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Explicit birational geometry of threefolds of general type, I

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EXPLICIT BIRATIONAL GEOMETRY OF THREEFOLDS OF GENERAL TYPE, I

BY JUNGKAI A. CHEN AND MENG CHEN

ABSTRACT. – Let V be a complex nonsingular projective 3-fold of general type. We prove $P_{12}(V) := \dim H^0(V, 12K_V) > 0$ and $P_{m_0}(V) > 1$ for some positive integer $m_0 \leq 24$. A direct consequence is the birationality of the pluricanonical map φ_m for all $m \geq 126$. Besides, the canonical volume $\mathrm{Vol}(V)$ has a universal lower bound $\nu(3) \geq \frac{1}{63 \cdot 126^2}$.

RÉSUMÉ. – Soit V une variété non singulière complexe de type général et de dimension 3. Nous montrons $P_{12}(V) := \dim H^0(V, 12K_V) > 0$ et $P_{m_0}(V) > 1$ pour un certain entier $m_0 \leq 24$. Une conséquence directe est la birationalité de l'application pluricanonique φ_m pour tout $m \geq 126$. De plus, le volume canonique $\mathrm{Vol}(V)$ a un minorant universel $\nu(3) \geq \frac{1}{63 \cdot 126^2}$.

1. Introduction

Let Y be a nonsingular projective variety of dimension n. It is said to be of general type if the pluricanonical map $\varphi_m := \Phi_{|mK_Y|}$ corresponding to the linear system $|mK_Y|$ is birational into a projective space for $m \gg 0$. Thus it is natural and important to ask:

PROBLEM 1. – Can one find a constant $c(n)$, so that φ_m is birational onto its image for all $m \geq c(n)$ and for all Y with $\dim Y = n$?

When $\dim Y = 1$, it was classically known that $|mK_Y|$ gives an embedding of Y into a projective space if $m \geq 3$. When $\dim Y = 2$, Kodaira-Bombieri's theorem [2] says that $|mK_Y|$ gives a birational map onto the image for $m \geq 5$. This theorem has essentially established the canonical classification theory for surfaces of general type.

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A natural approach to study this problem in higher dimensions is an induction on the dimension by utilizing vanishing theorems. This amounts to estimating the plurigenus, for which purpose the greatest difficulty seems to be to bound from below the canonical volume

$$\mathrm{Vol}(Y) := \limsup_{\{m\in\mathbb{Z}^+\}}\{\frac{n!}{m^n}\dim_{\mathbb{C}} H^0(Y, \mathcal{O}_Y(mK_Y))\}.$$

The volume is an integer when $\dim Y \leq 2$. However it is only a rational number in general, which may account for the complexity of high dimensional birational geometry. In fact, it is almost an equivalent question to study the lower bound of the canonical volume.

Problem 2. – Can one find a constant $\nu(n)$ such that $\mathrm{Vol}(Y) \geq \nu(n)$ for all varieties Y of general type with $\dim Y = n$?

A recent result of Hacon and M^c^Kernan [13], Takayama [24] and Tsuji [25] shows the existence of both $c(n)$ and $\nu(n)$. An explicit constant $c(n)$ or $\nu(n)$ is, however, mysterious at least up to now. Notice that similar questions were asked by Kollár and Mori [19, 7.74].

Here we mainly deal with $c(3)$ and $\nu(3)$. For known results under extra assumptions, one may refer to [3, 4, 5, 6, 8, 9, 10, 14, 18, 20] and others. In this series of papers, we would like to present two realistic constants $c(3)$ and $\nu(3)$. In fact, our method can help us to prove some sharp results. Being worried that a very long paper would tire the readers, we decided to only explain our key technique and rough statements in the first part whereas more refined and some sharp statements will be presented in the subsequent papers. Our main result in this paper is the following:

Theorem 1.1. – *Let V be a nonsingular projective 3-fold of general type. Then*

(1) $P_{12} > 0$*;*
(2) $P_{m_0} \geq 2$ *for some positive integer* $m_0 \leq 24$.

With Kollár's result [18, Corollary 4.8] and its improved form [7, Theorem 0.1], we immediately get the following:

Corollary 1.2. – *Let V be a nonsingular projective 3-fold of general type. Then*

(1) φ_m *is birational onto its image for all* $m \geq 126$.
(2) *Vol*$(V) \geq \frac{1}{63\cdot 126^2}$.

Example 1.3 (see [14, p. 151, No. 23]). – The "worst" known example is a general weighted hypersurface $X = X_{46} \subset \mathbb{P}(4, 5, 6, 7, 23)$. The 3-fold X has invariants: $p_g(X) = P_2(X) = P_3(X) = 0$, $P_4(X) = \cdots = P_9(X) = 1$, $P_{10}(X) = 2$ and $\mathrm{Vol}(X) = \frac{1}{420}$. Moreover, it is known that φ_m is birational for all $m \geq 27$, but φ_{26} is not birational.

Now we explain the main idea of our paper. It is very natural to investigate the plurigenus P_m, which can be calculated using Reid's Riemann-Roch formula in [21, 23]. However the most difficult point is to control the contribution from singularities due to the combinatorial complexity of baskets of singularities on the 3-fold.

Indeed, given a minimal 3-fold X with at worst canonical singularities, a known fact is that the canonical volume and all plurigenera are determined by the basket (of singularities) B, $\chi = \chi(\mathcal{O}_X)$ and $P_2 = P_2(X)$. We call the triple (B, χ, P_2) a *formal basket*. First we will define a partial ordering (called "packing") between formal baskets. (In this paper, we are only concerned about the numerical behavior of "packing", rather than its geometric meaning. More details on its geometric aspect will be explored in our subsequent works.) Then we introduce the "canonical sequence of prime unpackings of a basket"

$$B^{(0)} \succcurlyeq B^{(5)} \succcurlyeq \ldots \succcurlyeq B^{(n)} \succcurlyeq \ldots \succcurlyeq B$$

and, furthermore, each step in the sequence can be calculated in terms of the datum of the given formal basket. The intrinsic properties of the canonical sequence tell us many new inequalities among the Euler characteristic and the plurigenus, of which the most interesting one is:

$$2P_5 + 3P_6 + P_8 + P_{10} + P_{12} \geq \chi + 10P_2 + 4P_3 + P_7 + P_{11} + P_{13}.$$

If $P_{m_0} \geq 2$ for some $m_0 \leq 12$, then one gets many interesting results by [18, Corollary 4.8] and [7, Theorem 0.1]. Otherwise one has $P_m \leq 1$ for all $m \leq 12$ and the above inequality gives $\chi \leq 8$. This essentially tells us that the number of formal baskets is finite! Thus, theoretically, we are able to obtain various effective results.

Here is the overview to the structure of this paper. In Section 2, we introduce the notion of packing and define some invariants of baskets. Then we define the canonical sequence of "prime unpackings" of a basket and give some examples. In Section 3, we define the notion of formal baskets. Then we study various relations among formal baskets, Euler characteristics and K^3. We calculate the first few elements in the canonical sequence of the given basket. This immediately gives many inequalities among Euler characteristics. We would like to remark that the method so far works for $\mathbb{Q}$-factorial threefolds (not only of general type) with canonical singularities. With all these preparations, we prove the main theorem on threefolds of general type in Section 4.

Another remark is that the method in Sections 2 and 3 is also valid for $\mathbb{Q}$-Fano threefolds. More precisely, there are similar relations among formal baskets, anti-plurigenera and the anti-canonical volume with proper sign alterations because of Serre dualities. We will explore some more applications of our method in a future work.

In our next paper of this series, we will work out some classification of formal baskets with given small Euler characteristics. Together with some more detailed study of the geometry of pluricanonical maps, we will prove the following theorem:

THEOREM A. – *Let V be a nonsingular projective 3-fold of general type. Then the following hold.*

(i) *φ_m is birational onto its image for all $m \geq 73$.*
(ii) $\mathrm{Vol}(V) \geq \frac{1}{2660}$.
(iii) *Suppose that $\chi(\mathcal{O}_V) \leq 1$. Then $\mathrm{Vol}(V) \geq \frac{1}{420}$, which is optimal. Moreover φ_m is birational for all $m \geq 40$.*

Throughout, we work over the complex number field $\mathbb{C}$. We prefer to use $\sim$ to denote the linear equivalence and $\equiv$ means numerical equivalence. We mainly refer to [17, 19, 22] for tool books on 3-dimensional birational geometry.

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2. Baskets of singularities

In this section, we introduce the notion of packing between baskets of singularities. This notion defines a partial ordering on the set of baskets. For a given basket, we define its canonical sequence of prime unpackings. The canonical sequence trick is a fundamental and effective tool in our arguments.

2.1. – Terminal quotient singularity and basket. By *a 3-dimensional terminal quotient singularity Q of type* $\frac{1}{r}(1,-1,b)$, we mean a singularity which is analytically isomorphic to the quotient of $(\mathbb{C}^3, \mathrm{o})$ by a cyclic group action ε:

$$\varepsilon(x,y,z) = (\varepsilon x, \varepsilon^{-1}y, \varepsilon^b z)$$

where r is a positive integer, ε is a fixed r-th primitive root of 1, the integer b is coprime to r and $0 < b < r$.

2.2. – Convention. By replacing ε with another primitive root of 1 and changing the ordering of coordinates, we may and will assume that $b \leq \frac{r}{2}$.

A basket B of singularities is a collection (allowing multiplicities) of terminal quotient singularities of type $\frac{1}{r_i}(1,-1,b_i)$, $i \in I$ where I is a finite index set. For simplicity, we will always denote a terminal quotient singularity $\frac{1}{r}(1,-1,b)$ by a pair of integers (b,r). So we will write a basket as:

$$B := \{n_i \times (b_i, r_i) \mid i \in J,\ n_i \in \mathbb{Z}^+\},$$

where n_i denotes the multiplicities.

Given baskets $B_1 = \{n_i \times (b_i, r_i)\}$ and $B_2 = \{m_i \times (b_i, r_i)\}$, we define

$$B_1 \cup B_2 := \{(n_i + m_i) \times (b_i, r_i)\}.$$

Definition 2.3. – A generalized basket means a collection of pairs of integers (b,r) with $0 < b < r$, not necessarily coprime and allowing multiplicities.

2.4. – Invariants of baskets. Given a generalized basket (b,r) with $b \leq \frac{r}{2}$ and a fixed integer $n > 0$. Let $\delta := \lfloor \frac{bn}{r} \rfloor$. Then $\frac{\delta+1}{n} > \frac{b}{r} \geq \frac{\delta}{n}$. We define

$$\Delta^n(b,r) := \delta b n - \frac{(\delta^2+\delta)}{2} r. \tag{2.1}$$

One can see that $\Delta^n(b,r)$ is a non-negative integer. For a generalized basket $B = \{(b_i, r_i)\}_{i\in I}$ and a fixed $n > 0$, we define $\Delta^n(B) := \sum_{i\in I} \Delta^n(b_i, r_i)$. By definition, $\Delta^2(B) = 0$ for any basket B. By a direct calculation, one gets the following relation:

$$\frac{\overline{jb_i}(r_i - \overline{jb_i})}{2r_i} - \frac{jb_i(r_i - jb_i)}{2r_i} = \Delta^j(b_i, r_i)$$

for all $j > 0$, $i \in I$. Define

$$\sigma(B) := \sum_{i \in I} b_i \text{ and } \sigma'(B) := \sum_{i \in I} \frac{b_i^2}{r_i}. \tag{2.2}$$

2.5. – Packing. Given a generalized basket

$$B = \{(b_1, r_1), (b_2, r_2), \cdots, (b_k, r_k)\},$$

we call the basket

$$B' := \{(b_1 + b_2, r_1 + r_2), (b_3, r_3), \cdots, (b_k, r_k)\}$$

a packing of B (and B is an unpacking of B'), written as $B \succ B'$. (The symbol $B \succcurlyeq B'$ means either $B \succ B'$ or $B = B'$.)

If, furthermore, $b_1 r_2 - b_2 r_1 = 1$, we call $B \succ B'$ *a prime packing*. A prime packing is said to have *level* n if $r_1 + r_2 = n$.

The seemingly mysterious notion of packings can indeed be realized in various elementary birational maps.

EXAMPLE 2.6. – We consider the Kawamata blowup [16]. Let $X = X_\Sigma$ be a toric threefold associated to the fan Σ. Suppose that there is a cone σ in Σ generated by $v_1 = (1,0,0), v_2 = (0,1,0)$ and $v_3 = (s, r-s, r)$ with $0 < s < r$ and $(s,r) = 1$. The cone σ gives rise to a quotient singularity $P \in X$ of type $\frac{1}{r}(r-s, s, 1)$.

Let $\pi : \tilde{X} \to X$ be the partial resolution obtained by the subdivision by adding $v_4 = (1,1,1)$. One sees that $\tilde{X}$ has two quotient singularities of type $\frac{1}{s}(\overline{r}, \overline{-r}, 1)$, and $\frac{1}{r-s}(\overline{r}, \overline{-r}, 1)$ respectively, where $\bar{\cdot}$ denotes the residue modulo s and $r-s$ respectively.

Then it is easy to verify that $B(X) = \{(b,r)\}$ and $B(\tilde{X}) = \{(b', s), (b-b', r-s)\}$ for some b, b' satisfying $b'r - bs = \pm 1$. One sees that

$$B(\tilde{X}) \succ B(X)$$

is a prime packing of baskets.

EXAMPLE 2.7. – Let $X = X_\Sigma$ be a toric threefold associated to the fan Σ. Suppose that there are two cones σ_4, σ_3 in the fan Σ such that

$$\sigma_4 \text{ is generated by } v_1 = (1,0,0), v_2 = (0,1,0), v_3 = (0,0,1)$$
$$\sigma_3 \text{ is generated by } v_1 = (1,0,0), v_2 = (0,1,0), v_4 = (s, r-s, -r).$$

with $0 < s < r$ and $(s,r) = 1$.

Let X^+ be the threefold obtained by replacing σ_4, σ_3 with σ_1, σ_2 that

$$\sigma_1 \text{ is generated by } v_2, v_3, v_4$$
$$\sigma_2 \text{ is generated by } v_1, v_3, v_4.$$

The birational map $X \dashrightarrow X^+$ is a toric flip. One can verify that $B(X) = \{(b,r)\}$ and $B(X^+) = \{(b', s), (b-b', r-s)\}$ for some b, b' satisfying $b'r - bs = \pm 1$. Similarly,

$$B(X^+) \succ B(X)$$

is again a prime packing of baskets.

We have the following basic properties.

