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# SINGULARITIES OF THE SCATTERING KERNEL FOR TRAPPING OBSTACLES

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ABSTRACT. – It is shown that for certain classes of trapping obstacles $K$ in $\mathbb{R}^n$ there exists a sequence of scattering rays in the exterior of $K$ with sojourn times $T_m \to \infty$ such that $-T_m$ is a singularity of the scattering kernel for all $m$.

## 1. Introduction

Let $\Omega \subset \mathbb{R}^n, n \geq 2$, be an open and connected domain with $C^\infty$ boundary $\partial\Omega$ and bounded complement

$$K = \mathbb{R}^n \setminus \Omega \subset \{x \in \mathbb{R}^n :| x |\leq \rho_0\}.$$

Consider the problem

$$(1) \qquad \begin{cases} (\partial_t^2 - \Delta_x)u = 0 \text{ in } \mathbb{R} \times \Omega, \\ u = 0 \text{ on } \mathbb{R} \times \partial\Omega, \\ u(0,x) = f_1(x), \partial_t u(0,x) = f_2(x). \end{cases}$$

Associated to (1) is a *scattering operator*

$$S(\lambda) : L^2(S^{n-1}) \longrightarrow L^2(S^{n-1}), \lambda \in \mathbb{R}.$$

The kernel $a(\lambda, \theta, \omega)$ of the operator $S(\lambda) - Id$, called *scattering amplitude*, depends analytically on $\omega, \theta \in S^{n-1}$ (*see* [LP1], [LP2]). For fixed $(\theta, \omega) \in S^{n-1} \times S^{n-1}$, $a(\lambda, \theta, \omega)$ is a tempered distribution in $\lambda$ and

$$a(\lambda, \theta, \omega) = \left(\frac{2\pi}{i\lambda}\right)^{(n-1)/2} \mathcal{F}_{t\to\lambda} s(t, \theta, \omega).$$

Here $\mathcal{F}_{t\to\lambda}$ denotes the *Fourier transform* and the distribution $s(t, \theta, \omega)$ is called the *scattering kernel* (*see* [Ma], [P]). For the applications concerning inverse scattering

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problems it is convenient to examine the singularities of the scattering kernel which one can observe for $t$ in some bounded interval, while the scattering amplitude is related to the Fourier transform taking account of the global behaviour of $s(t, \theta, \omega)$ on $\mathbb{R}$.For $n$ odd the operator $S(\lambda)$ and the distribution $a(\lambda, \theta, \omega)$ admit meromorphic continuation in $\mathbf{C}$ with poles $\lambda_j$, Im $\lambda_j < 0$, which are independent of $\theta$ and $\omega$. For $n$ even the operator $S(\lambda)$ admits a meromorphic continuation on the Riemann logarithmic surface $\Xi = \{z \in \mathbf{C} : -\infty < \arg z < +\infty\}$ (*see* [LP1], [LP2]).

One can characterize the poles $\lambda_j$ using the *modified resolvent* of the Laplacian in $\Omega$ given by

$$R_{\varphi,\psi}(\lambda) = \varphi(x) R(\lambda) \psi(x).$$

Here the operator

$$R(\lambda) : C_0^\infty(\overline{\Omega}) \ni f \mapsto u(x, \lambda) \in C^\infty(\overline{\Omega}), \qquad \mathrm{Im}\lambda \geq 0,$$

is determined by the $(-i\lambda)$-outgoing solution $u(x, \lambda)$ of the Dirichlet problem for the reduced wave equation

$$\begin{cases} (\Delta + \lambda^2) u(x, \lambda) = f & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega \end{cases}$$

and the functions $\varphi(x)$, $\psi(x) \in C_0^\infty(\mathbf{R}^n)$ are chosen to be equal to 1 in some neighbourhood of the obstacle $K$. Then $R_{\varphi,\psi}(\lambda)$ admits a meromorphic continuation in $\mathbf{C}$ for $n$ odd and in $\Xi$ for $n$ even the poles of which and their multiplicities coincide with those of the $\lambda_j$'s. Moreover, the poles $\lambda_j$ do not depend on the choice of $\varphi$ and $\psi$ (*see* [LP1], [V], [Vo1], [Vo2]). Below we denote by $\Lambda$ the *set of scattering poles*.Given $(x, \xi) \in T^*(\partial K) \setminus \{0\}$, consider a geodesic segment $c(t)$ on $\partial K$ (with respect to the standard metric) with $c(0) = x$ and $\dot{c}(0) = \xi$, and let $\kappa(t)$ be its curvature at $c(t)$ with respect to normal to $\partial K$ pointing into the interior of $K$. The normal (sectional) curvature of $\partial K$ at $x$ in direction $\xi$ is said to *vanish of infinite order*, if $\kappa(t)$ and all its derivatives vanish at $t = 0$. If some of the derivatives of $\kappa(t)$ (this may be the 0th derivative, *i.e.* the function $\kappa(t)$ itself) does not vanish at $t = 0$ and the first non-zero derivative at $t = 0$ is positive, then $(x, \xi)$ is called a *diffractive point*. Finally, if $\kappa(t) \geq 0$ on some open interval $t \in (-\epsilon, \epsilon)$, $\epsilon > 0$, we will say that $\partial K$ is *convex at $x$ in direction $\xi$*.

Denote by $\mathcal{K}$ the class of obstacles $K$ having the property: for each $(x, \xi) \in T^*(\partial K)$ if the normal sectional curvature of $\partial K$ at $x$ in direction $\xi$ vanishes of infinite order, then $\partial K$ is convex at $x$ in direction $\xi$. Clearly $\mathcal{K}$ contains the class $\mathcal{K}_0$ of all obstacles $K$ the normal sectional curvature of which does not vanish of infinite order.

In what follows we assume that $K \in \mathcal{K}$. Fix an open ball $B_0$ of radius $\rho_0$ containing $K$. For each $\xi \in S^{n-1}$ denote by $Z_\xi$ the hyperplane tangent to $B_0$ and orthogonal to $\xi$ such that the halfspace $H_\xi$, determined by $Z_\xi$ and having $\xi$ as an inner normal, contains $K$. It follows from $K \in \mathcal{K}$ that the generalized Hamiltonian flow $F_t$ related to the wave operator $\partial_t^2 - \Delta_x$ is well defined in $S^*(\Omega)$ (*see* [MS] or Section 24.3 in [H]) and for $(x, \xi) \in S^*(\Omega)$ we denote by

$$\gamma(x, \xi) = \{F_t(x, \xi) : t \in [0, \infty)\}$$

the *generalized bicharacteristic* passing trough $(x, \xi)$ for $t = 0$. Let $\pi : S^*(\Omega) \longrightarrow \Omega$ be the standard projection. Given $z = (x, \xi) \in S^*(\Omega)$, we say that $\gamma(z)$ is a *trapping ray* for $K$, if $\pi(\gamma(z)) \subset B_0$, that is the geodesic issued from $z$ stays in $B_0$ for $t \in [0, \infty)$. Denote by $Z^{(\infty)}$ the set of all $z \in S^*(\Omega)$ so that $\gamma(z)$ is trapping. We shall say that $\gamma(z)$ is a *regular trapping ray* for $K$ if $z = (x, \xi) \in Z^{(\infty)}$, $x$ is not an interior point of $\{y \in \mathbb{R}^n : (y, \xi) \in Z^{(\infty)}\}$, and there exists an open neighbourhood $\mathcal{O}$ of $x$ in $\mathbb{R}^n$ such that for almost all $y \in \mathcal{O}$ (with respect to the Lebesgue measure in $\mathbb{R}^n$) the bicharacteristic $\gamma(y, \xi)$ does not contain diffractive points.

DÉFINITION. – *An obstacle $K \in \mathcal{K}$ is called* trapping *if the set $Z^{(\infty)}$ is not empty. If $K \in \mathcal{K}$ and there exist a regular trapping ray for $K$, then $K$ will be called a* regular trapping obstacle.

Notice that if there exists a generalized geodesic of $\partial_t^2 - \Delta_x$ which stays in $\overline{\Omega}$ for $t \geq 0$, then the set $Z^{(\infty)}$ is not empty and the obstacle K is trapping. This follows from the continuity of the generalized Hamiltonian flow $F_t$ (*see* [MS] or Section 5 in [PS2]).

For $\epsilon > 0$, $d > 0$ introduce the domain

$$U_{\epsilon,d} = \{z \in \mathbf{C} : d - \epsilon \log(1 + |z|) \leq \Im z \leq 0\}.$$

For $n$ even in the definition of $U_{\epsilon,d}$ we add the condition $-\pi/2 < \arg z < 3\pi/2$. It is well known (*see* [V]) that for non-trapping obstacles there exist $\epsilon > 0$, $d > 0$ so that $U_{\epsilon,d} \cap \Lambda = \emptyset$ and with some constants $C > 0$, $\alpha \geq 0$ for all $\lambda \in U_{\epsilon,d}$ we have

$$\|R_{\varphi,\psi}(\lambda) f\|_{H^1(\Omega)} \leq C e^{\alpha|\Im \lambda|} \|f\|_{L^2(\Omega)}. \tag{2}$$

For $n$ odd and obstacles having at least one ordinary reflecting ray $\gamma(z)$ with $z \in Z^{(\infty)}$, Ralston [Ra] proved that for all $t \geq 0$ we have $\|Z(t)\| = 1$, where $Z(t)$ is the semi-group introduced in Chapter 3 in [LP1]. This leads to

$$\sup_{\lambda \in \mathbb{R}, \|f\|_{L^2(\Omega)} = 1} \|R_{\varphi,\psi}(\lambda) f\|_{H^1(\Omega)} = +\infty.$$

One expects that for trapping obstacles we have $U_{\epsilon,d} \cap \Lambda \neq \emptyset$ for all $\epsilon > 0$, $d > 0$. This fact has been proved in some cases (*see* [BGR], [G], [I1], [I2], [I3], [Fa1],[Fa2]).

