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q -SELBERG INTEGRALS AND MACDONALD POLYNOMIALS

BY JYOICHI KANEKO

ABSTRACT. — We consider a Jackson integral with special integrand (q -Selberg integral) and give an explicit formula of a system of q -difference equations satisfied by it. We also define a kind of hypergeometric function having series expansions in terms of Macdonald polynomials and show that this function satisfies a q -difference equation formed by summing up equations of the q -difference system above after multiplying each by a suitable factor. We can thus conclude the q -Selberg integral to be the hypergeometric function in our sense. This implies, in particular, the q -integration formula of Macdonald polynomials due to Kadell [Kad2]. These results reproduce our previous ones [Kan2] if we put $q = t^\alpha$ and let $t \rightarrow 1$.

1. Introduction

The purpose of this paper is to give q -analogues of our previous results in [Kan2]. Fix q with $0 < q < 1$ and set $(x)_\infty = (x; q)_\infty = \prod_{i=0}^{\infty} (1 - xq^i)$ and $(x)_a = (x; q)_a = (x)_\infty / (xq^a)_\infty$. For $x = (x_1, \dots, x_m) \in \mathbb{C}^m$ and $t = (t_1, \dots, t_n) \in (\mathbb{C}^*)^n$ put

$$(1.1) \quad \Phi(t) = \prod_{j=1}^n t_j^{\alpha + (j-1)(1-2\gamma)} \frac{(qt_j)_\infty}{(q^\beta t_j)_\infty} \prod_{1 \leq i < j \leq n} \frac{(q^{1-\gamma} t_j / t_i)_\infty}{(q^\gamma t_j / t_i)_\infty} f(x, t)$$

$$(1.2) \quad f(x, t) = \prod_{1 \leq i \leq m, 1 \leq j \leq n} \frac{(x_i t_j)_\infty}{(q^\mu x_i t_j)_\infty}$$

$$(1.3) \quad \Phi_0(t) = \Phi(t) D(t)$$

where $D(t) = \prod_{1 \leq i < j \leq n} (t_i - t_j)$.

For $\xi \in (\mathbb{C}^*)^n$ we put $[0, \xi\infty]_q = \{(q^{s_1} \xi_1, \dots, q^{s_n} \xi_n) \mid (s_1, \dots, s_n) \in \mathbb{Z}^n\}$. The Jackson integral of a function f on $(\mathbb{C}^*)^n$ over $[0, \xi\infty]_q$ is defined by

$$\int_{[0, \xi\infty]_q} f(t_1, \dots, t_n) \tilde{\omega} = (1 - q)^n \sum_{s_i \in \mathbb{Z}} f(\xi_1 q^{s_1}, \dots, \xi_n q^{s_n})$$

$$\tilde{\omega} = \frac{d_q t_1}{t_1} \wedge \dots \wedge \frac{d_q t_n}{t_n}$$

provided it exists. Similarly, the integral over $[0, \xi] = [0, \xi_1] \times \cdots \times [0, \xi_n]$ is defined by

$$\int_{[0, \xi]} f(t_1, \dots, t_n) \tilde{\omega} = (1 - q)^n \sum_{s_i \in \mathbb{Z}_{\geq 0}} f(\xi_1 q^{s_1}, \dots, \xi_n q^{s_n}).$$

We consider the integral

$$(1.4) \quad {}_q S_{n,m}(\alpha, \beta, \gamma, \mu; x_1, \dots, x_m; \xi) \quad ({}_q S_{n,m}(x) \text{ for short}) = \int_{[0, \xi \infty]_q} \Phi_0(t) \tilde{\omega}.$$

Write $q^{\mathbb{Z}} = \{q^k; k \in \mathbb{Z}\}$. We assume the following condition which assures that $\Phi(t)$ has no poles on $[0, \xi \infty]_q$.

$$(C_1) \quad \begin{cases} q^\gamma \xi_j / \xi_i \notin q^{\mathbb{Z}} \text{ for } 1 \leq i \leq j \leq n \text{ or } 2\gamma - 1 \in \mathbb{Z}_{\geq 0}; \\ q^\beta \xi_j \notin q^{\mathbb{Z}} \text{ for } 1 \leq j \leq n \text{ or } \beta - 1 \in \mathbb{Z}_{\geq 0}; \\ q^\mu x_i \xi_j \notin q^{\mathbb{Z}} \text{ for } 1 \leq i \leq m, 1 \leq j \leq n. \end{cases}$$

For convergence of the integral we assume also

$$(C_2) \quad \begin{cases} \operatorname{Re} \alpha + n - 1 > 4(n - 1) \max\{\operatorname{Re} \gamma, 0\}, \\ \operatorname{Re} \alpha + n - 1 + \operatorname{Re} \beta - 1 + m \operatorname{Re} \mu < -2(n - 1) |\operatorname{Re} \gamma|. \end{cases}$$

For the proof of convergence under the conditions $(C_1), (C_2)$, see the Appendix A.

One of our main results (Theorem 4.11) states that if $\mu = 1$ or $-\gamma$, then ${}_q S_{n,m}(x)$ has an explicit series expansion in terms of A -type Macdonald polynomials [Ma2] (in the case $\mu = -\gamma$, we need to choose $\xi = \xi_F =: (1, q^\gamma, \dots, q^{(n-1)\gamma})$). This precisely corresponds to our previous result that $S_{n,m}(x) := \lim_{q=t^\alpha, t \rightarrow 1} {}_q S_{n,m}(x)$ has an explicit series expansion in terms of Jack polynomials [Kan2, Theorem 5, p. 1106] (see also [Ko]).

In the case $f(x, t) \equiv 1$ ($m = 0$), ${}_q S_{n,0}(\alpha, \beta, \gamma; \xi)$ has been evaluated by K. Aomoto in [Ao2] (cf. also [Ao3]):

$$(1.5) \quad \begin{aligned} & {}_q S_{n,0}(\alpha, \beta, \gamma; \xi) \\ &= q^{\frac{n(n-1)^2}{2}\gamma} \prod_{j=1}^n \xi_j^{\alpha-2(j-1)\gamma} \frac{\vartheta(\xi_j q^{\alpha+\beta-(n-1)\gamma}) \vartheta(q^{\beta+(j-1)\gamma}) \vartheta(q^{j\gamma})}{\vartheta(q^{\alpha+\beta-(n-j)\gamma}) \vartheta(\xi_j q^\beta) \vartheta(q^\gamma)} \\ &\quad \times \prod_{1 \leq i < j \leq n} \frac{\vartheta(\xi_j / \xi_i)}{\vartheta(q^\gamma \xi_j / \xi_i)} \\ &\quad \times \prod_{j=1}^n \frac{\Gamma_q(\alpha + n - 1 - (n + j - 2)\gamma) \Gamma_q(\beta + (j - 1)\gamma) \Gamma_q(j\gamma)}{\Gamma_q(\alpha + \beta + n - 1 - (n - j)\gamma) \Gamma_q(\gamma)}. \end{aligned}$$

Here $\vartheta(x)$ denotes the Jacobi elliptic theta function $(x)_\infty (q/x)_\infty (q)_\infty$ and $\Gamma_q(x)$ denotes the q -gamma function $(1 - q)^{1-x} (q)_\infty / (q^x)_\infty$. We notice that when $n = 1$, this integral is

nothing but the Ramanujan's ${}_1\psi_1$ sum [As1]. In Appendix B we shall give a self-contained proof of (1.5) based on q -difference equation (the $m = 1$ case of Theorem 4.11). If $\xi = \xi_F$, one can simplify the formula (1.5) to get

$$(1.6) \quad {}_qS_{n,0}(\alpha, \beta, \gamma; \xi_F) = q^{A_n} \prod_{j=1}^n \frac{\Gamma_q(\alpha + n - 1 - (n + j - 2)\gamma) \Gamma_q(\beta + (j - 1)\gamma) \Gamma_q(j\gamma)}{\Gamma_q(\alpha + \beta + n - 1 - (n - j)\gamma) \Gamma_q(\gamma)},$$

where $A_n = \sum_{j=1}^n (\alpha - 2(j - 1)\gamma + n - 1)(j - 1)\gamma$. If γ is equal to a positive integer k , (1.6) reduces to the Askey-Habsieger-Kadell's formula [As, H, Kad1] (see Proposition 5.2) :

$$\begin{aligned} & \int_{[0,1]^n} \prod_{j=1}^n t_j^x \frac{(qt_j)_\infty}{(q^y t_j)_\infty} \prod_{1 \leq i < j \leq n} t_i^{2k} \left(q^{1-k} \frac{t_j}{t_i} \right)_{2k} \tilde{\omega} \\ &= q^{kx} \binom{n}{2} + 2k^2 \binom{n}{3} \prod_{i=1}^n \frac{\Gamma_q(x + (n - j)k) \Gamma_q(y + (n - j)k) \Gamma_q(1 + jk)}{\Gamma_q(x + y + (2n - j - 1)k) \Gamma_q(1 + k)}, \end{aligned}$$

where α and β are identified with $x + (n - 1)(2k - 1)$ and y respectively.