LEMMA 2.8. – *Let $B \succ B'$ be any packing between generalized baskets. Keep the same notation as above. Then:*

(1) $\Delta^n(B) \geq \Delta^n(B')$ *for all* $n \geq 2$*;*
(2) *the equality in (1) holds if and only if both* $\frac{b_1}{r_1}$ *and* $\frac{b_2}{r_2}$ *are in the closed interval* $[\frac{\delta}{n}, \frac{\delta+1}{n}]$ *for some* δ*;*
(3) $\sigma(B') = \sigma(B)$ *and* $\sigma'(B) = \sigma'(B') + \frac{(r_1 b_2 - r_2 b_1)^2}{r_1 r_2 (r_1+r_2)} \geq \sigma'(B')$*. Thus equality holds only when* $\frac{b_1}{r_1} = \frac{b_2}{r_2}$.

Proof. – First, if both $\frac{b_1}{r_1}$ and $\frac{b_2}{r_2}$ are in the closed interval $[\frac{\delta}{n}, \frac{\delta+1}{n}]$ for some δ, then a direct calculation shows $\Delta^n(B) = \Delta^n(B')$.

Suppose, for some $\delta > j$,

$$\frac{\delta+1}{n} > \frac{b_2}{r_2} \geq \frac{\delta}{n} \geq \frac{j+1}{n} > \frac{b_1}{r_1} \geq \frac{j}{n}$$

and $\frac{j_1+1}{n} > \frac{b_1+b_2}{r_1+r_2} \geq \frac{j_1}{n}$ for some $j_1 \in [j, \delta]$. Then

$$\begin{aligned}\Delta^n(b_1+b_2, r_1+r_2) &= j_1 n(b_1+b_2) - \frac{1}{2}(j_1^2+j_1)(r_1+r_2)\\ &= \Delta^n(b_2, r_2) + \Delta^n(b_1, r_1) + \nabla_2 + \nabla_1,\end{aligned}$$

where $\nabla_2 = (j_1-\delta)nb_2 + \frac{1}{2}(\delta^2+\delta-j_1^2-j_1)r_2$ and $\nabla_1 = (j_1-j)nb_1 + \frac{1}{2}(j^2+j-j_1^2-j_1)r_1$. Now since $nb_2 \geq \delta r_2$, one gets

$$\nabla_2 \leq \frac{1}{2}(\delta - j_1)(j_1+1-\delta)r_2.$$

When $j_1 = \delta$, $\nabla_2 = 0$; when $j_1 = \delta - 1$, $\nabla_2 = -nb_1 + \delta r_2 \leq 0$; when $j_1 < \delta - 1$, $\nabla_2 < 0$.

Similarly the relation $nb_1 < (j+1)r_1$ implies

$$\nabla_1 \leq \frac{1}{2}(j_1 - j)(j+1-j_1)r_1.$$

When $j_1 = j$, $\nabla_1 = 0$; when $j_1 = j+1$, $\nabla_1 = nb_1 - (j+1)r_1 < 0$; when $j_1 > j+1$, $\nabla_1 < 0$.

Thus in any case, we see $\Delta^n(B) \geq \Delta^n(B')$, which implies (1). Furthermore we see $\Delta^n(B) = \Delta^n(B')$ if, and only if, $\nabla_2 = \nabla_1 = 0$; if, and only if, $j_1 = j$ and $\delta = j_1 + 1 = j+1$. We have proved (2).

The inequality (3) is obtained by a direct calculation. □

COROLLARY 2.9. – *If* $B = \{m \times (b, r) \mid b \leq \frac{r}{2},\ b \text{ coprime to } r\}$ *and* $B' = \{(mb, mr)\}$ *for an integer* $m > 1$*, then*

(i) $\sigma(B') = \sigma(B)$*;* $\sigma'(B') = \sigma'(B)$*;*
(ii) $\Delta^n(B') = \Delta^n(B)$ *for any* $n > 0$.

Proof. – This can be obtained by the definition of σ and Lemma 2.8. □

REMARK 2.10. – The additive properties in Corollary 2.9 allow us to regard the generalized single basket $\{(mb, mr)\}$ as a basket $\{m \times (b, r)\}$.

Besides, a prime packing has the following property:

LEMMA 2.11. – *Let* $B = \{(b_1, r_1), (b_2, r_2)\} \succ \{(b_1 + b_2, r_1 + r_2)\} = B'$ *be a prime packing as in* 2.5*, i.e.* $b_1 r_2 - b_2 r_1 = 1$*. Then*

$$\Delta^{r_1+r_2}(b_1 + b_2, r_1 + r_2) = \Delta^{r_1+r_2}(b_1, r_1) + \Delta^{r_1+r_2}(b_2, r_2) - 1.$$

Proof. – When $b_1 r_2 - b_2 r_1 = 1$, since $r_1 > 1, r_2 > 1$, one has

$$\frac{b_1 + b_2 + 1}{r_1 + r_2} > \frac{b_1}{r_1} > \frac{b_1 + b_2}{r_1 + r_2} > \frac{b_2}{r_2} > \frac{b_1 + b_2 - 1}{r_1 + r_2}.$$

We set $n = r_1 + r_2$. A direct calculation gives the equality

$$\Delta^n(b_1 + b_2, r_1 + r_2) = \Delta^n(b_1, r_1) + \Delta^n(b_2, r_2) - 1.$$

□

2.12. – Initial basket and limiting process. Given a basket $B = \{(b_j, r_j) \mid b_j$ coprime to $r_j, b_j \leq \frac{r_j}{2}\}_{j\in J}$, we define a sequence of baskets $\{\mathscr{B}^{(n)}(B)\}$ as follows.

Take the set $S^{(0)} := \{\frac{1}{n}\}_{n\geq 2}$. For any element $B_j = (b_j, r_j) \in B$, we can find a unique $n > 0$ such that $\frac{1}{n} > \frac{b_j}{r_j} \geq \frac{1}{n+1}$. The element (b_j, r_j) can be regarded as finite step successive packings beginning from the basket $B_j^{(0)} := \{(nb_j + b_j - r_j) \times (1, n), (r_j - nb_j) \times (1, n + 1)\}$. Adding up those $B_j^{(0)}$, one obtains the basket $\mathscr{B}^{(0)}(B) = \{n_{1,2} \times (1, 2), n_{1,3} \times (1, 3), \cdots, n_{1,r} \times (1, r)\}$, called *the initial basket* of B. Clearly $\mathscr{B}^{(0)}(B) \succcurlyeq B$. Defined in this way, $\mathscr{B}^{(0)}(B)$ is uniquely determined by the given basket B.

We begin to construct other baskets $\{\mathscr{B}^{(n)}(B)\}$ for $n > 1$. Consider the sets $S^{(4)} = S^{(3)} = S^{(2)} = S^{(1)} = S^{(0)}$ and

$$S^{(5)} := S^{(0)} \cup \left\{\frac{2}{5}\right\}$$

and inductively, $S^{(n)} = S^{(n-1)} \cup \{\frac{i}{n}\}_{i=2,\cdots,\lfloor\frac{n}{2}\rfloor}$. Reordering elements in $S^{(n)}$ and writing $S^{(n)} = \{w_i^{(n)}\}_{i\in I}$ such that $w_i^{(n)} > w_{i+1}^{(n)}$ for all i, then we see that the interval $(0, \frac{1}{2}] = \cup_i [w_{i+1}^{(n)}, w_i^{(n)}]$. Note that $w_i^{(n)} = \frac{q_i}{p_i}$ with p_i coprime to q_i and $p_i \leq n$ unless $w_i^{(n)} = \frac{1}{m}$ for some $m > n$. First we prove the following:

Claim A

$u_1 v_2 - u_2 v_1 = 1$ *for any two endpoints of* $[w_{i+1}^{(n)}, w_i^{(n)}] = [\frac{v_1}{u_1}, \frac{v_2}{u_2}]$.

Proof. – We can prove this inductively. Suppose that this property holds for $S^{(n-1)}$. Now, for any $\frac{j}{n} \in S^{(n)} - S^{(n-1)}$, $\frac{j}{n} \in [w_{i+1}^{(n-1)}, w_i^{(n-1)}] = [\frac{q_1}{p_1}, \frac{q_2}{p_2}]$ for some i. Thus $\frac{q_1}{p_1} < \frac{j}{n} < \frac{q_2}{p_2}$. If $p_2 \geq n$, then $\frac{q_2}{p_2} = \frac{1}{m}$ and $\frac{q_1}{p_1} = \frac{1}{m+1}$ for some $m \geq n$ which contradicts to $\frac{j}{n} < \frac{q_2}{p_2}$. Therefore, we must have $p_2 < n$. Then we consider $\frac{j-q_2}{n-p_2}$ and it is easy to see that

$$\frac{q_1}{p_1} \leq \frac{j - q_2}{n - p_2} < \frac{j}{n} < \frac{q_2}{p_2}.$$

Clearly, $\frac{j-q_2}{n-p_2} \in S^{(n-1)}$ and hence $\frac{j-q_2}{n-p_2} = \frac{q_1}{p_1}$. It follows that $n = p_2 + \alpha p_1, j = q_2 + \alpha q_1$ for some integer $\alpha > 0$.

If $\alpha \geq 2$, then $\frac{q_1}{p_1} < \frac{q_2+(\alpha-1)q_1}{p_2+(\alpha-1)p_1} < \frac{q_2}{p_2}$, and $\frac{q_2+(\alpha-1)q_1}{p_2+(\alpha-1)p_1} \in S^{(n-1)}$, which is absurd. Thus $\alpha = 1$ and then $n = p_2 + p_1, j = q_2 + q_1$. It is then clear that $\frac{j}{n}$ is the only element of $S^{(n)}$

inside the interval $[\frac{q_1}{p_1}, \frac{q_2}{p_2}]$. Moreover, $jp_1 - nq_1 = 1, nq_2 - jp_2 = 1$. This completes the proof of the claim. $\square$

Now for an element $B_i = (b_i, r_i) \in B$, if $\frac{b_i}{r_i} \in S^{(n)}$, then we set $B_i^{(n)} := \{(b_i, r_i)\}$. If $\frac{b_i}{r_i} \notin S^{(n)}$, then $\frac{q_1}{p_1} < \frac{b_i}{r_i} < \frac{q_2}{p_2}$ for some interval $[\frac{q_1}{p_1}, \frac{q_2}{p_2}]$ due to $S^{(n)}$. In this situation, we can unpack (b_i, r_i) to $B_i^{(n)} := \{(r_i q_2 - b_i p_2) \times (q_1, p_1), (-r_i q_1 + b_i p_1) \times (q_2, p_2)\}$. Adding up those $B_i^{(n)}$, we get a new basket $\mathscr{B}^{(n)}(B)$. Clearly $\mathscr{B}^{(n)}(B)$ is uniquely determined according to our construction and $\mathscr{B}^{(n)}(B) \succcurlyeq B$ for all n.

Claim B

$\mathscr{B}^{(n-1)}(B) = \mathscr{B}^{(n-1)}(\mathscr{B}^{(n)}(B)) \succcurlyeq \mathscr{B}^{(n)}(B)$ *for all* $n \geq 1$.

Proof. – Since we have already seen $\mathscr{B}^{(n-1)}(\mathscr{B}^{(n)}(B)) \succcurlyeq \mathscr{B}^{(n)}(B)$ by definition, it suffices to show the first equality of the claim. By the definition of $\mathscr{B}^{(n)}$, we only need to verify the statement for each element $B_i = \{(b_i, r_i)\} \subset B$ and for $n \geq 5$.

If $\frac{b_i}{r_i} \in S^{(n-1)} \subset S^{(n)}$, then there is nothing to prove since the equality follows from the definition of $\mathscr{B}^{(n)}$ and $\mathscr{B}^{(n-1)}$.

If $\frac{b_i}{r_i} \in S^{(n)} - S^{(n-1)}$, then this is also clear since $\mathscr{B}^{(n)}(B_i) = B_i$.

Suppose finally that $\frac{b_i}{r_i} \notin S^{(n)}$. Then $\frac{q_1}{p_1} < \frac{b_i}{r_i} < \frac{q_2}{p_2}$ for some $\frac{q_1}{p_1} = w_{i+1}^{(n)}$ and $\frac{q_2}{p_2} = w_i^{(n)}$.

Subcase (i). – If both $\frac{q_1}{p_1}$ and $\frac{q_2}{p_2}$ are in $S^{(n)} - S^{(n-1)}$, then $p_1 = p_2 = n$ and hence $p_1 q_2 - p_2 q_1 \neq 1$, a contradiction to Claim A.

Subcase (ii). – If both $\frac{q_1}{p_1}$ and $\frac{q_2}{p_2}$ are in $S^{(n-1)}$, then by definition

$$\mathscr{B}^{(n-1)}(B_i) = \mathscr{B}^{(n)}(B_i) = \mathscr{B}^{(n-1)}(\mathscr{B}^{(n)}(B_i)).$$

Subcase (iii). – We are left to consider the situation that one of $\frac{q_1}{p_1}$, $\frac{q_2}{p_2}$ is in $S^{(n-1)}$, but another one is in $S^{(n)} - S^{(n-1)}$. Let us assume, for example, $\frac{q_1}{p_1} = w_{j+1}^{(n-1)} \in S^{(n-1)}$. Then $\frac{q_2}{p_2} < w_j^{(n-1)} = \frac{q}{p} \in S^{(n-1)}$. The proof for the other case is similar. Notice that by the proof of Claim A, we have $q_2 = q_1 + q, p_2 = p_1 + p$. By definition,

$$\begin{aligned}\mathscr{B}^{(n)}(B_i) &= \{(r_i q_2 - b_i p_2) \times (q_1, p_1), (-r_i q_1 + b_i p_1) \times (q_2, p_2)\},\\ \mathscr{B}^{(n-1)}(B_i) &= \{(r_i q - b_i p) \times (q_1, p_1), (-r_i q_1 + b_i p_1) \times (q, p)\}.\end{aligned}$$

Since $\mathscr{B}^{(n-1)}(q_2, p_2) = \{(q_1, p_1), (q, p)\}$, we get the following by the direct computation:

$$\begin{aligned}\mathscr{B}^{(n-1)}(\mathscr{B}^{(n)}(B_i)) &= \{(r_i q_2 - b_i p_2) \times (q_1, p_1)\} \cup \{(-r_i q_1 + b_i p_1) \times (q_1, p_1),\\ &\qquad (-r_i q_1 + b_i p_1) \times (q, p)\}\\ &= \{(r_i q - b_i p) \times (q_1, p_1), (-r_i q_1 + b_i p_1) \times (q, p)\}.\end{aligned}$$

So we can see that $\mathscr{B}^{(n-1)}(B_i) = \mathscr{B}^{(n-1)}(\mathscr{B}^{(n)}(B_i))$. We are done. $\square$

By Claim B, we have a sequence $\{\mathscr{B}^{(n)}(B)\}$ of baskets with the following relation:

$$\mathscr{B}^{(0)}(B) = \cdots = \mathscr{B}^{(4)}(B) \succcurlyeq \mathscr{B}^{(5)}(B) \succcurlyeq \cdots \succcurlyeq \mathscr{B}^{(n)}(B) \succcurlyeq \cdots \succcurlyeq B. \tag{2.3}$$

Clearly, by definition, $B = \mathscr{B}^{(w)}(B)$ for some $w \gg 0$ for a given finite basket B. Thus, in some sense, B can be realized as the limit of the sequence $\{\mathscr{B}^{(n)}(B)\}$, which is called *the canonical sequence of* B.