It is common to the works just cited that one obtains complete information on the dynamics of the rays sufficiently close to trapping ones, and the existence of periodic rays plays an essential role in the analysis of the singularities of the trace of the kernel $E(t, x, y)$ of $\cos(t\sqrt{-\Delta})$. Assuming only the condition $Z^{(\infty)} \neq \emptyset$, in general one can deal with generalized trapping rays and some rays $\gamma(z)$ with $z$ sufficiently close to $\partial Z^{(\infty)}$ should produce singularities $-T_m \to -\infty$ of the scattering kernel $s(t, \theta_m, \omega_m)$. An obstacle $K$ will be said to have the *property (S)* if there exists a sequence $(\omega_m, \theta_m) \in \mathbf{S}^{n-1} \times \mathbf{S}^{n-1}$ and reflecting $(\omega_m, \theta_m)$-rays $\gamma_m$ with sojourn times $T_m \longrightarrow +\infty$ so that

$$-T_m \in \operatorname{sing\,supp} s(t, \theta_m, \omega_m), \qquad \forall m \in \mathbf{N}. \tag{3}$$

It is natural to make the following

CONJECTURE. – *Every regular trapping obstacle $K \in \mathcal{K}$ has the property $(S)$.*

We refer the reader to Section 2 for the definitions of reflecting rays, sojourn times, etc. Notice that if $-T_m$ is isolated in sing supp $s(t, \theta_m, \omega_m)$ and if $T_m$ is the sojourn time of a non-degenerated ordinary reflecting $(\omega_m, \theta_m)$ ray $\gamma_m$, one can determine explicitely the leading singularity of the scattering kernel near $-T_m$, provided there are no $(\omega_m, \theta_m)$-rays different from $\gamma_m$ with sojourn time $T_m$ (*see* Section 9.1 in [PS1] for $n$ odd and (4) for $n$ even). Therefore, if $(S)$ holds, according to Theorem 2.3 in [PS2], one concludes that either for all $\epsilon > 0$ and $d > 0$ we have $U_{\epsilon,d} \cap \Lambda \neq \emptyset$ or there exist $\epsilon > 0$ and $d > 0$ so that $R_{\varphi,\psi}(\lambda)$ is analytic in $U_{\epsilon,d}$ but for all $\alpha \geq 0$, $p \in \mathbb{N}$, $k \in \mathbb{N}$ we have

$$\sup_{\lambda \in U_{\epsilon,d},\ \|f\|_{H^k(\Omega)}=1} (1+|\lambda|)^{-p} e^{-\alpha|\Im\lambda|} \|R_{\varphi,\psi}(\lambda) f\|_{H^1(\Omega)} = +\infty.$$

The latter leads either to the existence of poles in $U_{\epsilon,d}$ or to a polynomial blow-up of the norm of $R_{\varphi,\psi}(\lambda)$. It seems that for *general trapping obstacles* this should be considered as optimal, provided we do not have precise information on the dynamics of the rays close to trapping ones and if the existence of periodic rays is not assumed.

The aim of this paper is to prove that a class of regular trapping obstacles $K \subset \mathcal{K}_0$ in $\mathbb{R}^n, n \geq 3$, satisfying an additional condition (*cf.* condition $(F)$ in Section 4), have the property (S). In particular, we show that all regular trapping obstacles in $\mathbb{R}^2$ have this property. Moreover, if $D$ is a regular trapping obstacle in $\mathbb{R}^2$ with smooth boundary $\partial D$ symmetric with respect to a line $L$, then the obstacle $K \subset \mathbb{R}^3$ obtained by rotating $D$ about $L$ has the property (S). For these obstacles one can also apply Theorem 2.3 in [PS2] mentioned above. In the special case when $K$ is a finite disjoint union of strictly convex bodies, (S) was established in [PS2]. Section 6 below contains another result concerning the case of several disjoint convex bodies.

Our first motivation to examine the property (S) came from Theorem 2.3 of [PS2]. Another motivation is related to the inverse scattering result obtained by one of the authors (*see* [St1], [St2]). This result says that for a large cass of obstacles the knowledge of all singularities of $s(t, \theta, \omega)$ for a dense set of directions $(\omega, \theta) \in S^{n-1} \times S^{n-1}$ determines uniquely the obstacle. Consequently, the sojourn times can be considered as *scattering data*. Clearly for obstacles satisfying $(S)$ some sojourn times can be observed only after a sufficiently large time. Moreover, if $K$ has an additional property (*see* condition (ND) in Section 2), then for each $m \in \mathbb{N}$ there exists a set $\Pi_m \subset S^{n-1} \times S^{n-1}$ with positive measure $\epsilon_m > 0$ so that the $(\omega, \theta)$-rays with $(\omega, \theta) \in \Pi_m$ produce singularities $-T_m \leq -m$. It is interesting to construct examples when some part of $\partial\Omega$ cannot be determined from the sojourn times in any bounded time interval.

The definition of *regular* trapping obstacles probably deserves a few comments. If for an open neighbourhood $\mathcal{O}$ of a point $x$, $(x, \xi) \in \partial Z^{(\infty)}$, all generalized rays $\gamma(y, \xi)$ with $y \in \mathcal{O}$ contain diffractive segments, then the map $J_\gamma(y)$ cannot be defined and we are unable to study the singularities related to these rays. On the other hand, it follows from the result in Section 3 that the points $u$ for which the rays $\gamma(u, \xi)$ contain gliding segments form a set of Lebesgue measure zero on $Z_\xi$. It is probably not a coincidence that in the analysis of the exact controlability of solutions of the wave equation with a control given on a part $(0,T)\times\{\omega\}\subset\mathbb{R}\times\partial\Omega$ (*see* [BLR]) the generalized rays containing diffractive points are exluded. The geometric condition established in [BLR] says that every generalized ray must pass over $(0,T)\times\{\omega\}$ either at a point of reflection or at a gliding point.

## 2. Preliminaries

Let $K\in\mathcal{K}$ be an obstacle in $\mathbb{R}^n$, $n\geq 2$. As in Section 1, fix an open ball $B_0$ of radius $\rho_0$ containing $K$. For $\xi\in S^{n-1}$ define the hyperplane $Z_\xi$ as before. Let $\omega\in S^{n-1},\theta\in S^{n-1}$. An $(\omega,\theta)$*-ray* in $\overline{\Omega}$ is a curve of the form $\gamma=\mathrm{Im}\,\Gamma$, where $\Gamma(t):\mathbb{R}\longrightarrow\overline{\Omega}$ is the natural projection on $\overline{\Omega}$ of a *generalized bicharacteristic* of the wave equation in $T^*(\overline{\Omega}\times\mathbb{R})$ (*cf.* [MS] or Section 24.3 in [H]) such that there exist constants $a<b$ with $\Gamma'(t)=\omega$ for $t\leq a$ and $\Gamma'(t)=\theta$ for $t\geq b$. Geometrically, such a curve $\gamma$ is the trajectory of a point incoming from infinity with direction $\omega$, moving with constant velocity in $\Omega$, and outgoing to infinity with direction $\theta$ (*cf.* [PS1], Chapter 2). If $\gamma$ meets the boundary $\partial\Omega$ transversally, then $\gamma$ is reflecting at $\partial\Omega$ following the usual law of geometrical optics. In general, an $(\omega,\theta)$-ray $\gamma$ may have segments lying entirely on $\partial\Omega$; these segments, called *gliding segments*, are geodesics with respect to the standard metric on $\partial\Omega$. If $\gamma$ does not contain gliding segments on $\partial\Omega$ and has only finitely many reflection points, it is called a *reflecting* $(\omega,\theta)$*-ray* in $\overline{\Omega}$. If moreover $\gamma$ has no segments tangent to $\partial K$, then it is called an *ordinary* reflecting $(\omega,\theta)$-ray.

The *sojourn time* $T_\gamma$ of an $(\omega,\theta)$-ray $\gamma$, introduced by Guillemin [Gu], is defined by $T_\gamma=T'_\gamma-2\rho_0$, where $T'_\gamma$ is the length of this part of $\gamma$ which is contained in $H_\omega\cap H_{-\theta}$.

Let $\gamma$ be an ordinary reflecting $(\omega,\theta)$-ray in $\overline{\Omega}$ with successive reflection points $x_1,\ldots,x_k$ on $\partial K$. In this case we have

$$T_\gamma=\langle\omega,x_1\rangle+\sum_{i=1}^{k-1}\|x_i-x_{i+1}\|-\langle\theta,x_k\rangle,$$

where $<\ ,\ >$ denotes the standard inner product in $\mathbb{R}^n$ (*see* [Gu] or Section 2.4 in [PS1]). Denote by $u_\gamma$ the orthogonal projection of $x_1$ on $Z=Z_\omega$. Then there exists a neighbourhood $W=W_\gamma$ of $u_\gamma$ in $Z$ such that for every $u\in W$ there are unique $\theta(u)\in S^{n-1}$ and points $x_1(u),\ldots,x_k(u)\in\partial K$ which are the successive reflection points of a reflecting $(\omega,\theta(u))$-ray in $\overline{\Omega}$ passing through $u$. Setting $J_\gamma(u)=\theta(u)$, we obtain a smooth map $J_\gamma:W_\gamma\longrightarrow S^{n-1}$ and the ray $\gamma$ is called *non-degenerate* if $\det dJ_\gamma(u_\gamma)\neq 0$.

For trapping obstacles it is not difficult to construct a sequence of rays $\gamma_m$ with $T_m\longrightarrow+\infty$. (*see* Section 5 in [PS2]). The problem is to construct the sequence in such a way that $-T_m$ are singularities, and a natural way to try to do that is to make all $\gamma_m$ non-degenerate. However in general the latter is also a difficult problem. The difficulty comes from the fact that (especially for rays $\gamma$ with many reflections) the map $J_\gamma$ depends in a very complicated way on the geometry of the boundary $\partial K$ near the reflection points.

It follows from the results in [CPS], [PS2], [St1], that to obtain (3) for a given trapping obstacle $K$, it is sufficient to establish the following property.