In Section 2 we show that, when $\mu = 1$ or $-\gamma$, ${}_qS_{n,m}(x)$ satisfies a system of q -difference equations (Theorem 2.3, (2.26)). This system tends to the holonomic system of differential equations in [Kan2, (9), p. 1088] when $q \rightarrow 1$. In Section 3 we define a kind of q -hypergeometric function ${}_r\Phi_s^{(q,t)}(a_1, \dots, a_r; b_1, \dots, b_s; x_1, \dots, x_m)$ using A -type Macdonald polynomials. By setting $q = t^\alpha$ and $t \rightarrow 1$, ${}_r\Phi_s^{(q,t)}(x)$ reduces to the hypergeometric function ${}_rF_s^{(\alpha)}(x)$ defined by using Jack polynomials [Kan2, Ko]. We notice that ${}_2F_1^{(\alpha)}$ is a special case of BC -type hypergeometric function of Heckman-Opdam [BO]. In Section 4 we prove that ${}_2\Phi_1^{(q,t)}(a, b; c; x)$ satisfies a q -difference equation formed by summing up equations of (2.26) multiplied each by a suitable factor. Therefore the uniqueness properties of solutions of this summed-up equation assures us that ${}_qS_{n,m}(x)$ with $\mu = 1$ or $-\gamma$ is nothing but ${}_2\Phi_1^{(q,t)}(a, b; c; x)$ if we adjust (q, t) and a, b, c suitably (Theorem 4.11). As a consequence we obtain an q -integration formula of Macdonald polynomials (Theorem 5.1). In the special case that $\xi = \xi_F$ and $\gamma = k$, a positive integer, this was conjectured and proved by Kadell [Kad2] in a different way from ours (though some details of the proof have been omitted in our copy of [Kad2]). This integration formula in turn gives explicit formulae of the values of ${}_2\Phi_1^{(q,t)}(a, b; c; x)$ at special points (Proposition 5.4). In a separate paper [Kan3] we shall show that Theorem 4.11 implies the constant term identities due to Forrester, Zeilberger and Cooper [F, Z, C].

In a recent preprint [BC], Barsky and Carpentier have given a different proof of our previous result [Kan2, Theorem 5] by employing a new method of G. Anderson. It would be interesting to know whether their argument has q -analogous counterpart.

Part of the results of this paper were announced in [Kan1]. The author thanks Prof. K. Aomoto for inspiring discussions and the referee for helpful remarks and suggestions.

2. q -difference system

2.1. Let $T_{q,t_i} = T_i$ denote the q -shift operator on the i -th coordinate: $T_i\varphi(t) = \varphi(t_1, \dots, qt_i, \dots, t_n)$ and set

$$\frac{\partial\varphi}{\partial_q t_i} = \frac{(T_i - 1)\varphi}{(q - 1)t_i}.$$

Note that

$$\frac{\partial(\varphi\psi)}{\partial_q t_i} = \frac{\partial\varphi}{\partial_q t_i}\psi + T_i\varphi\frac{\partial\psi}{\partial_q t_i}$$

which will be of frequent use. Put $b_i(t) = T_i\Phi(t)/\Phi(t)$. In particular

$$b_1(t) = q^\alpha \frac{1 - q^\beta t_1}{1 - qt_1} \prod_{j=2}^n \frac{t_1 - q^{-\gamma} t_j}{t_1 - q^{\gamma-1} t_j} \prod_{k=1}^m \frac{1 - q^\mu x_k t_1}{1 - x_k t_1}.$$

Define the covariant q -difference operator ∇_i by

$$\nabla_i\varphi(t) = \varphi(t) - b_i(t)T_i\varphi(t).$$

Clearly

$$\int_{[0, \xi\infty]_q} \Phi(t)\varphi(t)\tilde{\omega} = \int_{[0, \xi\infty]_q} T_i(\Phi(t)\varphi(t))\tilde{\omega},$$

provided the integral is convergent. Hence we have

$$\int_{[0, \xi\infty]_q} \Phi(t)\nabla_i\varphi(t)\tilde{\omega} = 0.$$

Let $\mathfrak{S}_n$ denote the symmetric group of degree n and for $\sigma \in \mathfrak{S}_n$ define

$$(\sigma\varphi)(t) = \varphi(t_{\sigma(1)}, \dots, t_{\sigma(n)}).$$

Put

$$U_\sigma(t) = \sigma\Phi(t)/\Phi(t).$$

Then we have

$$U_\sigma(t) = \prod_{\substack{1 \leq i < j \leq n \\ \sigma^{-1}(i) > \sigma^{-1}(j)}} \left(\frac{t_j}{t_i}\right)^{2\gamma-1} \frac{\vartheta(q^\gamma t_j/t_i)}{\vartheta(q^{1-\gamma} t_j/t_i)}.$$

By using $\vartheta(qx) = -1/x \vartheta(x)$ one can easily verify that $T_i U_\sigma(t) = U_\sigma(t)$ for every i . We assert that

$$\int_{[0, \xi\infty]_q} \Phi(t)\sigma(\nabla_1\varphi(t))\tilde{\omega} = 0$$

provided the integral is convergent. In fact

$$\begin{aligned} 0 &= \int_{[0, \xi\infty]_q} \sigma(\Phi(t)) \sigma(\nabla_1 \varphi(t)) \tilde{\omega} \\ &= U_\sigma(\xi) \int_{[0, \xi\infty]_q} \Phi(t) \sigma(\nabla_1 \varphi(t)) \tilde{\omega}. \end{aligned}$$

Hence for the alternation $\mathcal{A}\varphi = \sum_{\sigma \in \mathfrak{S}_n} \text{sgn } \sigma \cdot (\sigma\varphi)$, we obtain the fundamental

$$(2.1) \quad \int_{[0, \xi\infty]_q} \Phi(t) \mathcal{A}(\nabla_1 \varphi(t)) \tilde{\omega} = 0.$$

For complex Q we shall denote by ${}_Q D_n(t) = {}_Q D(t)$ the product $\prod_{1 \leq i < j \leq n} (t_i - Qt_j)$. The following lemma ([Kad1, (4.10), p. 976], cf. also [Ma1, chapter 3, (1.3)]) is crucial to our calculations.

LEMMA 2.1. – *Let $M \subset \{1, \dots, n\}$. Then*

$$(2.2) \quad \mathcal{A}\left(\prod_{j \in M} t_j {}_Q D(t)\right) = Q^{e(M)} \frac{(Q; Q)_{|M|} (Q; Q)_{n-|M|}}{(1-Q)^n} e_{|M|}(t) D(t)$$

where $e(M) = |\{(i, j) | 1 \leq i < j \leq n, i \notin M, j \in M\}|$ and $e_r(t)$ denotes the elementary symmetric function of degree r .

