}_1(E_i, X) = c_X$ and $\text{codim } \mathcal{N}_1(D \cap E_i, X) = c_X + 1$ for every $i = 1, \dots, s$. In particular, Lemma 3.1.7 applies; let $L \subset \mathcal{N}_1(X)$ be the linear

subspace such that $\text{codim } L = c_X + 1$ and $L = \mathcal{N}_1(D \cap E_i, X) = \mathcal{N}_1(D, X) \cap E_i^\perp$ for every $i = 1, \dots, s$.

Fix $i \in \{1, \dots, s\}$. Since $E_i \cdot f_i = -1$, we have $\mathcal{N}_1(E_i, X) \not\subseteq E_i^\perp$, therefore $\dim \mathcal{N}_1(E_i, X) \cap E_i^\perp = \dim \mathcal{N}_1(E_i, X) - 1 = \rho_X - c_X - 1 = \dim L$. On the other hand we have $L \subseteq E_i^\perp$ and $L = \mathcal{N}_1(D \cap E_i, X)$, in particular $L \subseteq \mathcal{N}_1(E_i, X)$. Thus $L \subseteq \mathcal{N}_1(E_i, X) \cap E_i^\perp$, so the two subspaces must coincide, and we get (3).

Finally, (2) follows from Remark 3.1.4 applied to D and E_i . □

LEMMA 3.2.5. – *Let X be a Fano manifold such that either $c_X \geq 4$, or $c_X = 3$ and X satisfies (3.6.a).*

Let $D \subset X$ be a prime divisor with $\text{codim } \mathcal{N}_1(D, X) = c_X$, and R an extremal ray of type $(n - 1, n - 2)^{sm}$ such that $D \cdot R > 0$, $R \not\subseteq \mathcal{N}_1(D, X)$, and the target of the contraction of R is Fano.

Set $E := \text{Locus}(R)$. Then $\mathcal{N}_1(D \cap E, X) = \mathcal{N}_1(D, X) \cap E^\perp = \mathcal{N}_1(E, X) \cap E^\perp$.

Proof. – Consider the contraction $\varphi: X \rightarrow Y$ of R , so that by the assumptions Y is a Fano manifold, and consider the prime divisor $\varphi(D) \subset Y$.

By Proposition 2.4, there exists a special Mori program for $-\varphi(D)$ in Y . Together with φ , this gives a special Mori program for $-D$ in X , where the first extremal ray is precisely $Q_0 = R$:

$$X \xrightarrow{\varphi} Y = Y_0 \xrightarrow{\sigma_0} Y_1 \dashrightarrow \dots \dashrightarrow Y_{k-1} \xrightarrow{\sigma_{k-1}} Y_k.$$

We apply Lemmas 2.7 and 3.2.4; since $R \not\subseteq \mathcal{N}_1(D, X)$, E is one of the \mathbb{P}^1 -bundles determined by this special Mori program for $-D$. Thus the statement follows from 3.2.4(3). □

REMARK 3.2.6. – Let X be a Fano manifold such that either $c_X \geq 4$, or $c_X = 3$ and X satisfies (3.6.a). Recall from Proposition 2.4 that there exists a special Mori program for any divisor in X .

The first consequence of Lemma 3.2.4 (applied to any prime divisor $D \subset X$ with $\text{codim } \mathcal{N}_1(D, X) = c_X$) is that X has an extremal ray R_0 of type $(n - 1, n - 2)^{sm}$ such that if $E_0 := \text{Locus}(R_0)$, then $\text{codim } \mathcal{N}_1(E_0, X) = c_X$, and the target of the contraction of R_0 is Fano.

In particular, we can consider a special Mori program for $-E_0$, and apply again Lemma 3.2.4. Let R_1, \dots, R_s be the extremal rays determined by the Mori program, with loci E_1, \dots, E_s . Since, by 2.7(3) and 2.7(4), E_1, \dots, E_s are pairwise disjoint and $E_0 \neq E_i$, $E_0 \cap E_i \neq \emptyset$ for $i = 1, \dots, s$, by Remark 3.1.6 we have two possibilities: either $E_1 \cdot R_0 = \dots = E_s \cdot R_0 = 0$, or $E_i \cdot R_0 > 0$ for every $i = 1, \dots, s$.

In the next lemma we are going to show that in the second case (*i.e.*, when $E_1 \cdot R_0 > 0$) the extremal rays R_0, \dots, R_s have very special properties, in particular that the divisors E_0, \dots, E_s are products.

LEMMA 3.2.7. – Let X be a Fano manifold such that either $c_X \geq 4$, or $c_X = 3$ and X satisfies (3.6.a).

Let R_0 be an extremal ray of X , of type $(n-1, n-2)^{sm}$, such that the target of the contraction of R_0 is Fano, and $\text{codim } \mathcal{N}_1(E_0, X) = c_X$, where $E_0 := \text{Locus}(R_0)$.

Consider a special Mori program for $-E_0$, let R_1, \dots, R_s be the extremal rays determined by the Mori program, and set $E_i := \text{Locus}(R_i)$ for $i = 1, \dots, s$.

Assume that $E_1 \cdot R_0 > 0$. Then we have the following:

- (1) $\text{codim } \mathcal{N}_1(E_i, X) = c_X$, and $E_i \cong \mathbb{P}^1 \times F$ with F an $(n-2)$ -dimensional Fano manifold, for $i = 0, \dots, s$. We set $F_i := \{\text{pt}\} \times F \subset E_i$;
- (2) R_i is the unique extremal ray of X having negative intersection with E_i , and the target of the contraction of R_i is Fano, for every $i = 0, \dots, s$;
- (3) E_1, \dots, E_s are pairwise disjoint, and $E_0 \cap E_i = \{\text{pts}\} \times F$ for every $i = 1, \dots, s$;
- (4) $E_i \cdot R_0 > 0$ and $E_0 \cdot R_i > 0$ for every $i = 1, \dots, s$;
- (5) there exists a linear subspace $L \subset \mathcal{N}_1(X)$, of codimension $c_X + 1$, such that

$$L = \mathcal{N}_1(E_0 \cap E_i, X) = \mathcal{N}_1(F_j, X) \text{ and } \mathcal{N}_1(E_j, X) = \mathbb{R}R_j \oplus L$$

for every $i = 1, \dots, s$ and $j = 0, \dots, s$, and moreover $\dim(\mathbb{R}(R_0 + \dots + R_s) + L) = s + 1 + \dim L$;

- (6) $L \subseteq E_0^\perp \cap \dots \cap E_s^\perp$, and equality holds if $s = c_X$.

Proof. – By 2.7(3) and 2.7(4) we know that $E_0 \cdot R_i > 0$ (in particular $E_0 \neq E_i$ and $E_0 \cap E_i \neq \emptyset$) and $R_i \notin \mathcal{N}_1(E_0, X)$ for $i = 1, \dots, s$, and that E_1, \dots, E_s are pairwise disjoint.

Secondly, Lemma 3.2.4 shows that $s \in \{c_X - 1, c_X\}$ and $s \geq 3$, that $\text{codim } \mathcal{N}_1(E_i, X) = c_X$ for $i = 1, \dots, s$, and that there exists a linear subspace $L \subset \mathcal{N}_1(X)$, of codimension $c_X + 1$, such that

$$(3.2.8) \quad L = \mathcal{N}_1(E_0 \cap E_i, X) = \mathcal{N}_1(E_0, X) \cap E_i^\perp = \mathcal{N}_1(E_i, X) \cap E_0^\perp$$

for every $i = 1, \dots, s$. Moreover Remark 3.1.6 yields $E_i \cdot R_0 > 0$ for every $i = 1, \dots, s$, because $E_1 \cdot R_0 > 0$, so we get (4).

Fix $i \in \{1, \dots, s\}$. We have $\dim \mathcal{N}_1(E_0 \cap E_i, X) = \dim L = \rho_X - c_X - 1 < \rho_X - c_X = \dim \mathcal{N}_1(E_0, X)$, and since $E_i \cdot R_0 > 0$, 3.1.3(2) gives $R_0 \notin \mathcal{N}_1(E_0 \cap E_i, X)$. Moreover

$$\mathcal{N}_1(E_0 \cap E_i, X) \subseteq \mathcal{N}_1(E_0, X) \cap \mathcal{N}_1(E_i, X) \subsetneq \mathcal{N}_1(E_0, X)$$

(because $R_i \notin \mathcal{N}_1(E_0, X)$), and since $\mathcal{N}_1(E_0 \cap E_i, X)$ has codimension 1 in $\mathcal{N}_1(E_0, X)$, we deduce that $\mathcal{N}_1(E_0 \cap E_i, X) = \mathcal{N}_1(E_0, X) \cap \mathcal{N}_1(E_i, X)$. This yields that $R_0 \notin \mathcal{N}_1(E_i, X)$.

Now we can apply Lemma 3.2.5 to E_i and R_0 , and deduce that

$$(3.2.9) \quad L = \mathcal{N}_1(E_0 \cap E_i, X) = \mathcal{N}_1(E_i, X) \cap E_0^\perp.$$

Thanks to (4), (3.2.8), and (3.2.9), we can use Lemma 3.1.5 to show (1). First of all we apply Lemma 3.1.5 with $D = E_i$ and $E = E_0$, and we deduce that $E_0 \cong \mathbb{P}^1 \times F$ where F is an $(n-2)$ -dimensional Fano manifold, and $E_0 \cap E_i = \{\text{pts}\} \times F \subset E_0$. Moreover we get (2) for R_0 .

Then we apply Lemma 3.1.5 again, with $D = E_0$ and $E = E_i$, and we get $E_i \cong \mathbb{P}^1 \times F^i$ and $E_0 \cap E_i = \{pts\} \times F^i \subset E_i$; in particular, $F^i = F$, and we have (3). Moreover we get (2) for R_i .

We have $L \subseteq E_0^\perp \cap \dots \cap E_s^\perp$ by (3.2.8) and (3.2.9). To get (5), it is enough to show that $[f_0], \dots, [f_s] \in \mathcal{N}_1(X)$ are linearly independent and that $(\mathbb{R}[f_0] + \dots + \mathbb{R}[f_s]) \cap L = \{0\}$. So suppose that there exist $\lambda_0, \dots, \lambda_s \in \mathbb{R}$ such that

$$\sum_{i=0}^s \lambda_i f_i \in L.$$

Intersecting with E_j for $j \in \{1, \dots, s\}$ we get $\lambda_j = \lambda_0 E_j \cdot f_0$, and intersecting with E_0 we get $\lambda_0 (\sum_{i=1}^s (E_i \cdot f_0)(E_0 \cdot f_i) - 1) = 0$. Since $E_i \cdot f_0$ and $E_0 \cdot f_i$ are positive integers by (4), and $s \geq 3$, we get $\lambda_0 = 0$ and hence $\lambda_i = 0$ for $i = 1, \dots, s$, and we are done.

We are left to show (6). Similarly to what we have done for $[f_0], \dots, [f_s]$, one checks that $[E_0], \dots, [E_s]$ are linearly independent in $\mathcal{N}^1(X)$, so that $\text{codim}(E_0^\perp \cap \dots \cap E_s^\perp) = s + 1$. Since $L \subseteq E_0^\perp \cap \dots \cap E_s^\perp$ and $\text{codim } L = c_X + 1$, if $s = c_X$ the two subspaces coincide. \square

LEMMA 3.2.10. – *Let X be a Fano manifold such that either $c_X \geq 4$, or $c_X = 3$ and X satisfies (3.6.a). Then X has an extremal ray R_0 with the following properties:*

- R_0 is of type $(n - 1, n - 2)^{sm}$, the target of the contraction of R_0 is Fano, and $\text{codim } \mathcal{N}_1(E_0, X) = c_X$, where $E_0 := \text{Locus}(R_0)$;
- there exists a special Mori program for $-E_0$ such that, if R_1, \dots, R_s are the extremal rays determined by the Mori program, we have $\text{Locus}(R_i) \cdot R_0 > 0$ for every $i = 1, \dots, s$.