(ND) $\begin{cases}\text{There exists a sequence } (\omega_m,\theta_m)\in\mathbf{S}^{n-1}\times\mathbf{S}^{n-1} \text{ and non-degenerate}\\ \text{reflecting } (\omega_m,\theta_m)\text{-rays } \gamma_m \text{ with sojourn times } T_m\longrightarrow+\infty.\end{cases}$

It is easy to prove that the property (S) follows from (ND). In fact, it is sufficient to construct a sequence of ordinary reflecting non-degenerate $(\omega_m, \theta_m)$-rays $\gamma_m$ with sojourn times $T_m \longrightarrow +\infty$ so that for each $m$ the pair $(\omega_m, \theta_m)$ has the followig properties:

(i) if $\delta$ and $\gamma$ are different ordinary reflecting $(\omega_m, \theta_m)$-rays, then $T_\delta \neq T_\gamma$;

(ii) there are no $(\omega_m, \theta_m)$-rays in $\overline{\Omega}$ containing tangent or gliding segments.

To arrange (i) we approximate $(\omega_m, \theta_m)$ by suitable directions using the results in [PS2], while for (ii) we make an approximation applying the results in [St1] concerning generalized rays with gliding segments. More precisely, there exists a dense set $\mathcal{R} \subset S^{n-1} \times S^{n-1}$ such that for all $(\omega, \theta) \in \mathcal{R}$ every $(\omega, \theta)$-ray in $\Omega$ is ordinary reflecting. Therefore, from the Poisson relation for the scattering kernel (established in [CPS] for $n$ odd and in Appendix for $n$ even) and the continuity of the generalized Hamiltonian flow (*see* [MS]), we obtain that for a sequence of directions $(\omega'_m, \theta'_m)$ there exist ordinary reflecting non-degenerate $(\omega'_m, \theta'_m)$-rays $\delta_m$ with sojourn times $T'_m \to \infty$. Moreover, $-T'_m$ are isolated in sing supp $s(t, \theta'_m, \omega'_m)$ and, following the argument in Section 9.1 in [PS1] which works without any change for all dimensions $n \geq 2$, the leading singularity of the scattering kernel at $-T'_m$ can be described as follows. Assume that $\gamma$ is non-degenerate ordinary reflecting $(\omega, \theta)$-ray with $m$ reflections. Let $-T_\gamma$ be an isolated singularity of $s(t, \theta, \omega)$ and assume that there are no $(\omega, \theta)$-rays different from $\gamma$ with sojourn time $T_\gamma$. Take a function $\rho(t) \in C_0^\infty(\mathbb{R})$ so that supp $\rho \subset (-1, 1)$, $\rho(0) = 1$. Then for all $n \geq 2$ and $\epsilon > 0$ sufficiently small we have

$$\langle s(t,\theta,\omega),\ \rho\Big(\frac{t+T_\gamma}{\epsilon}\Big)e^{-i\lambda t}\rangle = (2\pi)^{(1-n)/2}(-1)^m exp\Big(i\frac{\pi}{2}\beta_\gamma + i\lambda T_\gamma\Big) \tag{4}$$

$$\times\left|\frac{\det dJ_\gamma(u_\gamma)\langle \nu(q_1),\omega\rangle}{\langle \nu(q_m),\theta\rangle}\right|^{-1/2} \lambda^{(n-1)/2} + \mathcal{O}\Big(|\lambda|^{(n-3)/3}\Big),$$

where $\beta_\gamma \in \mathbf{Z}$ is related to a Maslov index and $q_1,\ \ q_m$ denote the first and the last reflection points of $\gamma$, respectively.

## 3. Tangent and gliding rays

Let $K \in \mathcal{K}_0$. Fix an open ball $B_0$ containing $K$ in its interior. Given $\omega \in S^{n-1}$, define $Z_\omega$ as in Section 1. For $u \in Z_\omega$ let $\gamma_\omega(u)$ be the generalized geodesic in $\Omega = \Omega_K$ issued from $(u, \omega)$. Denote by $Z_\omega^{(\infty)}$ the set of those $u \in Z_\omega$ such that $\gamma_\omega(u)$ is contained in a compact subset of $\mathbb{R}^n$, that is $Z_\omega^{(\infty)} = Z_\omega \cap Z^{(\infty)}$. Then $Z_\omega^{(\infty)}$ is a compact subset of $Z_\omega$, so

$$U_\omega = Z_\omega \setminus Z_\omega^{(\infty)}$$

is an open unbounded subset of $Z_\omega$. Clearly for each $u \in U_\omega$ there exists a (unique) $\theta_\omega(u) \in S^{n-1}$ such that $\gamma_\omega(u)$ is part of an $(\omega, \theta_\omega(u))$-ray in $\Omega$. Denote by $T_\omega(u)$ the *sojourn time* of this ray. It follows from [MS] that the two maps

$$J_\omega : U_\omega \longrightarrow S^{n-1} \ \ ,\ J_\omega(u) = \theta_\omega(u),$$

and $T_\omega : U_\omega \longrightarrow \mathbb{R}$ are continuous.

Next, denote by $Z_\omega^{(t)}$ the set of those $u \in Z_\omega$ such that the ray $\gamma_\omega(u)$ contains a point $(x, \xi) \in S^*(\partial K)$ (that is, $\gamma_\omega(u)$ is tangent to $\partial K$ at $x$). Notice that $Z_\omega^{(t)}$ contains all $u \in Z_\omega$ such that $\gamma_\omega(u)$ has at least one non-trivial gliding segment on $\partial K$.

Clearly for $u \in U_\omega \setminus Z_\omega^{(t)}$, the ray $\gamma_\omega(u)$ consists of finitely many straightline segments and has only transversal reflections at $\partial K$.

Denote by $U_\omega^{(t)}$ the set of these $u \in U_\omega \cap Z_\omega^{(t)}$ such that all tangent points of the $(\omega, \theta_\omega(u))$-ray $\gamma_\omega(u)$ are diffractive points. Thus, for $u \in U_\omega^{(t)}$, $\gamma_\omega(u)$ is a reflecting ray which does not contain gliding segments on $\partial K$.

It follows from Section 3 of [PS2] that there exists a subset $\Lambda$ of full Lebesgue measure in $S^{n-1}$ such that whenever $\omega \in \Lambda$, the set $U_\omega^{(t)}$ has Lebesgue measure zero in $U_\omega$. Moreover, for such $\omega$, $U_\omega^{(t)}$ is a $\sigma$-compact set, *i.e.* it is a countable union of compact sets of measure zero.

LEMMA 3.1. – *Let $\omega \in S^{n-1}$ be arbitrary. There exist a countable family of $(n-2)$-dimensional submanifolds $\mathcal{I}_m$ of $Z_\omega$ such that $Z_\omega^{(t)} \setminus U_\omega^{(t)} \subset \bigcup_m \mathcal{I}_m$.*

*Proof.* – Given integers $s \geq 0$, $k \geq 1$, denote by $\Sigma_{s,k}(\omega)$ the set of those $u \in U_\omega$ such that there exists a point $\sigma(u) = (y(u), \eta(u)) \in \gamma_\omega(u) \cap S^*(\partial K)$ such that the normal curvature of $\partial K$ at $y(u)$ in direction $\eta(u)$ vanishes exactly of order $k$ and that part of $\gamma_\omega(u)$ which is between $(u, \omega)$ and $\sigma(u)$ has exactly $s$ transversal reflection points and no gliding segments (however it may have some tangencies to $\partial K$). Clearly,

$$Z_\omega^{(t)} \setminus U_\omega^{(t)} \subset \bigcup_{s \geq 0, k \geq 1} \Sigma_{s,k}(\omega),$$

so it is enough to show that each $\Sigma_{s,k}(\omega)$ is contained in a countable union of $(n-2)$-dimensional submanifolds of $Z_\omega$.

Fix integers $s, k$ and a point $u' \in \Sigma_{s,k}(\omega)$. Let $F_t$ be the generalized geodesic flow in $S^*(\Omega)$ and let $t_0 > 0$ be such that

$$F_{t_0}(u', \omega) = \sigma(u').$$

It follows by [MS] (*cf.* also Section 24.3 in [H]) that there exist an open neighbourhood $\mathcal{O}$ of $\sigma(u')$ in $T^*(\mathbb{R}^n)$ and local symplectic coordinates $(x, \xi) = (x_1, \ldots, x_n; \xi_1, \ldots, \xi_n)$ in $\mathcal{O}$ such that $\sigma(u') = 0$,

$$T^*(\Omega) \cap \mathcal{O} = \{(x, \xi) : x_1 \geq 0\}, \quad \partial T^*(\Omega \cap \mathcal{O}) = T^*_{\partial\Omega}(\Omega \cap \mathcal{O}) = \{(x, \xi) : x_1 = 0\},$$

and there exists a smooth (Hamiltonian) function of the form

$$p(x, \xi) = \xi_1^2 - r(x, \xi')$$

such that the generalized bicharacteristics in $T^*(\Omega)$ (possibly changing the natural parametrization along them) are precisely the integral curves of the generalized Hamiltonian flow of $p$. Here and in what follows we use the notation

$$x' = (x_2, \ldots, x_n), \qquad \xi' = (\xi_2, \ldots, \xi_n).$$

Also we have

$$S^*(\Omega) \cap \mathcal{O} = p^{-1}(0)$$

and the set of glancing points $G$ is given in $\mathcal{O}$ by

$$G = \{(x, \xi) : x_1 = \xi_1 = 0\} \cap p^{-1}(0).$$

For $(0, x'; 0, \xi') \in \mathcal{O}$ set

$$r_0(x', \xi') = r(0, x'; \xi'), \qquad r_1(x', \xi') = \frac{\partial r}{\partial x_1}(0, x'; \xi').$$

Below we assume that $k \geq 1$. It follows from [MS] (*see* also Lemma 24.3.1 in [H]) that in $\mathcal{O}$ the set $G^{k+2}$ of points $(y, \eta) \in T^*(\Omega)$ so that the curvature of $\partial\Omega$ at $y$ in direction $\eta$ vanishes of order at least $k$ has the form

$$G^{k+2} = \{(0, x'; 0, \xi') : r_0(x', \xi') = 0 \text{ and } H_{r_0}^j r_1(x', \xi') = 0, j = 0, 1, \ldots, k-1\}.$$

By assumption $\sigma(u') \in G^{k+2} \setminus G^{k+3}$, so $H_{r_0}^k r_1(0) \neq 0$ which (*cf.* again Lemma 24.3.1 in [H]) is equivalent to $H_p^{k+2} x_1(0) \neq 0$. We may assume that $\mathcal{O}$ is so small that

$$H_p^{k+2} x_1(x, \xi) \neq 0, \qquad (x; \xi) \in \mathcal{O}.$$

Then

$$S = \{(x; \xi) \in \mathcal{O} : p(x, \xi) = H_p^{k+1} x_1(x, \xi) = 0\}$$

is a symplectic submanifold of $T^*(\Omega)$ with $\dim S = 2n - 2$ and $S \subset p^{-1}(0) = S^*(\Omega)$.