Another direct consequence of Claim B is the following property:

$$\mathscr{B}^{(i)}(\mathscr{B}^{(j)}(B)) = \mathscr{B}^{(i)}(B) \tag{2.4}$$

for $i \le j$.

2.13. – The quantity $\epsilon_n(B)$. Now let us consider the step $\mathscr{B}^{(n-1)}(B) \succ \mathscr{B}^{(n)}(B)$. For an element $w \in S^{(n)}$, let $m(w)$ be the number of baskets (b,r) in $\mathscr{B}^{(n)}(B)$ with b coprime to r and $\frac{b}{r} = w$. Thus we can write $\mathscr{B}^{(n)}(B) = \{m(w) \times (b,r)\}_{w=\frac{b}{r} \in S^{(n)}}$.

Suppose that $S^{(n)} - S^{(n-1)} = \{\frac{j_s}{n}\}_{s=1,\cdots,t}$. We have $w_{i_s}^{(n-1)} = \frac{q_{i_s}}{p_{i_s}} > \frac{j_s}{n} > w_{i_s+1}^{(n-1)} = \frac{q_{i_s+1}}{p_{i_s+1}}$ for some i_s. We remark that by the proof of Claim A, $j_s = q_{i_s} + q_{i_s+1}$, $n = p_{i_s} + p_{i_s+1}$. Since $\mathscr{B}^{(n-1)}(B) = \mathscr{B}^{(n-1)}(\mathscr{B}^{(n)}(B))$ by Claim B, we may write

$$\mathscr{B}^{(n)}(B) = \{m(w) \times (b,r)\}_{w=\frac{b}{r} \in S^{(n-1)}} \cup \{m(\frac{j_s}{n}) \times (j_s, n)\}_{\frac{j_s}{n}}.$$

Then

$$\begin{aligned}\mathscr{B}^{(n-1)}(B) = \ & \{m(w) \times (b,r)\}_{w=\frac{b}{r} \in S^{(n-1)}} \cup \{m(\frac{j_s}{n}) \times (q_{i_s}, p_{i_s}), \\ & m(\frac{j_s}{n}) \times (q_{i_s+1}, p_{i_s+1})\}_{\frac{j_s}{n}}.\end{aligned}$$

We define $\epsilon_n(B) := \sum_{s=1}^{t} m(\frac{j_s}{n})$, which is the number of type (j_s, n) single baskets with $\frac{j_s}{n} \in S^{(n)} - S^{(n-1)}$. In other words, $\epsilon_n(B)$ counts the number of elements $\{(j_s, n)\}$ contained in $\mathscr{B}^{(n)}(B)$ with $(j_s, n) = 1$ and $j_s > 1$. By Claim A, we conclude that $\mathscr{B}^{(n-1)}(B) \succcurlyeq \mathscr{B}^{(n)}(B)$ consists of $\epsilon_n(B)$ prime packings of level n. This is going to be an important quantity in our arguments.

Definition 2.14. – Given a basket B. The sequence defined as in (2.3) is called the canonical sequence of prime unpackings of B, or canonical sequence of B for short.

2.15. – Notation. When no confusion is likely, we will simply write $B^{(n)}$ for $\mathscr{B}^{(n)}(B)$.

Lemma 2.16. – *For the canonical sequence* $\{B^{(n)}\}$ *, the following statements hold.*

(i) $\Delta^j(B^{(0)}) = \Delta^j(B)$ *for* $j = 3, 4$*;*
(ii) $\Delta^j(B^{(n-1)}) = \Delta^j(B^{(n)})$ *for all* $j < n$*;*
(iii) $\Delta^n(B^{(n-1)}) = \Delta^n(B^{(n)}) + \epsilon_n(B)$.
(iv) $\Delta^n(B^{(n)}) = \Delta^n(B)$.

Proof. – From $B^{(0)}$ to B, via $B^{(n)}$, the whole process can be realized through a composition of finite number of prime packings. Each step is of the form $\{(q_1,p_1),(q_2,p_2)\} \succ \{(q_1+q_2,p_1+p_2)\}$. Notice that either $\frac{q_1}{p_1}, \frac{q_2}{p_2} \le \frac{1}{3}$ or $\frac{q_1}{p_1}, \frac{q_2}{p_2} \ge \frac{1}{3}$. By Lemma 2.8(2), one gets $\Delta^3(B^{(0)}) = \Delta^3(B)$. The proof for Δ^4 is similar.

Now we consider the typical step $B^{(n-1)} \succ B^{(n)}$. By Lemma 2.11 and a direct computation, one has:

$$\begin{aligned}
&\Delta^n(B^{(n-1)}) - \Delta^n(B^{(n)})\\
&\quad = \textstyle\sum_{s=1}^t m(\frac{j_s}{n})(\Delta^n(q_{i_s},p_{i_s} + \Delta^n(q_{i_s+1},p_{i_s+1}) - \Delta^n(j_s,n))\\
&\quad = \textstyle\sum_{s=1}^t m(\frac{j_s}{n})(\Delta^n(q_{i_s},p_{i_s}) + \Delta^n(q_{i_s+1},p_{i_s+1}) - \Delta^n(q_{i_s}+q_{i_s+1},p_{i_s}+p_{i_s+1}))\\
&\quad = \textstyle\sum_{s=1}^t m(\frac{j_s}{n})\\
&\quad = \epsilon_n(B),
\end{aligned}$$

where one notices $n = p_{i_s} + p_{i_s+1}$.

Finally, for any $j < n$, suppose that $\frac{k+1}{j} \ge \frac{q_{i_s}}{p_{i_s}} = w_{i_s}^{(n-1)} > \frac{k}{j}$ for some k. Then $\frac{k+1}{j} \in S^{(n-1)}$ by definition. Thus $\frac{q_{i_s+1}}{p_{i_s+1}} = w_{i_s+1}^{(n-1)} \ge \frac{k}{j}$. By Lemma 2.8, we have

$$\Delta^j(q_{i_s},p_{i_s}) + \Delta^j(q_{i_s+1},p_{i_s+1}) = \Delta^j(q_{i_s}+q_{i_s+1},p_{i_s}+p_{i_s+1}).$$

The last statement is due to (ii) and the fact that $B = B^{(n)}$ for a sufficiently large n. This completes the proof. □

Let us go back to investigate the canonical sequence (2.3)

$$B^{(0)} \succcurlyeq B^{(5)} \succcurlyeq \ldots \succcurlyeq B^{(n)} \succcurlyeq \ldots \succcurlyeq B.$$

We see that $\Delta^j(B^{(n)}) = \Delta^j(B)$ for all $j < n$. Thus we can informally view $B^{(n)}$ as an n-th order approximation of B. Also each approximation step $B^{(n-1)} \succcurlyeq B^{(n)}$ is nothing but the composition of prime packings of ϵ_n pairs of baskets of type (b,n) with b coprime to n, $b \le \frac{r}{2}$ and $b > 1$.

3. Formal baskets

In this section, we are going to introduce the notion of formal baskets. A formal basket is a basket together with a choice of K^3 and χ. The purpose of this section is to classify all formal baskets with a given initial sequence $(\chi_1,\ldots,\chi_k)$.

Given a 3-fold X with canonical singularities, there is an associated basket $B := \mathscr{B}(X)$ [(1)] according to Reid.

3.1. – Euler characteristic. Let us recall Reid's Riemann-Roch formula ([23, Page 143]) for a $\mathbb{Q}$-factorial terminal 3-fold X: for all $m > 1$,

$$\chi(X,\mathcal{O}_X(mK_X)) = \frac{1}{12}m(m-1)(2m-1)K_X^3 - (2m-1)\chi(\mathcal{O}_X) + l(m) \tag{3.1}$$

[(1)] Iano-Fletcher [12] has shown that Reid's virtual basket $\mathscr{B}(X)$ is uniquely determined by X.

where the correction term $l(m)$ can be computed as:

$$l(m) := \sum_{Q\in\mathscr{B}(X)} l_Q(m) := \sum_{Q\in\mathscr{B}(X)} \sum_{j=1}^{m-1} \frac{\overline{jb_Q}(r_Q - \overline{jb_Q})}{2r_Q}$$

where the sum $\sum_Q$ runs through all single baskets Q in $\mathscr{B}(X)$ with type $\frac{1}{r_Q}(1,-1,b_Q)$ and $\overline{jb_Q}$ means the smallest residue of jb_Q mod r_Q.

For brevity, $\chi(X, \mathcal{O}_X(mK_X))$ is usually denoted by $\chi_m(X)$ or simply χ_m.

We are going to analyze the above formula and Reid's virtual basket $\mathscr{B}(X)$.

3.2. – Euler characteristic in terms of baskets. Take $B = \mathscr{B}(X)$ and set $\Delta := \Delta(B)$, $\sigma := \sigma(B)$, $\sigma' := \sigma'(B)$ (cf. 2.2). We can now rewrite Reid's Riemann-Roch formula as the following:

$$\begin{cases} \chi_2 & = \frac{1}{2}(K_X^3 - \sigma') + \frac{1}{2}\sigma - 3\chi, \\ \chi_3 - \chi_2 & = \frac{4}{2}(K_X^3 - \sigma') + \frac{2}{2}\sigma - 2\chi, \\ \chi_{m+1} - \chi_m & = \frac{m^2}{2}(K_X^3 - \sigma') + \frac{m}{2}\sigma - 2\chi + \Delta^m, \text{ for } m \geq 3. \end{cases} \tag{3.2}$$

Notice that, by the equalities (3.2), all χ_m are determined by $\sigma, \sigma' - K^3, \chi, \Delta^j$ for all $j < m$. These, in turn, are determined by B, χ and χ_2 by virtue of the first equality in (3.2). This leads us to consider a more general setting.

Definition 3.3. – Assume that B is a basket, $\tilde{\chi}$ and $\tilde{\chi}_2$ are integers. We call the triple $\mathbf{B} := (B, \tilde{\chi}, \tilde{\chi}_2)$ a *formal basket*.

We can define the Euler characteristic and K^3 of a formal basket formally by the Riemann-Roch formula. First we define

$$\begin{cases} \chi_2(\mathbf{B}) := \tilde{\chi}_2, \\ \chi_3(\mathbf{B}) := -\sigma(B) + 10\tilde{\chi} + 5\tilde{\chi}_2 \end{cases}$$

and the volume

$$\begin{aligned} K^3(\mathbf{B}) &:= \sigma'(B) - 4\tilde{\chi} - 3\tilde{\chi}_2 + \chi_3(\mathbf{B}) \\ &= -\sigma + \sigma' + 6\tilde{\chi} + 2\tilde{\chi}_2. \end{aligned} \tag{3.3}$$

For $m \geq 4$, the Euler characteristic $\chi_m(\mathbf{B})$ is defined inductively by

$$\chi_{m+1}(\mathbf{B}) - \chi_m(\mathbf{B}) := \frac{m^2}{2}(K^3(\mathbf{B}) - \sigma'(B)) + \frac{m}{2}\sigma(B) - 2\tilde{\chi} + \Delta^m(B). \tag{3.4}$$

Clearly, by definition, $\chi_m(\mathbf{B})$ is an integer for all $m \geq 4$ because $K^3(\mathbf{B}) - \sigma'(B) = -4\tilde{\chi} - 3\tilde{\chi}_2 + \chi_3(\mathbf{B})$ and $\sigma = 10\tilde{\chi} + 5\tilde{\chi}_2 - \chi_3(\mathbf{B})$ have the same parity.

Given a $\mathbb{Q}$-factorial canonical 3-fold X, one can associate to X a triple $\mathbf{B}(X) := (B, \tilde{\chi}, \tilde{\chi}_2)$ where $B = \mathscr{B}(X)$, $\tilde{\chi} = \chi(\mathcal{O}_X)$ and $\tilde{\chi}_2 = \chi_2(X)$. It is clear that such a triple is a formal basket. The Euler characteristic and K^3 of the formal basket $\mathbf{B}(X)$ are nothing but the Euler characteristic and K^3 of the variety X.

3.4. – Notations. For simplicity, we denote $\chi_m(\mathbf{B})$ by $\tilde{\chi}_m$ for all $m \geq 2$. Also denote $K^3(\mathbf{B})$ by $\tilde{K}^3$, $\sigma = \sigma(B)$, $\sigma' = \sigma'(B)$ and $\Delta^m = \Delta^m(B)$.

DEFINITION 3.5. – Let $\mathbf{B} := (B, \tilde{\chi}, \tilde{\chi}_2)$ and $\mathbf{B}' := (B', \tilde{\chi}, \tilde{\chi}_2)$ be two formal baskets.

(1) We say that $\mathbf{B}'$ is a packing of $\mathbf{B}$ (written as $\mathbf{B} \succ \mathbf{B}'$) if $B \succ B'$. Clearly "packing" between formal baskets gives a partial ordering.

(2) A formal basket $\mathbf{B}$ is called *positive* if $K^3(\mathbf{B}) > 0$.

(3) A formal basket $\mathbf{B}$ is said to be minimal positive if it is positive and minimal with regard to packing relation.

By definition and Lemma 2.8(1), we immediately get the following:

LEMMA 3.6. – *Assume* $\mathbf{B} := (B, \tilde{\chi}, \tilde{\chi}_2) \succ \mathbf{B}' := (B', \tilde{\chi}, \tilde{\chi}_2)$. *Then*

(1) $K^3(\mathbf{B}) \geq K^3(\mathbf{B}')$;

(2) $\chi_m(\mathbf{B}) \geq \chi_m(\mathbf{B}')$ *for all* $m \geq 2$.

In what follows, we would like to classify all baskets with a given initial sequence $(\tilde{\chi}, \tilde{\chi}_2, \tilde{\chi}_3, \cdots, \tilde{\chi}_m)$.