This lemma implies

$$(2.3) \quad \sum_{M \in \{1, \dots, n\}, |M|=r} Q^{e(M)} = \frac{(Q; Q)_n}{(Q; Q)_r (Q; Q)_{n-r}}$$

$$(2.4) \quad \sum_{M \in \{2, \dots, n\}, |M|=r} Q^{e(M)} = Q^r \frac{(Q; Q)_{n-1}}{(Q; Q)_r (Q; Q)_{n-r-1}}.$$

LEMMA 2.2.

$$\begin{aligned} (2.5) \quad \mathcal{A}({}_Q D(t) \prod_{k=2}^n (1 - xt_k)) \\ = D(t) \left\{ \frac{(Q; Q)_{n-1}}{(1-Q)^{n-1}} (-x) \frac{d(\prod_{k=1}^n (1 - xt_k))}{d_Q x} + \frac{(Q; Q)_n}{(1-Q)^n} \prod_{k=1}^n (1 - xt_k) \right\}. \end{aligned}$$

$$\begin{aligned}
 (2.6) \quad & \mathcal{A}\left(\prod_{k=2}^n (t_1 - Q^{-1}t_k) \prod_{2 \leq h < k \leq n} (t_h - Qt_k) \prod_{k=2}^n (1 - xt_k)\right) \\
 &= Q^{-(n-1)} \frac{(Q; Q)_{n-1}}{(1-Q)^n} D(t) \left\{ \prod_{k=1}^n (1 - xt_k) - \prod_{k=1}^n (Q - xt_k) \right\}.
 \end{aligned}$$

Proof. – From (2.2) and (2.4) we have

$$\mathcal{A}({}_Q D(t) \sum_{M \in \{2, \dots, n\}, |M|=r} \prod_{j \in M} t_j) = Q^r (1 - Q^{n-r}) \frac{(Q; Q)_{n-1}}{(1-Q)^n} e_r(t) D(t),$$

from which follows (2.5). For the proof of (2.6), observe that

$$\text{LHS of (2.6)} = Q^{-(n-1)} \mathcal{A}({}_Q D(t) \prod_{k=1}^{n-1} (1 - xt_k)).$$

Then (2.6) follows at once since (2.2) and (2.3) imply

$$\mathcal{A}({}_Q D(t) \sum_{M \in \{1, \dots, n-1\}, |M|=r} \prod_{j \in M} t_j) = (1 - Q^{n-r}) \frac{(Q; Q)_{n-1}}{(1-Q)^n} e_r(t) D(t).$$

Put

$$A_i(x_1, \dots, x_m; t) = \prod_{j=1, j \neq i}^m \frac{tx_i - x_j}{x_i - x_j}.$$

Expansion in partial fractions gives

$$(2.7) \quad \prod_{j=1}^m \frac{x_j - tz}{x_j - z} = (1-t) \sum_{j=1}^m \frac{x_j A_j(x; t)}{x_j - z} + t^m.$$

Replacing z by $1/z$ and m by $m-1$, we have also

$$(2.8) \quad \prod_{j=1, j \neq i}^m \frac{1 - x_j z/t}{1 - x_j z} = (t-1)t^{1-m} \sum_{j=1, j \neq i}^m \frac{A_j(x; t)(x_j - x_i)}{(1 - x_j z)(tx_j - x_i)} + t^{1-m}.$$

Specializing z suitably in (2.7) and (2.8), we obtain

$$(2.9) \quad \sum_{i=1}^m A_i(x; t) = \frac{1 - t^m}{1 - t}$$

$$(2.10) \quad \sum_{j=1, j \neq i}^m A_j(x; t) \frac{x_j - x_i}{tx_j - x_i} = \frac{1 - t^{m-1}}{1 - t}$$

$$(2.11) \quad \sum_{j=1, j \neq i}^m A_j(x; t) \frac{x_j}{tx_j - x_i} = \frac{A_i(x; t) - t^{m-1}}{1 - t}$$

$$(2.12) \quad \sum_{j=1, j \neq i}^m A_j(x; t) \frac{x_i}{tx_j - x_i} = \frac{A_i(x; t) - 1}{1 - t}.$$

Multiplying both sides of (2.8) by $(1 - x_i z)^{-1}$ and using

$$\frac{x_j - x_i}{(1 - x_i z)(1 - x_j z)} = \frac{x_j}{1 - x_j z} - \frac{x_i}{1 - x_i z}$$

and (2.12), we get

$$(2.13) \quad \frac{1}{1 - x_i z} \prod_{j=1, j \neq i}^m \frac{1 - x_j z/t}{1 - x_j z} = (t - 1)t^{1-m} \\ \times \sum_{j=1, j \neq i}^m \frac{A_j(x; t)x_j}{(1 - x_j z)(tx_j - x_i)} + t^{1-m} \frac{A_i(x; t)}{1 - x_i z}.$$

In what follows Q stands for q^γ .

2.2. CASE OF $\mu = 1$. – In this case we see

$$f(x, t) = \prod_{1 \leq j \leq m, 1 \leq k \leq n} (1 - x_j t_k).$$

Put

$$\varphi_i(x, t) = \frac{1 - t_1}{1 - x_i t_1} \prod_{j=1, j \neq i}^m \frac{1 - q^{-1} x_j t_1}{1 - x_j t_1} Q D(t),$$

so that

$$b_1(t) T_1 \varphi_i(x, t) = q^{\alpha+n-1} \frac{1 - q^\beta t_1}{1 - x_i t_1} \prod_{k=2}^n (t_1 - Q^{-1} t_k) \prod_{2 \leq h < k \leq n} (t_h - Q t_k).$$

We want to calculate $\mathcal{A}(\varphi_i)$ and $\mathcal{A}(b_1 T_1 \varphi_i)$. Since

$$1 - t_1 = \frac{1 - x_i t_1}{x_i} + 1 - \frac{1}{x_i}$$

we see

$$(2.14) \quad \mathcal{A}(\varphi_i) = \frac{1}{x_i} \mathcal{A} \left(\prod_{j=1, j \neq i}^m \frac{1 - q^{-1} x_j t_1}{1 - x_j t_1} {}_Q D(t) \right) \\ + \left(1 - \frac{1}{x_i} \right) \mathcal{A} \left(\frac{1}{1 - x_i t_1} \prod_{j=1, j \neq i}^m \frac{1 - q^{-1} x_j t_1}{1 - x_j t_1} {}_Q \mathcal{D}(t) \right).$$

Substituting $t = q$ and $z = t_1$ in (2.8) and (2.13) gives

$$(2.15) \quad \prod_{j=1, j \neq i}^m \frac{1 - q^{-1} x_j t_1}{1 - x_j t_1} = (q - 1) q^{1-m} \sum_{j=1, j \neq i}^m \frac{A_j(x; q)(x_j - x_i)}{(1 - x_j t_1)(q x_j - x_i)} + q^{1-m}$$

$$(2.16) \quad \frac{1}{1 - x_i t_1} \prod_{j=1, j \neq i}^m \frac{1 - q^{-1} x_j t_1}{1 - x_j t_1} = (q - 1) q^{1-m} \sum_{j=1, j \neq i}^m \frac{A_j(x; q) x_j}{(1 - x_j t_1)(q x_j - x_i)} \\ + q^{1-m} \frac{A_i(x; q)}{1 - x_i t_1}.$$

Substituting (2.15) and (2.16) into (2.14) and applying (2.5), (2.10) and (2.11), we have

$$\frac{f(x, t)}{D(t)} \mathcal{A}(\varphi_i) = \frac{(Q; Q)_n}{(1 - Q)^n} f(x, t) \\ + q^{1-m} (1 - q) \frac{(Q; Q)_{n-1}}{(1 - Q)^{n-1}} \sum_{j=1, j \neq i}^m \frac{x_j(x_j - 1)}{q x_j - x_i} A_j(x; q) \frac{\partial f(x, t)}{\partial_Q x_j} \\ + q^{1-m} \frac{(Q; Q)_{n-1}}{(1 - Q)^{n-1}} (1 - x_i) A_i(x; q) \frac{\partial f(x, t)}{\partial_Q x_i}.$$

Hence

$$(2.17) \quad T_{Q, x_i} \left(\frac{f(x, t)}{D(t)} \mathcal{A}(\varphi_i) \right) \\ = \frac{(Q; Q)_{n-1}}{(1 - Q)^{n-1}} \left\{ \frac{(1 - Q^n)}{1 - Q} \left(f(x, t) + (Q - 1) x_i \frac{\partial f}{\partial_Q x_i} \right) \right. \\ + q^{1-m} (1 - q) \sum_{j=1, j \neq i}^m \frac{x_j(x_j - 1)}{q x_j - Q x_i} T_{Q, x_i}(A_j(x; q)) \\ \left(\frac{\partial f}{\partial_Q x_j} + (Q - 1) x_i \frac{\partial^2 f}{\partial_Q x_i \partial_Q x_j} \right) \\ \left. + q^{1-m} (1 - Q x_i) T_{Q, x_i}(A_i(x; q)) \left(\frac{\partial f}{\partial_Q x_i} + (Q - 1) x_i \frac{\partial^2 f}{\partial_Q x_i^2} \right) \right\}.$$