We claim that $\mathcal{M} = S \cap G$ is a symplectic submanifold of $S$ with $\dim \mathcal{M} = 2n - 4$. Indeed,

$$\mathcal{M} = \{(0, x'; 0, \xi') \in \mathcal{O} : r_0(x', \xi') = H_{r_0}^{k-1} r_1(x', \xi') = 0\} \subset G,$$

and in $G$ we have $\{r_0, H_{r_0}^{k-1} r_1\} = H_{r_0}^k r_1 \neq 0$. Now the Darboux lemma implies that $\mathcal{M}$ is a symplectic submanifold of $G$ (and therefore of $S$) of codimension 2.

Take small open neighbourhoods $U'$ of $u'$ in $Z_\omega$, $V'$ of $\omega$ in $S^{n-1}$. Choose a number $t' \in (0, t_0)$ so close to $t_0$ that the segment $\{F_t(u', \omega) : t' \leq t < t_0\}$ of $\gamma_\omega(u')$ is contained in $\mathcal{O}$ and has no common points with $\partial K$. Let

$$F_{t'}(u', \omega) = (u'', \eta)$$

and let $A$ be a hyperplane in $\mathbb{R}^n$ containing $u''$ and transversal to $\eta$. There exist $\lambda > t_0 - t'$ close to $t_0 - t'$ and an open neighbourhood $W'' = U'' \times V''$ of $(u'', \eta)$ in $S^*(A)$ such that $F_t(W'') \subset \mathcal{O}$ for all $|t| < \lambda$.

Next, let $x_1, \ldots, x_s$ be the consecutive transversal reflection points of $\gamma_\omega(u')$. For each $i \leq s$, let $\Gamma_i$ be an open neighbourhood of $x_i$ in $\partial K$ so that

$$\Gamma_i \cap \{F_t(u', \omega) : 0 \leq t \leq t'\} = \{x_i\}.$$

We may assume that these neighbourhoods and the neighbourhood $W' = U' \times V'$ of $(u', \omega)$ are so small that whenever $(u, \xi) \in W'$, the trajectory $\{F_t(u, \xi) : 0 \leq t \leq t'\}$ has exactly $s$

transversal reflections $y_1, \ldots, y_s$ at $\partial K$ and $y_i \in \Gamma_i$ for each $i = 1, \ldots, s$ and this trajectory has no tangent points to $\Gamma = \Gamma_1 \cup \ldots \cup \Gamma_s$. Now for $(u, \xi) \in W'$ define the trajectory $\tilde{\gamma}(u, \xi)$ to be the billiard trajectory issued from $(u, \xi)$ which has reflections at $\Gamma$ only (*i.e.* the rest of $\partial K$ is disregarded). Let $\mathcal{P}_1(u, \xi)$ be the first intersection point of $\tilde{\gamma}(u, \xi)$ with the set $W''$. Assuming $W'$ is small enough, we get a well-defined symplectic map

$$\mathcal{P}_1 : W' \longrightarrow W''.$$

Notice that for $(u, \xi) \in W' \cap \Sigma_{s,k}(\omega)$, $\mathcal{P}_1(u, \xi)$ coincides with the first intersection point of the trajectory $\{F_t(u, \xi) : t \geq 0\}$ with $W''$.

Since

$$\mathcal{L}_0 = \{(u, \omega) : u \in U'\}$$

is a Lagrangian submanifold of $W' \subset S^*(Z_\omega)$, it follows that $\mathcal{L}' = \mathcal{P}_1(\mathcal{L}_0)$ is a Lagrangian submanifold of $W''$.

Next, we define the map

$$\mathcal{P}_2 : W'' \longrightarrow S$$

in the following way. Given $\rho \in W''$, consider the integral curve of the vector field $H_p$ in $T^*(\mathbb{R}^n)$ (this curve is actually in $S^*(\mathbb{R}^n)$) issued from $\rho$ and denote by $\mathcal{P}_2(\rho)$ its intersection point with $S$. If $W''$ (resp. $W'$) is small enough, $\mathcal{P}_2$ is a well-defined smooth symplectic map. Hence $\mathcal{L}'' = \mathcal{P}_2(\mathcal{L}')$ is a Lagrangian submanifold of $S$. It now follows from Proposition 3.6 in [St1] that there exists an open neighbourhood $\mathcal{O}'$ of $\sigma(u')$ in $\mathcal{O}$ with

$$\mathcal{L}'' \cap \mathcal{M} \cap \mathcal{O}' \subset \mathcal{L}$$

for some Lagrangian submanifold $\mathcal{L}$ of $\mathcal{M}$. In particular $\dim \mathcal{L} = n - 2$. Set

$$W = (\mathcal{P}_2 \circ \mathcal{P}_1)^{-1}(\mathcal{O}'), \qquad \mathcal{I} = (\mathcal{P}_2 \circ \mathcal{P}_1)^{-1}(\mathcal{L}).$$

Then $W$ is an open neighbourhood of $(u', \omega)$ in $S^*(Z_\omega)$ with $W \subset W'$, while $\mathcal{I}$ is an $(n-2)$-dimensional submanifold of $\mathcal{L}_0$. Finally, notice that for the set $\Sigma_{s,k}(\omega)$, defined in the beginning of this proof, we have $(\Sigma_{s,k}(\omega) \times \{\omega\}) \cap W \subset \mathcal{I}$. So, there exist a neighbourhood $U_1 = \mathrm{pr}_1(W)$ of $u'$ in $Z_\omega$ and a smooth $(n-2)$-dimensional submanifold $\mathcal{I}_1 = \mathrm{pr}_1(\mathcal{I})$ of $Z_\omega$ such that $\Sigma_{s,k}(\omega) \cap U_1 \subset \mathcal{I}_1$.

The above local argument shows that $\Sigma_{s,k}(\omega)$ can be covered by a finite union of $(n-2)$-dimensional submanifolds of $Z_\omega$. This completes the proof of the assertion. □

## 4. Trapping obstacles

Throughout this section we assume that the obstacle $K \in \mathcal{K}_0$ satisfies the following condition:

$$(F)\quad\left\{\begin{array}{l}\text{There exist } \omega_0 \in S^{n-1} \text{ and a boundary point } u_0 \text{ of } Z^{(\infty)}_{\omega_0} \text{ such that } \gamma(u_0,\omega_0)\\ \text{is a regular trapping ray for } K \text{ and there exists an open ball } U_0 \text{ with}\\ \text{center } u_0 \text{ in } Z_{\omega_0} \text{ such that for almost all } u \in U_0, \text{ if the map } u' \mapsto J_{\omega_0}(u')\\ \text{is defined, differentiable and singular on a whole neighbourhood } V \text{ of } u\\ \text{in } Z_{\omega_0}, \text{ then } J_{\omega_0} = \text{const on some neighbourhood of } u \text{ in } Z_{\omega_0}.\end{array}\right.$$

*Remark.* – The above condition emerged from our efforts to find a geometrical condition that implies (ND). As one can easily convince himself, in general this would be quite a difficult task. It is natural to expect that (F) would be satisfied if $K$ admits a regular trapping ray with reflections from cylindrical (or close to cylindrical) parts of the boundary $\partial K$. So it certainly determines a non-trivial class of obstacles. Especially when the dimension $n$ is relatively small, it does not look so restrictive. In fact, as we shall see in the next section, every regular trapping obstacle in the plane satisfies the condition (F).

Fix $\omega_0, u_0$ and the ball $U_0$ with the properties in (F). According to the regularity of the trapping ray $\gamma(u_0,\omega_0)$, we may assume that $U_0$ is so small that for almost all $u \in U_0$ the ray $\gamma(u,\omega_0)$ has no diffractive tangent points to $\partial K$. The fact that $u_0$ is a boundary point of $Z^{(\infty)}_{\omega_0}$ implies

$$u_0 \in \overline{U_0 \cap U_{\omega_0}}.$$

Recall from Section 3 that $U_{\omega_0} = Z_{\omega_0} \setminus Z^{(\infty)}_{\omega_0}$. Thus, $U_{\omega_0} \setminus Z^{(t)}_{\omega_0} = Z_{\omega_0} \setminus (Z^{(\infty)}_{\omega_0} \cup Z^{(t)}_{\omega_0})$ is an open subset of $Z_{\omega_0}$.