First of all, by the definition of $\tilde{K}^3$ and $\tilde{\chi}_m$, we get:

$$
\tag{3.5}
\begin{aligned}
\tau := \sigma' - \tilde{K}^3 &= 4\tilde{\chi} + 3\tilde{\chi}_2 - \tilde{\chi}_3,\\
\sigma &= 10\tilde{\chi} + 5\tilde{\chi}_2 - \tilde{\chi}_3\\
\Delta^3 &= 5\tilde{\chi} + 6\tilde{\chi}_2 - 4\tilde{\chi}_3 + \tilde{\chi}_4\\
\Delta^4 &= 14\tilde{\chi} + 14\tilde{\chi}_2 - 6\tilde{\chi}_3 - \tilde{\chi}_4 + \tilde{\chi}_5\\
\Delta^5 &= 27\tilde{\chi} + 25\tilde{\chi}_2 - 10\tilde{\chi}_3 - \tilde{\chi}_5 + \tilde{\chi}_6\\
\Delta^6 &= 44\tilde{\chi} + 39\tilde{\chi}_2 - 15\tilde{\chi}_3 - \tilde{\chi}_6 + \tilde{\chi}_7\\
\Delta^7 &= 65\tilde{\chi} + 56\tilde{\chi}_2 - 21\tilde{\chi}_3 - \tilde{\chi}_7 + \tilde{\chi}_8\\
\Delta^8 &= 90\tilde{\chi} + 76\tilde{\chi}_2 - 28\tilde{\chi}_3 - \tilde{\chi}_8 + \tilde{\chi}_9\\
\Delta^9 &= 119\tilde{\chi} + 99\tilde{\chi}_2 - 36\tilde{\chi}_3 - \tilde{\chi}_9 + \tilde{\chi}_{10}\\
\Delta^{10} &= 152\tilde{\chi} + 125\tilde{\chi}_2 - 45\tilde{\chi}_3 - \tilde{\chi}_{10} + \tilde{\chi}_{11}\\
\Delta^{11} &= 189\tilde{\chi} + 154\tilde{\chi}_2 - 55\tilde{\chi}_3 - \tilde{\chi}_{11} + \tilde{\chi}_{12}\\
\Delta^{12} &= 230\tilde{\chi} + 186\tilde{\chi}_2 - 66\tilde{\chi}_3 - \tilde{\chi}_{12} + \tilde{\chi}_{13}.
\end{aligned}
$$

Recall that $B^{(0)} = \{n^0_{1,2} \times (1,2), \cdots, n^0_{1,r} \times (1,r)\}$ is the initial basket of B. Then by Lemma 2.16 and the definition of $\sigma(B)$, we have

$$
\begin{aligned}
\sigma(B) = \sigma(B^{(0)}) &= \sum n^0_{1,r},\\
\Delta^3(B) = \Delta^3(B^{(0)}) &= n^0_{1,2}\\
\Delta^4(B) = \Delta^4(B^{(0)}) &= 2n^0_{1,2} + n^0_{1,3}.
\end{aligned}
$$

Therefore, the initial basket has the coefficients:

$$
\tag{3.6}
B^{(0)} \begin{cases}
n^0_{1,2} = 5\tilde{\chi} + 6\tilde{\chi}_2 - 4\tilde{\chi}_3 + \tilde{\chi}_4\\
n^0_{1,3} = 4\tilde{\chi} + 2\tilde{\chi}_2 + 2\tilde{\chi}_3 - 3\tilde{\chi}_4 + \tilde{\chi}_5\\
n^0_{1,4} = \tilde{\chi} - 3\tilde{\chi}_2 + \tilde{\chi}_3 + 2\tilde{\chi}_4 - \tilde{\chi}_5 - \sum_{r\geq 5} n^0_{1,r}\\
n^0_{1,r}, r \geq 5.
\end{cases}
$$

By Lemma 2.16, we see that

$$\begin{aligned}\epsilon_5 &:= \Delta^5(B^{(0)}) - \Delta^5(B) = 4n^0_{1,2} + 2n^0_{1,3} + n^0_{1,4} - \Delta^5(B)\\ &= 2\tilde{\chi} - \tilde{\chi}_3 + 2\tilde{\chi}_5 - \tilde{\chi}_6 - \sigma_5 \text{ where}\\ \sigma_5 &:= \textstyle\sum_{r\geq 5} n^0_{1,r}.\end{aligned} \tag{3.7}$$

Thus we can write

$$B^{(5)} = \{n^5_{1,2}\times(1,2), n^5_{2,5}\times(2,5), n^5_{1,3}\times(1,3), n^5_{1,4}\times(1,4), n^5_{1,5}\times(1,5), \ldots\}$$

with

$$B^{(5)}\begin{cases} n^5_{1,2} = 3\tilde{\chi} + 6\tilde{\chi}_2 - 3\tilde{\chi}_3 + \tilde{\chi}_4 - 2\tilde{\chi}_5 + \tilde{\chi}_6 + \sigma_5,\\ n^5_{2,5} = 2\tilde{\chi} - \tilde{\chi}_3 + 2\tilde{\chi}_5 - \tilde{\chi}_6 - \sigma_5\\ n^5_{1,3} = 2\tilde{\chi} + 2\tilde{\chi}_2 + 3\tilde{\chi}_3 - 3\tilde{\chi}_4 - \tilde{\chi}_5 + \tilde{\chi}_6 + \sigma_5\\ n^5_{1,4} = \tilde{\chi} - 3\tilde{\chi}_2 + \tilde{\chi}_3 + 2\tilde{\chi}_4 - \tilde{\chi}_5 - \sigma_5\\ n^5_{1,r} = n^0_{1,r}, r\geq 5,\end{cases} \tag{3.8}$$

noting that this is obtained from $B^{(0)}$ by taking ϵ_5 prime packings of type $\{(1,2),(1,3)\} \succ \{(2,5)\}$.

Clearly, $B^{(5)} = B^{(6)}$ by our construction. Thus by Lemma 2.16 we have $\Delta^6(B^{(5)}) = \Delta^6(B^{(6)}) = \Delta^6(B)$. Computation shows that

$$\begin{aligned}\Delta^6(B^{(5)}) &= 6n^5_{1,2} + 9n^5_{2,5} + 3n^5_{1,3} + 2n^5_{1,4} + n^5_{1,5}\\ &= 44\tilde{\chi} + 36\tilde{\chi}_2 - 16\tilde{\chi}_3 + \tilde{\chi}_4 + \tilde{\chi}_5 - \epsilon,\end{aligned}$$

where

$$\epsilon := n^0_{1,5} + 2\sum_{r\geq 6} n^0_{1,r} = 2\sigma_5 - n^0_{1,5} \geq 0. \tag{3.9}$$

Comparing this with (3.5), we see that

$$\epsilon_6 = -3\tilde{\chi}_2 - \tilde{\chi}_3 + \tilde{\chi}_4 + \tilde{\chi}_5 + \tilde{\chi}_6 - \tilde{\chi}_7 - \epsilon = 0. \tag{3.10}$$

Next, by similar computation, we get

$$\begin{aligned}\epsilon_7 :&= \Delta^7(B^{(6)}) - \Delta^7(B) = \Delta^7(B^{(5)}) - \Delta^7(B)\\ &= 9n^5_{1,2} + 13n^5_{2,5} + 5n^5_{1,3} + 3n^5_{1,4} + 2n^5_{1,5} + n^5_{1,6} - \Delta^7(B)\\ &= \tilde{\chi} - \tilde{\chi}_2 - \tilde{\chi}_3 + \tilde{\chi}_6 + \tilde{\chi}_7 - \tilde{\chi}_8 - 2\sigma_5 + 2n^0_{1,5} + n^0_{1,6}.\end{aligned} \tag{3.11}$$

Since $S^{(7)} - S^{(6)} = \{\frac{2}{7}, \frac{3}{7}\}$, there are two ways of prime packings into type $(b,7)$ baskets. Let $\eta \geq 0$ be the number of prime packings of type $\{(1,3),(1,4)\} \succ \{(2,7)\}$. Then $\epsilon_7 - \eta \geq 0$ is the number of prime packings of type $\{(1,2),(2,5)\} \succ \{(3,7)\}$. Thus we can write

$B^{(7)} = \{n^7_{b,r} \times (b,r)\}_{\frac{b}{r}\in S^{(7)}}$ with

$$(3.12)\qquad B^{(7)} \begin{cases} n^7_{1,2} = 2\tilde{\chi} + 7\tilde{\chi}_2 - 2\tilde{\chi}_3 + \tilde{\chi}_4 - 2\tilde{\chi}_5 - \tilde{\chi}_7 + \tilde{\chi}_8 + 3\sigma_5 - 2n^0_{1,5} - n^0_{1,6} + \eta \\ n^7_{3,7} = \tilde{\chi} - \tilde{\chi}_2 - \tilde{\chi}_3 + \tilde{\chi}_6 + \tilde{\chi}_7 - \tilde{\chi}_8 - 2\sigma_5 + 2n^0_{1,5} + n^0_{1,6} - \eta \\ n^7_{2,5} = \tilde{\chi} + \tilde{\chi}_2 + 2\tilde{\chi}_5 - 2\tilde{\chi}_6 - \tilde{\chi}_7 + \tilde{\chi}_8 + \sigma_5 - 2n^0_{1,5} - n^0_{1,6} + \eta \\ n^7_{1,3} = 2\tilde{\chi} + 2\tilde{\chi}_2 + 3\tilde{\chi}_3 - 3\tilde{\chi}_4 - \tilde{\chi}_5 + \tilde{\chi}_6 + \sigma_5 - \eta \\ n^7_{2,7} = \eta \\ n^7_{1,4} = \tilde{\chi} - 3\tilde{\chi}_2 + \tilde{\chi}_3 + 2\tilde{\chi}_4 - \tilde{\chi}_5 - \sigma_5 - \eta \\ n^7_{1,r} = n^0_{1,r}, r \geq 5. \end{cases}$$

From $B^{(7)}$, we can compute ϵ_8 and then $B^{(8)}$, and inductively $B^{(n)}$ for all $n \geq 9$. But notice that one can even compute ϵ_9, ϵ_{10} and ϵ_{12} directly from $B^{(7)}$, thanks to Lemma 2.8.

To see this, let us consider $\epsilon_9 := \Delta^9(B^{(8)}) - \Delta^9(B)$ for example. Note that $B^{(7)} \succ B^{(8)}$ is obtained by some prime packings into $\{(3,8)\}$. Every such packing, which is $\{(2,5),(1,3)\} \succ \{(3,8)\}$, happens inside a closed interval $[\frac{3}{9}, \frac{4}{9}]$. Thus by Lemma 2.8(2), $\Delta^9(B^{(8)}) = \Delta^9(B^{(7)})$ and hence

$$\epsilon_9 := \Delta^9(B^{(8)}) - \Delta^9(B) = \Delta^9(B^{(7)}) - \Delta^9(B).$$

Similarly we can see that $\Delta^{10}(B^{(9)}) = \Delta^{10}(B^{(7)})$ and $\Delta^{12}(B^{(10)}) = \Delta^{12}(B^{(7)})$. Unfortunately, $\Delta^{11}(B^{(10)}) \neq \Delta^{11}(B^{(7)})$.

In summary, we have the following by direct calculations:

$$\begin{aligned} \Delta^8(B^{(7)}) &= 12n^7_{1,2} + 30n^7_{3,7} + 18n^7_{2,5} + 7n^7_{1,3} + 11n^7_{2,7} + 4n^7_{1,4} + 3n^7_{1,5} + 2n^7_{1,6} + n^7_{1,7} \\ &= 90\tilde{\chi} + 74\tilde{\chi}_2 - 29\tilde{\chi}_3 - \tilde{\chi}_4 + \tilde{\chi}_5 + \tilde{\chi}_6 - 3\sigma_5 + 3n^0_{1,5} + 2n^0_{1,6} + n^0_{1,7}; \\ \Delta^9(B^{(8)}) &= \Delta^9(B^{(7)}) \\ &= 16n^7_{1,2} + 39n^7_{3,7} + 24n^7_{2,5} + 9n^7_{1,3} + 15n^7_{2,7} + 6n^7_{1,4} + 4n^7_{1,5} + 3n^7_{1,6} + 2n^7_{1,7} + n^7_{1,8} \\ &= 119\tilde{\chi} + 97\tilde{\chi}_2 - 38\tilde{\chi}_3 + \tilde{\chi}_4 + \tilde{\chi}_5 - \tilde{\chi}_7 + \tilde{\chi}_8 - 3\sigma_5 + \eta \\ &\quad +2n^0_{1,5} + 2n^0_{1,6} + 2n^0_{1,7} + n^0_{1,8}; \\ \Delta^{10}(B^{(9)}) &= \Delta^{10}(B^{(8)}) = \Delta^{10}(B^{(7)}) \\ &= 20n^7_{1,2} + 50n^7_{3,7} + 30n^7_{2,5} + 12n^7_{1,3} + 19n^7_{2,7} + 8n^7_{1,4} \\ &\quad +5n^7_{1,5} + 4n^7_{1,6} + 3n^7_{1,7} + 2n^7_{1,8} + n^7_{1,9} \\ &= 152\tilde{\chi} + 120\tilde{\chi}_2 - 46\tilde{\chi}_3 + 2\tilde{\chi}_6 - 6\sigma_5 - \eta \\ &\quad +5n^0_{1,5} + 4n^0_{1,6} + 3n^0_{1,7} + 2n^0_{1,8} + n^0_{1,9}; \\ \Delta^{12}(B^{(11)}) &= \Delta^{12}(B^{(10)}) = \cdots = \Delta^{12}(B^{(7)}) \\ &= 30n^7_{1,2} + 75n^7_{3,7} + 46n^7_{2,5} + 18n^7_{1,3} + 30n^7_{2,7} + 12n^7_{1,4} \\ &\quad +9n^7_{1,5} + 6n^7_{1,6} + 5n^7_{1,7} + 4n^7_{1,8} + 3n^7_{1,9} + 2n^7_{1,10} + n^7_{1,11} \\ &= 229\tilde{\chi} + 181\tilde{\chi}_2 - 69\tilde{\chi}_3 + 2\tilde{\chi}_5 + \tilde{\chi}_6 - \tilde{\chi}_7 + \tilde{\chi}_8 - 8\sigma_5 + \eta \\ &\quad +7n^0_{1,5} + 5n^0_{1,6} + 5n^0_{1,7} + 4n^0_{1,8} + 3n^0_{1,9} + 2n^0_{1,10} + n^0_{1,11}. \end{aligned}$$