Proof. – Let $\phi = \{S^1, \dots, S^h\}$ be an ordered set of extremal rays of X , and set $E^i := \text{Locus}(S^i)$. Consider the following properties:

- (P1) S^i is of type $(n - 1, n - 2)^{sm}$, the target of the contraction of S^i is Fano, and $\text{codim } \mathcal{N}_1(E^i, X) = c_X$, for every $i = 1, \dots, h$;
- (P2) $E^{i-1} \cdot S^i > 0$ and $S^i \notin \mathcal{N}_1(E^{i-1}, X)$, for every $i = 2, \dots, h$;
- (P3) for every $1 \leq j < i \leq h$ we have $E^i \cdot S^j = 0$ and $E^i \cap E^j \neq \emptyset$.

We notice first of all that by Remark 3.2.6, there exists an extremal ray S^1 of X , of type $(n - 1, n - 2)^{sm}$, such that $\text{codim } \text{Locus}(S^1) = c_X$, and the target of the contraction of S^1 is Fano. Then $\phi = \{S^1\}$ satisfies properties (P1), (P2), and (P3).

Consider now an arbitrary ordered set of extremal rays $\phi = \{S^1, \dots, S^h\}$ satisfying properties (P1), (P2), and (P3). We show that $h \leq \rho_X$.

Let $\gamma_i \in S^i$ be a non-zero element, for $i = 1, \dots, h$. We have $E^i \cdot \gamma_i \neq 0$ for every $i = 1, \dots, h$, and $E^i \cdot \gamma_j = 0$ for every $1 \leq j < i \leq h$ by (P3). This shows that $\gamma_1, \dots, \gamma_h$ are linearly independent in $\mathcal{N}_1(X)$: indeed if there exist $a_1, \dots, a_h \in \mathbb{R}$ such that $\sum_{i=1}^h a_i \gamma_i = 0$, then intersecting with E_h we get $a_h = 0$, and so on. Thus $h \leq \rho_X$.

Then Lemma 3.2.10 is a consequence of the following claim. \square

CLAIM 3.2.11. – *Assume that $\phi = \{S^1, \dots, S^h\}$ is an ordered set of extremal rays having properties (P1), (P2), and (P3). Then either $R_0 := S^h$ satisfies the statement of Lemma 3.2.10, or there exists an extremal ray S^{h+1} such that $\phi' := \{S^1, \dots, S^h, S^{h+1}\}$ still has properties (P1), (P2), and (P3).*

Proof of Claim 3.2.11. – By (P1) the ray S^h is of type $(n-1, n-2)^{sm}$, the target of its contraction is Fano, and $\text{codim } \mathcal{N}_1(E^h, X) = c_X$. Consider a special Mori program for $-E^h$ (which exists by Proposition 2.4), and let $S_1^{h+1}, \dots, S_s^{h+1}$ be the extremal rays determined by the Mori program, as in Lemma 3.2.4. Notice that $s \geq 3$ by 3.2.4(1). We set $E_l^{h+1} := \text{Locus}(S_l^{h+1})$ for $l = 1, \dots, s$, so that $E_1^{h+1}, \dots, E_s^{h+1}$ are the \mathbb{P}^1 -bundles determined by the Mori program. By 2.7(3) we have

$$(3.2.12) \quad E^h \cdot S_l^{h+1} > 0 \text{ and } S_l^{h+1} \notin \mathcal{N}_1(E^h, X) \text{ for every } l = 1, \dots, s,$$

and $E_1^{h+1}, \dots, E_s^{h+1}$ are pairwise disjoint by 2.7(4).

Remark 3.2.6 shows that the intersections $E_l^{h+1} \cdot S^h$ (for $l = 1, \dots, s$) are either all zero, or all positive. In the latter case, S^h satisfies the statement of Lemma 3.2.10.

Thus let us assume that $E_1^{h+1} \cdot S^h = \dots = E_s^{h+1} \cdot S^h = 0$, and set $S^{h+1} := S_1^{h+1}$ and $E^{h+1} := E_1^{h+1}$.

Since by assumption \mathcal{J} has properties (P1) and (P2), in order to show that \mathcal{J}' still satisfies (P1) and (P2), we just have to consider the case $i = h+1$. Then (P2) is given by (3.2.12), and (P1) follows from 3.2.4(2).

Now let us show the following:

$$(3.2.13) \quad E_l^{h+1} \cdot S^j = 0 \text{ and } E_l^{h+1} \cap E^j \neq \emptyset \text{ for every } j = 1, \dots, h \text{ and } l = 1, \dots, s.$$

In particular, for $l = 1$, (3.2.13) implies that \mathcal{J}' satisfies (P3).

Let $l \in \{1, \dots, s\}$. Since $E^h \cdot S_l^{h+1} > 0$ by (3.2.12), we have $E^h \cap E_l^{h+1} \neq \emptyset$; moreover we have assumed that $E_l^{h+1} \cdot S^h = 0$. Therefore (3.2.13) holds for $j = h$ and $l = 1, \dots, s$.

We proceed by decreasing induction on j : we assume that (3.2.13) holds for some $j \in \{2, \dots, h\}$ and for every $l = 1, \dots, s$, and we show that $E_l^{h+1} \cdot S^{j-1} = 0$ and $E_l^{h+1} \cap E^{j-1} \neq \emptyset$ for every $l = 1, \dots, s$.

Fix $l \in \{1, \dots, s\}$. Since $E_l^{h+1} \cdot S^j = 0$ and $E_l^{h+1} \cap E^j \neq \emptyset$ by the induction assumption, E_l^{h+1} contains a curve C with class in S^j , in particular

$$(3.2.14) \quad S^j \subset \mathcal{N}_1(E_l^{h+1}, X).$$

Since $E^{j-1} \cdot S^j > 0$ by (P2), we have $E^{j-1} \cap C \neq \emptyset$ and hence $E_l^{h+1} \cap E^{j-1} \neq \emptyset$. Moreover $E_l^{h+1} \cdot S^j = 0$ implies that $E_l^{h+1} \neq E^{j-1}$, thus $E_l^{h+1} \cdot S^{j-1} \geq 0$.

Recall from (P1) that E^{j-1} is the locus of the extremal ray S^{j-1} , of type $(n-1, n-2)^{sm}$; in particular E^{j-1} is a \mathbb{P}^1 -bundle. Since $E_1^{h+1}, \dots, E_s^{h+1}$ are pairwise disjoint, by Remark 3.1.6 the intersections $E_l^{h+1} \cdot S^{j-1}$ (for $l = 1, \dots, s$) are either all zero or all positive.

By contradiction, suppose that $E_l^{h+1} \cdot S^{j-1} > 0$ for every $l = 1, \dots, s$. We have $\text{codim } \mathcal{N}_1(E^{j-1}, X) = c_X$ by (P1), hence 3.1.3(1) gives

$$\text{codim } \mathcal{N}_1(E^{j-1} \cap E_l^{h+1}, X) \leq \text{codim } \mathcal{N}_1(E^{j-1}, X) + 1 = c_X + 1 \text{ for every } l = 1, \dots, s.$$

Since $s \geq 3$, we can apply Lemma 3.1.7 to E^{j-1} and $E_1^{h+1}, \dots, E_s^{h+1}$, and deduce that $\text{codim } \mathcal{N}_1(E^{j-1} \cap E^{h+1}, X) = c_X + 1$ and $\mathcal{N}_1(E^{j-1} \cap E^{h+1}, X) \subseteq (E^{h+1})^\perp$. In particular

$$\mathcal{N}_1(E^{j-1} \cap E^{h+1}, X) \subseteq \mathcal{N}_1(E^{h+1}, X) \cap (E^{h+1})^\perp.$$

On the other hand $\mathcal{N}_1(E^{h+1}, X) \not\subseteq (E^{h+1})^\perp$ because $E^{h+1} \cdot S^{h+1} < 0$, therefore

$$\text{codim } (\mathcal{N}_1(E^{h+1}, X) \cap (E^{h+1})^\perp) = c_X + 1 = \text{codim } \mathcal{N}_1(E^{j-1} \cap E^{h+1}, X),$$

and the two subspaces coincide.

By (3.2.14) and by the induction assumption we have $S^j \subset \mathcal{N}_1(E^{h+1}, X) \cap (E^{h+1})^\perp$, therefore $S^j \subset \mathcal{N}_1(E^{j-1}, X)$, and this contradicts property (P2). \square

Proof of Proposition 3.2.1. – Let R_0 be the extremal ray of X given by Lemma 3.2.10, and set $E_0 := \text{Locus}(R_0)$. Then $\text{codim } \mathcal{N}_1(E_0, X) = c_X$, and there exists a special Mori program for $-E_0$ which determines extremal rays R_1, \dots, R_s such that $E_i \cdot R_0 > 0$ for all $i = 1, \dots, s$, where $E_i := \text{Locus}(R_i)$. Thus Lemma 3.2.7 applies.

If R is an extremal ray of X different from R_1, \dots, R_s , by 3.2.7(2) we have $E_i \cdot R \geq 0$ for every $i = 1, \dots, s$, hence $(-K_X + E_1 + \dots + E_s) \cdot R > 0$. On the other hand $(-K_X + E_1 + \dots + E_s) \cdot R_i = 0$ for every $i = 1, \dots, s$ (recall from 3.2.7(3) that E_1, \dots, E_s are pairwise disjoint), therefore $-K_X + E_1 + \dots + E_s$ is nef and

$$(-K_X + E_1 + \dots + E_s)^\perp \cap \text{NE}(X) = R_1 + \dots + R_s$$

is a face of $\text{NE}(X)$, of dimension s by 3.2.7(5).

Let $\sigma: X \rightarrow X_s$ be the associated contraction, so that $\ker \sigma_* = \mathbb{R}(R_1 + \dots + R_s)$. Since E_1, \dots, E_s are pairwise disjoint, we see that $\text{Exc}(\sigma) = E_1 \cup \dots \cup E_s$, X_s is smooth, and σ is the blow-up of s smooth, pairwise disjoint, irreducible subvarieties $T_1, \dots, T_s \subset X_s$ of codimension 2, where $T_i := \sigma(E_i)$ for $i = 1, \dots, s$. Moreover X_s is again Fano, because $-K_X + E_1 + \dots + E_s = \sigma^*(-K_{X_s})$. Recall from 3.2.7(1) that $E_i \cong \mathbb{P}^1 \times F$, and notice that $\sigma|_{E_i}$ is the projection onto $F \cong T_i$.

Set $(E_0)_s := \sigma(E_0) \subset X_s$. Since $E_0 \cong \mathbb{P}^1 \times F$ and $E_0 \cap E_i = \{pts\} \times F$ for $i = 1, \dots, s$ by 3.2.7(1) and 3.2.7(3), the morphism $\sigma|_{E_0}: E_0 \rightarrow (E_0)_s$ is birational and finite, *i.e.*, it is the normalization. Moreover for $i = 1, \dots, s$ we have $T_i = \sigma(E_0 \cap E_i) \subset (E_0)_s$, so that

$$(3.2.15) \quad \mathcal{N}_1(T_i, X_s) = \sigma_*(\mathcal{N}_1(E_0 \cap E_i, X)) = \sigma_*(L),$$

where $L \subset \mathcal{N}_1(X)$ is the linear subspace defined in 3.2.7(5). Again by 3.2.7(5) we know that $\mathcal{N}_1(E_0, X) = \mathbb{R}R_0 \oplus L$, and that $\dim(\ker \sigma_* + \mathcal{N}_1(E_0, X)) = \dim \ker \sigma_* + \dim \mathcal{N}_1(E_0, X)$, therefore:

$$(3.2.16) \quad \ker \sigma_* \cap \mathcal{N}_1(E_0, X) = \{0\} \text{ and } \mathcal{N}_1((E_0)_s, X_s) = \mathbb{R}\sigma_*(R_0) \oplus \sigma_*(L).$$

Finally, since $\sigma^*((E_0)_s) = E_0 + \sum_{i=1}^s (E_0 \cdot f_i)E_i$ (as usual we denote by $f_i \subseteq E_i$ a fiber of the \mathbb{P}^1 -bundle), by 3.2.7(4) and 3.2.7(6) we see that

$$(3.2.17) \quad (E_0)_s \cdot \sigma(f_0) = \sum_{i=1}^s (E_0 \cdot f_i)(E_i \cdot f_0) - 1 > 0 \text{ and } \sigma_*(L) \subseteq (E_0)_s^\perp$$

(recall that $s \geq 3$ and $s \in \{c_X - 1, c_X\}$ by 3.2.4(1)).