PROPOSITION 4.1. – *Let $u' \in U_0$ and let $V$ be a connected open subset of $U_{\omega_0} \setminus Z^{(t)}_{\omega_0}$. If $J_{\omega_0}(u) = const$ for $u \in V$, then $T_{\omega_0}(u) = const$ on $V$.*

*Proof.* – It is enough to show that $\nabla T_{\omega_0} = 0$ on $V$. Let $\theta = J_{\omega_0}(u)$ for $u \in V$. Fix $v \in V$ and take a neighbourhood $V'$ of $v$ in $V$ such that $k(u) = k = \text{const}$ for $u \in V'$. Then

$$T_{\omega_0}(u) = \langle \omega_0, x_1(u)\rangle + \sum_{i=1}^{k-1} \|x_i(u) - x_{i+1}(u)\| - \langle x_k(u), \theta\rangle$$

for each $u \in V'$. Using this and the reflection law at each reflection point $x_i(u)$ of $\gamma(u,\omega_0)$, we get:

$$\begin{aligned}\frac{\partial T_{\omega_0}}{\partial u_j}(u) =& \left\langle \omega, \frac{\partial x_1}{\partial u_j}(u)\right\rangle\\ &+ \sum_{i=1}^{k-1}\left\langle \frac{x_i(u)-x_{i+1}(u)}{\|x_i(u)-x_{i+1}(u)\|}, \frac{\partial x_i(u)}{\partial u_j}(u) - \frac{\partial x_{i+1}}{\partial u_j}(u)\right\rangle - \left\langle \frac{\partial x_k}{\partial u_j}(u), \theta\right\rangle\\ =& \sum_{i=2}^{k-1}\left\langle \frac{\partial x_i}{\partial u_j}(u), -\frac{x_{i-1}(u)-x_i(u)}{\|x_{i-1}(u)-x_i(u)\|} + \frac{x_i(u)-x_{i+1}(u)}{\|x_i(u)-x_{i+1}(u)\|}\right\rangle\\ &+ \left\langle \frac{\partial x_1}{\partial u_j}(u), \omega_0 + \frac{x_1(u)-x_2(u)}{\|x_1(u)-x_2(u)\|}\right\rangle\\ &+ \left\langle \frac{\partial x_k}{\partial u_j}(u), -\frac{x_{k-1}(u)-x_k(u)}{\|x_{k-1}(u)-x_k(u)\|} + \theta\right\rangle = 0.\end{aligned}$$

This holds for all $j = 1, \dots, n-1$ which shows that $\nabla T_{\omega_0} = 0$ on $V'$. Consequently, $\nabla T_{\omega_0} = 0$ on $V$ and so $T_{\omega_0} = \text{const}$ on $V$. $\square$

THEOREM 4.2. – *Let the obstacle $K \in \mathcal{K}_0$ satisfy the condition* (F). *Then the property* (ND) *holds.*

*Proof.* – Take $\omega_0, u_0$ and $U_0$ as above. Without loss of generality we may assume that $Z_{\omega_0} = \mathbb{R}^{n-1} = \mathbb{R}^{n-1} \times \{0\} \subset \mathbb{R}^n$.

It follows from the assumptions about $u_0$ and Lemma 3.1 that the set $Z_{\omega_0}^{(t)} \cap U_0$ has Lebesgue measure zero in $Z_{\omega_0}$. On the other hand, $U_0 \setminus Z_{\omega_0}^{(\infty)}$ is a countable union of open connected components. Slightly changing $u_0$, if necessary (and possibly taking a smaller ball $U_0$), we may assume that $u_0$ is a boundary point of some connected component of the open subset $U_0 \setminus Z_{\omega_0}^{(\infty)}$ of $U_0$ and moreover $\omega_0$, $u_0$ and $U_0$ have the properties in (F). Using this, Fubini's theorem and (5), one gets that there exists a smooth curve $f : [0,1] \longrightarrow U_0$ such that $f(0) = u_0$, $f(s) \in U_0 \setminus Z_{\omega_0}^{(\infty)}$ for each $s \in (0,1]$, and the set

$$L = \{s \in (0,1) : f(s) \in Z_{\omega_0}^{(t)}\}$$

has Lebesgue measure zero in $[0,1]$. Since $f(L) \cap Z_{\omega_0}^{(\infty)} = \emptyset$, it is easily seen that $L$ is closed in $(0,1)$. Thus, $I = (0,1) \setminus L$ is an open subset of $(0,1)$.

We claim that there exists an infinite sequence

$$1 \geq s_1 > s_2 > \dots > s_m \to 0 \tag{6}$$

such that for all $m$, $f(s_m) \in U_0 \setminus Z_{\omega_0}^{(t)}$ and $f(s_m)$ can be approximated by points $u \in Z_{\omega_0}$ with $\det dJ_{\omega_0}(u) \neq 0$. Indeed, assume the contrary. Then there exists $\epsilon > 0$ such that for each $s \in (0,\epsilon) \cap I$ there is a connected open neighbourhood $V$ of $f(s)$ in $U_0$ such that $\det dJ_{\omega_0}(u) = 0$ for all $u \in V$. Now Proposition 4.1 implies $T_{\omega_0}(u) = \text{const}$ on $V$. So, the function

$$T : (0,\epsilon] \longrightarrow \mathbb{R}, \qquad T(s) = T_{\omega_0}(s),$$

is constant on each connected component of the open set $I \cap (0,\epsilon)$. Thus, $T(I \cap (0,\epsilon))$ is a countable subset of $\mathbb{R}$ and so it has Lebesgue measure zero in $\mathbb{R}$.

Though in general the map $T$ is not smooth at the points of $L$, it turns out that $T(L)$ also has Lebesgue measure zero in $\mathbb{R}$. This follows from $K \in \mathcal{K}_0 \subset \mathcal{K}$ and Section 3 in [St1] (*see* Lemma 3.5 there). Indeed [St1] implies that $U_{\omega_0} \setminus Z_{\omega_0}^{(\infty)}$ can be represented as a countable disjoint union $\cup_\alpha S_\alpha$ such that for each $\alpha$ and each $u \in S_\alpha$ there exists an open neighbourhood $W$ of $u$ in $U_0$ and a smooth map $T_\alpha$ such that $T_\alpha = T_{\omega_0}$ on $W \cap S_\alpha$. Consequently

$$T(L \cap f^{-1}(W \cap S_\alpha)) = (T_\alpha \circ f)(L \cap f^{-1}(W \cap S_\alpha))$$

has Lebesgue measure zero in $\mathbb{R}$. This holds for each $\alpha$ and therefore $T(L \cap f^{-1}(W))$ has Lebesgue measure zero in $\mathbb{R}$. From this it follows that $T(L)$ also has Lebesgue measure zero in $\mathbb{R}$.

Now, since $(0,\epsilon) \subset L \cup I$, we get that $f((0,\epsilon))$ has Lebesgue measure zero in $\mathbb{R}$. On the other hand, $T$ is continuous on $(0,\epsilon]$, so $T = \text{const}$ on $(0,\epsilon)$. This is a contradiction, because $u_0 \in Z_{\omega_0}^{(\infty)}$ shows that $T(s) \to \infty$ as $s \to 0$. Thus, there exists a sequence (6) having the required properties. Now taking for each $m$ a point $u'_m \in U_0 \setminus Z_{\omega_0}^{(t)}$ close to $f(s_m)$ such that $\det dJ_{\omega_0}(u'_m) \neq 0$, one proves the assertion. □

As an immediate consequence of the above one gets the following

THEOREM 4.3. – *Every obstacle $K \in \mathcal{K}_0$ satisfying the condition* (F) *has the property* (S). □

## 5. Trapping obstacles in the plane

In this section we consider obstacles $K \in \mathcal{K}$ in $\mathbb{R}^2$.

THEOREM 5.1. – *Every regular trapping obstacle $K \in \mathcal{K}$ in $\mathbb{R}^2$ has the property* (S).

Given a regular trapping $K$, there exist $\omega_0 \in S^{n-1}$ and $u_0 \in Z_{\omega_0}$ such that $\gamma(u_0, \omega_0)$ is a regular trapping ray for $K$. Then there is an open segment $U_0$ (notice that in the present case $Z_{\omega_0}$ is a line in $\mathbb{R}^2$) containing $u_0$ such that for almost all $u \in U_0$ the ray $\gamma(u, \omega_0)$ has no diffractive tangent points to $\partial K$. Fix an arbitrary $u_1 \in U_\omega$ close to $u_0$. We may assume that

$$Z_{\omega_0}^{(\infty)} \cap [u_0, u_1] = \{u_0\};$$

otherwise one can replace $u_0$ by the closest point $u' \in [u_0, u_1]$ to $u_1$ such that $u' \in Z_{\omega_0}^{(\infty)}$.

We are going to show that $Z_{\omega_0}^{(t)}$ has Lebesgue measure zero in $Z_{\omega_0}$. This will be derived from the following

LEMMA 5.2. – *Let $X$ and $Y$ be smooth curves without common points in $\mathbb{R}^2$ and let $N(x), x \in X$, be a smooth field of normal unit vectors to $X$. Given $x \in X$, denote*

$$l(x) = \{x + tN(x) : t \geq 0\}.$$

*Let $X_0$ be the set of those $x \in X$ such that $l(x)$ is tangent to the curve $Y$ at some point $y \in Y$ and the curvature of $Y$ vanishes at $y$. Then $X_0$ has Lebesgue measure zero in $X$.*

*Proof.* – Let $y(s)$ be a smooth natural parametrization of $Y$ so that $\|y'(s)\| = 1$ for all $s$. Then the curvature of $Y$ vanishes at $y(s)$ iff $y''(s) = 0$. Applying Sard's Theorem to the map $f$, $f(s) = y'(s) \in S^1$, one gets that the set

$$D = \{\xi \in S^1 : \xi = y'(s) \text{ for some } s \text{ with } y''(s) = 0\}$$

has Lebesgue measure zero in $S^1$.

Consider an arbitrary $x_0 \in X_0$. It is enough to show that there exists a neighbourhood $W$ of $x_0$ in $X$ so that $X_0 \cap W$ has Lebesgue measure zero in $X$. There are two cases.

*Case 1.* $l(x_0)$ contains focal points of the map $N$. This means that for some $t > 0$ the map $X \ni x \mapsto x + tN(x)$ is singular at $x_0$. Since $\dim X = 1$ this implies that the map

$N(x)$ is regular at $x_0$. Hence there exists an open neighbourhood $W$ of $x_0$ in $X$ such that $N$ induces a diffeomorphism $N : W \longrightarrow N(W) \subset S^1$. Consequently $N^{-1}(D) \cap W$ has Lebesgue measure zero in $X$. Since $X_0 \cap W \subset N^{-1}(D) \cap W$, it now follows that $X_0 \cap W$ has Lebesgue measure zero in $X$.