We thus have:

$$
\begin{aligned}
\epsilon_8 = {} & -2\tilde{\chi}_2 - \tilde{\chi}_3 - \tilde{\chi}_4 + \tilde{\chi}_5 + \tilde{\chi}_6 + \tilde{\chi}_8 - \tilde{\chi}_9 - 3\sigma_5 \\
& +3n^0_{1,5} + 2n^0_{1,6} + n^0_{1,7}; \\
\epsilon_9 = {} & -2\tilde{\chi}_2 - 2\tilde{\chi}_3 + \tilde{\chi}_4 + \tilde{\chi}_5 - \tilde{\chi}_7 + \tilde{\chi}_8 + \tilde{\chi}_9 - \tilde{\chi}_{10} - 3\sigma_5 + \eta \\
& +2n^0_{1,5} + 2n^0_{1,6} + 2n^0_{1,7} + n^0_{1,8}; \\
\epsilon_{10} = {} & -5\tilde{\chi}_2 - \tilde{\chi}_3 + 2\tilde{\chi}_6 + \tilde{\chi}_{10} - \tilde{\chi}_{11} - 6\sigma_5 - \eta \\
& +5n^0_{1,5} + 4n^0_{1,6} + 3n^0_{1,7} + 2n^0_{1,8} + n^0_{1,9}; \\
\epsilon_{12} = {} & -\tilde{\chi} - 5\tilde{\chi}_2 - 3\tilde{\chi}_3 + 2\tilde{\chi}_5 + \tilde{\chi}_6 - \tilde{\chi}_7 + \tilde{\chi}_8 + \tilde{\chi}_{12} - \tilde{\chi}_{13} - 8\sigma_5 + \eta \\
& +7n^0_{1,5} + 5n^0_{1,6} + 5n^0_{1,7} + 4n^0_{1,8} + 3n^0_{1,9} + 2n^0_{1,10} + n^0_{1,11}.
\end{aligned}
\tag{3.13}
$$

Since both ϵ_{10} and ϵ_{12} are non-negative, we have $\epsilon_{10} + \epsilon_{12} \geq 0$. This gives rise to:

$$
2\tilde{\chi}_5 + 3\tilde{\chi}_6 + \tilde{\chi}_8 + \tilde{\chi}_{10} + \tilde{\chi}_{12} \geq \tilde{\chi} + 10\tilde{\chi}_2 + 4\tilde{\chi}_3 + \tilde{\chi}_7 + \tilde{\chi}_{11} + \tilde{\chi}_{13} + R,
\tag{3.14}
$$

where

$$
\begin{aligned}
R := {} & 14\sigma_5 - 12n^0_{1,5} - 9n^0_{1,6} - 8n^0_{1,7} - 6n^0_{1,8} - 4n^0_{1,9} - 2n^0_{1,10} - n^0_{1,11} \\
= {} & 2n^0_{1,5} + 5n^0_{1,6} + 6n^0_{1,7} + 8n^0_{1,8} + 10n^0_{1,9} + 12n^0_{1,10} + 13n^0_{1,11} + 14\sum_{r\geq 12} n^0_{1,r}.
\end{aligned}
$$

REMARK 3.7. – By definition, $\epsilon_n \geq 0$. This gives rise to various new inequalities among Euler characteristics. For example, $\epsilon_5 \geq 0$ (cf. 3.7) gives

$$2\tilde{\chi} - \tilde{\chi}_3 + 2\tilde{\chi}_5 - \tilde{\chi}_6 \geq 0.$$

In particular, for a $\mathbb{Q}$-factorial threefold X with canonical singularities, one has $2\chi(X) - \chi_3(X) + 2\chi_5(X) - \chi_6(X) \geq 0$.

Among those we have presented above, the equation (3.10) and the inequality (3.14) will play the most important roles in the context.

In practice, we will frequently end up with situations (see Lemma 4.8 and the proof of Theorem 4.12) satisfying the following assumption and then our computation will be comparatively simpler.

3.8. – Assumption. $\tilde{\chi}_2 = 0$ and $n^0_{1,r} = 0$ for all $r \geq 6$.

Under Assumption 3.8, we list our datum in details as follows. First,

$$\epsilon_7 = \tilde{\chi} - \tilde{\chi}_3 + \tilde{\chi}_6 + \tilde{\chi}_7 - \tilde{\chi}_8$$

and $B^{(7)} = \{n^7_{b,r} \times (b,r)\}_{\frac{b}{r}\in S^{(7)}}$ has coefficients:

$$B^{(7)} \begin{cases} n^7_{1,2} = 2\tilde{\chi} - 2\tilde{\chi}_3 + \tilde{\chi}_4 - 2\tilde{\chi}_5 - \tilde{\chi}_7 + \tilde{\chi}_8 + n^0_{1,5} + \eta \\ n^7_{3,7} = \tilde{\chi} - \tilde{\chi}_3 + \tilde{\chi}_6 + \tilde{\chi}_7 - \tilde{\chi}_8 - \eta \\ n^7_{2,5} = \tilde{\chi} + 2\tilde{\chi}_5 - 2\tilde{\chi}_6 - \tilde{\chi}_7 + \tilde{\chi}_8 - n^0_{1,5} + \eta \\ n^7_{1,3} = 2\tilde{\chi} + 3\tilde{\chi}_3 - 3\tilde{\chi}_4 - \tilde{\chi}_5 + \tilde{\chi}_6 + n^0_{1,5} - \eta \\ n^7_{2,7} = \eta \\ n^7_{1,4} = \tilde{\chi} + \tilde{\chi}_3 + 2\tilde{\chi}_4 - \tilde{\chi}_5 - n^0_{1,5} - \eta \\ n^7_{1,5} = n^0_{1,5}. \end{cases}$$

We have already known

$$\epsilon_8 = -\tilde{\chi}_3 - \tilde{\chi}_4 + \tilde{\chi}_5 + \tilde{\chi}_6 + \tilde{\chi}_8 - \tilde{\chi}_9.$$

Thus, taking some prime packings into account, $B^{(8)} = \{n^8_{b,r} \times (b,r)\}_{\frac{b}{r}\in S^{(8)}}$ has the coefficients:

$$B^{(8)} \begin{cases} n^8_{1,2} = 2\tilde{\chi} - 2\tilde{\chi}_3 + \tilde{\chi}_4 - 2\tilde{\chi}_5 - \tilde{\chi}_7 + \tilde{\chi}_8 + n^0_{1,5} + \eta \\ n^8_{3,7} = \tilde{\chi} - \tilde{\chi}_3 + \tilde{\chi}_6 + \tilde{\chi}_7 - \tilde{\chi}_8 - \eta \\ n^8_{2,5} = \tilde{\chi} + \tilde{\chi}_3 + \tilde{\chi}_4 + \tilde{\chi}_5 - 3\tilde{\chi}_6 - \tilde{\chi}_7 + \tilde{\chi}_9 - n^0_{1,5} + \eta \\ n^8_{3,8} = -\tilde{\chi}_3 - \tilde{\chi}_4 + \tilde{\chi}_5 + \tilde{\chi}_6 + \tilde{\chi}_8 - \tilde{\chi}_9 \\ n^8_{1,3} = 2\tilde{\chi} + 4\tilde{\chi}_3 - 2\tilde{\chi}_4 - 2\tilde{\chi}_5 - \tilde{\chi}_8 + \tilde{\chi}_9 + n^0_{1,5} - \eta \\ n^8_{2,7} = \eta \\ n^8_{1,4} = \tilde{\chi} + \tilde{\chi}_3 + 2\tilde{\chi}_4 - \tilde{\chi}_5 - n^0_{1,5} - \eta \\ n^8_{1,5} = n^0_{1,5}. \end{cases}$$

We know that

$$\epsilon_9 = -2\tilde{\chi}_3 + \tilde{\chi}_4 + \tilde{\chi}_5 - \tilde{\chi}_7 + \tilde{\chi}_8 + \tilde{\chi}_9 - \tilde{\chi}_{10} - n^0_{1,5} + \eta.$$

Moreover $S^{(9)} - S^{(8)} = \{\frac{4}{9}, \frac{2}{9}\}$. Let ζ be the number of prime packings of type $\{(1,2),(3,7)\} \succ \{(4,9)\}$, then the number of type $\{(1,4),(1,5)\} \succ \{(2,9)\}$ prime packings is $\epsilon_9 - \zeta$. We can get $B^{(9)}$ consisting of the following coefficients.

$$B^{(9)} \begin{cases} n^9_{1,2} = 2\tilde{\chi} - 2\tilde{\chi}_3 + \tilde{\chi}_4 - 2\tilde{\chi}_5 - \tilde{\chi}_7 + \tilde{\chi}_8 + n^0_{1,5} + \eta - \zeta \\ n^9_{4,9} = \zeta \\ n^9_{3,7} = \tilde{\chi} - \tilde{\chi}_3 + \tilde{\chi}_6 + \tilde{\chi}_7 - \tilde{\chi}_8 - \eta - \zeta \\ n^9_{2,5} = \tilde{\chi} + \tilde{\chi}_3 + \tilde{\chi}_4 + \tilde{\chi}_5 - 3\tilde{\chi}_6 - \tilde{\chi}_7 + \tilde{\chi}_9 - n^0_{1,5} + \eta \\ n^9_{3,8} = -\tilde{\chi}_3 - \tilde{\chi}_4 + \tilde{\chi}_5 + \tilde{\chi}_6 + \tilde{\chi}_8 - \tilde{\chi}_9 \\ n^9_{1,3} = 2\tilde{\chi} + 4\tilde{\chi}_3 - 2\tilde{\chi}_4 - 2\tilde{\chi}_5 - \tilde{\chi}_8 + \tilde{\chi}_9 + n^0_{1,5} - \eta \\ n^9_{2,7} = \eta \\ n^9_{1,4} = \tilde{\chi} + 3\tilde{\chi}_3 + \tilde{\chi}_4 - 2\tilde{\chi}_5 + \tilde{\chi}_7 - \tilde{\chi}_8 - \tilde{\chi}_9 + \tilde{\chi}_{10} - 2\eta + \zeta \\ n^9_{2,9} = -2\tilde{\chi}_3 + \tilde{\chi}_4 + \tilde{\chi}_5 - \tilde{\chi}_7 + \tilde{\chi}_8 + \tilde{\chi}_9 - \tilde{\chi}_{10} - n^0_{1,5} + \eta - \zeta \\ n^9_{1,5} = 2\tilde{\chi}_3 - \tilde{\chi}_4 - \tilde{\chi}_5 + \tilde{\chi}_7 - \tilde{\chi}_8 - \tilde{\chi}_9 + \tilde{\chi}_{10} + 2n^0_{1,5} - \eta + \zeta \end{cases}$$

One has

$$\epsilon_{10} = -\tilde{\chi}_3 + 2\tilde{\chi}_6 + \tilde{\chi}_{10} - \tilde{\chi}_{11} - n^0_{1,5} - \eta$$

and then $B^{(10)}$ consists of the following coefficients:

$$B^{(10)} \begin{cases} n^{10}_{1,2} = 2\tilde{\chi} - 2\tilde{\chi}_3 + \tilde{\chi}_4 - 2\tilde{\chi}_5 - \tilde{\chi}_7 + \tilde{\chi}_8 + n^0_{1,5} + \eta - \zeta \\ n^{10}_{4,9} = \zeta \\ n^{10}_{3,7} = \tilde{\chi} - \tilde{\chi}_3 + \tilde{\chi}_6 + \tilde{\chi}_7 - \tilde{\chi}_8 - \eta - \zeta \\ n^{10}_{2,5} = \tilde{\chi} + \tilde{\chi}_3 + \tilde{\chi}_4 + \tilde{\chi}_5 - 3\tilde{\chi}_6 - \tilde{\chi}_7 + \tilde{\chi}_9 - n^0_{1,5} + \eta \\ n^{10}_{3,8} = -\tilde{\chi}_3 - \tilde{\chi}_4 + \tilde{\chi}_5 + \tilde{\chi}_6 + \tilde{\chi}_8 - \tilde{\chi}_9 \\ n^{10}_{1,3} = 2\tilde{\chi} + 5\tilde{\chi}_3 - 2\tilde{\chi}_4 - 2\tilde{\chi}_5 - 2\tilde{\chi}_6 - \tilde{\chi}_8 + \tilde{\chi}_9 - \tilde{\chi}_{10} + \tilde{\chi}_{11} + 2n^0_{1,5} \\ n^{10}_{3,10} = -\tilde{\chi}_3 + 2\tilde{\chi}_6 + \tilde{\chi}_{10} - \tilde{\chi}_{11} - n^0_{1,5} - \eta \\ n^{10}_{2,7} = \tilde{\chi}_3 - 2\tilde{\chi}_6 - \tilde{\chi}_{10} + \tilde{\chi}_{11} + n^0_{1,5} + 2\eta \\ n^{10}_{1,4} = \tilde{\chi} + 3\tilde{\chi}_3 + \tilde{\chi}_4 - 2\tilde{\chi}_5 + \tilde{\chi}_7 - \tilde{\chi}_8 - \tilde{\chi}_9 + \tilde{\chi}_{10} - 2\eta + \zeta \\ n^{10}_{2,9} = -2\tilde{\chi}_3 + \tilde{\chi}_4 + \tilde{\chi}_5 - \tilde{\chi}_7 + \tilde{\chi}_8 + \tilde{\chi}_9 - \tilde{\chi}_{10} - n^0_{1,5} + \eta - \zeta \\ n^{10}_{1,5} = 2\tilde{\chi}_3 - \tilde{\chi}_4 - \tilde{\chi}_5 + \tilde{\chi}_7 - \tilde{\chi}_8 - \tilde{\chi}_9 + \tilde{\chi}_{10} + 2n^0_{1,5} - \eta + \zeta. \end{cases}$$

By computing $\Delta^{11}(B^{(10)})$, we get

$$\epsilon_{11} = \tilde{\chi} - \tilde{\chi}_3 + \tilde{\chi}_4 - \tilde{\chi}_7 + \tilde{\chi}_9 + \tilde{\chi}_{11} - \tilde{\chi}_{12} - n^0_{1,5} - \zeta.$$