Next we calculate $\mathcal{A}(b_1 T_1 \varphi_i)$. Since

$$1 - q^\beta t_1 = q^\beta \frac{1 - x_i t_1}{x_i} + 1 - \frac{q^\beta}{x_i},$$

we see

$$\begin{aligned} \mathcal{A}(b_1 T_1 \varphi_i) &= q^{\alpha+\beta+n-1} \frac{1}{x_i} \mathcal{A} \left(\prod_{k=2}^n (t_1 - Q^{-1} t_k) \prod_{2 \leq h < k \leq n} (t_h - t_k) \right) \\ &\quad + q^{\alpha+n-1} \left(1 - \frac{q^\beta}{x_i} \right) \mathcal{A} \left((1 - x_i t_1)^{-1} \prod_{k=2}^n (t_1 - Q^{-1} t_k) \prod_{2 \leq h < k \leq n} (t_h - t_k) \right). \end{aligned}$$

Applying (2.6), we have

$$\begin{aligned} \frac{f(x, t)}{D(t)} \mathcal{A}(b_1 T_1 \varphi_i) &= q^{\alpha+\beta+n-1} Q^{-(n-1)} \frac{(Q; Q)_n}{(1 - Q)^n} \frac{f(x, t)}{x_i} \\ &\quad + q^{\alpha+n-1} Q^{-(n-1)} \frac{(Q; Q)_{n-1}}{(1 - Q)^{n-1}} \\ &\quad \times \left(1 - \frac{q^\beta}{x_i} \right) \prod_{\substack{1 \leq j \leq m, j \neq i \\ 1 \leq k \leq n}} (1 - x_j t_k) \left\{ \prod_{k=1}^n (1 - x_i t_k) - \prod_{k=1}^n (Q - x_i t_k) \right\}. \end{aligned}$$

Hence

$$\begin{aligned} (2.18) \quad T_{Q, x_i} \left(\frac{f(x, t)}{D(t)} \mathcal{A}(b_1 T_1 \varphi_i) \right) &= q^{\alpha+\beta+n-1} Q^{-n} \frac{(Q; Q)_n}{(1 - Q)^n} \frac{1}{x_i} \left(f(x, t) + (Q - 1) x_i \frac{\partial f}{\partial_Q x_i} \right) \\ &\quad + q^{\alpha+n-1} Q^{-(n-1)} \frac{(Q; Q)_{n-1}}{(1 - Q)^n} \left(1 - \frac{q^\beta}{Q x_i} \right) \\ &\quad \left((Q - 1) x_i \frac{\partial f}{\partial_Q x_i} + (1 - Q^n) f(x, t) \right) \\ &= q^{\alpha+n-1} Q^{-(n-1)} \frac{(Q; Q)_{n-1}}{(1 - Q)^{n-1}} \left\{ \frac{1 - Q^n}{1 - Q} f(x, t) + (q^\beta Q^{n-1} - x_i) \frac{\partial f}{\partial_Q x_i} \right\}. \end{aligned}$$

It is clear from (2.1) that

$$(2.19) \quad \int_{[0, \xi \infty]_q} T_{Q, x_i} (\Phi(t) \mathcal{A}(\nabla_1 \varphi(t))) \tilde{\omega} = 0.$$

Substituting (2.17) and (2.18) into (2.19), we arrive at ($S = {}_qS_{n,m}(x)$)

$$\begin{aligned}
 (2.20) \quad 0 = & q^{-(\alpha+n-1)}Q^{-1}x_i(1-Qx_i)T_{Q,x_i}(A_i(x;q))\frac{\partial^2 S}{\partial_Q x_i^2} \\
 & + q^{-(\alpha+n-1)}(1-q)\sum_{j=1, j \neq i}^m \frac{Q^{-1}x_i x_j(1-x_j)}{Qx_i - qx_j}T_{Q,x_i}(A_j(x;q))\frac{\partial^2 S}{\partial_Q x_i \partial_Q x_j} \\
 & + \left\{ \frac{q^{m-1}-c}{1-Q} + \frac{1}{1-Q}((1-a)(1-b)q^{m-1} \right. \\
 & \left. - (q^{m-1} - abQ))q^{-\beta}Qx_i \right\} q^{\beta}Q^{-1}\frac{\partial S}{\partial_Q x_i} \\
 & + \frac{1-q}{1-Q} \left\{ \frac{1-T_{Q,x_i}(A_i(x;q))}{1-q}(c - abQq^{-\beta}Qx_i)q^{\beta}Q^{-1}\frac{\partial S}{\partial_Q x_i} \right. \\
 & \left. - \sum_{j=1, j \neq i}^m \frac{q^{-(\alpha+n-1)}Q^{-1}x_j(1-x_j)}{Qx_i - qx_j}T_{Q,x_i}(A_j(x;q))\frac{\partial S}{\partial_Q x_j} \right\} \\
 & - \frac{(1-a)(1-b)q^{m-1}}{(1-Q)^2}S
 \end{aligned}$$

where

$$a = Q^{-n}, b = q^{-(\alpha+n-1)}Q^{n-1}, c = q^{-(\alpha+\beta+n-1)}.$$

Now we change the variables:

$$x_i = q^{\beta}Q^{-1}y_i, \quad i = 1, \dots, m.$$

Note that

$$T_{Q,x_i} = T_{Q,y_i}, \quad \frac{\partial}{\partial_Q x_i} = q^{-\beta}Q \frac{\partial}{\partial_Q y_i},$$

and

$$A_i(x;q) = A_i(y;q).$$

(2.20) is transformed into

$$\begin{aligned}
 (2.21) \quad & y_i(c - abQy_i)T_{Q,y_i}(A_i(y;q))\frac{\partial^2 S}{\partial_Q y_i^2} + (1-q) \\
 & \times \sum_{j=1, j \neq i}^m \frac{y_i y_j(c - aby_j)}{Qy_i - qy_j}T_{Q,y_i}(A_j(y;q))\frac{\partial^2 S}{\partial_Q y_i \partial_Q y_j} \\
 & + \left\{ \frac{q^{m-1}-c}{1-Q} + \frac{1}{1-Q}((1-a)(1-b)q^{m-1} - (q^{m-1} - abQ))y_i \right\} \frac{\partial S}{\partial_Q y_i}
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{1-q}{1-Q} \left\{ \frac{1-T_{Q,y_i}(A_i(y;q))}{1-q} (c-abQy_i) \frac{\partial S}{\partial_Q y_i} \right. \\
 & - \sum_{j=1, j \neq i}^m \frac{y_j(c-ab y_j)}{Qy_i - qy_j} T_{Q,y_i}(A_j(y;q)) \frac{\partial S}{\partial_Q y_j} \left. \right\} \\
 & - \frac{(1-a)(1-b)q^{m-1}}{(1-Q)^2} S = 0.
 \end{aligned}$$

2.3. CASE OF $\mu = -\gamma$. – In this case we have

$$f(x, t) = \prod_{1 \leq j \leq m, 1 \leq k \leq n} \frac{(x_j t_k)_\infty}{(Q^{-1} x_j t_k)_\infty}.$$

Put

$$\varphi_i(x, t) = \frac{1-t_1}{1-(qQ)^{-1}x_i t_1} {}_q D(t),$$

so that

$$b_1(t) T_1 \varphi_i(x, t) = q^{\alpha+n-1} \frac{1-q^\beta t_1}{1-Q^{-1}x_i t_1} \prod_{k=2}^n \frac{t_1 - q^{-\gamma} t_k}{t_1 - q^\gamma t_k} \prod_{j=1}^m \frac{1-Q^{-1}x_j t_1}{1-x_j t_1} {}_q D(t).$$