Factoring σ as a sequence of s blow-ups, we can view $\sigma: X \rightarrow X_s$ as a part of a special Mori program for $-E_0$ in X , with s steps, and by (3.2.16) at each step we have $Q_i \notin \mathcal{N}_1((E_0)_i, X_i)$. In particular 2.6(3) yields that

$$\text{codim } \mathcal{N}_1((E_0)_s, X_s) = \text{codim } \mathcal{N}_1(E_0, X) - s = c_X - s,$$

hence either $s = c_X$ and $\mathcal{N}_1((E_0)_s, X_s) = \mathcal{N}_1(X_s)$, or $s = c_X - 1$ and $\text{codim } \mathcal{N}_1((E_0)_s, X_s) = 1$.

3.2.18. – Suppose that there exists an extremal ray R of X_s with $(E_0)_s \cdot R > 0$ and $\text{Locus}(R) \subsetneq X_s$. Then $s = c_X - 1$ and $R \notin \mathcal{N}_1((E_0)_s, X_s)$.

Since we have shown that $\mathcal{N}_1((E_0)_s, X_s) = \mathcal{N}_1(X_s)$ when $s = c_X$, it is enough to show that $R \notin \mathcal{N}_1((E_0)_s, X_s)$.

We first show that $R \notin \text{NE}((E_0)_s, X_s)$. Otherwise, since $\text{NE}((E_0)_s, X_s) \subseteq \text{NE}(X_s)$, R should be a one-dimensional face of $\text{NE}((E_0)_s, X_s)$. We have $\text{NE}(E_0, X) = R_0 + \text{NE}(F_0, X)$ and $\text{NE}((E_0)_s, X_s) = \sigma_*(R_0) + \sigma_*(\text{NE}(F_0, X))$. On the other hand 3.2.7(5) and (3.2.17) give

$$\sigma_*(\text{NE}(F_0, X)) \subset \sigma_*(\mathcal{N}_1(F_0, X)) = \sigma_*(L) \subseteq (E_0)_s^\perp,$$

while $(E_0)_s \cdot R > 0$, therefore we get $R = \sigma_*(R_0)$. But $(E_0)_s$ is covered by the curves $\sigma(f_0)$, so that $\text{Locus}(R) \supseteq (E_0)_s$, which is impossible.

Therefore $R \notin \text{NE}((E_0)_s, X_s)$, and in particular the contraction of R is finite on $(E_0)_s$. Since $(E_0)_s \cdot R > 0$, this means that the contraction of R has fibers of dimension ≤ 1 , therefore R is of type $(n-1, n-2)^{sm}$ by [1, Theorem 2.3] and [21, Theorem 1.2].

In particular, $E_R := \text{Locus}(R)$ is a prime divisor covered by curves of anticanonical degree 1. Moreover these curves have class in R , thus they cannot be contained in $T_1 \cup \dots \cup T_s$, because $T_1 \cup \dots \cup T_s \subset (E_0)_s$. By a standard argument (see for instance [10, Remark 2.3]) we deduce that $E_R \cap (T_1 \cup \dots \cup T_s) = \emptyset$, hence by (3.2.15) and Remark 3.1.2 we have

$$\sigma_*(L) = \mathcal{N}_1(T_1, X_s) \subseteq E_R^\perp.$$

Moreover $E_R \cdot \sigma(f_0) \geq 0$, because $E_R \neq (E_0)_s$ (as $(E_0)_s \cdot R > 0$).

We show that $R \notin \mathcal{N}_1((E_0)_s, X_s)$. By contradiction, suppose that $R \subset \mathcal{N}_1((E_0)_s, X_s)$, and let C be an irreducible curve with class in R . Then by (3.2.16) we have $[C] = \lambda[\sigma(f_0)] + \gamma$, with $\lambda \in \mathbb{R}$ and $\gamma \in \sigma_*(L)$. Using (3.2.17) we get $0 < (E_0)_s \cdot C = \lambda(E_0)_s \cdot \sigma(f_0)$ and $(E_0)_s \cdot \sigma(f_0) > 0$, thus $\lambda > 0$. On the other hand $-1 = E_R \cdot C = \lambda E_R \cdot \sigma(f_0)$, which gives a contradiction. Thus $R \notin \mathcal{N}_1((E_0)_s, X_s)$.

3.2.19. – We show that we can assume that there exists an extremal ray R of X_s such that $(E_0)_s \cdot R > 0$ and $\text{Locus}(R) = X_s$.

This is clear if $s = c_X$, by 3.2.18. Suppose that $s = c_X - 1$, and consider an extremal ray R of X_s with $(E_0)_{c_X-1} \cdot R > 0$. If $\text{Locus}(R) = X_{c_X-1}$, we are done; otherwise, by 3.2.18, we have $R \notin \mathcal{N}_1((E_0)_{c_X-1}, X_{c_X-1})$.

Let $\sigma_{c_X-1}: X_{c_X-1} \rightarrow X_{c_X}$ be the contraction of R , and consider the sequence

$$X \xrightarrow{\sigma} X_{c_X-1} \xrightarrow{\sigma_{c_X-1}} X_{c_X}.$$

Again, factoring σ as a sequence of $c_X - 1$ blow-ups, we can view this as a part of a special Mori program for $-E_0$ in X , with c_X steps, and at each step $Q_i \notin \mathcal{N}_1((E_0)_i, X_i)$.

The \mathbb{P}^1 -bundles determined by this special Mori program are E_1, \dots, E_{c_X-1} , and the transform of E_R in X ; the associated extremal rays (see Lemma 3.2.4) are R_1, \dots, R_{c_X-1} , and an additional extremal ray R_{c_X} .

Since $E_1 \cdot R_0 > 0$, Lemma 3.2.7 still applies, thus we can just replace R_1, \dots, R_{c_X-1} with R_1, \dots, R_{c_X} , and restart. Since now the extremal rays are c_X (instead of $c_X - 1$), we are done by what precedes.

3.2.20. – By 3.2.19 there exists an elementary contraction of fiber type $\varphi: X_s \rightarrow Y$ such that $(E_0)_s \cdot \text{NE}(\varphi) > 0$; set $\psi := \varphi \circ \sigma: X \rightarrow Y$, and notice that $\varphi((E_0)_s) = \psi(E_0) = Y$.

$$\begin{array}{ccccc}
 & & \psi & & \\
 & & \curvearrowright & & \\
 X & \xrightarrow{\sigma} & X_s & \xrightarrow{\varphi} & Y
 \end{array}$$

The sequence above is a Mori program for $-E_0$, with s steps, and at each step $Q_i \notin \mathcal{N}_1((E_0)_i, X_i)$. By 2.7(2) we have two possibilities: either $\mathcal{N}_1((E_0)_s, X_s) = \mathcal{N}_1(X_s)$ and $s = c_X$, or $\text{NE}(\varphi) \notin \mathcal{N}_1((E_0)_s, X_s)$ and $s = c_X - 1$.

Since $\mathcal{N}_1(T_1, X_s) \subseteq (E_0)_s^\perp$ by (3.2.15) and (3.2.17), φ must be finite on T_1 , so that $\dim Y \geq n - 2$.

3.2.21. – *First case:* φ is not finite on $(E_0)_s$. In this case $\text{NE}(\varphi) \subset \mathcal{N}_1((E_0)_s, X_s)$, therefore $\mathcal{N}_1((E_0)_s, X_s) = \mathcal{N}_1(X_s)$ and $s = c_X$. This also shows that $L = E_0^\perp \cap \dots \cap E_{c_X}^\perp$, by 3.2.7(6).

Recall that $Y = \psi(E_0)$, and that $E_0 \cong \mathbb{P}^1 \times F$ is smooth and Fano by 3.2.7(1). Moreover if $F_0 := \{pt\} \times F \subset E_0$, then $\mathcal{N}_1(F_0, X) = L$ by 3.2.7(5). Finally $\mathcal{N}_1(\sigma(F_0), X_{c_X}) = \sigma_*(L) \subseteq (E_0)_{c_X}^\perp$ by (3.2.17), so that φ is finite on $\sigma(F_0)$. Since σ is finite on E_0 , we deduce that ψ is finite on F_0 .

Let $E_0 \xrightarrow{\alpha} \tilde{Y} \rightarrow Y$ be the Stein factorization of $\psi|_{E_0}$. Since φ is not finite on $(E_0)_{c_X}$, α is a non-trivial contraction of E_0 . On the other hand α is finite on F_0 : the only possibility is that $\tilde{Y} \cong F$ and α is the projection.

We deduce that $\dim Y = n - 2$ and that ψ contracts f_0 , hence $\text{NE}(\varphi) = \sigma_*(R_0)$. Thus $\text{NE}(\psi)$ is a $(c_X + 1)$ -dimensional face of $\text{NE}(X)$ containing R_0, \dots, R_{c_X} ; in particular $\rho_Y = \rho_X - c_X - 1$.

Let us consider the divisor

$$H := 2E_0 + \sum_{i=1}^{c_X} E_i$$

on X . By 3.2.7(4) we have $H \cdot R_i > 0$ for every $i = 0, \dots, c_X$, and

$$L = E_0^\perp \cap \dots \cap E_{c_X}^\perp \subseteq H^\perp.$$

Recall from 3.2.7(1) and 3.2.7(5) that for every $i = 0, \dots, c_X$ we have $E_i \cong \mathbb{P}^1 \times F$, and if $F_i := \{pt\} \times F \subset E_i$, then $\mathcal{N}_1(F_i, X) = L \subset H^\perp$. In particular $\text{NE}(E_i, X) = R_i + \text{NE}(F_i, X) \subset R_i + L$.

Let $C \subset X$ be an irreducible curve with $C \subset \text{Supp } H = E_0 \cup \dots \cup E_{c_X}$. Then $C \subseteq E_i$ for some $i \in \{0, \dots, c_X\}$, hence $[C] \in R_i + L$ and $H \cdot C \geq 0$.

On the other hand, since H is effective, we have $H \cdot C' \geq 0$ for every irreducible curve C' not contained in $\text{Supp } H$. Therefore H is nef and defines a contraction $\xi: X \rightarrow S$ such that $\text{NE}(\xi) = H^\perp \cap \text{NE}(X)$.

$$\begin{array}{ccc}
 X & \xrightarrow{\sigma} & X_{c_X} \\
 \xi \swarrow & & \downarrow \varphi \\
 S & & Y
 \end{array}$$

Let $i \in \{0, \dots, c_X\}$. Since $\mathcal{N}_1(F_i, X) \subset H^\perp$, the image $\xi(F_i)$ is a point, and $\xi(E_i) = \xi(f_i)$ is an irreducible rational curve (because $H \cdot f_i > 0$). Therefore $\xi|_{E_i}: E_i \rightarrow \xi(f_i)$ factors

through the projection $E_i \rightarrow \mathbb{P}^1$. In particular $\dim \xi(\text{Supp } H) = 1$, hence S is a surface by 3.1.1(1).