*Case 2.* $l(x_0)$ does not contain focal points of $N$. Consider an arbitrary $y_0 \in Y \cap l(x_0)$. Then there exists an open neighbourhood $G$ of $y_0$ in $\mathbb{R}^2$ such that the orthogonal projection $g : G \longrightarrow W$ is well-defined and smooth. Denote by $h$ the restriction of $g$ to $Y \cap G$. Then critical values of $h$ are those $x \in W$ for which there exists $y \in l(x) \cap Y \cap G$ such that $l(x)$ is tangent to $Y$ at $y$. Hence $X_0 \cap W$ consists of critical values of $h$ and Sard's Theorem implies that $X_0 \cap W$ has Lebesgue measure zero in $X$. $\square$

LEMMA 5.3. – *$Z^{(t)}_{\omega_0} \cap U_0$ has Lebesgue measure zero in $Z_{\omega_0}$.*

*Proof.* – Given an integer $s \geq 0$, denote by $\Sigma_s$ the set of those $u \in U_0$ for which there exists a point $\sigma(u) = (y(u), \eta(u)) \in \gamma(u, \omega_0)$ such that the normal curvature of $\partial K$ at $y(u)$ in direction $\eta(u)$ vanishes of order $\geq 1$ and that part of $\gamma(u)$ which is between $(u, \omega_0)$ and $\sigma(u)$ has no tangencies at $\partial K$ and exactly $s$ (transversal) reflection points. It follows from

$$(U_0 \cap Z^{(t)}_{\omega_0}) \setminus (\cup_s \Sigma_s) \subset U_0 \cap U^{(t)}_{\omega}$$

and the choice of $U_0$ that $(U_0 \cap Z^{(t)}_{\omega_0})$ has Lebesgue measure zero in $Z_{\omega_0}$. So it remains to show that each $\Sigma_s$ has Lebesgue measure zero in $Z_{\omega_0}$. This time we cannot use Lemma 3.1 because we only know that $K \in \mathcal{K}$ (Lemma 3.1 requires $K \in \mathcal{K}_0$ which is a stronger assumption). Fix $s$ and $u' \in \Sigma_s$. Let $x_1, \ldots, x_s$ be the first $s$ reflection points of the ray $\gamma(u', \omega_0)$ and let $x_{s+1}$ be the next common point of $\gamma(u', \omega_0)$ with $\partial K$. It follows from $u' \in \Sigma_s$ that every $x_i$ $(1 \leq i \leq s)$ is a transversal reflection point while $\gamma(u', \omega_0)$ is tangent to $\partial K$ at $x_{s+1}$ and the curvature of $\partial K$ vanishes at $x_{s+1}$. Choose an arbitrary point $x_0$ on the open segment $(x_s, x_{s+1})$ and denote by $T$ the distance between $u'$ and $x_0$ along the ray $\gamma(u')$, *i.e.* $F_T(u', \omega_0) = (x_0, *)$, where $F_t$ is the generalized geodesic flow on $S^*(\Omega)$. Taking a sufficiently small open neighbourhood $W$ of $u'$ in $Z$, we get a smooth curve $X = \mathrm{pr}_1(F_T(W \times \{\omega_0\}))$ in $\Omega$ which contains the point $x_0$ and is transversal to $\gamma(u')$ at $x_0$. Moreover for $x = F_T(u, \omega_0)$, $u \in W$, the vector $N(x) = \mathrm{pr}_2(F_T(u, \omega_0))$ is a unit normal vector to $X$ at $x$ smoothly depending on $x$. Applying Lemma 5.2 to $X$, $N$ and $Y = \partial K$, we see that the set $X_0$ of those $x \in X$ such that $l(x)$ is tangent to $\partial K$ at some $y \in \partial K$ and the curvature of $\partial K$ vanishes at $Y$ has Lebesgue measure zero in $X$. Let $f : W \longrightarrow X$ be the map induced by $F_T$. Then $f$ is a diffeomorphism and $f^{-1}(X_0) = W \cap \Sigma_s$. Therefore $W \cap \Sigma_s$ has Lebesgue measure zero in $Z_{\omega_0}$. This easily implies that $\Sigma_s$ has Lebesgue measure zero in $Z_{\omega_0}$. $\square$

Denote $T = T_{\omega_0}$. Our next aim is to show that $T(Z^{(t)}_{\omega_0} \cap [u_0, u_1])$ is a subset of Lebesgue measure zero in $\mathbb{R}$. This would be a trivial consequence of Lemma 5.3 if the function $T$ were smooth in $U_0$ near $Z^{(t)}_{\omega_0}$. However the latter is not so, and to overcome this difficulty, as in the proof of Theorem 4.2, we use an argument from [St1].

LEMMA 5.4. – *$T(Z^{(t)}_{\omega_0} \cap [u_0, u_1])$ has Lebesgue measure zero in $\mathbb{R}$.*

*Proof.* – Since $K \in \mathcal{K}$, it follows from Section 3 in [St1] (*see* Lemma 3.5 there) that $(u_0, u_1)$ can be represented as a countable disjoint union $\cup_\alpha S_\alpha$ such that for each $\alpha$ and each $u \in S_\alpha$ there exists an open neighbourhood $W$ of $u$ in $(u_0, u_1)$ and a smooth map $T_\alpha$ such that $T_\alpha = T$ on $W \cap S_\alpha$. Consequently $T(Z_{\omega_0}^{(t)} \cap W \cap S_\alpha) = T_\alpha(Z_{\omega_0}^{(t)} \cap W \cap S_\alpha)$ has Lebesgue measure zero in $\mathbb{R}$. This holds for each $\alpha$ and therefore $T(Z_{\omega_0}^{(t)} \cap W)$ has Lebesgue measure zero in $\mathbb{R}$ which proves the assertion. $\square$

*Proof of Theorem 5.1.* – As in the proof of Theorem 4.2, it is enough to show that there exists an infinite sequence of points $u_m$ in $(u_0, u_1) \cap U_{\omega_0} \setminus Z_{\omega_0}^{(t)}$ with $u_m \to u_0$ as $m \to \infty$ such that for each $m$ the point $u_m$ can be approximated by points $u \in U_0$ with $J'_{\omega_0}(u) \neq 0$. Suppose such a sequence does not exist. Possibly taking $u_1$ closer to $u_0$, we may assume that for each $u \in (u_0, u_1) \cap U_{\omega_0} \setminus Z_{\omega_0}^{(t)}$ we have $J'_{\omega_0} = 0$ on a whole neighbourhood $W(u)$ of $u$ in $U_0$. Since $\dim Z_{\omega_0} = 1$, this implies $J_{\omega_0} = \text{const}$ on $W(u)$ and by Proposition 4.1, $T = \text{const}$ on $W(u)$. Hence $T((u_0, u_1) \cap U_{\omega_0} \setminus Z_{\omega_0}^{(t)})$ is a countable subset of $\mathbb{R}$. Combining this with Lemma 5.4 gives that the set $T((u_0, u_1))$ has measure zero in $\mathbb{R}$. Since the function $T$ is continuous on $(u_0, u_1)$, this is only possible if $T = \text{const}$ on $(u_0, u_1)$. The latter is a contradiction with $u_0 \in Z_\infty$. Hence there exists a sequence $\{u_m\}$ with the required properties. $\square$

Applying Theorem 5.1, one gets immediately that the conjecture (S) holds for the following special class of obstacles $K$ in $\mathbb{R}^3$.

PROPOSITION 5.5. – *Let $K \subset \mathbb{R}^3$ be an obstacle obtained by rotating an obstacle $D \in \mathcal{K}$ in $\mathbb{R}^2$ about a line $L$ of symmetry of $D$. Assume that for one of the vectors $\omega_0 \in S^{n-1}$ parallel to $L$ there exists a regular trapping ray $\gamma(u_0, \omega_0)$ for $D$. Then $K$ has the property* (S). $\square$

## 6. Several convex disjoint obstacles

Throughout this section we consider the case when $K \subset \mathbb{R}^3$ and

(7) $K = \cup_{j=1}^N K_j, \qquad K_i \cap K_j = \emptyset, \qquad \text{for} \quad i \neq j\ , \quad K_j$ convex for all $j = 1, \dots, N$.

First notice that if for an ordinary reflecting ray $\gamma$ the Gauss curvature of $\partial K$ does not vanish at least at one reflection point of $\gamma$, then $\gamma$ is non-degenerate (*see* [PS2]). Consequently, $K$ has the property (S) provided the Gauss curvature $\mathcal{K}(u)$ of $\partial K$ does not vanish on non-trivial open subsets of $\partial K$. On the other hand, it is well known that if the Gauss curvatutre vanishes on a neighbourhood of some point $z \in \partial K$, then the standard metric on $\partial K$ is locally flat around $z$. The latter means that there exists a neighbourhood $V_z$ of $z$ such that $V_z \cap \partial K$ is contained either in a plane or in a cylinder. By definition, a *cylinder* is a surface of the form

$$\mathcal{S} = \bigcup_{x \in l} L(x) \cap U,$$

where $l$ is a smooth planar curve, $U$ is an open subset of $\mathbb{R}^3$ containing $l$, and for each $x \in l$, $L(x)$ is a line containing $x$ and parallel to a constant line $L$ (called the *generator* of the cylinder) so that $L$ is transversal to the plane of $l$. A point $z \in \mathcal{S}$ will be called *non-degenerate* if $z \in L(x)$ for some $x \in l$ such that the curvature of $l$ does not vanish at $x$. Notice that in general a cylinder $\mathcal{S}$ may contain some flat (planar) pieces of $\partial K$.

A point $y \in \partial K$ will be called *planar* (resp. *cylindrical*) if there exists an open neighbourhood $V_y$ of $y$ such that $V_y \cap \partial K$ is contained in a plane (resp. cylinder). Let $\mathcal{P}$ and $\mathcal{C}$ be the sets of planar and cylindrical points on $\partial K$, respectively. Denote by $\mathcal{C}_0 \subset \mathcal{C}$ the set of all *non-degenerate* cylindrical points of $\partial K$. Notice that $\mathcal{P}$, $\mathcal{C}$ and $\mathcal{C}_0$ are open subsets of $\partial K$, and each point in $\mathcal{C} \setminus \mathcal{P}$ can be approximated by points of $\mathcal{C}_0$.

Fix $\omega_0 \in S^{n-1}$ and denote $Z = Z_{\omega_0}$ (*cf.* Section 2). For $u \in Z$ denote by $\gamma(u)$ the reflecting ray in $\Omega$ issued from $(u, \omega_0)$. In what follows we assume that $Z_{\omega_0}^{(\infty)}$ is not empty.