Then one can proceed in a similar way as in the case of $\mu = 1$. We have (we omit the details of calculation)

$$\frac{f(x, t)}{D(t)} \mathcal{A}(\varphi_i) = \frac{(Q; Q)_n}{(1-Q)^n} \frac{qQ}{x_i} f(x, t) + \frac{(Q; Q)_{n-1}}{(1-Q)^n} \left(1 - \frac{qQ}{x_i}\right) (T_{q^{-1}, x_i} f(x, t) - Q^n f(x, t)),$$

so that

$$(2.22) \quad T_{q, x_i} \left(\frac{f(x, t)}{D(t)} \mathcal{A}(\varphi_i) \right) = \frac{(Q; Q)_{n-1}}{(1-Q)^n} \left\{ (1-Q^n) f(x, t) + (q-1) Q (1-Q^{n-1} x_i) \frac{\partial f}{\partial_Q x_i} \right\}.$$

We have also

$$\begin{aligned}
 & \frac{f(x, t)}{D(t)} \mathcal{A}(b_1 T_1 \varphi_i) \\
 & = q^{\alpha+n-1} Q^{2-m-n} \left\{ Q^{m-1} \frac{(Q; Q)_n}{(1-Q)^n} f(x, t) \right. \\
 & - (1-q) Q^n \frac{(Q; Q)_{n-1}}{(1-Q)^n} (q^\beta - x_i) A_i(x; Q) \frac{\partial f}{\partial_Q x_i} \\
 & \left. - (1-q) Q^n \frac{(Q; Q)_{n-1}}{(1-Q)^{n-1}} \sum_{j=1, j \neq i}^m \frac{x_j (q^\beta - x_j)}{x_i - Q x_j} A_j(x; Q) \frac{\partial f}{\partial_Q x_j} \right\}.
 \end{aligned}$$

Hence

$$\begin{aligned}
 (2.23) \quad & T_{q,x_i} \left(\frac{f(x,t)}{D(t)} \mathcal{A}(b_1 T_1 \varphi_i) \right) \\
 &= q^{\alpha+n-1} Q^{2-m-n} \frac{(Q; Q)_{n-1}}{(1-Q)^n} \left\{ (1-Q^n) Q^{m-1} \left(f(x,t) + (q-1)x_i \frac{\partial f}{\partial_q x_i} \right) \right. \\
 &\quad - q(1-q) Q^n (q^{\beta-1} - x_i) T_{q,x_i}(A_i(x; Q)) \left(\frac{\partial f}{\partial_q x_i} + (q-1)x_i \frac{\partial^2 f}{\partial_q x_i^2} \right) \\
 &\quad - (1-q) Q^n (1-Q) \sum_{j=1, j \neq i}^m \frac{x_j (q^\beta - x_j)}{qx_i - Qx_j} T_{q,x_i}(A_j(x; Q)) \\
 &\quad \left. \times \left(\frac{\partial f}{\partial_q x_j} + (q-1)x_i \frac{\partial^2 f}{\partial_q x_i \partial_q x_j} \right) \right\}.
 \end{aligned}$$

Substituting (2.22) and (2.23) into (2.19) gives the equation corresponding to (2.20):

$$\begin{aligned}
 (2.24) \quad 0 &= q^{\alpha+n} Q x_i (q^{\beta-1} - x_i) T_{q,x_i}(A_i(x; Q)) \frac{\partial^2 S}{\partial_q x_i^2} \\
 &\quad + q^{\alpha+n-1} (1-Q) \sum_{j=1, j \neq i}^m \frac{Q x_i x_j (q^\beta - x_j)}{qx_i - Qx_j} T_{q,x_i}(A_j(x; Q)) \frac{\partial^2 S}{\partial_q x_i \partial_q x_j} \\
 &\quad + \left\{ \frac{Q^m}{1-q} + \frac{Q^{m-1}}{1-q} (-Q^n - q^{\alpha+n-1} Q^{-(n-1)} + q^{\alpha+n-1} Q) x_i \right\} \frac{\partial S}{\partial_q x_i} \\
 &\quad - \frac{q^{\alpha+n} Q}{1-q} (q^{\beta-1} - x_i) T_{q,x_i}(A_i(x; Q)) \frac{\partial S}{\partial_q x_i} \\
 &\quad - \frac{1-Q}{1-q} q^{\alpha+n-1} Q \sum_{j=1, j \neq i}^m \frac{x_j (q^\beta - x_j)}{qx_i - Qx_j} T_{q,x_i}(A_j(x; Q)) \frac{\partial S}{\partial_q x_j} \\
 &\quad - \frac{(1-Q^n)(1-q^{\alpha+n-1} Q^{-(n-1)}) Q^{m-1}}{(1-q)^2} S.
 \end{aligned}$$

Hence changing variables as

$$x_i = Q y_i, \quad i = 1, \dots, m$$

yields

$$\begin{aligned}
 (2.25) \quad & y_i (c - abq y_i) T_{q,y_i}(A_i(y; Q)) \frac{\partial^2 S}{\partial_q y_i^2} \\
 &\quad + (1-Q) \sum_{j=1, j \neq i}^m \frac{y_i y_j (c - aby_j)}{qy_i - Qy_j} T_{q,y_i}(A_j(y; Q)) \frac{\partial^2 S}{\partial_q y_i \partial_q y_j}
 \end{aligned}$$

$$\begin{aligned}
 & + \left\{ \frac{Q^{m-1} - c}{1 - q} + \frac{1}{1 - q} ((1 - a)(1 - b)Q^{m-1} - (Q^{m-1} - abq))y_i \right\} \frac{\partial S}{\partial_q y_i} \\
 & + \frac{1 - Q}{1 - q} \left\{ \frac{1 - T_{q,y_i}(A_i(y; Q))}{1 - Q} (c - abqy_i) \frac{\partial S}{\partial_q y_i} \right. \\
 & \quad \left. - \sum_{j=1, j \neq i}^m \frac{y_j(c - aby_j)}{qy_i - Qy_j} T_{q,y_i}(A_j(y; Q)) \frac{\partial S}{\partial_q y_j} \right\} \\
 & - \frac{(1 - a)(1 - b)Q^{m-1}}{(1 - q)^2} S = 0
 \end{aligned}$$

where

$$a = Q^n, \quad b = q^{\alpha+n-1}Q^{-(n-1)}, \quad c = q^{\alpha+\beta+n-1}.$$

We have thus proved

THEOREM 2.3. – Assume $\mu = 1$ or $-\gamma$. Then ${}_qS_{n,m}(\alpha, \beta, \gamma, \mu; x; \xi)$ satisfies the following system of q -difference equations ($T_i = T_{q,x_i}$).

$$\begin{aligned}
 (2.26) \quad & x_i(c - abqx_i)T_i(A_i(x; t)) \frac{\partial^2 S}{\partial_q x_i^2} \\
 & + (1 - q) \sum_{j=1, j \neq i}^m \frac{x_i x_j (c - abx_j)}{qx_i - tx_j} T_i(A_j(x; t)) \frac{\partial^2 S}{\partial_q x_i \partial_q x_j} \\
 & + \left\{ \frac{t^{m-1} - c}{1 - q} + \frac{1}{1 - q} ((1 - a)(1 - b)t^{m-1} - (t^{m-1} - abq))x_i \right\} \frac{\partial S}{\partial_q x_i} \\
 & + \frac{1 - t}{1 - q} \left\{ \frac{1 - T_i(A_i(x; t))}{1 - t} (c - abqx_i) \frac{\partial S}{\partial_q x_i} \right. \\
 & \quad \left. - \sum_{j=1, j \neq i}^m \frac{x_j(c - abx_j)}{qx_i - tx_j} T_i(A_j(x; t)) \frac{\partial S}{\partial_q x_j} \right\} \\
 & - \frac{(1 - a)(1 - b)t^{m-1}}{(1 - q)^2} S = 0, \quad i = 1, \dots, m,
 \end{aligned}$$

where if $\mu = 1$, then put $(q, t) = (Q, q)$, $a = Q^{-n}$, $b = q^{-(\alpha+n-1)}Q^{n-1}$, $c = q^{-(\alpha+\beta+n-1)}$ and change x_i with $q^{-\beta}Qx_i$, $1 \leq i \leq m$. If $\mu = -\gamma$, then put $(q, t) = (q, Q)$, $a = Q^n$, $b = q^{\alpha+n-1}Q^{-(n-1)}$, $c = q^{\alpha+\beta+n-1}$ and change x_i with $Q^{-1}x_i$, $1 \leq i \leq m$.