Let us show that

$$(3.2.22) \quad \text{NE}(\xi) = L \cap \text{NE}(X).$$

We already have $\text{NE}(\xi) = H^\perp \cap \text{NE}(X) \supseteq L \cap \text{NE}(X)$. Conversely, let $C_1 \subset X$ be an irreducible curve such that $\xi(C_1) = \{pt\}$, i.e., $H \cdot C_1 = 0$.

If C_1 is disjoint from $\text{Supp } H = E_0 \cup \dots \cup E_{c_X}$, then $C_1 \cdot E_i = 0$ for $i = 0, \dots, c_X$, hence $[C_1] \in L$.

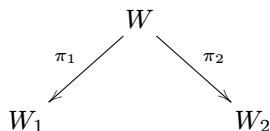
If instead C_1 intersects $E_0 \cup \dots \cup E_{c_X}$, then it must be contained in it, and we have $C_1 \subset E_i$ for some i . Since $\xi|_{E_i}$ factors as the projection onto \mathbb{P}^1 followed by a finite map, we get $C_1 \subset F_i$, and again $[C_1] \in \mathcal{N}_1(F_i, X) = L$. Therefore we have (3.2.22).

In particular, for every $i = 0, \dots, c_X$ we have $\text{NE}(\xi) \subseteq E_i^\perp$, therefore $E_i = \xi^*(\xi(E_i))$ by 3.1.1(2).

Let $\pi: X \rightarrow S \times Y$ be the morphism induced by ξ and ψ . We have $\ker \psi_* = \mathbb{R}(R_0 + \dots + R_{c_X})$, and $\ker \psi_* \cap L = \{0\}$ by 3.2.7(5). Moreover $\ker \xi_* \subseteq L$ by (3.2.22), therefore π is finite.

In particular, ξ must be equidimensional, hence S is smooth by [2, Proposition 1.4.1] and [10, Lemma 3.10]. We need the following remark.

REMARK 3.2.23. – Let W be a smooth Fano variety and suppose we have two contractions



such that W_1 is smooth and the induced morphism $\pi: W \rightarrow W_1 \times W_2$ is finite. Consider the relative canonical divisor $K_{W/W_1} := K_W - \pi_1^* K_{W_1}$. If $\ker(\pi_2)_* \subseteq (K_{W/W_1})^\perp$ in $\mathcal{N}_1(W)$, then π is an isomorphism.

This is rather standard, we give a proof for the reader’s convenience. Let d be the degree of π , and $F \subset W$ a general fiber of π_2 ; the restriction $f := (\pi_1)|_F: F \rightarrow W_1$ is finite of degree d . We observe that F is Fano, hence numerical and linear equivalence for divisors in F coincide, and by assumption $(K_{W/W_1})|_F \equiv 0$. Then

$$K_F = (K_W)|_F = (\pi_1^* K_{W_1})|_F = f^* K_{W_1},$$

so that f is étale. Therefore W_1 is Fano too, in particular it is simply connected, thus f is an isomorphism and $d = 1$.

We carry on with the proof of Proposition 3.2.1. We want to apply Remark 3.2.23 to deduce that $\pi: X \rightarrow S \times Y$ is an isomorphism; for this we just need to show that $K_{X/S} \cdot R_i = 0$ for $i = 0, \dots, c_X$, because $\ker \psi_* = \mathbb{R}(R_0 + \dots + R_{c_X})$. But this follows easily because E_i are products.

Indeed since both S and E_i are smooth, 3.1.1(4) yields that $\xi(E_i)$ is a smooth curve. Therefore $\xi(E_i) \cong \mathbb{P}^1$ and $\xi|_{E_i}$ is the projection, hence

$$K_{X/S} \cdot f_i = (K_{X/S})|_{E_i} \cdot f_i = K_{E_i/\xi(E_i)} \cdot f_i = 0.$$

Thus we conclude that π is an isomorphism and $X \cong S \times Y$. Moreover since $\rho_Y = \rho_X - c_X - 1$, we have $\rho_S = c_X + 1$.

3.2.24. – *Second case: φ is finite on $(E_0)_s$.* Then $\dim Y = n - 1$ and every fiber of φ is one-dimensional; moreover every fiber of ψ has an irreducible component of dimension 1. Since X and X_s are Fano, [3, Lemma 2.12 and Theorem 4.1] show that Y is smooth and that φ and ψ are conic bundles.

$$X \begin{array}{c} \xrightarrow{\psi} \\ \xrightarrow{\sigma} X_s \xrightarrow{\varphi} Y \end{array}$$

Set $Z_i := \varphi(T_i) = \psi(E_i) \subset Y$ for $i = 1, \dots, s$. By standard arguments on conic bundles (as at the end of the proof of Lemma 2.8), we see that $Z_1, \dots, Z_s \subset Y$ are pairwise disjoint smooth prime divisors, and that φ is smooth over $Z_1 \cup \dots \cup Z_s$. For $i = 1, \dots, s$ let $\widehat{E}_i \subset X$ be the transform of $\varphi^{-1}(Z_i) \subset X_s$, so that $\psi^{-1}(Z_i) = E_i \cup \widehat{E}_i$. Then \widehat{E}_i is a smooth \mathbb{P}^1 -bundle with fiber $\widehat{f}_i \subset \widehat{E}_i$, such that $\widehat{E}_i \cdot \widehat{f}_i = -1$. Moreover $f_i + \widehat{f}_i$ is numerically equivalent to a general fiber of ψ , and $E_i \cdot \widehat{f}_i = \widehat{E}_i \cdot f_i = 1$.

In particular, the divisors $E_0, E_1, \dots, E_s, \widehat{E}_1, \dots, \widehat{E}_s$ are all distinct (recall that $\psi(E_0) = Y$), and $E_1 \cup \widehat{E}_1, \dots, E_s \cup \widehat{E}_s$ are pairwise disjoint.

Let us show that $[E_0], [E_1], \dots, [E_s], [\widehat{E}_1]$ are linearly independent in $\mathcal{N}^1(X)$. Indeed suppose that

$$aE_0 + \sum_{i=1}^s b_i E_i + d\widehat{E}_1 \equiv 0,$$

with $a, b_i, d \in \mathbb{R}$. Intersecting with a general fiber of $\psi: X \rightarrow Y$, we get $a = 0$. Intersecting with f_2, \dots, f_s , we get $b_2 = \dots = b_s = 0$. Finally intersecting with f_1 we get $d = b_1$, that is, $d(E_1 + \widehat{E}_1) \equiv 0$, which yields $d = 0$, and we are done.

If $i, j \in \{1, \dots, s\}$ with $i \neq j$, we have $E_i \cap \widehat{E}_j = \emptyset$, and hence $L \subseteq \mathcal{N}_1(E_i, X) \subseteq \widehat{E}_j^\perp$ (see Remark 3.1.2). Therefore by 3.2.7(6)

$$L \subseteq E_0^\perp \cap E_1^\perp \cap \dots \cap E_s^\perp \cap \widehat{E}_1^\perp \cap \dots \cap \widehat{E}_s^\perp \subseteq E_0^\perp \cap E_1^\perp \cap \dots \cap E_s^\perp \cap \widehat{E}_1^\perp.$$

Since the classes of $E_0, \dots, E_s, \widehat{E}_1$ in $\mathcal{N}^1(X)$ are linearly independent and $s \geq c_X - 1$, we get

$$c_X + 1 = \text{codim } L \geq s + 2 \geq c_X + 1,$$

which yields $s = c_X - 1$ and

$$L = E_0^\perp \cap E_1^\perp \cap \dots \cap E_{c_X-1}^\perp \cap \widehat{E}_1^\perp = E_0^\perp \cap E_1^\perp \cap \dots \cap E_{c_X-1}^\perp \cap \widehat{E}_1^\perp \cap \dots \cap \widehat{E}_{c_X-1}^\perp.$$

Let $i \in \{1, \dots, c_X - 1\}$. Observe that $[\widehat{f}_i] \notin \mathcal{N}_1(E_i, X)$: otherwise by 3.2.7(5) we would have $\widehat{f}_i \equiv \lambda f_i + \gamma$, with $\lambda \in \mathbb{R}$ and $\gamma \in L \subseteq E_0^\perp \cap E_i^\perp$. Intersecting with E_i we get $\lambda = -1$, hence $E_0 \cdot \widehat{f}_i = -E_0 \cdot f_i < 0$, which is impossible because $E_0 \neq \widehat{E}_i$. We also notice that E_0 cannot contain any curve \widehat{f}_i , because $\sigma(\widehat{f}_i)$ is a fiber of φ , and φ is finite on $(E_0)_{c_X-1} = \sigma(E_0)$.

Therefore we can apply Lemma 3.1.10 to E_0 and $E_1, \dots, E_{c_X-1}, \widehat{E}_1, \dots, \widehat{E}_{c_X-1}$, and we get:

$$\text{codim } \mathcal{N}_1(\widehat{E}_i, X) = c_X \text{ and } R_i \not\subset \mathcal{N}_1(\widehat{E}_i, X) \text{ for every } i = 1, \dots, c_X - 1.$$

Fix again $i \in \{1, \dots, c_X - 1\}$. Lemma 3.2.5, applied to \widehat{E}_i and R_i , yields that

$$\mathcal{N}_1(E_i \cap \widehat{E}_i, X) = \mathcal{N}_1(\widehat{E}_i, X) \cap E_i^\perp = \mathcal{N}_1(E_i, X) \cap E_i^\perp = L$$

(see (3.2.8) for the last equality). Finally we apply Lemma 3.1.5 to $D = E_i$ and $E = \widehat{E}_i$, and we deduce that $\widehat{R}_i := \mathbb{R}_{\geq 0}[\widehat{f}_i]$ is an extremal ray of type $(n - 1, n - 2)^{sm}$, $\widehat{E}_i \cong \mathbb{P}^1 \times \widehat{F}^i$, and $E_i \cap \widehat{E}_i = \{pts\} \times \widehat{F}^i \subset \widehat{E}_i$. On the other hand again Lemma 3.1.5, applied now to $D = \widehat{E}_i$ and $E = E_i$, shows that $E_i \cap \widehat{E}_i = \{pts\} \times F \subset E_i \cong \mathbb{P}^1 \times F$, hence $\widehat{F}^i = F$.

Observe that $\text{NE}(\psi) = R_1 + \widehat{R}_1 + \dots + R_{c_X-1} + \widehat{R}_{c_X-1}$ has dimension c_X , and that $\psi|_{E_0}: E_0 \cong \mathbb{P}^1 \times F_0 \rightarrow Y$ is finite. We need the following lemma.

LEMMA 3.2.25. – *Let E be a projective manifold and $\pi: E \rightarrow W$ a \mathbb{P}^1 -bundle with fiber $f \subset E$. Moreover let $\psi_0: E \rightarrow Y$ be a morphism onto a projective manifold Y , such that $\dim \psi_0(f) = 1$. Suppose that there exists a prime divisor $Z_1 \subset Y$ such that $\mathcal{N}_1(Z_1, Y) \subsetneq \mathcal{N}_1(Y)$ and $\psi_0^*(Z_1) \cdot f > 0$. Then there is a commutative diagram:*

$$\begin{array}{ccc} E & \xrightarrow{\psi_0} & Y \\ \pi \downarrow & & \downarrow \zeta \\ W & \longrightarrow & Y' \end{array}$$

where Y' is smooth and ζ is a smooth morphism with fibers isomorphic to \mathbb{P}^1 .

Proof of Lemma 3.2.25. – Consider the morphism $\phi: E \rightarrow W \times Y$ induced by π and ψ_0 , set $E' := \phi(E) \subset W \times Y$, and let $\pi': E' \rightarrow W$ be the projection. For every $p \in W$ we have $\pi^{-1}(p) = \phi^{-1}((\pi')^{-1}(p))$, hence $(\pi')^{-1}(p) = \psi_0(\pi^{-1}(p)) \subset Y$ is an irreducible and reduced rational curve in Y .