Let α be the number of prime packings of type $\{(1,2),(4,9)\} \succ \{(5,11)\}$ and β be the number of prime packings of type $\{(1,3),(3,8)\} \succ \{(4,11)\}$. Then we get $B^{(11)}$ with

$$B^{(11)} \begin{cases} n^{11}_{1,2} = 2\tilde{\chi} - 2\tilde{\chi}_3 + \tilde{\chi}_4 - 2\tilde{\chi}_5 - \tilde{\chi}_7 + \tilde{\chi}_8 + n^0_{1,5} + \eta - \zeta - \alpha \\ n^{11}_{5,11} = \alpha \\ n^{11}_{4,9} = \zeta - \alpha \\ n^{11}_{3,7} = \tilde{\chi} - \tilde{\chi}_3 + \tilde{\chi}_6 + \tilde{\chi}_7 - \tilde{\chi}_8 - \eta - \zeta \\ n^{11}_{2,5} = \tilde{\chi} + \tilde{\chi}_3 + \tilde{\chi}_4 + \tilde{\chi}_5 - 3\tilde{\chi}_6 - \tilde{\chi}_7 + \tilde{\chi}_9 - n^0_{1,5} + \eta \\ n^{11}_{3,8} = -\tilde{\chi}_3 - \tilde{\chi}_4 + \tilde{\chi}_5 + \tilde{\chi}_6 + \tilde{\chi}_8 - \tilde{\chi}_9 - \beta \\ n^{11}_{4,11} = \beta \\ n^{11}_{1,3} = 2\tilde{\chi} + 5\tilde{\chi}_3 - 2\tilde{\chi}_4 - 2\tilde{\chi}_5 - 2\tilde{\chi}_6 - \tilde{\chi}_8 + \tilde{\chi}_9 - \tilde{\chi}_{10} + \tilde{\chi}_{11} + 2n^0_{1,5} - \beta \\ n^{11}_{3,10} = -\tilde{\chi}_3 + 2\tilde{\chi}_6 + \tilde{\chi}_{10} - \tilde{\chi}_{11} - n^0_{1,5} - \eta \\ n^{11}_{2,7} = -\tilde{\chi} + 2\tilde{\chi}_3 - \tilde{\chi}_4 - 2\tilde{\chi}_6 + \tilde{\chi}_7 - \tilde{\chi}_9 - \tilde{\chi}_{10} + \tilde{\chi}_{12} + 2n^0_{1,5} + 2\eta + \zeta + \alpha + \beta \\ n^{11}_{3,11} = \tilde{\chi} - \tilde{\chi}_3 + \tilde{\chi}_4 - \tilde{\chi}_7 + \tilde{\chi}_9 + \tilde{\chi}_{11} - \tilde{\chi}_{12} - n^0_{1,5} - \zeta - \alpha - \beta \\ n^{11}_{1,4} = 4\tilde{\chi}_3 - 2\tilde{\chi}_5 + 2\tilde{\chi}_7 - \tilde{\chi}_8 - 2\tilde{\chi}_9 + \tilde{\chi}_{10} - \tilde{\chi}_{11} + \tilde{\chi}_{12} + n^0_{1,5} - 2\eta + 2\zeta + \alpha + \beta \\ n^{11}_{2,9} = -2\tilde{\chi}_3 + \tilde{\chi}_4 + \tilde{\chi}_5 - \tilde{\chi}_7 + \tilde{\chi}_8 + \tilde{\chi}_9 - \tilde{\chi}_{10} - n^0_{1,5} + \eta - \zeta \\ n^{11}_{1,5} = 2\tilde{\chi}_3 - \tilde{\chi}_4 - \tilde{\chi}_5 + \tilde{\chi}_7 - \tilde{\chi}_8 - \tilde{\chi}_9 + \tilde{\chi}_{10} + 2n^0_{1,5} - \eta + \zeta. \end{cases}$$

Finally since

$$\epsilon_{12} = -\tilde{\chi} - 3\tilde{\chi}_3 + 2\tilde{\chi}_5 + \tilde{\chi}_6 - \tilde{\chi}_7 + \tilde{\chi}_8 + \tilde{\chi}_{12} - \tilde{\chi}_{13} - n^0_{1,5} + \eta,$$

we get $B^{(12)}$ with

(3.15)

$$B^{(12)}\begin{cases} n^{12}_{1,2}=2\tilde{\chi}-2\tilde{\chi}_3+\tilde{\chi}_4-2\tilde{\chi}_5-\tilde{\chi}_7+\tilde{\chi}_8+n^0_{1,5}+\eta-\zeta-\alpha\\ n^{12}_{5,11}=\alpha\\ n^{12}_{4,9}=\zeta-\alpha\\ n^{12}_{3,7}=2\tilde{\chi}+2\tilde{\chi}_3-2\tilde{\chi}_5+2\tilde{\chi}_7-2\tilde{\chi}_8-\tilde{\chi}_{12}+\tilde{\chi}_{13}-2\eta-\zeta+n^0_{1,5}\\ n^{12}_{5,12}=-\tilde{\chi}-3\tilde{\chi}_3+2\tilde{\chi}_5+\tilde{\chi}_6-\tilde{\chi}_7+\tilde{\chi}_8+\tilde{\chi}_{12}-\tilde{\chi}_{13}+\eta-n^0_{1,5}\\ n^{12}_{2,5}=2\tilde{\chi}+4\tilde{\chi}_3+\tilde{\chi}_4-\tilde{\chi}_5-4\tilde{\chi}_6-\tilde{\chi}_8+\tilde{\chi}_9-\tilde{\chi}_{12}+\tilde{\chi}_{13}\\ n^{12}_{3,8}=-\tilde{\chi}_3-\tilde{\chi}_4+\tilde{\chi}_5+\tilde{\chi}_6+\tilde{\chi}_8-\tilde{\chi}_9-\beta\\ n^{12}_{4,11}=\beta\\ n^{12}_{1,3}=2\tilde{\chi}+5\tilde{\chi}_3-2\tilde{\chi}_4-2\tilde{\chi}_5-2\tilde{\chi}_6-\tilde{\chi}_8+\tilde{\chi}_9-\tilde{\chi}_{10}+\tilde{\chi}_{11}+2n^0_{1,5}-\beta\\ n^{12}_{3,10}=-\tilde{\chi}_3+2\tilde{\chi}_6+\tilde{\chi}_{10}-\tilde{\chi}_{11}-n^0_{1,5}-\eta\\ n^{12}_{2,7}=-\tilde{\chi}+2\tilde{\chi}_3-\tilde{\chi}_4-2\tilde{\chi}_6+\tilde{\chi}_7-\tilde{\chi}_9-\tilde{\chi}_{10}+\tilde{\chi}_{12}+2n^0_{1,5}+2\eta+\zeta+\alpha+\beta\\ n^{12}_{3,11}=\tilde{\chi}-\tilde{\chi}_3+\tilde{\chi}_4-\tilde{\chi}_7+\tilde{\chi}_9+\tilde{\chi}_{11}-\tilde{\chi}_{12}-n^0_{1,5}-\zeta-\alpha-\beta\\ n^{12}_{1,4}=4\tilde{\chi}_3-2\tilde{\chi}_5+2\tilde{\chi}_7-\tilde{\chi}_8-2\tilde{\chi}_9+\tilde{\chi}_{10}-\tilde{\chi}_{11}+\tilde{\chi}_{12}+n^0_{1,5}-2\eta+2\zeta+\alpha+\beta\\ n^{12}_{2,9}=-2\tilde{\chi}_3+\tilde{\chi}_4+\tilde{\chi}_5-\tilde{\chi}_7+\tilde{\chi}_8+\tilde{\chi}_9-\tilde{\chi}_{10}-n^0_{1,5}+\eta-\zeta\\ n^{12}_{1,5}=2\tilde{\chi}_3-\tilde{\chi}_4-\tilde{\chi}_5+\tilde{\chi}_7-\tilde{\chi}_8-\tilde{\chi}_9+\tilde{\chi}_{10}+2n^0_{1,5}-\eta+\zeta. \end{cases}$$

To recall the meaning of several symbols, η is the number of prime packings of type $\{(1,3),(1,4)\}\succ\{(2,7)\}$, ζ is the number of prime packings of type $\{(1,2),(3,7)\}\succ\{(4,9)\}$, α is the number of prime packings of type $\{(1,2),(4,9)\}\succ\{(5,11)\}$ and β is the number of prime packings of type $\{(1,3),(3,8)\}\succ\{(4,11)\}$.

4. Main results on general type 3-folds

In this section, we would like to utilize those equalities and inequalities of formal baskets to study 3-folds of general type. Let V be a nonsingular projective 3-fold of general type. The 3-dimensional Minimal Model Program (cf. [17, 19, 22]) says that V has a minimal model X with $\mathbb{Q}$-factorial terminal singularities. Therefore to study the birational geometry of V is equivalent to study that of X.

Let us begin with recalling some known relevant results. The following theorem was proved by the first author and Hacon.

THEOREM 4.1 ([5]). – *Assume $q(X):=h^1(\mathcal{O}_X)>0$. Then $P_m>0$ for all $m\geq 2$ and φ_m is birational for all $m\geq 7$.*

Thus we do not need to worry about irregular 3-folds in the following discussion. The following result is due to Kollár.

THEOREM 4.2 ([18, Corollary 4.8]). – *Assume $P_{m_0}:=P_{m_0}(X)\geq 2$ for some integer $m_0>0$. Then φ_{11m_0+5} is birational onto its image.*

Kollár's result was improved by the second author.

THEOREM 4.3 ([7, Theorem 0.1]). – *Assume* $P_{m_0} := P_{m_0}(X) \geq 2$ *for some integer* $m_0 > 0$. *Then* φ_m *is birational onto its image for all* $m \geq 5m_0 + 6$.

4.4. – Other known results.

(i) When X is Gorenstein, it is proved in [4] that φ_m is birational for all $m \geq 5$.
(ii) When $\chi(\mathcal{O}_X) < 0$, Reid's formula (4.1) says $P_2 \geq 4$ and $P_m > 0$ for all $m \geq 2$. It is proved in [9, Corollary 1.3] that φ_m is birational for all $m \geq 8$.
(iii) When $\chi(\mathcal{O}_X) = 0$, since one can verify $l_Q(3) \geq l_Q(2)$ for any basket Q, Reid's formula (4.1) says: $P_3(X) > P_2(X) > 0$. Moreover, $P_{m+1} \geq P_m$ for all $m \geq 2$. So $P_3(X) \geq 2$. It is proved in [9, Theorem 1.4] that φ_m is birational for all $m \geq 14$.

4.5. – From now on, we only study minimal 3-fold X of general type with $\chi(\mathcal{O}_X) > 0$. Recall that X is always attached the formal basket $\mathbf{B}(X)$. Moreover, since X is minimal and of general type, the vanishing theorem ([15, 26]) on X gives $\chi_m(X) = P_m(X)$ for $m \geq 2$. Therefore we have various equalities and inequalities among plurigenera by the results in the previous sections. Furthermore, the canonical volume $\mathrm{Vol}(V) = \mathrm{Vol}(X)$ is nothing but K_X^3.

The following result is due to Iano-Fletcher.

THEOREM 4.6 ([11]). – *Assume* $\chi(\mathcal{O}_X) = 1$. *Then* $P_{12} \geq 1$ *and* $P_{24} \geq 2$.

Combining all known results, we only need to consider the 3-fold X satisfying $\chi(\mathcal{O}_X) \geq 2$ and $P_m \leq 1$ for all $2 \leq m \leq 12$.

THEOREM 4.7. – *There are only finitely many formal baskets of minimal threefolds of general type satisfying* $\chi \geq 2$ *and* $P_m \leq 1$ *for all* $2 \leq m \leq 12$.

Proof. – By looking at inequality (3.14), we have

$$8 \geq \chi(\mathcal{O}_X) + R \geq \chi(\mathcal{O}_X)$$

since $1 \geq \chi_m(X) = P_m(X) \geq 0$ for all $2 \leq m \leq 12$. Moreover, $8 \geq R$ implies that $n_{1,r}^0 = 0$ for all $r \geq 9$. By equality (3.5), one has $\sigma = \sum_{r=2}^{8} n_{1,r}^0 = 10\chi + 5P_2 - P_3 \leq 85$. It is clear that there are finitely many initial baskets $B^0 = \{n_{1,r}^0\}$ satisfying $\sigma \leq 85$ and $n_{1,r}^0 = 0$ for all $r \geq 9$. Each initial basket allows finite ways of packings. Hence it follows that there are only finitely many formal baskets satisfying the given conditions. $\square$

By Theorem 4.7, one can obtain various effective results by working out the classification of formal baskets with small plurigenera. Indeed, by some more careful usage of those inequalities in the previous section, we are able to obtain our main results without too much extra works.

LEMMA 4.8. – *If* $P_m \leq 1$ *for all* $m \leq 12$, *then* $P_2 = 0$.

Proof. – Recalling Equation (3.10), we have:

$$\epsilon_6 = -3P_2 - P_3 + P_4 + P_5 + P_6 - P_7 - \epsilon = 0$$

which is equivalent to

$$P_4 + P_5 + P_6 = 3P_2 + P_3 + P_7 + \epsilon.$$

If $P_2 = 1$, then $P_4 = P_5 = P_6 = 1$. It follows that $P_3 = P_7 = \epsilon = 0$. But this is impossible since $P_2 = P_5 = 1$ implies $P_7 \geq 1$. □

LEMMA 4.9. – *Assume that $\chi(\mathcal{O}_X) \geq 2$ and $P_m \leq 1$ for $m \leq 12$. Then $\chi(\mathcal{O}_X) \leq 6$.*

Proof. – If $P_m \leq 1$ for all $m \leq 12$, we have seen $P_2 = 0$. Then by inequality (3.14), we get $8 \geq \chi = \chi(\mathcal{O}_X)$. If $\chi = 7$ or 8, then $P_5 = P_6 = 1$. It follows that $P_{10} = P_{11} = P_{12} = 1$. Hence $8 \geq \chi + 1$ gives $\chi = 7$ and $P_8 = 1$ as well. Then $P_{13} = 1$. This leads to $8 \geq \chi + 2 = 9$, a contradiction. □

THEOREM 4.10. – *Let X be a projective minimal 3-fold of general type. Then $P_{12} \geq 1$.*

Proof. – It suffices to prove this for the situation $\chi \geq 2$ by 4.4(ii), (iii) and Theorem 4.6. We assume $P_{12} = 0$ and will deduce a contradiction. It is then clear that $P_2 = P_3 = P_4 = P_6 = 0$.

Step 1

If $P_5 = 0$, then the equality (3.10) for ϵ_6 gives $P_7 = \epsilon = 0$. This also means $\sigma_5 = 0$. Hence Assumption 3.8 is satisfied. Now since $\epsilon_7 \geq \eta$ and $\epsilon_{12} \geq 0$ (cf. (3.11), (3.13)), one gets

$$\chi \geq P_8 + \eta \geq \chi + P_{13}.$$

It follows that $\chi = P_8 + \eta$, $\epsilon_7 = \eta$ and $n^7_{3,7} = 0$. Since $n^9_{3,7} = -\zeta$, we have $\zeta = 0$. Now $n^{11}_{4,9} = \zeta - \alpha \geq 0$ gives $\alpha = 0$.

Hence since $n^0_{1,5} = 0$ and so $n^9_{2,9} = -n^9_{1,5} = 0$, we have $n^9_{2,9} = 0$ and $\epsilon_9 = n^9_{2,9} + \zeta = 0$ which gives $P_{10} = P_8 + P_9 + \eta$.

Now $n^{12}_{3,8} + n^{12}_{2,7} \geq 0$ gives $\eta \geq \chi + 3P_9 = \eta + P_8 + 3P_9$. Hence $P_8 = P_9 = 0$, and also $P_{10} = \eta = \chi$. However, $n^{12}_{3,8} + n^{12}_{1,4} = P_{10} - 2\eta - P_{11} = -\chi - P_{11} < 0$, which is a contradiction.