Remark. – In the theorem above the change of variables means that we change only the variables of the q -difference equations. We *do not* change the variables of the unknown function S . The same remark applies also to the following Theorem 2.4 and 2.5.

2.4. VARIANTS. – One can calculate a system of q -difference equations satisfied by the integral ${}_qS_{n,m}(\alpha, \beta, \gamma; \mu; x_1^{-1}, \dots, x_m^{-1}; \xi)$ provided $\mu = 1$ or $-\gamma$ in the same way as in the case of Theorem 2.3. If $\mu = 1$, then put

$$\varphi_i(x, t) = \frac{1 - t_1}{1 - t_1/x_i} Q D(t).$$

We have

$$\frac{f(x, t)}{D(t)} \mathcal{A}(\varphi_i) = \frac{(Q; Q)_{n-1}}{(1 - Q)^n} \left\{ (1 - Q^n) x_i f(x, t) + (1 - x_i) (T_{Q^{-1}, x_i} f(x, t) - Q^n f(x, t)) \right\}$$

and

$$\begin{aligned} & \frac{f(x, t)}{D(t)} \mathcal{A}(b_1 T_1 \varphi_i) \\ &= q^{\alpha+n-1} Q^{1-n} \frac{(Q; Q)_{n-1}}{(1 - Q)^n} \left\{ (1 - Q^n) f(x, t) + (1 - Q) Q^n x_i (1 - q^\beta x_i) A_i(x; q) \frac{\partial f}{\partial_Q x_i} \right. \\ & \quad \left. + (1 - q)(1 - Q) Q^n \sum_{j=1, j \neq i}^m \frac{x_i x_j (1 - q^\beta x_j)}{x_i - q x_j} A_j(x; q) \frac{\partial f}{\partial_Q x_j} \right\}. \end{aligned}$$

Substituting these into (2.19) gives

$$\begin{aligned} (2.27) \quad 0 &= q^{\alpha+n-1} Q^2 x_i^2 (1 - q^\beta Q x_i) T_{Q, x_i} (A_i(x; q)) \frac{\partial^2 S}{\partial_Q x_i^2} \\ &+ q^{\alpha+n-1} (1 - q) Q^2 \sum_{j=1, j \neq i}^m \frac{x_i^2 x_j (1 - q^\beta x_j)}{Q x_i - q x_j} T_{Q, x_i} (A_j(x; q)) \frac{\partial^2 S}{\partial_Q x_i \partial_Q x_j} \\ &+ \frac{x_i}{1 - Q} \left\{ Q(Q^{n-1} - x_i) + q^{\alpha+n-1} Q^{-(n-1)} (1 - Q^n) \right\} \frac{\partial S}{\partial_Q x_i} \\ &- \frac{q^{\alpha+n-1} Q^2}{1 - Q} x_i (1 - q^\beta Q x_i) T_{Q, x_i} (A_i(x; q)) \frac{\partial S}{\partial_Q x_i} \\ &- \frac{(1 - q) q^{\alpha+n-1} Q^2}{1 - Q} \sum_{j=1, j \neq i}^m \frac{x_i x_j (1 - q^\beta x_j)}{Q x_i - q x_j} T_{Q, x_i} (A_j(x; q)) \frac{\partial S}{\partial_Q x_j} \\ &+ \frac{(1 - Q^n)(1 - q^{\alpha+n-1} Q^{-(n-1)})}{(1 - Q)^2} S. \end{aligned}$$

If $\mu = -\gamma$, then put

$$\varphi_i(x, t) = \frac{1 - t_1}{1 - (qQ)^{-1} t_1/x_i} \prod_{j=1, j \neq i}^m \frac{1 - q^{-1} t_1/x_j}{1 - (qQ)^{-1} t_1/x_j} Q D(t).$$

We have

$$\begin{aligned} \frac{f(x, t)}{D(t)} \mathcal{A}(\varphi_i) &= \frac{(Q; Q)_{n-1}}{(1-Q)^n} \left\{ (1-Q^n) f(x, t) - (1-q)x_i(1-qQx_i) A_i(x; Q) \frac{\partial f}{\partial_q x_i} \right. \\ &\quad \left. - (1-q)(1-Q) \sum_{j=1, j \neq i}^m \frac{x_i x_j (1-qQx_j)}{x_i - Qx_j} A_j(x; Q) \frac{\partial f}{\partial_q x_j} \right\} \end{aligned}$$

and

$$\begin{aligned} \frac{f(x, t)}{D(t)} \mathcal{A}(b_1 T_1 \varphi_i) &= q^{\alpha+n-1} Q^{-(n-1)} \frac{(Q; Q)_{n-1}}{(1-Q)^n} \left\{ (1-Q^n) q^\beta x_i f(x, t) \right. \\ &\quad \left. + (1-q^\beta x_i) \frac{f(x, t)}{\prod_{k=1}^n (1-t_k/x_i)} \left(\prod_{k=1}^n (1-t_k/x_i) - \prod_{k=1}^n (Q-t_k/x_i) \right) \right\}. \end{aligned}$$

Substituting these into (2.19) yields

$$\begin{aligned} 0 &= qx_i^2 (1-q^2 Qx_i) T_{q, x_i}(A_i(x; Q)) \frac{\partial^2 S}{\partial_q x_i^2} \\ &\quad + (1-Q) \sum_{j=1, j \neq i}^m \frac{qx_i^2 x_j (1-qQx_j)}{qx_i - Qx_j} T_{q, x_i}(A_j(x; Q)) \\ &\quad \times \frac{\partial^2 S}{\partial_q x_i \partial_q x_j} + \frac{x_i}{1-q} \left\{ q^{\alpha+n-1} Q^{-(n-1)} (1-q^{\beta+1} Q^n x_i) \right. \\ &\quad \left. - (1-Q^n + (1-q^2 Qx_i)q) \right\} \frac{\partial S}{\partial_q x_i} \\ &\quad + \frac{qx_i(1-q^2 Qx_i)}{1-q} (1-T_{q, x_i}(A_i(x; Q))) \frac{\partial S}{\partial_q x_i} \\ &\quad - \frac{1-Q}{1-q} \sum_{j=1, j \neq i}^m \frac{qx_i x_j (1-qQx_j)}{qx_i - Qx_j} T_{q, x_i}(A_j(x; Q)) \frac{\partial S}{\partial_q x_j} \\ &\quad + \frac{(1-Q^n)(1-q^{\alpha+n-1} Q^{-(n-1)})}{(1-q)^2} S. \end{aligned}$$

From (2.27) and (2.28) we can conclude

THEOREM 2.4. – Assume $\mu = 1$ or $-\gamma$. Then ${}_q S_{n,m}(\alpha, \beta, \gamma, \mu; x_1^{-1}, \dots, x_m^{-1}; \xi)$ satisfies the following system of q -difference equations.

$$x_i(c - abqx_i) T_i(A_i(x; t)) \frac{\partial^2 S}{\partial_q x_i^2} + (1-q) \sum_{j=1, j \neq i}^m \frac{x_i x_j (c - abx_j)}{qx_i - tx_j} T_i(A_j(x; t)) \frac{\partial^2 S}{\partial_q x_i \partial_q x_j}$$

$$\begin{aligned}
& + \left\{ \frac{at^{m-1} - c/q + ac/q - c}{1-q} + \frac{1}{1-q} (abq - a^2t^{m-1})x_i \right\} \frac{\partial S}{\partial_q x_i} \\
& + \frac{1-t}{1-q} \left\{ \frac{1 - T_i(A_i(x;t))}{1-t} (c - abqx_i) \frac{\partial S}{\partial_q x_i} - \sum_{j=1, j \neq i}^m \frac{x_j(c - abx_j)}{qx_i - tx_j} T_i(A_j(x;t)) \frac{\partial S}{\partial_q x_j} \right\} \\
& - \frac{(1-a)(at^{m-1} - c/q)}{(1-q)^2} \frac{1}{x_i} S = 0, \quad i = 1, \dots, m,
\end{aligned}$$

where if $\mu = 1$, then put $(q, t) = (Q, q)$, $a = Q^{-n}$, $b = q^{-(\alpha+n-1)}Q^{n-1}$, $c = q^{-(\alpha+\beta+n-1)}$ and change x_i with $q^{-\beta}Qx_i$, $1 \leq i \leq m$. If $\mu = -\gamma$, then put $(q, t) = (q, Q)$, $a = Q^n$, $b = q^{\alpha+n-1}Q^{-(n-1)}$, $c = q^{\alpha+\beta+n-1}$ and change x_i with $Q^{-1}x_i$, $1 \leq i \leq m$.