Now $\pi': E' \rightarrow W$ is a well defined family of algebraic one-cycles on Y over W (see [15, Definition I.3.11 and Theorem I.3.17]), and induces a morphism $\iota: W \rightarrow \text{Chow}(Y)$. Set $V := \iota(W) \subset \text{Chow}(Y)$. Then V is a proper, covering family of irreducible and reduced rational curves on Y , so that V is an *unsplit* family (see [15, Definition IV.2.1]).

The family V induces an equivalence relation on Y as a set, called V -equivalence; we refer the reader to [12, §5] and references therein for the related definitions and properties.

We have $Z_1 \cdot \psi_0(f) > 0$; in particular Z_1 intersects every V -equivalence class in Y . This implies that

$$\mathcal{N}_1(Y) = \mathbb{R}[\psi_0(f)] + \mathcal{N}_1(Z_1, Y)$$

(see for instance [18, Lemma 3.2]). On the other hand by assumption $\mathcal{N}_1(Z_1, Y) \subsetneq \mathcal{N}_1(Y)$, therefore $[\psi_0(f)] \notin \mathcal{N}_1(Z_1, Y)$.

Let $T \subseteq Y$ be a V -equivalence class; notice that T is either a closed subset, or a countable union of closed subsets. Let $T_1 \subseteq T$ be an irreducible closed subset with $\dim T_1 = \dim T$. We have $\mathcal{N}_1(T_1, Y) = \mathbb{R}[\psi_0(f)]$ by [15, Proposition IV.3.13.3], and $T_1 \cap Z_1 \neq \emptyset$. This implies that $\dim(T_1 \cap Z_1) = 0$ and $\dim T = \dim T_1 = 1$, that is: *every V -equivalence class has dimension 1*. Then by [8, Proposition 1] there exists a contraction $\zeta: Y \rightarrow Y'$ whose fibers coincide with V -equivalence classes.

Since Y is smooth, Y' is irreducible, and ζ has connected fibers, the general fiber l of ζ is irreducible and smooth. But l is a V -equivalence class and $\dim l = 1$, hence $l \cong \mathbb{P}^1$ and $-K_Y \cdot l = 2$. Moreover $\text{NE}(\zeta) = \mathbb{R}_{\geq 0}[l]$, so $-K_{Y'}$ is ζ -ample; this implies that ζ is an

elementary contraction and a conic bundle, and that Y' is smooth (see [1, Theorem 3.1]). Finally ζ cannot have singular fibers, because the family V is unsplit. \square

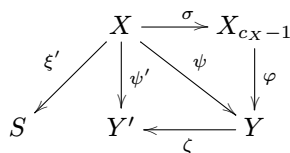
Let us carry on with the proof of Proposition 3.2.1. We have $\psi^*(Z_1) \cdot f_0 = (E_1 + \widehat{E}_1) \cdot f_0 > 0$, and $\mathcal{N}_1(Z_1, Y) \subseteq Z_2^\perp \subsetneq \mathcal{N}_1(Y)$ because $Z_1 \cap Z_2 = \emptyset$ (see Remark 3.1.2). Therefore we can apply Lemma 3.2.25 to E_0 and $\psi_0 := (\psi)|_{E_0}: E_0 \rightarrow Y$. This shows that $[\psi(f_0)]$ belongs to an extremal ray of Y , whose contraction is a smooth conic bundle $\zeta: Y \rightarrow Y'$.

We consider the composition $\psi' := \zeta \circ \psi: X \rightarrow Y'$; the cone $\text{NE}(\psi')$ is a $(c_X + 1)$ -dimensional face of $\text{NE}(X)$ containing $R_0, R_1, \dots, R_{c_X-1}, \widehat{R}_1, \dots, \widehat{R}_{c_X-1}$, and $\rho_{Y'} = \rho_X - c_X - 1$.

Now we proceed similarly to the previous case. Let us consider the divisor

$$H' := 2E_0 + 2 \sum_{i=1}^{c_X-1} E_i + \sum_{i=1}^{c_X-1} \widehat{E}_i$$

on X . We have $H' \cdot R_0 > 0, H' \cdot R_i > 0$ and $H' \cdot \widehat{R}_i > 0$ for every $i = 1, \dots, c_X - 1$, and $(H')^\perp \supseteq L$. As before, H' is nef and defines a contraction onto a surface $\xi': X \rightarrow S$, such that $\xi'(E_0), \xi'(E_i)$, and $\xi'(\widehat{E}_i)$ are irreducible rational curves and $E_0 = (\xi')^*(\xi'(E_0)), E_i = (\xi')^*(\xi'(E_i)), \widehat{E}_i = (\xi')^*(\xi'(\widehat{E}_i))$ for all $i = 1, \dots, c_X - 1$.



Then we consider the morphism $\pi': X \rightarrow S \times Y'$ induced by ξ' and ψ' . As in the previous case, one sees first that π' is finite, and then that it is an isomorphism, applying Remark 3.2.23. Finally we have $\rho_S = c_X + 1$, because $\rho_{Y'} = \rho_X - c_X - 1$.

3.2.26. – We have shown in 3.2.21 and 3.2.24 that $X \cong S \times T$, where S is a Del Pezzo surface with $\rho_S = c_X + 1$ (and $T = Y$ in 3.2.21, while $T = Y'$ in 3.2.24). In particular $c_X \leq 8$, as $\rho_S \leq 9$. Finally $c_T \leq c_X$ by Example 3.1, and this concludes the proof of Proposition 3.2.1. \square

3.3. The case of codimension 3

In this section we show the following.

PROPOSITION 3.3.1. – *Let X be a Fano manifold with $c_X = 3$. Then there exists a flat, quasi-elementary contraction $X \rightarrow T$ where T is an $(n - 2)$ -dimensional Fano manifold, $\rho_X - \rho_T = 4$, and $c_T \leq 3$.*

Proof. – By Corollary 3.2.2, there exist a prime divisor $D \subset X$ with $\text{codim } \mathcal{N}_1(D, X) = 3$, and a special Mori program for $-D$, such that $Q_k \notin \mathcal{N}_1(D_k, X_k)$.

$$(3.3.2) \quad \begin{array}{ccccccc} X = X_0 & \xrightarrow{\sigma_0} & X_1 & \dashrightarrow & \dots & \dashrightarrow & X_{k-1} & \xrightarrow{\sigma_{k-1}} & X_k \\ & & & & & & & & \downarrow \varphi \\ & & & & & & & & Y \end{array}$$

σ (dashed arrow from X_0 to X_k)
 ψ (dashed arrow from X_0 to Y)

We apply Lemmas 2.7 and 2.8. By 2.7(2) and 2.7(3), there exist exactly two indices $i_1, i_2 \in \{0, \dots, k - 1\}$ such that $Q_{i_j} \notin \mathcal{N}_1(D_{i_j}, X_{i_j})$; the \mathbb{P}^1 -bundles $E_1, E_2 \subset X$ determined by the Mori program are the transforms of $\text{Exc}(\sigma_{i_1}), \text{Exc}(\sigma_{i_2})$ respectively. Let moreover $\widehat{E}_1, \widehat{E}_2 \subset X$ be as in 2.8(4). Recall that for $i = 1, 2$ E_i (respectively, \widehat{E}_i) is a smooth \mathbb{P}^1 -bundle with fiber $f_i \subset E_i$ (respectively, $\widehat{f}_i \subset \widehat{E}_i$), such that $E_i \cdot f_i = \widehat{E}_i \cdot \widehat{f}_i = -1$, $E_i \cdot \widehat{f}_i > 0$, and $\widehat{E}_i \cdot f_i > 0$. Moreover $(E_1 \cup \widehat{E}_1) \cap (E_2 \cup \widehat{E}_2) = \emptyset$.

3.3.3. – Before going on, let us give an outline of what we are going to do.

Our goal is to show that $k = 2$ and σ is just the composition of two smooth blow-ups with exceptional divisors E_1 and E_2 . The proof of this fact is quite technical, and will be achieved in several steps.

We first show in 3.3.4 some properties of $\mathcal{N}_1(E_i, X)$ and $\mathcal{N}_1(\widehat{E}_i, X)$ which are needed in the sequel.

In 3.3.6 we prove that if $F \subset X$ is a prime divisor whose class in $\mathcal{N}^1(X)$ spans a one-dimensional face of the cone of effective divisors $\text{Eff}(X) \subset \mathcal{N}^1(X)$ (see 3.3.5), then F must intersect both $E_1 \cup \widehat{E}_1$ and $E_2 \cup \widehat{E}_2$.

Then we show in 3.3.7 that the Mori program (3.3.2) contains only two divisorial contractions, the ones with exceptional divisors E_1 and E_2 . We proceed by contradiction, applying 3.3.6 to the exceptional divisor of a divisorial contraction (different from σ_{i_1} and σ_{i_2}) in the Mori program.

In 3.3.9 and 3.3.10 we prove the existence of two disjoint prime divisors $F, \widehat{F} \subset X$, which are smooth \mathbb{P}^1 -bundles with fibers $l \subset F, \widehat{l} \subset \widehat{F}$ such that $F \cdot l = \widehat{F} \cdot \widehat{l} = -1$, which are horizontal for the rational conic bundle $\psi: X \dashrightarrow Y$, and intersect the divisors $E_1, E_2, \widehat{E}_1, \widehat{E}_2$ in a suitable way.

Finally in 3.3.11 and 3.3.13 we use F and \widehat{F} to show that the Mori program (3.3.2) contains no flips. This means that $k = 2$, X_2 and Y are smooth, σ is just a smooth blow-up with exceptional divisors E_1 and E_2 , and φ and ψ are conic bundles.

The situation is now analogous to the one in 3.2.24, and similarly to that case we prove that there is a smooth conic bundle $Y \rightarrow Y'$, where $\dim Y' = n - 2$ (see 3.3.15). We have $\rho_X - \rho_{Y'} = 4$, and the contraction $X \rightarrow Y'$ is flat and quasi-elementary.

To conclude, in 3.3.16 we show that the conic bundle $\varphi: X_2 \rightarrow Y$ is smooth. This implies that every fiber of the conic bundle $\psi: X \rightarrow Y$ is reduced, and hence by a result in [21] both Y and Y' are Fano.

3.3.4. – For $i = 1, 2$ we have:

$\text{codim } \mathcal{N}_1(E_i, X) = \text{codim } \mathcal{N}_1(\widehat{E}_i, X) = 3$, $[\widehat{f}_i] \notin \mathcal{N}_1(E_i, X)$, and $[f_i] \notin \mathcal{N}_1(\widehat{E}_i, X)$; in particular $\mathcal{N}_1(E_i, X) \neq \mathcal{N}_1(\widehat{E}_i, X)$.

Indeed $[\widehat{f}_i] \notin \mathcal{N}_1(E_i, X)$ by 2.8(4). Moreover D cannot contain any curve \widehat{f}_i , because $\sigma(\widehat{f}_i)$ is a fiber of φ , and φ is finite on $D_k \subset X_k$. Therefore Lemma 3.1.10 yields the statement.

3.3.5. – Let Z be a Mori dream space, and $\text{Eff}(Z) \subset \mathcal{N}^1(Z)$ the convex cone spanned by classes of effective divisors. By [13, Proposition 1.11(2)] $\text{Eff}(Z)$ is a closed, convex polyhedral cone. If $F \subset Z$ is a prime divisor covered by a family of curves with which F has negative intersection, then it is easy to see that $[F] \in \mathcal{N}^1(Z)$ spans a one-dimensional face of $\text{Eff}(Z)$, and that the only prime divisor whose class belongs to this face is F itself. In particular, this is true for $E_1, E_2, \widehat{E}_1, \widehat{E}_2 \subset X$ (recall that X is a Mori dream space by Theorem 2.1).

3.3.6. – Consider a prime divisor $F \subset X$ such that $[F]$ spans a one-dimensional face of $\text{Eff}(X)$. We show that if F is different from $E_1, E_2, \widehat{E}_1, \widehat{E}_2$, then F must intersect both $E_1 \cup \widehat{E}_1$ and $E_2 \cup \widehat{E}_2$.