Step 2

If $P_5 > 0$, then we have $P_7 = 0$. First of all, (3.10) gives $P_5 = \epsilon := n^0_{1,5} + 2\sum_{r\geq 6} n^0_{1,r}$. Since $n^7_{1,4} \geq 0$, one has

$$\chi \geq P_5 + \eta + \sigma_5.$$

Again $\epsilon_{12} \geq 0$ (cf. (3.13)) gives the inequality:

$$2P_5 + P_8 + \eta \geq \chi + P_{13} + (8\sigma_5 - 7n^0_{1,5} - 5n^0_{1,6} - 5n^0_{1,7} - \cdots - n^0_{1,11}).$$

Combining these two inequalities, we get

$$2\epsilon + P_8 + \eta = 2P_5 + P_8 + \eta \geq P_5 + P_{13} + \eta + R',$$

where $R' = 2n^0_{1,5} + 4n^0_{1,6} + 4n^0_{1,7} + \cdots + 8n^0_{1,11} + 8\sum_{r\geq 12} n^0_{1,r} \geq 2\epsilon$. It follows that $P_8 \geq P_5 + P_{13}$. Since $P_5 > 0$, we get $P_8 > 0$ and thus $P_{13} \geq P_8$. This means $P_5 = 0$, a contradiction. □

LEMMA 4.11. – *Let W be a projective variety with at worst canonical singularities. Given positive integers m and n, let $l := \mathrm{lcm}(m,n)$ and $d := \gcd(m,n)$. Suppose that $P_m = P_n = P_l = 1$. Then $P_d = 1$.*

Proof. – Let $\pi : \tilde{W} \to W$ be a resolution of singularities. It is clear that $P_k(\tilde{W}) = P_k(W)$ for all $k \geq 1$. We may thus assume that W is nonsingular. The same argument as in [1, Lemma VIII.1.c] concludes the statement. □

THEOREM 4.12. – *Let X be a projective minimal 3-fold of general type. Then either $P_{10} \geq 2$ or $P_{24} \geq 2$.*

Proof. – By 4.4 and Theorem 4.6, we may only study those 3-folds with $\chi = \chi(\mathcal{O}_X) \geq 2$. Suppose, on the contrary, that $P_{24} \leq 1$ and $P_{10} \leq 1$. By Theorem 4.10, one has $P_{12} = P_{24} = 1$. We will deduce a contradiction.

Claim 1

If $P_8 > 0$, then $P_4 = P_8 = 1$.

In fact, this follows from Lemma 4.11 by taking $m = 12$ and $n = 8$.

Set $d := \min\{m \mid P_m(X) > 0, m \in \mathbb{Z}^+\}$. Clearly, one has $d \leq 12$.

Claim 2

If $d \mid 24$, then $P_n = 0$ for any positive integer $n \leq \frac{24}{d}$ with $\gcd(n,d) < d$.

To see this, suppose that $P_n > 0$ for some $n \leq \frac{24}{d}$ with $d \nmid n$. Since $P_d > 0$ and $d \mid 24$, we see that $1 = P_{24} \geq P_{nd} \geq P_n$. Thus, for $l := \mathrm{lcm}(n,d)$, $P_l = 1$. Now Lemma 4.11 gives $P_{(n,d)} = 1$, contradicting the minimality of d.

Claim 3

We may assume that $d \geq 3$, i.e. $P_2 = 0$.

If $d = 1$, then $P_m = 1$ for all $m \leq 12$. But equality (3.10) gives $\epsilon_6 = -2 - \epsilon = 0$, a contradiction.

If $d = 2$, then $P_4 = P_6 = 1$ and Claim 2 tells that $P_3 = P_5 = P_7 = 0$. Again equality (3.10) gives $\epsilon_6 = -1 - \epsilon = 0$, a contradiction.

In what follows, we are going to apply those formulae in Section 4. Recall, from equality (3.9), that $\epsilon := n^0_{1,5} + 2\sum_{r\geq 6} n^0_{1,r}$. We will frequently use the following:

Observation

If $\epsilon + P_7 = 1$, then one of the following situations occurs:

(A) $P_7 = 1$ and $n^0_{1,r} = 0$ for all $r \geq 5$.
(B) $P_7 = 0$, $n^0_{1,5} = 1$ and $n^0_{1,r} = 0$ for all $r \geq 6$.

Thus Assumption 3.8 is satisfied under both situations.

Now we are ready for the proof, which is the case-by-case analysis though it is slightly long.

Case 1

If $d = 3$, then, since $P_9 \leq P_{12}$, one has $P_3 = P_6 = P_9 = 1$. By Claim 2, one gets $P_4 = P_5 = P_7 = P_8 = 0$. Now equality (3.10) gives $\epsilon_6 = -\epsilon = 0$. It follows that $n^0_{1,r} = 0$ for all $r \geq 5$ and hence Assumption 3.8 is satisfied. But then, one will get $\epsilon_8 = -1$, a contradiction.

Case 2

If $d = 4$, then $P_4 = P_8 = 1$. One has $P_5 = P_6 = 0$ by Claim 2. Now equality (3.10) gives $P_7 + \epsilon = 1$. Thus Assumption 3.8 is satisfied and so $P_9 = 0$ by the inequality $\epsilon_8 = -P_9 \geq 0$. We discuss the two cases in Observation:

(2-A). – If $P_7 = 1$ and $\epsilon = 0$, then we have $P_{11} \geq P_7 \geq 1$. Now $\epsilon_{10} \geq 0$ yields

$$P_{10} \geq P_{11} + n^0_{1,5} + \eta \geq 1.$$

This means, by our assumption on P_{10}, that $P_{10} = 1$ and $n^0_{1,5} = \eta = 0$. So inequality (4.1) gives

$$3 = P_8 + P_{10} + P_{12} \geq \chi + 1 + P_{11} + P_{13} + R \geq \chi + 2,$$

contradicting our assumption $\chi \geq 2$.

(2-B). – If $P_7 = 0$ and $\epsilon = 1$, then $n^0_{1,5} = 1$. Again, $\epsilon_{10} \geq 0$ gives

$$P_{10} \geq P_{11} + n^0_{1,5} + \eta \geq P_{11} + \eta + 1.$$

Thus $P_{10} = 1$ and $P_{11} = \eta = 0$. So inequality (3.14) yields

$$3 = P_8 + P_{10} + P_{12} \geq \chi + P_{13} + R \geq \chi + 2,$$

contradicting the assumption $\chi \geq 2$.

Case 3

If $d = 7$, then $P_2 = \cdots = P_6 = 0$. But then equality (3.10) gives $\epsilon_6 = -P_7 - \epsilon < 0$, a contradiction.

Case 4

If $d = 8$, then, by Claim 1, $P_4 = 1$, a contradiction.

Case 5

If $d = 9$, then (3.10) gives $\epsilon = 0$. Hence Assumption 3.8 is satisfied. Now $\epsilon_8 = -P_9 < 0$ yields a contradiction.

Case 6

If $d = 10$, then, similarly, (3.10) gives $\epsilon = 0$ and thus Assumption 3.8 is satisfied. Now $\epsilon_9 \geq 0$ and $\epsilon_{10} \geq 0$ imply:

$$\eta \geq P_{10} \geq P_{11} + \eta.$$

It follows that $\eta = P_{10} = 1$ and $P_{11} = 0$. So inequality (3.14) gives $2 \geq \chi + P_{13}$, which implies $P_{13} = 0$ and $\chi = 2$. Now the direct computation shows $\epsilon_{12} = 0$ and thus

$$B^{(11)} = B^{(12)} = \{5 \times (1,2), (3,7), 3 \times (2,5), 3 \times (1,3), (3,11)\}.$$

But now we see that $B^{(12)}$ admits no non-trivial prime packing of level > 12. This already means that $B^{(12)} = B^{(13)} = \cdots = B$. Therefore, there is only one of the formal baskets $\mathbf{B} = (B, \chi(\mathcal{O}_X), P_2) = (B, 2, 0)$ in this case. By the direct computation, we see that $P_{10}(\mathbf{B}) = 0$ and $P_{24}(\mathbf{B}) = 8$, a contradiction.

Case 7

If $d = 11$, then (3.10) gives $\epsilon = 0$ and hence Assumption 3.8 is satisfied. But then $\epsilon_{10} = -P_{11} - \eta < 0$, a contradiction.

Case 8

If $d = 12$, then similarly (3.10) gives $\epsilon = 0$ and hence Assumption 3.8 is satisfied. But then inequality (3.14) yields $1 = P_{12} \geq \chi + P_{13} \geq 2$, a contradiction to the assumption $\chi \geq 2$.

Notation

In what follows, we will abuse the notation of a basket B with its associated formal basket $\mathbf{B} = (B, \chi, \chi_2) = (B, \chi, 0)$.

Case 9

If $d = 6$, then $P_8 = 0$ by Claim 1. Since $0 < P_6 \leq P_{18} \leq P_{24} = 1$, we have $P_9 = 0, 1$. Suppose $P_9 = 1$, then Lemma 4.11 gives $P_3 = 1$, a contradiction to $d = 6$. Hence we have seen that $P_9 = 0$. Now $\epsilon_6 = 0$ implies $P_7 + \epsilon = 1$. Thus we get two situations as follows:

(11-A). – $(P_7, \epsilon) = (0, 1)$. Then $\epsilon_9 \geq 0$ and $\epsilon_{10} \geq 0$ give

$$\eta + 1 \geq P_{10} + 2 \geq P_{11} + \eta + 1.$$

In particular, one has $P_{11} = 0$ and $\eta = P_{10} + 1$. Recall that $P_{10} \leq 1$ by our assumption.

(11-B). – $(P_7, \epsilon) = (1, 0)$. Then $\epsilon_9 \geq 0$ and $\epsilon_{10} \geq 0$ give

$$\eta + 1 \geq P_{10} + 2 \geq P_{11} + \eta.$$

In particular, one has $1 \geq P_{11}$ and $P_{10} + 2 \geq \eta \geq P_{10} + 1$.

The following table is the summary on the possible value of (P_7, P_{10}, P_{11}). Note, however, that all items should be non-negative by our definition.

(P_7, P_{10}, P_{11})	$(0,0,0)$	$(0,1,0)$	$(1,0,0)$	$(1,0,1)$	$(1,1,0)$	$(1,1,1)$
ϵ_7	$\chi + 1$	$\chi + 1$	$\chi + 2$	$\chi + 2$	$\chi + 2$	$\chi + 2$
ϵ_8	1	1	1	1	1	1
ϵ_9	$-1 + \eta$	$-2 + \eta$	$-1 + \eta$	$-1 + \eta$	$-2 + \eta$	$-2 + \eta$
ϵ_{10}	$1 - \eta$	$2 - \eta$	$2 - \eta$	$1 - \eta$	$3 - \eta$	$2 - \eta$
$\epsilon_{10} + \epsilon_{12}$	$2 - \chi - P_{13}$	$3 - \chi - P_{13}$	$3 - \chi - P_{13}$	$2 - \chi - P_{13}$	$4 - \chi - P_{13}$	$3 - \chi - P_{13}$

We are going to discuss it case by case.

Subcase 9-I. – $(P_7, P_{10}, P_{11}) = (0, 0, 0)$.

The table shows that $2 \geq \chi$ and $\eta = 1$, hence $\chi = 2$. But then $n^8_{2,5} = -1$, a contradiction.

Subcase 9-II. – $(P_7, P_{10}, P_{11}) = (0, 1, 0)$.

The table shows that $\eta = 2$ and $3 \geq \chi$. If $\chi = 2$, then $n^7_{1,4} = -1$, a contradiction. Hence $\chi = 3$. Then we see that $\epsilon_{12} = -P_{13}$, which means $P_{13} = 0$ and thus $\epsilon_{12} = 0$. Also $n^{11}_{2,9} = -\zeta$ implies $\zeta = 0$. Then $n^{11}_{4,9} = \zeta - \alpha \geq 0$ gives $\alpha = 0$. Since $n^{11}_{1,4} = \beta - 1 \geq 0$ and $n^{11}_{3,8} = 1 - \beta \geq 0$, we have $\beta = 1$. Now we have,

$$B^{(12)} = B^{(11)} = \{9 \times (1,2), 2 \times (3,7), (2,5), (4,11), 4 \times (1,3), 2 \times (2,7), (1,5)\}.$$

The only 1-step prime packing of level > 12 of $B^{(12)}$ can only happen between $(4, 11)$ and $(1, 3)$. We obtained

$$\hat{B} = \{9 \times (1,2), 2 \times (3,7), (2,5), (5,14), 3 \times (1,3), 2 \times (2,7), (1,5)\}.$$

We see that $K^3(\hat{B})=0$, and thus $0>K^3(B')$ for any $\hat{B}\succ B'$ by Lemma 3.6. Therefore, we get $B=B^{(12)}$. Thus $P_{24}(X)=P_{24}(B^{(12)})=6$, a contradiction.

Subcase 9-III. – $(P_7,P_{10},P_{11})=(1,0,0)$.

We have $P_{13}\geq P_7\geq 1$ since $P_6>0$. Thus the table shows that $\eta=1,2$ and that $\chi\leq 2$, hence $\chi=2$.

If $\eta=1$, then $n^8_{2,5}=-1$, a contradiction. If $\eta=2$, then $\epsilon_9=1$. Since $n^9_{1,4}=-1+\zeta\geq 0$ while $\epsilon_9\geq\zeta$, one sees that $\zeta=1$. It follows that $\epsilon_{11}=-1<0$, a contradiction.

Subcase 9-IV. – $(P_7,P_{10},P_{11})=(1,0,1)$.

Since $P_{13}\geq P_7\geq 1$, the table gives $\chi\leq 1$, a contradiction to $\chi\geq 2$.

Subcase 9-V. – $(P_7,P_{10},P_{11})=(1,1,0)$.

Since $P_{13}>0$, the table shows that $\chi\leq 3$ and $2\leq\eta\leq 3$.

If $\chi=2$ and $\eta=3$, then $n^7_{1,4}=2-\eta=-1<0$, a contradiction.

If $\chi=3$ and $\eta=2$, then $\epsilon_{10}=1$ and $\epsilon_{10}+\epsilon_{12}=0$. Thus $\epsilon_{12}=-1<0$, a contradiction.