It is known that if $\gamma = \gamma(u_\gamma)$ is an ordinary $(\omega_0, \theta)$-ray (for some $\theta \in S^{n-1}$), then $dJ_\gamma(u_\gamma)$ has the following representation

$$dJ_\gamma(u_\gamma)u = M_k\sigma_k(I + \lambda_k M_{k-1})\sigma_{k-1}(I + \lambda_{k-1}M_{k-2})\ldots\sigma_2(I + \lambda_2 M_1)\sigma_1 u.$$

Here $u_\gamma = x_0$, $\lambda_i = \|x_{i-1} - x_i\|$, $\sigma_i$ is a linear map determined by the symmetry with respect to the tangent plane $\alpha_i$ to $\partial K$ at $x_i$, and

$$M_i = \sigma_i M_{i-1}(I + \lambda_i M_{i-1})^{-1}\sigma_i + \tilde{\psi}_i, \qquad i = 2, \ldots, k, \quad M_1 = \tilde{\psi}_1,$$

where $\tilde{\psi}_i \geq 0$ is a linear symmetric map related to the second fundamemtal form of $\partial K$ at $x_i$ $(i = 1, \ \ldots, k)$. We refer to [PS1], Chapter 2, for details concerning the above representation of $dJ_\gamma(u_\gamma)$. Clearly, $M_i \geq 0$ and $M_i f = 0$ yields $M_{i-1}\sigma_i f = 0, \ \ \tilde{\psi}_i f = 0$. Consequently, if $\gamma$ is degenerate, *i.e.* $\det dJ_\gamma(u_\gamma) = 0$, then there exists $w = \sigma_k \ldots \sigma_1 v \neq 0$ such that

$$M_k(w) = 0, \ M_{k-1}\sigma_k(w) = 0, \ \ldots, \ M_1\sigma_2\ldots\sigma_k(w) = 0,$$

$$\tilde{\psi}_k\sigma_k\ldots\sigma_1(v) = 0, \ldots, \ \tilde{\psi}_1\sigma_1(v) = 0.$$

Fix for a moment an ordinary ray $\gamma = \gamma(u_\gamma)$ and denote by $x_1, x_2, \ldots$ its successive reflection points.

First, assume that $x_1, x_2 \in \mathcal{C}_0$. Then $x_1$ and $x_2$ lie on cylinders with generators $l_1$ and $l_2$, passing through $x_1$ and $x_2$, respectively. Set

$$\omega_1 = \frac{(x_2 - x_1)}{\|x_2 - x_1\|}$$

and let $G_1 : \alpha_1 \longrightarrow \alpha_1$ be the differential at $x_1$ of the Gauss map of $\partial K$. Let $\pi_1$ be the projection along $\omega_0$ onto $\alpha_1$. According to the definition of $\tilde{\psi}_1$ (*see* [PS1], Chapter 2), $\tilde{\psi}_1\sigma_1(v) = 0$ implies $G_1(\pi_1 v) = 0$. The latter is only possible if $\pi_1(v)$ lies on $l_1$. Therefore, $l_1$ belongs to the plane $\beta_0$ determined by $v$ and $\omega_0$. In the same way we conclude that $\tilde{\psi}_2\sigma_2\sigma_1(v) = 0$ implies that $l_2$ belongs to the plane $\beta_1$ determined by $\sigma_1(v)$ and $\omega_1$. On the other hand, $\sigma_1(\beta_0) = \beta_1$, hence $l_1$ and $l_2$ lie both on $\beta_1$. Thus, the plane $\beta_1$ is determined by $l_1$ and $l_2$ and $\omega_1$ must be contained in it. Notice that $\beta_1$ does not depend on $\omega_1$, since

$$\beta_1 = \{\mu_1 w_1 + \mu_2 w_2 \in \mathbb{R}^3 : \mu_1, \mu_2 \in \mathbb{R}\},$$

where $w_1$ and $w_2$ are unit vectors lying on the generators $l_1$ and $l_2$, respectively. Choose a direction $\omega'$ close to $\omega_0$ and consider the ray $\gamma'$ through $x_1$ with incoming direction

$\omega'$. We may arrange $\omega_1' \notin \beta_1$ for the reflecting direction $\omega_1'$ at $x_1$. Consequently, the ray $\gamma'$ will be non-degenerate.

Next, assume that $x_1 \in \mathcal{C}_0$ lies on a cylinder with generator $l_1$, the reflection points $x_2, \dots, x_p$ are situated on planes $\alpha_2, \dots, \alpha_p$, the point $x_{p+1} \in \mathcal{C}_0$ lies on a cylinder with generator $l_2$ and each point $x_i$ with $i = 2, \dots, p$ has a neghbourhood $W_i$ such that $W_i \cap \partial K \subset \alpha_i$. Suppose that

$$L_p = \sigma_p \dots \sigma_2(l_1) \neq l_2. \tag{8}$$

As above, if $\gamma$ is degenerate, we conclude that $l_2$ and $L_p$ belong to the plane $\beta_p$ determined by $\sigma_p \dots \sigma_1(v)$ and $\omega_p = \sigma_p \dots \sigma_1(\omega_0)$. The condition (8) shows that $\beta_p$ is determined by $L_p$ and $l_2$. Since $\beta_p$ does not depend on $\omega_0$, replacing $\gamma$ by a ray $\gamma'$ through $x_1$ with incoming direction $\omega'$ close to $\omega_0$, we arrange $\omega_p' = \sigma_p \dots \sigma_1(\omega') \notin \beta_p$ and conclude that $\gamma'$ will be non-degenerate.

To satisfy the condition (8) we shall change $\omega_0$. In fact, (8) is equivalent to

$$l_1 \neq \sigma_2 \dots \sigma_p(l_2) = M_p,$$

where the line $M_p$ depends on $l_2$ and the planes $\alpha_2, \dots, \alpha_p$, only. Take $\omega_p'$ in some small conic neighbourhood $\Sigma$ of $\omega_p$ and consider the rays issued from $x_{p+1}$ with direction $-\omega_p'$. These rays, after $p-1$ reflections on $\alpha_p, \dots, \alpha_2$, hit $\partial K$ in some set with positive measure on $\partial K$. Hence, we can find a reflection point $y_1$ sufficiently close to $x_1$ so that $y_1 \notin M_p$. Now, replacing $x_1$ by $y_1$ and the generator $l_1$ by a generator $l_1'$ parallel to $l_1$ and passing through $y_1$, we arrange (8) with $l_1$ instead of $l_1'$. Going back, we find a suitable direction $\omega'$ close to $\omega_0$ related to the choice of $\omega_p' \in \Sigma$ above. Then the ray $\gamma'$ through $y_1$ with incoming direction $\omega'$ will be non-degenerate.

Finally, we may assume that the first $m$ reflection points $x_1, \dots, x_m$ of $\gamma$ are situated on $W_i \cap \partial K \subset \alpha_i$, $i = 1, \dots, m$, the reflection points $x_{m+1} \in \mathcal{C}_0$ and $x_{m+p+1} \in \mathcal{C}_0$ with $p \geq 1$ lie on cylinders, and $x_{m+2}, \dots, x_{m+p}$ are reflecting on $W_j \cap \partial K \subset \alpha_j$ $(j = m+2, \dots, m+p)$. The above argument works without any change replacing $\omega_0$ by $\tilde{\omega}_0 = \sigma_m \dots \sigma_1(\omega_0)$. Thus, we obtain the following.

PROPOSITION 6.1. – *Assume that $K$ has the form* (7). *Let $\gamma$ be an ordinary reflecting $(\omega, \theta)$-ray in $\overline{\Omega}$ with sojourn time $T$ having at least two reflection points in $\mathcal{C} \setminus \mathcal{P}$. Then there exists a sequence $(\omega_m, \theta_m) \longrightarrow (\omega, \theta)$ and non-degenerate ordinary reflecting $(\omega_m, \theta_m)$-rays $\gamma_m$ with sojourn times $T_m \longrightarrow T$.* $\square$

Using the above argument, one also gets that a large class of obstacles of the form (7) have the property (S).

THEOREM 6.2. – *Let $K$ be of the form* (7). *Suppose that one of the following conditions is satisfied:*

(i) *There exists a reflecting ray $\gamma_0$ in $\Omega$ with infinitely many reflections which has at least one transversal reflection at a point $x \notin \mathcal{P}$.*

(ii) *There exists a reflecting ray $\gamma_0$ in $\Omega$ with infinitely many reflections and*

$$\partial \mathcal{P} \cap \mathcal{C} = \emptyset. \tag{9}$$

*Then $K$ has the property* (S).

*Proof.* – We may assume that $\gamma_0$ is generated by some point $u_0 \in Z$, where $Z$ is as above. It is not difficult to see that, for $K$ of the form (7), the set $Z^{(t)}_{\omega_0}$ has Lebesgue measure zero in $Z$ (*cf.* Section 3 for the definition of $Z^{(t)}_{\omega_0}$). To check this, one can use for example the argument from the proof of Lemma 10.1.2 in [PS1] (this argument shows that $Z^{(t)}_{\omega_0}$ has empty interior in $Z$ but a very slight modification of its gives that $Z^{(t)}_{\omega_0}$ has Lebesgue measure zero in $Z$).

(i) Let $x_1, x_2, \ldots$ be the reflection points of $\gamma_0 = \gamma(u_0)$. Take an arbitrary segment $[u_1, u_0]$ in $Z$ such that $[u_1, u_0] \cap Z^{(t)}_{\omega_0}$ has (one-dimensional) Lebesgue measure zero in $[u_1, u_0]$. Denote

$$I = [u_1, u_0] \setminus Z^{(t)}_{\omega_0}.$$

Then for $u \in I$, the reflecting ray $\gamma(u)$ is ordinary. Let $x_1(u), x_2(u), \ldots$ be its reflection points. Our aim is to show that there exist $u \in I$ arbitrarily close to $u_0$ so that $u$ can be approximated by points $u' \in Z$ such that $\gamma(u')$ is ordinary and has either a reflection point $x_j(u')$ so that $\partial K$ is strictly convex at $x_j(u')$ or two reflection points in $\mathcal{C}$. According to Proposition 2.3 in [St1], almost every $(u'', \omega) \in Z \times S^{n-1}$ generates an ordinary reflecting ray with only finitely many reflections. Using this and the above Proposition 6.1, one gets that there exists $(u'', \omega)$ arbitrarily close to $(u, \omega_0)$ so that $(u'', \omega)$ generates an ordinary non-degenerate reflecting ray $\gamma(u'', \omega)$. Clearly, taking $u$ and $u''$ sufficiently close to $u_0$ and $\omega$ to $\omega_0$, one can make the sojourn time of $\gamma(u'', \omega)$ arbitrarily large.