Indeed if for instance F is disjoint from $E_1 \cup \widehat{E}_1$, then

$$\mathcal{N}_1(E_1, X) \cup \mathcal{N}_1(\widehat{E}_1, X) \subseteq E_2^\perp \cap \widehat{E}_2^\perp \cap F^\perp$$

(see Remark 3.1.2). However this is impossible, because since $[E_2], [\widehat{E}_2], [F] \in \mathcal{N}^1(X)$ span three distinct one-dimensional faces of $\text{Eff}(X)$, they must be linearly independent, thus $E_2^\perp \cap \widehat{E}_2^\perp \cap F^\perp$ has codimension 3, while $\mathcal{N}_1(E_1, X)$ and $\mathcal{N}_1(\widehat{E}_1, X)$ are distinct subspaces of codimension 3 by 3.3.4.

3.3.7. – Let us show that σ_i is a flip for every $i \in \{0, \dots, k-1\} \setminus \{i_1, i_2\}$, namely that σ_{i_1} and σ_{i_2} are the unique divisorial contractions in the Mori program (3.3.2).

By contradiction, suppose that there exists $i \in \{0, \dots, k-1\} \setminus \{i_1, i_2\}$ such that σ_i is a divisorial contraction. By 3.3.5 $\text{Exc}(\sigma_i) \subset X_i$ is a prime divisor whose class spans a one-dimensional face of $\text{Eff}(X_i)$, and it is the unique prime divisor in X_i with class in $\mathbb{R}_{\geq 0}[\text{Exc}(\sigma_i)]$.⁽³⁾

Let $G \subset X$ be the transform of $\text{Exc}(\sigma_i)$. By 2.8(3) and 2.8(4) there exists an open subset $U \subseteq X$, containing $E_1, E_2, \widehat{E}_1, \widehat{E}_2$, such that σ is regular on U , and $\text{Exc}(\sigma_i)$ is disjoint from the image of U in X_i . Therefore $G \cap U = \emptyset$, in particular the divisor G is disjoint from $E_1, E_2, \widehat{E}_1, \widehat{E}_2$.

Then 3.3.6 shows that $[G] \in \mathcal{N}^1(X)$ cannot span an extremal ray of $\text{Eff}(X)$. This means that $[G] = \sum_j \lambda_j [G_j]$ with $\lambda_j \in \mathbb{R}_{>0}$ and $G_j \subset X$ prime divisors such that $[G] \notin \mathbb{R}_{\geq 0}[G_j]$; in particular $G_j \neq G$.

On the other hand, the map $\xi := \sigma_{i-1} \circ \dots \circ \sigma_0 : X \dashrightarrow X_i$ induces a surjective linear map $\xi_* : \mathcal{N}^1(X) \rightarrow \mathcal{N}^1(X_i)$ such that $\xi_*(\text{Eff}(X)) = \text{Eff}(X_i)$. Then in $\mathcal{N}^1(X_i)$ we get

$$[\text{Exc}(\sigma_i)] = [\xi_*(G)] = \sum_j \lambda_j [\xi_*(G_j)],$$

⁽³⁾ Notice that X_i is again a Mori dream space.

hence $[\xi_*(G_j)] \in \mathbb{R}_{\geq 0}[\text{Exc}(\sigma_i)]$ for every j . If $\xi_*(G_j) \neq 0$ for some j , then $\xi_*(G_j)$ is a prime divisor, and we get $\xi_*(G_j) = \text{Exc}(\sigma_i)$ and hence $G_j = G$, a contradiction. Thus $\xi_*(G_j) = 0$ for every j , therefore $[\text{Exc}(\sigma_i)] = 0$, again a contradiction.

3.3.8. – Let $F \subset X$ be a smooth prime divisor which is a \mathbb{P}^1 -bundle with $F \cdot l = -1$, where $l \subset F$ is a fiber. Suppose that F is different from $E_1, E_2, \widehat{E}_1, \widehat{E}_2$. Then:

- F must intersect both $E_1 \cup \widehat{E}_1$ and $E_2 \cup \widehat{E}_2$;
- either $E_1 \cdot l = \widehat{E}_1 \cdot l = E_2 \cdot l = \widehat{E}_2 \cdot l = 0$, or $(E_1 + \widehat{E}_1) \cdot l > 0$ and $(E_2 + \widehat{E}_2) \cdot l > 0$.

By 3.3.5 $[F]$ spans a one-dimensional face of $\text{Eff}(X)$, so that 3.3.6 gives the first statement.

Recall that $(E_1 \cup \widehat{E}_1) \cap (E_2 \cup \widehat{E}_2) = \emptyset$. If $(E_1 + \widehat{E}_1) \cdot l = 0$, since F intersects $E_1 \cup \widehat{E}_1$, there exists a fiber \bar{l} of the \mathbb{P}^1 -bundle structure of F which is contained in $E_1 \cup \widehat{E}_1$. Thus $\bar{l} \cap (E_2 \cup \widehat{E}_2) = \emptyset$, and we get $(E_2 + \widehat{E}_2) \cdot l = 0$. In this way we see that the intersections $(E_1 + \widehat{E}_1) \cdot l$, $(E_2 + \widehat{E}_2) \cdot l$ are either both zero or both positive, and this gives the second statement.

3.3.9. – We show that there exist two disjoint smooth prime divisors $F, \widehat{F} \subset X$, different from $E_1, E_2, \widehat{E}_1, \widehat{E}_2$, such that:

- F and \widehat{F} are \mathbb{P}^1 -bundles, with fibers $l \subset F$ and $\widehat{l} \subset \widehat{F}$ respectively, such that $F \cdot l = \widehat{F} \cdot \widehat{l} = -1$;
- the intersections $(E_1 + \widehat{E}_1) \cdot l$, $(E_1 + \widehat{E}_1) \cdot \widehat{l}$, $(E_2 + \widehat{E}_2) \cdot l$, $(E_2 + \widehat{E}_2) \cdot \widehat{l}$ are all positive.

We have $\text{codim } \mathcal{N}_1(E_1, X) = 3$ (see 3.3.4). Consider a special Mori program for $-E_1$ (which exists by Proposition 2.4), and let $G_1, \dots, G_s \subset X$ be the \mathbb{P}^1 -bundles determined by the Mori program. Recall from Lemma 2.7 that G_1, \dots, G_s are pairwise disjoint smooth prime divisors, with $2 \leq s \leq 3$, such that every G_i is a \mathbb{P}^1 -bundle with $G_i \cdot r_i = -1$, where $r_i \subset G_i$ is a fiber; moreover $E_1 \cdot r_i > 0$. In particular $G_i \neq E_1$ and $G_i \cap E_1 \neq \emptyset$, thus $G_i \neq E_2$ and $G_i \neq \widehat{E}_2$. Finally, if $G_i \neq \widehat{E}_1$, by 3.3.8 we have $(E_1 + \widehat{E}_1) \cdot r_i > 0$ and $(E_2 + \widehat{E}_2) \cdot r_i > 0$.

Suppose that $\{G_1, \dots, G_s\}$ contains at least two divisors distinct from \widehat{E}_1 , say G_1 and G_2 . Then we set $F := G_1$ and $\widehat{F} := G_2$, and we are done.

Otherwise, we have $s = 2$ and $G_2 = \widehat{E}_1$. Then Lemma 2.8 applies, and by 2.8(4) there exists a smooth prime divisor \widehat{G}_2 , having a \mathbb{P}^1 -bundle structure with fiber \widehat{r}_2 , such that:

$$\widehat{G}_2 \cdot \widehat{r}_2 = -1, \quad G_1 \cap \widehat{G}_2 = \emptyset, \quad \widehat{G}_2 \neq E_1, \quad \text{and} \quad \widehat{E}_1 \cdot \widehat{r}_2 = 1.$$

In particular $\widehat{G}_2 \neq \widehat{E}_1$ and $\widehat{G}_2 \cap \widehat{E}_1 \neq \emptyset$, therefore $\widehat{G}_2 \neq E_2$ and $\widehat{G}_2 \neq \widehat{E}_2$. By 3.3.8 we have $(E_1 + \widehat{E}_1) \cdot \widehat{r}_2 > 0$ and $(E_2 + \widehat{E}_2) \cdot \widehat{r}_2 > 0$, thus we set $F := G_1$ and $\widehat{F} := \widehat{G}_2$.

3.3.10. – As soon as F (respectively \widehat{F}) intersects one of the divisors E_i , then $F \cdot f_i > 0$ and $E_i \cdot l > 0$ (respectively $\widehat{F} \cdot f_i > 0$ and $E_i \cdot \widehat{l} > 0$), and similarly for \widehat{E}_i . In particular we have $F \cdot f > 0$ and $\widehat{F} \cdot f > 0$, where f is a general fiber of ψ .

Suppose for instance that $F \cap E_1 \neq \emptyset$. If $E_1 \cdot l = 0$, then E_1 contains some curve l , but this is impossible because $(E_2 + \widehat{E}_2) \cdot l > 0$ while $E_1 \cap (E_2 \cup \widehat{E}_2) = \emptyset$; thus $E_1 \cdot l > 0$.

If $F \cdot f_1 = 0$, then F contains an irreducible curve \bar{f}_1 which is a fiber of the \mathbb{P}^1 -bundle structure on E_1 . Let $\pi: F \rightarrow G$ be the \mathbb{P}^1 -bundle structure on F , and $\pi_*: \mathcal{N}_1(F) \rightarrow \mathcal{N}_1(G)$ the push-forward. Notice that $\pi(\bar{f}_1)$ is a curve, because \bar{f}_1 and l are not numerically equivalent in X , and hence neither in F .

Consider the surface $S := \pi^{-1}(\pi(\widehat{f}_1))$. Then $\pi_*(\mathcal{N}_1(S, F)) = \mathbb{R}\pi_*([\widehat{f}_1]_F)$, hence $\mathcal{N}_1(S, F) = \ker \pi_* \oplus \mathbb{R}[\widehat{f}_1]_F = \mathbb{R}[l]_F \oplus \mathbb{R}[\widehat{f}_1]_F$, and $\mathcal{N}_1(S, X) = \mathbb{R}[l] \oplus \mathbb{R}[\widehat{f}_1]$.

Since $\widehat{E}_1 \cdot \widehat{f}_1 > 0$, we have $S \cap \widehat{E}_1 \neq \emptyset$, and there exists an irreducible curve $C \subset S \cap \widehat{E}_1$. Thus $[C] \in \mathcal{N}_1(S, X)$, so that $C \equiv \lambda l + \mu \widehat{f}_1$ with $\lambda, \mu \in \mathbb{R}$. On the other hand $C \cap (E_2 \cup \widehat{E}_2) = \emptyset$ (because $C \subset \widehat{E}_1$) and

$$0 = (E_2 + \widehat{E}_2) \cdot C = \lambda(E_2 + \widehat{E}_2) \cdot l,$$

which by 3.3.9 yields $\lambda = 0, \mu \neq 0$ and $[f_1] = (1/\mu)[C] \in \mathcal{N}_1(\widehat{E}_1, X)$, a contradiction with 3.3.4.

Therefore $F \cdot f_1 > 0$. We have $f \equiv f_1 + \widehat{f}_1$ (see 2.8(4)), and $F \cdot \widehat{f}_1 \geq 0$ because $F \neq \widehat{E}_1$ (see 3.3.9), hence $F \cdot f > 0$.

3.3.11. – For every $i \in \{0, \dots, k\}$ let $F_i, \widehat{F}_i \subset X_i$ be the transforms of F, \widehat{F} . Let us show that for any $i \in \{0, \dots, k-1\} \setminus \{i_1, i_2\}$, the divisors F_i and \widehat{F}_i are disjoint from $\text{Locus}(Q_i)$.