From this theorem one can derive the following theorem by straightforward calculation.

THEOREM 2.5. — Assume $\mu = 1$, or $-\gamma$. Then $(x_1 \cdots x_m)^{\mu n} {}_q S_{n,m}(\alpha, \beta, \gamma, \mu; x_1^{-1}, \dots, x_m^{-1}; \xi)$ satisfies the system (2.26) in which if $\mu = 1$, then put $(q, t) = (Q, q)$, $a = Q^{-n}$, $b = q^{-(\alpha+n-1)}Q^{n-1}$, $c = q^{-(\alpha+\beta+n-1)}$ and change x_i with $q^{-\beta}Qx_i$, $1 \leq i \leq m$. If $\mu = -\gamma$, then put $(q, t) = (q, Q)$, $a = Q^n$, $b = q^{\alpha+n-1}Q^{-(n-1)}$, $c = q^{\alpha+\beta+n-1}$ and change x_i with $Q^{-1}x_i$, $1 \leq i \leq m$.

3. q -Hypergeometric functions

3.1. MACDONALD POLYNOMIALS. — We first recall the definition of Macdonald polynomials [Ma2]. Let $\lambda = (\lambda_1, \lambda_2, \dots)$ be a partition and $\lambda' = (\lambda'_1, \lambda'_2, \dots)$ the conjugate partition. The number λ'_1 of parts of λ is denoted by $\ell(\lambda)$, called the length of λ . If λ has m_1 parts equal to 1, m_2 parts equal to 2, and so on, we write $\lambda = (1^{m_1} 2^{m_2} \dots)$ and denote $\prod_{r \geq 1} (r^{m_r} m_r!) = z_\lambda$. We write $|\lambda| = \sum \lambda_i$. If μ is another partition, then write $\mu \leq \lambda$ when $|\lambda| = |\mu|$ and $\mu_1 + \dots + \mu_i \leq \lambda_1 + \dots + \lambda_i$ for all i . Given a partition $\lambda = (\lambda_1, \dots, \lambda_m)$ of length $\leq m$, the monomial symmetric polynomial $m_\lambda(x_1, \dots, x_m)$ is defined by

$$m_\lambda(x_1, \dots, x_m) = \sum x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_m^{\lambda_m},$$

where the sum is over all distinct monomials obtainable from $x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_m^{\lambda_m}$ by permutations of the x 's. In particular when $\lambda = (r)$ we have the r -th power sum:

$$m_{(r)} = p_r(x_1, \dots, x_m) = \sum_{i=1}^m x_i^r.$$

For each partition λ , we set $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots$.

Let q, t be independent indeterminates and $\mathbb{Q}(q, t)$ be the field of rational functions in q and t . We have the fundamental ([Ma2, (2.8)]).

THEOREM 3.1. — For each partition λ of length $\leq m$ there exists a unique symmetric polynomial $P_\lambda(x_1, \dots, x_m; q, t)$ with coefficients in $\mathbb{Q}(q, t)$ satisfying

$$(3.1) \quad P_\lambda = \sum_{\mu \leq \lambda} u_{\lambda\mu} m_\mu$$

where $u_{\lambda\mu} \in \mathbb{Q}(q, t)$ and $u_{\lambda\lambda} = 1$; and

$$(3.2) \quad D_1^{(q,t)} P_\lambda = e_\lambda P_\lambda$$

where $D_1^{(q,t)}$ and $e_\lambda(q, t)$ are defined by

$$(3.3) \quad D_1^{(q,t)} = \sum_{i=1}^m A_i(x; t) x_i \frac{\partial}{\partial_q x_i}, \quad e_\lambda(q, t) = \sum_{i=1}^m \frac{1 - q^{\lambda_i}}{1 - q} t^{m-i}.$$

We understand that $P_\lambda(x_1, \dots, x_m) = 0$ if $\ell(\lambda) \geq m + 1$. One can readily verify that $P_\lambda(x_1, \dots, x_r, 0, \dots, 0) = P_\lambda(x_1, \dots, x_r)$.

Denote the ring of symmetric polynomials in $x_1, \dots, x_m$ over the field $F = \mathbb{Q}(q, t)$ by $\Lambda_{m,F}$. Define a scalar product $\langle \cdot, \cdot \rangle$ on $\Lambda_{m,F}$ by

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda(q, t)$$

where

$$z_\lambda(q, t) = z_\lambda \prod_{i=1}^{\ell(\lambda)} \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}}.$$

Note that $D_1^{(q,t)}$ has the following properties ([Ma2, (2.7.1)-(2.7.3)]):

$$D_1^{(q,t)} m_\lambda = \sum_{\mu \leq \lambda} c_{\lambda\mu} m_\mu$$

for each partition λ of length $\leq m$;

$$\langle D_1^{(q,t)} f, g \rangle = \langle g, D_1^{(q,t)} f \rangle$$

for all $f, g \in \Lambda_{m,F}$;

$$\lambda \neq \mu \Rightarrow c_{\lambda\lambda} \neq c_{\mu\mu}.$$

From these properties one can deduce that the condition (3.2) in the Theorem 3.1 can be replaced by

$$\langle P_\lambda, P_\mu \rangle = 0 \quad \text{if } \lambda \neq \mu.$$

We shall need a kind of specialization formula of P_λ . Let u be a new indeterminate and define a homomorphism

$$\epsilon_{u,t} : \Lambda_{m,F} \rightarrow F[u]$$

by

$$(3.4) \quad \epsilon_{u,t}(p_r) = \frac{1 - u^r}{1 - t^r}$$

for each $r \geq 1$. Then we see

$$\epsilon_{t^m,t}(f) = f(1, t, \dots, t^{m-1}).$$

Consider the *diagram* of λ in which the rows and columns are arranged as in a matrix, with the i th row consisting of λ_i boxes. For each square $s = (i, j)$ in the diagram of λ , let

$$\begin{aligned} a(s) &= \lambda_i - j, & a'(s) &= j - 1, \\ l(s) &= \lambda'_j - i, & l'(s) &= i - 1, \end{aligned}$$

and put

$$h_\lambda(q, t) = \prod_{s \in \lambda} (1 - q^{a(s)} t^{l(s)+1}), \quad h'_\lambda(q, t) = \prod_{s \in \lambda} (1 - q^{a(s)+1} t^{l(s)})$$

and

$$b_\lambda = b_\lambda(q, t) = \frac{h_\lambda(q, t)}{h'_\lambda(q, t)}.$$

One has different expressions of h_λ and h'_λ .

PROPOSITION 3.2. – *Let λ be a partition of length $\leq n$. Then*

$$(3.5) \quad h'_\lambda(q, t) = (q)_\infty^n \left(\prod_{i=1}^n (q^{\lambda_i+1} t^{n-i})_\infty \right)^{-1} \prod_{1 \leq i < j \leq n} \frac{(q^{\lambda_i - \lambda_j + 1} t^{j-i})_\infty}{(q^{\lambda_i - \lambda_j + 1} t^{j-i-1})_\infty}.$$

$$(3.6) \quad h_\lambda(q, t) = (t)_\infty^n \left(\prod_{i=1}^n (q^{\lambda_i} t^{n-i+1})_\infty \right)^{-1} \prod_{1 \leq i < j \leq n} \frac{(q^{\lambda_i - \lambda_j} t^{j-i+1})_\infty}{(q^{\lambda_i - \lambda_j} t^{j-i})_\infty}.$$

Proof. – We prove (3.5). One can prove (3.6) in the same way. Put

$$C = \{i \mid \lambda_{i+1} < \lambda_i\},$$

so that

$$\prod_{s \in \lambda} (1 - q^{a(s)+1} t^{l(s)}) = \prod_{i \in C} \prod_{r=0}^{i-1} \prod_{j=\lambda_{i+1}+1}^{\lambda_i} (1 - q^{\lambda_i - r - j + 1} t^r).$$