By assumption, there exists $k$ so that $x_k \notin \mathcal{P}$ and $x_k$ is a transversal reflection point for $\gamma_0$. Taking $u_1$ sufficiently close to $u_0$, we may assume that for each $u \in [u_1, u_0]$, $\gamma(u)$ has at least $k$ reflections and $x_k(u)$ is a transversal reflection point. If $x_j \notin \mathcal{C}$ for some $j$, then $x_j$ can be approximated by points $y$ so that $\partial K$ is strictly convex at $y$. In this case the assertion follows from the above remark. This is so also in the case when there exists $j < k$ with $x_j \notin \mathcal{P}$.

Thus, the only case that has to be considered is the one when $x_j \in \mathcal{P}$ for all $j < k$ and $x_k \in \mathcal{C} \setminus \mathcal{P}$. Then locally near $x_k$, $\partial K$ is a cylinder. We may assume that $[u_1, u_0]$ is chosen in such a way that $x_k(u) \notin \mathcal{P}$ for all $u \in [u_1, u_0]$.

It is clear that the set $Z^{(t)}_{\omega_0} \cap [u_1, u_0)$ contains points arbitrarily close to $u_0$ – otherwise it would follow that for all $u \in [u_1, u_0]$ sufficiently close to $u_0$, the ray $\gamma(u)$ has infinitely many reflections. Take $v \in Z^{(t)}_{\omega_0} \cap [u_1, u_0)$ close to $u_0$ and let $x_j(v)$ be a tangent reflection point of $\gamma(v)$. Then $j \neq k$. It is now clear that there exists $u \in I$ arbitrarily close to $v$ such that $x_j(u) \notin \mathcal{P}$. So, the ray $\gamma(u)$ has two reflection points $x_j(u)$ and $x_k(u)$ which do not lie in $\mathcal{P}$ and using Proposition 6.1, we see that there exists $u' \in Z$ arbitrarily close to $u$ such that $\gamma(u')$ is ordinary and non-degenerate. This proves the assertion.

(ii) Without loss of generality we may assume that $u_0$ is a boundary point of the set $Z^{(\infty)}_{\omega_0}$ in $Z$ (one may even assume that $u_0$ is an extremal point of $Z^{(\infty)}_{\omega_0}$).

Given $y \in \mathcal{C}$, consider the *maximal* connected (open) component $M_y$ of $y$ contained in $\mathcal{C}$. A simple argument shows that $y \in \mathcal{C} \setminus \mathcal{P}$ implies $M_y \cap \mathcal{P} = \emptyset$, while $y \in \mathcal{P}$ yields $M_y \subset \mathcal{P}$. Obviously, for each $y \in \mathcal{C}$ we have $\partial M_y \cap \mathcal{C} = \emptyset$.

For a fixed $T_0 > 0$, there exists a neighbourhood $U \subset Z_\omega$ of $u_0$ such that for every $u \in U$ the ray $\gamma(u)$ issued from $u$ in direction $\omega$ is either trapping or its sojourn time $T_{\gamma(u)} \geq T_0$. Obviously, for $T_0$ large enough the rays $\gamma(u)$ must have many transversal reflecting points. Our goal is to find a non-degenerate ordinary reflecting ray $\gamma(u)$ with $u \in U$. To do this for a suitably chosen $u \in U$ we shall replace $\gamma(u)$ by some ordinary reflecting ray $\delta$ with sojourn time close to $T_0$.

Let $v_0 \in U \setminus Z^{(\infty)}$ and let $\gamma(v_0)$ be an $(\omega, \theta)$-ray such that $x_i \in \overline{\mathcal{P}}, i = 1, \dots, m$ are some of the transversal reflecting points of $\gamma(v_0)$. Notice that between $x_i$ and $x_{i+1}$ there may be other reflecting or tangent points of $\gamma(v_0)$. If for some $i = 1, \dots, m$ we have $x_i \in \partial\mathcal{P}$, then $x_i \notin \mathcal{C}$ and we can replace $\gamma(v_0)$ by a non-degenerate ordinary reflecting ray $\delta$ reflecting at a point $z \in \partial K$ with $\mathcal{K}(z) \neq 0$. Hence we may suppose that $x_i \in M_{x_i} = M_i, i = 1, \dots, m$. Consider the linear segment $l_0 = (v_0, u_0) \subset U$ connecting $v_0$ and $u_0$. For $l_0 \ni u \to u_0$ the sojourn times of $\gamma(u)$ increase and the number of reflections of $\gamma(u)$ increase, too. For $u$ in a suffficently small neighbourhood of $v_0$ let $z_i(u) = x_{j_i}(u) \in \partial M_i, i = 1, \dots, m$ be ordinary reflection points of $\gamma(u)$ with $x_{j_i}(v_0) = x_i, i = 1, \dots, m$. As $u \to u_0$, the points $z_i(u)$ move over $M_i$ and for $u = u_1$ the ray $\gamma(u_1) = \gamma_1$ will be tangent to $M_i$ or for some $u = u_2$ the ray $\gamma(u_2) = \gamma_2$ will have an ordinary reflecting point $z_i(u_2) \in \partial M_i$. In both cases $\gamma_1$ or $\gamma_2$ passes over a point $z \notin \mathcal{C}$ and we can replace $\gamma_i,\ i = 1, 2$, by a non-degenerate ordinary reflecting ray $\delta$.

Next, assume that for all $u \in l_0$ we have $z_i(u) \in M_i,\ i = 1, \dots, m$. When the number of reflections of $\gamma(u)$ increase, the ray $\gamma(u)$ must be tangent to some obstacle $K_j, j = 1, \dots, N$ before to be reflecting on it. If for $u = u_3$ the ray $\gamma(u_3) = \gamma_3$ is tangent to some component $M_y$ with $y \in \mathcal{P}$, then $\gamma_3$ will be tangent to $\partial K$ at a point $z \notin \mathcal{C}$ and, as above, we replace $\gamma_3$ by an ordinary reflecting ray $\delta$.

It remains to treat the case when for $u = u_4$ the ray $\gamma(u_4) = \gamma_4$ is tangent to $\partial K$ at some point $z \in \overline{M_y}$ with $y \in \mathcal{C} \setminus \mathcal{P}$. In this situation we have either $z \notin \mathcal{C}$ or $z \in \mathcal{C} \setminus \mathcal{P}$. In the first case we replace $\gamma_4$ by a non-degenerate ordinary reflecting ray $\delta$, while in the second one we replace $\gamma_4$ by a ray $\delta$ reflecting at a point $\tilde{z} \in \mathcal{C} \setminus \mathcal{P}$. Since the number of reflections increase, for some $u_5 \in l_0$ we will obtain a ray $\gamma(u_5) = \gamma_5$ with at least two ordinary reflections points $y_i \in \mathcal{C} \setminus \mathcal{P},\ i = 1, 2$. Therefore, applying Proposition 6.1, we approximate $\gamma_5$ by a non-denegerate ordinary reflecting ray $\delta$. $\square$

## Appendix

In this appendix we discuss the modifications for $n$ even in the proof of the Poisson relation for the scattering kernel. First notice that for $n \geq 2$ the scattering kernel admits the representation

$$(10)\qquad s(\sigma, \theta, \omega) = \frac{(-1)^{(n+1)/2}}{2(2\pi)^{n-1}} \int_{\infty}^{\infty} \int_{\partial\Omega} \partial_t^{n-2} \partial_\nu w(\langle x, \theta \rangle - \sigma; \omega)\ dt\ dS_x,$$

where $\nu$ denotes the unit normal to $\partial\Omega$ pointing into $\Omega$ and $w(t, x; \omega)$ is the solution of the problem

$$\begin{cases} (\partial_t^2 - \Delta_x) w = 0 \text{ in } \mathbb{R} \times \Omega, \\ w = 0 \quad \text{on } \mathbb{R} \times \partial\Omega, \\ w|_{t < \rho_0} = \delta(t - \langle x, \omega \rangle). \end{cases}$$

The reader may consult [So], [P] for the proof of (10). To obtain the Poisson relation for $s(t, \theta, \omega)$ we follow the argument of Chapter 8 in [PS1]. The only point, where the parity of $n$ was used, is the anaysis of the asymptotic behaviour of the solution of the problem

$$\begin{cases} (\Delta + \lambda^2)V(x, \lambda) = -f(x, \lambda) \text{ in } \Omega, \\ V(x, \lambda) = 0 \quad \text{on } \partial\Omega, \\ V(x, \lambda) \text{ is } \ (i\lambda) \ - \text{outgoing}. \end{cases} \tag{11}$$

For $n$ even the $(i\lambda)$-outgoing Green function related to the operator $(\Delta + \lambda^2)$ in $\mathbb{R}^n$ has the form

$$G_{i\lambda}^{(+)}(x) = -\frac{i}{4}\left(\frac{\lambda}{2\pi|x|}\right)^{(n-2)/2} H_{(n-2)/2}^{(2)}(\lambda|x|),$$

where $H_\mu^{(2)}(z)$ is the Hankel function of order $\mu$ with asymptotic

$$H_\mu^{(2)}(z) \sim \left(\frac{2}{\pi z}\right) \exp\left(-\frac{i}{4}(4z - 2\mu\pi - \pi)\right)\left(1 + \mathcal{O}\left(\frac{1}{|z|}\right)\right)$$

as $|z| \to \infty$. Thus for $r = |x| \to \infty$ we get

$$G_{i\lambda}^{+}(x) = -\frac{(i\lambda)^{(n-3)/2}}{2(2\pi r)^{(n-1)/2}} e^{-i\lambda r} + \mathcal{O}\left(\frac{e^{-i\lambda r}}{r^{(n+1)/2}}\right).$$

The solution of the problem (11) can be expressed by integrals involving $G_{i\lambda}^{(+)}$, so repeating without any change the argument of Section 8.3 in [PS1], we obtain the Poisson formula for $s(t, \theta, \omega)$.

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