By contradiction, suppose for instance that this is not true for F , and let $j \in \{0, \dots, k-1\} \setminus \{i_1, i_2\}$ be the smallest index such that F_j intersects $\text{Locus}(Q_j)$. Recall from 3.3.7 that σ_i is a flip for every $i \in \{0, \dots, k-1\} \setminus \{i_1, i_2\}$; in particular, Q_j is a small extremal ray, and σ_j is a flip.

After 2.8(3), σ is regular on the divisors $E_1, E_2, \widehat{E}_1, \widehat{E}_2$, and $\text{Locus}(Q_j)$ is disjoint from their images in X_j .

Recall from 2.6(4) the definition of $A_i \subset X_i$, for $i \in \{1, \dots, k\}$: $A_1 \subset X_1$ is the indeterminacy locus of σ_0^{-1} , and for $i \geq 2$, if σ_{i-1} is a divisorial contraction (respectively, if σ_{i-1} is a flip), A_i is the union of $\sigma_{i-1}(A_{i-1})$ (respectively, the transform of A_{i-1}) and the indeterminacy locus of σ_{i-1}^{-1} .

If $j > 0$, by the minimality of j , F_j does not intersect the loci of the previous flips, hence it can intersect A_j only along the images of E_1 and E_2 . Therefore

$$(3.3.12) \quad \text{Locus}(Q_j) \cap F_j \cap A_j = \emptyset.$$

Let $\alpha_j : X_j \rightarrow Y_j$ be the contraction of Q_j . Suppose first that α_j is finite on F_j . Then $\text{Locus}(Q_j) = \text{Exc}(\alpha_j) \not\subset F_j$, and since $F_j \cap \text{Locus}(Q_j) \neq \emptyset$, we have $F_j \cdot Q_j > 0$. Hence every non trivial fiber of α_j must have dimension 1, otherwise α_j would not be finite on F_j .

If $j = 0$, then α_0 is a small contraction of a smooth Fano variety with one-dimensional fibers, which is impossible, see [3, Theorem 4.1].

Suppose that $j > 0$. If $C_0 \subset X_j$ is an irreducible curve in a fiber of α_j , then C_0 must intersect F_j , hence $C_0 \not\subset A_j$ by (3.3.12); in particular $C_0 \not\subset \text{Sing}(X_j)$ (recall that $\text{Sing}(X_j) \subseteq A_j$ by 2.6(4)). Then [14, Lemma 1] yields $-K_{X_j} \cdot C_0 \leq 1$, and [11, Lemma 3.8] implies that $C_0 \cap A_j = \emptyset$. We conclude that $\text{Locus}(Q_j) \subseteq X_j \setminus A_j$, and this is again impossible by [3, Theorem 4.1], because $-K_{X_j} \cdot Q_j > 0$ and $(\alpha_j)|_{X_j \setminus A_j} : X_j \setminus A_j \rightarrow Y_j \setminus \alpha_j(A_j)$ is a small contraction of a smooth variety with one-dimensional fibers.

Suppose now that α_j is not finite on F_j . Then there exists an irreducible curve $C_1 \subset F_j$ with $[C_1] \in Q_j$; in particular C_1 is disjoint from the images of $E_1, E_2, \widehat{E}_1, \widehat{E}_2$ in X_j . Consider the transform $\widetilde{C}_1 \subset F \subset X$ of C_1 , so that \widetilde{C}_1 is disjoint from $E_1, E_2, \widehat{E}_1, \widehat{E}_2$.

Recall that F intersects both $E_1 \cup \widehat{E}_1$ and $E_2 \cup \widehat{E}_2$ by 3.3.8. We assume that F intersects E_1 and E_2 , the other cases being analogous. Then $E_1 \cdot l > 0$ by 3.3.10, so that using 3.1.3(3) we get

$$\widetilde{C}_1 \equiv \lambda l + \mu C_2,$$

where $C_2 \subset F \cap E_1$ is a curve, $\lambda, \mu \in \mathbb{R}$, and $\mu \geq 0$. In particular $C_2 \cap E_2 = \emptyset$, therefore $0 = E_2 \cdot \widetilde{C}_1 = \lambda E_2 \cdot l$. On the other hand $E_2 \cdot l > 0$ by 3.3.10, and this implies that $\lambda = 0$ and $\widetilde{C}_1 \equiv \mu C_2$. Recall that the map $X \dashrightarrow X_j$ is regular on F by the minimality of j , and call C'_2 the image of C_2 in X_j . We deduce that $C_1 \equiv \mu C'_2$ in X_j , so that $[C'_2] \in Q_j$. But C'_2 is contained in the image of E_1 , which is disjoint from $\text{Locus}(Q_j)$, and we have a contradiction.

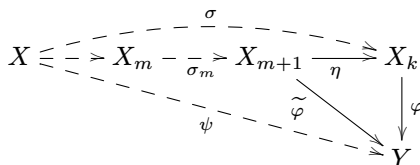
3.3.13. – We show that $k = 2$ in (3.3.2), so that $i_1 = 0$ and $i_2 = 1$.

By contradiction, suppose that $k > 2$. Recall from 3.3.7 that σ_i is a flip for every $i \in \{0, \dots, k - 1\} \setminus \{i_1, i_2\}$, so equivalently we are assuming that the Mori program (3.3.2) contains some flips.

We define an integer $m \in \{k - 3, k - 2, k - 1\}$ and a morphism $\eta: X_{m+1} \rightarrow X_k$ as follows:

- if σ_{k-1} is a flip, set $m := k - 1$ and $\eta := \text{Id}_{X_k}$;
- if σ_{k-1} is a contraction and σ_{k-2} is a flip, set $m := k - 2$ and $\eta := \sigma_{k-1}: X_{k-1} \rightarrow X_k$;
- if both σ_{k-1} and σ_{k-2} are contractions, set $m := k - 3$ and $\eta := \sigma_{k-2} \circ \sigma_{k-1}: X_{k-2} \rightarrow X_k$.

It follows from these definitions that Q_m is a small extremal ray of X_m and $\sigma_m: X_m \dashrightarrow X_{m+1}$ is a flip; let Q'_{m+1} be the corresponding small extremal ray of X_{m+1} . Set moreover $\widetilde{\varphi} := \varphi \circ \eta: X_{m+1} \rightarrow Y$.



We keep the same notations as in the proof of Lemma 2.8; in particular we set $T_i := \sigma(E_i) \subset X_k$ for $i = 1, 2$. Notice that when $m = k - 2$ (respectively, $m = k - 3$), η is just the smooth blow-up of $T_2 \subset X_k$ (respectively, of $T_1 \cup T_2 \subset X_k$).

Recall that for $j = 1, 2$ we have $Q_{i_j} \notin \mathcal{N}_1(D_{i_j}, X_{i_j})$, in particular σ_{i_j} is finite on D_{i_j} ; this implies that η is finite on D_{m+1} .

We show that every fiber of $\widetilde{\varphi}$ has dimension 1. Indeed this is true for φ by 2.8(1). Moreover η is an isomorphism over $X_k \setminus (T_1 \cup T_2)$, therefore $\widetilde{\varphi}$ has one-dimensional fibers over $Y \setminus \varphi(T_1 \cup T_2)$. On the other hand, we know by 2.8(3) that there exist open subsets $U \subseteq X$ and $V \subseteq Y$ such that $\varphi(T_1 \cup T_2) \subset V$, both $\psi: U \rightarrow V$ and $\varphi|_{\varphi^{-1}(V)}: \varphi^{-1}(V) \rightarrow V$ are conic bundles, and $\sigma|_U: U \rightarrow \varphi^{-1}(V)$ is just the blow-up of T_1 and T_2 . This implies that $\widetilde{\varphi}|_{\widetilde{\varphi}^{-1}(V)}: \widetilde{\varphi}^{-1}(V) \rightarrow V$ is a conic bundle, in particular it has one-dimensional fibers over $\varphi(T_1 \cup T_2) \subset V$.

Recall from 2.8(1) that φ is finite on $D_k = \eta(D_{m+1})$, and since η is finite on D_{m+1} , we deduce that $\widetilde{\varphi}$ must be finite on D_{m+1} . Notice also that $D_{m+1} \supset A_{m+1} \supseteq \text{Locus}(Q'_{m+1})$ (see 2.6(4)). As in the proof of Lemma 2.8, using [11, Lemma 3.8] we see that every fiber of $\widetilde{\varphi}$ which intersects $\text{Locus}(Q'_{m+1})$ is an integral rational curve.

Let $C \subset X_{m+1}$ be an irreducible curve with $[C] \in Q'_{m+1}$, and set $S := \tilde{\varphi}^{-1}(\tilde{\varphi}(C))$, so that S is an irreducible surface.

Since, by 3.3.10, F and \widehat{F} have positive intersection with a general fiber of ψ in X , F_{m+1} and \widehat{F}_{m+1} have positive intersection with every fiber of $\tilde{\varphi}$ in X_{m+1} . In particular, F_{m+1} and \widehat{F}_{m+1} intersect S .

On the other hand by 3.3.11 the divisors F_m and \widehat{F}_m in X_m are disjoint from $\text{Locus}(Q_m)$, therefore F_{m+1} and \widehat{F}_{m+1} are disjoint from $\text{Locus}(Q'_{m+1})$. We deduce that:

$$(3.3.14) \quad F_{m+1} \cap C = \widehat{F}_{m+1} \cap C = \emptyset \text{ and } \dim(F_{m+1} \cap S) = \dim(\widehat{F}_{m+1} \cap S) = 1.$$

For $i = 1, 2$ call G_i the image of E_i in X_{m+1} , so that $T_i = \eta(G_i)$ and $\varphi(T_i) = \tilde{\varphi}(G_i)$. Notice that $A_k \setminus (T_1 \cup T_2) = \eta(A_{m+1} \setminus (G_1 \cup G_2))$.

Recall that the open subset $V \subseteq Y$ was defined in (2.10) as

$$V := Y \setminus \varphi(A_k \setminus (T_1 \cup T_2)) = Y \setminus \tilde{\varphi}(A_{m+1} \setminus (G_1 \cup G_2)).$$

By 2.7(1) and 2.8(2) we have $\text{Locus}(Q'_{m+1}) \cap (G_1 \cup G_2) = \emptyset$. In particular $C \subseteq \text{Locus}(Q'_{m+1}) \subseteq A_{m+1} \setminus (G_1 \cup G_2)$, thus

$$\tilde{\varphi}(C) \subseteq Y \setminus V.$$

On the other hand we also have $\tilde{\varphi}(G_1 \cup G_2) = \varphi(T_1 \cup T_2) \subset V$, therefore we deduce that $\tilde{\varphi}(G_1 \cup G_2) \cap \tilde{\varphi}(C) = \emptyset$ and hence

$$(G_1 \cup G_2) \cap S = \emptyset.$$

Finally by 3.3.9 we have $F \cap \widehat{F} = \emptyset$ in X , and by 3.3.11 the divisors F and \widehat{F} are disjoint from the locus of every flip in the Mori program (3.3.2). This implies that $F_{m+1} \cap \widehat{F}_{m+1} \subseteq G_1 \cup G_2$, therefore:

$$F_{m+1} \cap \widehat{F}_{m+1} \cap S = \emptyset.$$

Together with (3.3.14), this yields that C , $F_{m+1} \cap S$, and $\widehat{F}_{m+1} \cap S$ are pairwise disjoint curves in S .

Let C' be an irreducible component of $\widehat{F}_{m+1} \cap S$. Since $\tilde{\varphi}|_S: S \rightarrow \tilde{\varphi}(C)$ is a fibration in integral rational curves, we have $C' \equiv \lambda C + \mu f$ where $\lambda, \mu \in \mathbb{R}$ and $f \subset S$ is a fiber. Then $0 = F_{m+1} \cdot C' = \mu F_{m+1} \cdot f$ while $F_{m+1} \cdot f > 0$, hence $\mu = 0$ and $[C'] \in Q'_{m+1}$. Therefore $C' \subseteq \text{Locus}(Q'_{m+1}) \cap F_{m+1}$, a contradiction because $\text{Locus}(Q'_{m+1}) \cap F_{m+1} = \emptyset$.