If $\chi=\eta=2$, we can determine other unknown quantities. First, $n^{12}_{2,5}=-1+P_{13}\geq 0$ gives $P_{13}=1$. Thus $\epsilon_{12}=0$ and $B^{(12)}=B^{(11)}$. Now $n^{11}_{2,9}=-\zeta\geq 0$ gives $\zeta=0$. Then $n^{11}_{4,9}\geq 0$ tells $\alpha=0$. Finally $n^{11}_{3,11}=-\beta\geq 0$ implies $\beta=0$. Hence we get:

$$B^{(12)}=\{5\times(1,2),2\times(3,7),(3,8),(1,3),(3,10),(2,7)\}.$$

It is clear that $B^{(12)}$ admits two 1-step prime packings of level >12:

$$B'=\{5\times(1,2),2\times(3,7),(3,8),(1,3),(5,17)\},$$

$$B''=\{5\times(1,2),2\times(3,7),(3,8),(4,13),(2,7)\}.$$

But $K^3(B'')<0$, $K^3(B')>0$ and B' is a minimal positive formal basket; we see that either $B^{(12)}\succcurlyeq B\succcurlyeq B'$ or $B^{(12)}=B$. By a direction calculation, we get $P_{24}(B^{(12)})=4$ and $P_{24}(B')=3$. Thus Lemma 3.6 implies $P_{24}=P_{24}(X)\geq 3$, a contradiction.

If $\chi=\eta=3$, then the table shows $\epsilon_{10}=\epsilon_{12}=0$ and $P_{13}=1$. We detect $B^{(11)}$ as before. First, $n^{11}_{2,9}\geq 0$ and $n^{11}_{1,5}\geq 0$ imply $\zeta=1$. Then $\epsilon_{11}=1-\zeta=0$ implies $\alpha=\beta=0$. So we get:

$$B^{(12)}=B^{(11)}=\{7\times(1,2),(4,9),(3,7),2\times(2,5),(3,8),3\times(1,3),3\times(2,7)\}.$$

We see that $B^{(12)}$ admits only two 1-step prime packings of level >12:

$$\hat{B}'=\{7\times(1,2),(7,16),2\times(2,5),(3,8),3\times(1,3),3\times(2,7)\},$$

$$\hat{B}''=\{7\times(1,2),(4,9),(3,7),(2,5),(5,13),3\times(1,3),3\times(2,7)\}.$$

By computation, both $\hat{B}'$ and $\hat{B}''$ are minimal positive (with regard to $B^{(12)}$). So we see that either $B^{(12)}\succcurlyeq B\succcurlyeq\hat{B}'$ or $B^{(12)}\succcurlyeq B\succcurlyeq\hat{B}''$. Since $P_{24}(B^{(12)})=8$, $P_{24}(\hat{B}')=6$ and $P_{24}(\hat{B}'')=4$, Lemma 3.6 implies $P_{24}\geq 4$, a contradiction.

Subcase 9-VI. – $(P_7, P_{10}, P_{11}) = (1, 1, 1)$.

Since $P_{13} > 0$, the table shows that $\chi = 2$, $\eta = 2$ and $\epsilon_{12} = 0$. Now $n^{11}_{2,9} = -\zeta \geq 0$ gives $\zeta = 0$. Further, $n^{11}_{4,9} \geq 0$ gives $\alpha = 0$. Finally, $n^{11}_{3,8} = 1 - \beta \geq 0$ and $n^{11}_{1,4} = -1 + \beta \geq 0$ implies $\beta = 1$. So we have:

$$B^{(12)} = B^{(11)} = \{5 \times (1,2), 2 \times (3,7), (4,11), (1,3), 2 \times (2,7)\}.$$

The only prime packing of $B^{(12)}$ of level > 12 is the following basket:

$$B' := \{5 \times (1,2), 2 \times (3,7), (5,14), 2 \times (2,7)\}$$

with $K^3(B') = 0$. This means that $B^{(12)}$ is already minimal positive and thus $B = B^{(12)}$. So $P_{24} = P_{24}(B^{(12)}) = 6 > 1$, a contradiction.

Case 10

If $d = 5$, then Claim 1 implies $P_8 = 0$. Also, we have $P_5 \leq P_{10} \leq 1$, which means $P_5 = 1$.

First we study P_6. Assume $P_6 > 0$, then $P_6 = 1$ since $P_6 \leq P_{12}$. Since $0 < P_6 \leq P_{18} \leq P_{24} = 1$, we have $P_9 = 0, 1$. Suppose $P_9 = 1$, then Lemma 4.11 gives $P_3 = 1$, a contradiction to $d = 5$. Hence we have seen that $P_9 = 0$. Similarly, if $P_8 > 0$, then $P_8 = 1$ since $P_8 \leq P_{24}$. Lemma 4.11 gives $P_2 = 1$, a contradiction to $d = 5$. Thus $P_8 = 0$. Noting that $P_{11} \geq P_6 = 1$, the inequality $\epsilon_9 + \epsilon_{10} \geq 0$ gives:

$$P_5 + 1 \geq P_7 + 9\sigma_5 - (7n^0_{1,5} + 6n^0_{1,6} + 5n^0_{1,7} + 3n^0_{1,8} + n^0_{1,9}). \tag{4.1}$$

On the other hand, equality (3.10) implies:

$$P_5 + 1 = P_7 + \epsilon = P_7 + \sigma_5 + \sum_{r \geq 6} n^0_{1,6}. \tag{4.2}$$

Now (4.1) and (4.2) imply $n^0_{1,r} = 0$ for all $r \geq 5$ and $P_7 = P_5 + 1 \geq 2$. It follows that $P_{12} \geq 2$, a contradiction. Therefore we have actually shown that $P_6 = 0$.

Next we study P_7. Clearly $P_7 \leq P_{12} = 1$. Assume $P_7 = 0$. Then equality (3.10) gives $\epsilon = 1$. This corresponds to Observation (B). Now $\epsilon_9 + \epsilon_{10} \geq 0$ implies that

$$1 + P_9 = P_5 + P_9 \geq P_{11} + 2.$$

Since $P_{15} > 0$, we see that $P_9 \leq P_{24} = 1$. Hence $P_9 = 1$,which implies $P_{11} = 0$. Now $\epsilon_{10} = -\eta$ gives $\eta = 0$. Thus we can see that $\epsilon_9 = 0$. It follows that $\zeta = 0$ since $\zeta \leq \epsilon_9$. Finally we can see that $n^{11}_{2,7} + n^{11}_{4,9} + n^{11}_{3,8} = -\chi + 1 \leq -1$, which is a contradiction. We have shown $P_7 = 1$.

To make a summary, we have: $P_5 = P_7 = P_{10} = P_{12} = 1$ and $P_2 = P_3 = P_4 = P_6 = P_8 = 0$. Note also that (3.10) gives $\epsilon = 0$, thus Assumption 3.8 is always satisfied. We need to study P_9, P_{11}.

Clearly, $P_9 \leq P_{24} = 1$ since $P_{15} > 0$. Again, $\epsilon_9 + \epsilon_{10} \geq 0$ gives $P_9 \geq P_{11}$. The next table is a summary for three possibilities of (P_9, P_{11}):

(P_9, P_{11})	$(0,0)$	$(1,0)$	$(1,1)$
ϵ_7	$\chi+1$	$\chi+1$	$\chi+1$
ϵ_8	1	0	0
ϵ_9	$-1+\eta$	η	$+\eta$
ϵ_{10}	$1-\eta$	$1-\eta$	$-\eta$
$\epsilon_{10}+\epsilon_{12}$	$3-\chi-P_{13}$	$3-\chi-P_{13}$	$2-\chi-P_{13}$

Subcase 10-I. – $(P_9, P_{11}) = (0,0)$.

The table shows that $\eta = 1$ and $\chi = 2, 3$.

When $\chi = 2$, $\epsilon_{11} = -\zeta \geq 0$ gives $\zeta = 0$ and thus $\epsilon_{11} = 0$. This implies $\alpha = \beta = 0$. Since $P_{13} \leq 1$ by the table, we first assume $P_{13} = 0$. Then we get

$$B^{(12)} = \{2 \times (1,2), (3,7), (5,12), 2 \times (2,5), (3,8), (1,3), (2,7)\}.$$

But we see that $K^3(B^{(12)}) < 0$, contradicting $K^3(B^{(12)}) \geq K^3(B) = K_X^3 > 0$. Thus $P_{13} = 1$, $\epsilon_{12} = 0$ and we get

$$B^{(12)} = \{2 \times (1,2), 2 \times (3,7), 3 \times (2,5), (3,8), (1,3), (2,7)\}.$$

Since any further prime packing dominated by $B^{(12)}$ has negative volume (due to the direct computation) and $B^{(12)} \succcurlyeq B$, we get $B = B^{(12)}$. So $P_{24} = P_{24}(B^{(12)}) = 4 > 1$, a contradiction.

When $\chi = 3$, the table shows that $P_{13} = 0$ and $\epsilon_{12} = 0$. Since $n_{2,9}^{11} = -\zeta \geq 0$, we have $\zeta = 0$. Thus by $n_{4,9}^{11} = \zeta - \alpha \geq 0$, we see that $\alpha = 0$. Finally $\epsilon_{11} = 1$ gives $\beta \leq 1$. If $\beta = 1$, then we get:

$$B^{(12)} = \{4 \times (1,2), 3 \times (3,7), 4 \times (2,5), (4,11), 2 \times (1,3), (2,7), (1,4)\}.$$

But we see that $K^3(B^{(12)}) < 0$, contradicting $K^3(B^{(12)}) \geq K^3(B) = K_X^3 > 0$. Thus we must have $\beta = 0$. Then we get:

$$B^{(12)} = \{4 \times (1,2), 3 \times (3,7), 4 \times (2,5), (3,8), 3 \times (1,3), (3,11)\}.$$

Since any further prime packing dominated by $B^{(12)}$ has negative volume (due to the direct computation) and $B^{(12)} \succcurlyeq B$, we see that $B = B^{(12)}$. So $P_{24} = P_{24}(B^{(12)}) = 2 > 1$, a contradiction.

Subcase 10-II. – $(P_9, P_{11}) = (1,0)$.

The table shows that $\eta = 0, 1$ and $\chi = 2, 3$.

If $\eta = 0$, then $n_{2,7}^{10} = -1$, a contradiction.

If $(\eta, \chi) = (1,2)$, then $n_{3,8}^{11} = -\beta \geq 0$ gives $\beta = 0$. Furthermore, $n_{4,9}^{11} + n_{3,11}^{11} = 1 - 2\alpha \geq 0$ implies $\alpha = 0$. Also, $n_{2,9}^{11} = 1 - \zeta \geq 0$ and $n_{1,5}^{11} = \zeta - 1 \geq 0$ imply $\zeta = 1$. Finally, the table shows that $\epsilon_{10} + \epsilon_{12} = 1 - P_{13}$ and so $P_{13} \leq 1$. When $P_{13} = 1$, we get:

$$B^{(12)} = \{(1,2), (4,9), (3,7), 4 \times (2,5), 2 \times (1,3), (2,7)\}.$$

Clearly, $B^{(12)}$ admits only one prime packing of level > 12:

$$\tilde{B} = \{(1,2),(7,16),4\times(2,5),2\times(1,3),(2,7)\}.$$

Thus we see that either $B^{(12)} = B$ or $B^{(12)} \succcurlyeq B \succcurlyeq \tilde{B}$. By computation, we know $P_{24}(B^{(12)}) = 5$ and $P_{24}(\tilde{B}) = 3$. Thus $P_{24} = P_{24}(B) \geq 3 > 1$, a contradiction. When $P_{13} = 0$, we get:

$$B^{(12)} = \{(1,2),(4,9),(5,12),3\times(2,5),2\times(1,3),(2,7)\}.$$

But we see that $K^3(B^{(12)}) < 0$, contradicting $K^3(B^{(12)}) \geq K^3(B) = K_X^3 > 0$.

If $(\eta,\chi) = (1,3)$, the table shows that $P_{13} = 0$ and $\epsilon_{12} = 0$. Also, $n^{11}_{2,9} = 1-\zeta \geq 0$ and $n^{11}_{1,5} = \zeta - 1 \geq 0$ imply $\zeta = 1$. Furthermore, $n^{11}_{3,8} = -\beta \geq 0$ gives $\beta = 0$. Finally, $n^{11}_{4,9} = 1-\alpha \geq 0$ imply $\alpha \leq 1$. When $\alpha = 1$, we get:

$$B^{(12)} = \{2\times(1,2),(5,11),2\times(3,7),5\times(2,5),4\times(1,3),(2,7),(1,4)\}.$$

But we see that $K^3(B^{(12)}) < 0$, contradicting $K^3(B^{(12)}) \geq K^3(B) = K_X^3 > 0$. When $\alpha = 0$, we get:

$$B^{(12)} = \{3\times(1,2),(4,9),2\times(3,7),5\times(2,5),4\times(1,3),(3,11)\}.$$

There is only one prime packing of level > 12:

$$\{3\times(1,2),(7,16),(3,7),5\times(2,5),4\times(1,3),(3,11)\},$$

which has $K^3 < 0$; we see that $B^{(12)} = B$. Thus $P_{24} = P_{24}(B) = P_{24}(B^{(12)}) = 3 > 1$, a contradiction.

Subcase 10-III. – $(P_9, P_{11}) = (1,1)$.

The table shows that $\eta = 0$ and $\chi = 2$. Also $\epsilon_{10} = 0$ implies $\zeta = 0$. But then $n^{11}_{2,7} + n^{11}_{4,9} + n^{11}_{3,8} = -2$, which is a contradiction.

This completes the proof. □

4.13. – Proof of Corollary 1.2

Proof. – (1) By virtue of 4.4, we may only study a minimal 3-fold X with $\chi(\mathcal{O}_X) > 0$. Then Theorem 4.6 and Theorem 4.12 imply that there is a positive integer $m_0 \leq 24$ such that $P_{m_0} \geq 2$. Thus, by Theorem 4.3, φ_m is birational for all $m \geq 126$.

(2) Set $\Phi := \varphi_{126}$. By taking a proper birational modification $\pi : \tilde{X} \to X$ (to resolve the indeterminacy of Φ), we may assume that $\Phi \circ \pi : \tilde{X} \longrightarrow \mathbb{P}^N$ is a birational morphism. Denote by M the movable part of $|126K_{\tilde{X}}|$. Then $126\pi^*(K_X) \geq M = \Phi\circ\pi^*(H)$ for a very ample divisor H on $\mathbb{P}^N$. Note that the image of $\tilde{X} \neq \mathbb{P}^3$; we see that $N > 3$ and that:

$$(126\pi^*(K_X))^3 \geq H^3 \geq 2,$$

which at least gives $K_X^3 \geq \frac{1}{63\cdot 126^2}$. We are done. □

Remark 4.14. – We will develop some more methods and more detail classification to estimate the lower bound of K_X^3 in our next paper, where a sharp bound is obtained. To curb the length of this paper, we have to cut out other details here.

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