3.3.15. – Since $k = 2$, X_2 is smooth and $\sigma: X \rightarrow X_2$ is just the blow-up of two disjoint smooth subvarieties $T_1, T_2 \subset X_2$, of codimension 2. In fact, we have $A_2 = T_1 \cup T_2$ (see 2.6(4)), and by (2.10) the description in 2.8(3) and 2.8(4) holds with $V = Y$ and $U = X$. In particular, Y is smooth, $\varphi: X_2 \rightarrow Y$ and $\psi: X \rightarrow Y$ are conic bundles, $\rho_X - \rho_Y = 3$, and the divisors $Z_1 = \psi(E_1)$ and $Z_2 = \psi(E_2)$ are disjoint in Y . Moreover we have $\psi(F) = Y$ by 3.3.10.

The situation is very similar to the case where φ is finite on $(E_0)_s$ in 3.2.24, with the difference that the E_i 's do not need to be products. In the same way we use Lemma 3.2.25 to show that $[\psi(l)] \in \text{NE}(Y)$ belongs to an extremal ray of Y , whose contraction is a smooth conic bundle $\zeta: Y \rightarrow Y'$, finite on Z_1 and Z_2 ; in particular Y' is smooth of dimension $n - 2$. The contraction $\psi' := \zeta \circ \psi: X \rightarrow Y'$ is equidimensional and hence flat, and $\rho_X - \rho_{Y'} = 4$.

Moreover the general fiber of ψ' is a Del Pezzo surface S containing curves $f_1, \widehat{f}_1, f_2, \widehat{f}_2, l$, hence $\mathcal{N}_1(S, X) = \ker(\psi')_*$ and ψ' is quasi-elementary.

$$\begin{array}{ccc} X & \xrightarrow{\sigma} & X_2 \\ \psi' \downarrow & \searrow \psi & \downarrow \varphi \\ Y' & \xleftarrow{\zeta} & Y \end{array}$$

3.3.16. – We show that the conic bundle $\varphi: X_2 \rightarrow Y$ is smooth.

By contradiction, suppose that this is not the case, and let $\Delta_\varphi \subset Y$ be the discriminant divisor of φ . Recall that this is an effective, reduced divisor in Y such that $\varphi^{-1}(y)$ is singular if and only if $y \in \Delta_\varphi$.

Consider also the discriminant divisor $\Delta_\psi \subset Y$ of the conic bundle $\psi: X \rightarrow Y$. Since φ is smooth over Z_1 and Z_2 , the divisors Δ_φ, Z_1, Z_2 are pairwise disjoint, and $\Delta_\psi = \Delta_\varphi \cup Z_1 \cup Z_2$.

The fibers of ψ over $Z_1 \cup Z_2$ are singular but reduced, hence $\psi^{-1}(y)$ is non-reduced if and only if $\varphi^{-1}(y)$ is. Let $W \subset \Delta_\varphi$ be the set of points y such that $\psi^{-1}(y)$ (equivalently, $\varphi^{-1}(y)$) is non-reduced. Then W is a closed subset of Y , and $W \subseteq \text{Sing}(\Delta_\varphi)$ (see for instance [19, Proposition 1.8(5.c)]). Moreover by [21, Proposition 4.3] we know that $-K_Y \cdot C > 0$ for every irreducible curve $C \subset Y$ not contained in W .

For $i = 1, 2$ we have $\text{codim } \mathcal{N}_1(Z_i, Y) \leq 1$, because $\zeta(Z_i) = Y'$ and hence $\zeta_*(\mathcal{N}_1(Z_i, Y)) = \mathcal{N}_1(Y')$. This yields $Z_1^\perp = Z_2^\perp = \Delta_\varphi^\perp = \mathcal{N}_1(Z_1, Y) = \mathcal{N}_1(Z_2, Y)$ (see Remark 3.1.2). The three divisors Δ_φ, Z_1, Z_2 are numerically proportional, nef, and cut a facet of $\text{NE}(Y)$, whose contraction $\beta: Y \rightarrow \mathbb{P}^1$ sends Δ_φ, Z_1, Z_2 to points (see [10, Lemma 2.6]). Even if a priori we do not know whether every curve contracted by β has positive anticanonical degree, the general fiber of β does not meet W , therefore it is a Fano manifold. Moreover $\text{NE}(\beta)$ is generated by finitely many classes of rational curves (see [10, Lemma 2.6]). Thus the same proof as [11, Lemma 4.9] yields that $Y \cong \mathbb{P}^1 \times Y'$, and $\Delta_\varphi = \{pts\} \times Y'$.

In particular Δ_φ is smooth, hence $W = \emptyset$ and Y is Fano. Because $Y \cong \mathbb{P}^1 \times Y'$, Y' is Fano too, so that each connected component of Δ_φ is simply connected. However this is impossible, because by a standard construction the conic bundle φ defines a double cover of every irreducible component of Δ_φ , obtained by considering the components of the fibers in the appropriate Hilbert scheme of lines, see [4, §1.5] and [19, §1.17]. Since φ is an elementary contraction, this double cover is non-trivial; on the other hand it is also étale, because every fiber of φ is reduced, and we have a contradiction.

3.3.17. – Since $\varphi: X_2 \rightarrow Y$ is smooth, every fiber of the conic bundle $\psi: X \rightarrow Y$ is reduced. Then [21, Proposition 4.3] shows that Y and Y' are Fano. Finally $c_{Y'} \leq 3$ by the following remark, which concludes the proof of Proposition 3.3.1. \square

REMARK 3.3.18. – Let X be a Fano manifold, $\varphi: X \rightarrow Y$ a surjective morphism, and $D \subset X$ a prime divisor. We have $\mathcal{N}_1(\varphi(D), Y) = \varphi_*(\mathcal{N}_1(D, X))$, hence $\text{codim } \mathcal{N}_1(D, X) \geq \text{codim } \mathcal{N}_1(\varphi(D), Y)$. In particular, if Y is a Fano manifold, then $c_Y \leq c_X$.

4. Applications

In this final section we prove the results stated in the introduction.

Proof of Theorem 1.1. – We have $c_X \geq \text{codim } \mathcal{N}_1(D, X) \geq 3$. If $c_X = 3$, Theorem 3.3 yields (ii). If instead $c_X \geq 4$, applying iteratively Theorem 3.3, we can write $X = S_1 \times \cdots \times S_r \times Z$, where S_i are Del Pezzo surfaces, $r \geq 1$, and Z is a Fano manifold with $c_Z \leq 3$.

If D dominates Z under the projection, up to reordering S_1, \dots, S_r we can assume that D dominates $S_2 \times \cdots \times S_r \times Z$. Then $\text{codim } \mathcal{N}_1(D, X) \leq \rho_{S_1} - 1$ (see Example 3.1), and we get (i).

Suppose instead that $D = S_1 \times \cdots \times S_r \times D_Z$, where $D_Z \subset Z$ is a prime divisor. Then

$$3 \geq c_Z \geq \text{codim } \mathcal{N}_1(D_Z, Z) = \text{codim } \mathcal{N}_1(D, X) \geq 3,$$

and the inequalities above are equalities. Therefore again by Theorem 3.3 we have a flat, quasi-elementary contraction $Z \rightarrow W$, where W is a Fano manifold with $\dim W = \dim Z - 2$, and $\rho_Z - \rho_W = 4$. Then the induced contraction $X \rightarrow S_1 \times \cdots \times S_r \times W$ satisfies (ii). □

Proof of Corollary 1.3. – We have $c_X \geq \text{codim } \mathcal{N}_1(D, X) \geq 3$. Suppose that X is not a product of a Del Pezzo surface with another variety. Then Theorem 3.3 shows that $c_X = 3$ and there is a quasi-elementary contraction $X \rightarrow T$ where T is a Fano manifold, $\dim T = n - 2$, and $\rho_X - \rho_T = 4$. If $n = 4$, [10, Theorem 1.1] implies that $\rho_T \leq 2$, hence $\rho_X \leq 6$. The case $n = 5$ follows similarly. □

LEMMA 4.1. – *Let X be a Fano manifold, $D \subset X$ a prime divisor, and $\varphi: X \rightarrow Y$ a contraction. Then $\text{codim } \mathcal{N}_1(\varphi(D), Y) \leq 8$.*

Suppose moreover that $\text{codim } \mathcal{N}_1(\varphi(D), Y) \geq 4$. Then $X \cong S \times T$ and $Y \cong W \times Z$, where S is a Del Pezzo surface, W is a blow-down of S , and $\text{codim } \varphi(D) \leq 2$. More precisely, one of the following holds:

- (i) $\varphi(D)$ is a divisor in Y , and dominates Z under the projection;
- (ii) $\varphi(D) = \{p\} \times Z$ and $D = C \times T$, where $C \subset S$ is a curve contracted to $p \in W$.

Proof. – We have $\text{codim } \mathcal{N}_1(\varphi(D), Y) \leq \text{codim } \mathcal{N}_1(D, X) \leq 8$ by Remark 3.3.18 and Theorem 1.1. Suppose that $\text{codim } \mathcal{N}_1(\varphi(D), Y) \geq 4$. Then, again by Theorem 1.1, $X \cong S \times T$ where S is a Del Pezzo surface, and D dominates T under the projection. Therefore $Y \cong W \times Z$, φ is induced by two contractions $S \rightarrow W$ and $f: T \rightarrow Z$, and $\varphi(D)$ dominates Z under the projection.

In particular $\dim W \leq 2$ and $\dim \mathcal{N}_1(\varphi(D), Y) \geq \rho_Z$, hence $\rho_W \geq \text{codim } \mathcal{N}_1(\varphi(D), Y) \geq 4$. This implies that $\dim W = 2$, thus W is a blow-down of S , and $\varphi(D)$ has codimension 1 or 2 in Y .

If $\varphi(D)$ is a divisor, we have (i). Suppose that $\text{codim } \varphi(D) = 2$, and consider the factorization of φ as $S \times T \xrightarrow{\psi} W \times T \xrightarrow{\xi} W \times Z$. Then $\xi = (\text{Id}_W, f)$ induces an isomorphism $W \times \{t\} \rightarrow W \times \{f(t)\}$ for every $t \in T$. If t is general, we have $\dim \varphi(D) \cap (W \times \{f(t)\}) = 0$ and $\psi(D) \cap (W \times \{t\}) \cong \varphi(D) \cap (W \times \{f(t)\})$. This implies that $\psi(D)$ has codimension 2 in $W \times T$, hence D is an exceptional divisor of ψ , which gives the statement. □

Corollary 1.4 follows from Theorem 1.1, while Corollary 1.5 is a straightforward application of Lemma 4.1 (just take the Stein factorization of φ).

Proof of Corollary 1.6. – By taking the Stein factorization, we can assume that φ is a contraction; we also assume that $\rho_Y \geq 6$. By [10, Lemma 2.6] we know that Y has some elementary contraction $\psi: Y \rightarrow Z$, and $\dim Z \geq 2$ because $\rho_Z \geq 5$.

We define a prime divisor $D \subset X$ depending on ψ , as follows. If $\dim Z = 2$, let $D \subset X$ be any prime divisor such that $\dim \psi(\varphi(D)) = 1$. If ψ is birational and divisorial, let $D \subset X$ be a prime divisor such that $\varphi(D) \subseteq \text{Exc}(\psi)$. If ψ is birational and small, its lifting in X (see [10, § 2.5]) must be an elementary contraction of type $(n-1, n-2)^{sm}$; let D be its exceptional divisor: then again we have $\varphi(D) \subseteq \text{Exc}(\psi)$.

In any case $\dim \mathcal{N}_1(\varphi(D), Y) \leq 2$, so that Lemma 4.1 implies the statement. \square

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