Observe that

$$\begin{aligned} \prod_{1 \leq i < j \leq n} (q^{\lambda_i - \lambda_j + 1} t^{j-i})_{\infty} &= \prod_{r=1}^{n-1} \prod_{i=r+1}^n (q^{\lambda_{i-r} - \lambda_i + 1} t^r)_{\infty} \\ &= (q)_{\infty}^{-n} \prod_{r=0}^{n-1} \prod_{i=r+1}^n (q^{\lambda_{i-r} - \lambda_i + 1} t^r)_{\infty}, \\ \prod_{1 \leq i < j \leq n} (q^{\lambda_i - \lambda_j + 1} t^{j-i-1})_{\infty} &= \prod_{r=0}^{n-2} \prod_{i=r+1}^{n-1} (q^{\lambda_{i-r} - \lambda_{i+1} + 1} t^r)_{\infty} \\ &= \left(\prod_{r=0}^{n-1} (q^{\lambda_{n-r} + 1} t^r)_{\infty} \right)^{-1} \prod_{r=0}^{n-1} \prod_{i=r+1}^n (q^{\lambda_{i-r} - \lambda_{i+1} + 1} t^r)_{\infty}. \end{aligned}$$

Hence we get

$$\begin{aligned} \text{RHS of (3.5)} &= \prod_{r=0}^{n-1} \prod_{i=r+1}^n \frac{(q^{\lambda_{i-r} - \lambda_i + 1} t^r)_{\infty}}{(q^{\lambda_{i-r} - \lambda_{i+1} + 1} t^r)_{\infty}} \\ &= \prod_{i \in C} \prod_{r=0}^{i-1} \frac{(q^{\lambda_{i-r} - \lambda_i + 1} t^r)_{\infty}}{(q^{\lambda_{i-r} - \lambda_{i+1} + 1} t^r)_{\infty}} \\ &= \prod_{i \in C} \prod_{r=0}^{i-1} \prod_{j=\lambda_{i+1}+1}^{\lambda_i} (1 - q^{\lambda_{i-r} - j + 1} t^r), \end{aligned}$$

as desired.

We define the *generalized factorial* $(a)_{\lambda}^{(q,t)}$ by

$$(a)_{\lambda}^{(q,t)} = \prod_{s \in \lambda} (t^{l'(s)} - q^{a'(s)} a).$$

The following explicit formula ([Ma2, (5.3)]) is essential:

THEOREM 3.3. – *We have*

$$\epsilon_{u,t}(P_{\lambda}(q, t)) = \frac{(u)_{\lambda}^{(q,t)}}{h_{\lambda}(q, t)}.$$

Remark. – This formula is equivalent to the q -binomial theorem for the q -hypergeometric functions defined in the next subsection (see Theorem 3.5). We shall treat this problem in a separate paper.

We shall also need ([Ma2, (3.11)]):

$$(3.7) \quad \sum_{\lambda} b_{\lambda}(q, t) P_{\lambda}(x; q, t) P_{\lambda}(y; q, t) = \prod_{i,j} \frac{(tx_i y_j; q)_{\infty}}{(x_i y_j; q)_{\infty}},$$

or its dual [Ma2, (3.12)]

$$(3.8) \quad \sum_{\lambda} P_{\lambda}(x; q, t) P_{\lambda'}(y; t, q) = \prod_{i,j} (1 + x_i y_j).$$

This is a consequence of [Ma2, Sect. 5, p. 159]:

$$(3.9) \quad \langle P_{\lambda}, P_{\lambda} \rangle = b_{\lambda}(q, t)^{-1}.$$

3.2. q -HYPERGEOMETRIC FUNCTION. – For a partition λ , denote $b(\lambda) = \sum (i-1) \lambda_i = \sum \lambda'_i (\lambda'_i - 1)/2$.

DEFINITION 3.4. – Let $a_1, \dots, a_r$ and $b_1, \dots, b_s$ be complex numbers such that $(b_j)_{\lambda}^{(q,t)} \neq 0, 1 \leq j \leq m$ for any partition of length $\leq m$. The q -hypergeometric function ${}_r\Phi_s^{(q,t)}(a_1, \dots, a_r; b_1, \dots, b_s; x_1, \dots, x_m)$ is defined by

$$(3.10) \quad {}_r\Phi_s^{(q,t)}(a_1, \dots, a_r; b_1, \dots, b_s; x) = \sum_{\lambda} \frac{\prod_{i=1}^r (a_i)_{\lambda}^{(q,t)} \left\{ (-1)^{|\lambda|} q^{b(\lambda')} \right\}^{s+1-r}}{\prod_{i=1}^s (b_i)_{\lambda}^{(q,t)} h'_{\lambda}(q, t)} P_{\lambda}(x; q, t).$$

As a consequence of the factor $\left\{ (-1)^{|\lambda|} q^{b(\lambda')} \right\}^{s+1-r}$, it follows that

$$(3.11) \quad \lim_{a \rightarrow \infty} {}_{r+1}\Phi_s^{(q,t)}(a_1, \dots, a_r, a; b_1, \dots, b_s; x_1/a, \dots, x_m/a) \\ = {}_r\Phi_s^{(q,t)}(a_1, \dots, a_r; b_1, \dots, b_s; x).$$

When $m = 1$, ${}_r\Phi_s^{(q,t)}(x)$ reduces to the ordinary q -hypergeometric function ${}_r\phi_s(x)$ (cf. [An, GR]), being independent of t .

THEOREM 3.5. – We have

$$(3.12) \quad {}_1\Phi_0^{(q,t)}(a; -; x_1, \dots, x_m) = \prod_{i=1}^m \frac{(ax_i; q)_{\infty}}{(x_i; q)_{\infty}}.$$

Proof. – Note first that [Ma2, (2.6)]:

$$(3.13) \quad \prod_{i,j} \frac{(tx_i y_j; q)_{\infty}}{(x_i y_j; q)_{\infty}} = \sum_{\lambda} z_{\lambda}(q, t)^{-1} p_{\lambda}(x) p_{\lambda}(y).$$

In particular we have

$$\prod_{i=1}^m \frac{(ax_i; q)_{\infty}}{(x_i; q)_{\infty}} = \sum_{\lambda} z_{\lambda}(q, a)^{-1} p_{\lambda}(x).$$

It follows from (3.7) and (3.13) that

$$(3.14) \quad \sum_{\lambda} b_{\lambda}(q, t) P_{\lambda}(x; q, t) P_{\lambda}(y; q, t) = \sum_{\lambda} z_{\lambda}(q, t)^{-1} p_{\lambda}(x) p_{\lambda}(y)$$

so

$$(3.15) \quad \sum_{|\lambda| \leq l} b_{\lambda}(q, t) P_{\lambda}(x; q, t) P_{\lambda}(y; q, t) = \sum_{|\lambda| \leq l} z_{\lambda}(q, t)^{-1} p_{\lambda}(x) p_{\lambda}(y)$$

for arbitrary l . In view of Theorem 3.3 and

$$\epsilon_{a,t}(p_{\lambda}(y)) = \prod_{i=1}^{\ell(\lambda)} \frac{1 - a^{\lambda_i}}{1 - t^{\lambda_i}},$$

applying $\epsilon_{a,t}$ to both sides of (3.15) considered as polynomials in y yields

$$(3.16) \quad \sum_{|\lambda| \leq l} \frac{(a)_{\lambda}^{(q,t)}}{h'_{\lambda}(q, t)} P_{\lambda}(x) = \sum_{|\lambda| \leq l} z_{\lambda}(q, a)^{-1} p_{\lambda}(x).$$

Since l is arbitrary, this clearly gives (3.12).

COROLLARY 3.6

$$(3.17) \quad {}_0\Phi_0^{(q,t)}(x_1, \dots, x_m) = \prod_{i=1}^m (x_i; q)_{\infty}.$$

Proof. – This follows at once from (3.11) because

$$\lim_{a \rightarrow \infty} \prod_{i=1}^m \frac{(x_i; q)_{\infty}}{(x_i/a; q)_{\infty}} = \prod_{i=1}^m (x_i; q)_{\infty}.$$

Next we consider the convergence of the series. We assume $0 < t < 1$.

LEMMA 3.7. – *Let $\|x\| = \max\{|x_1|, \dots, |x_m|\}$. There exists a positive constant C depending only on q, t and m such that*

$$|P_{\lambda}(x; q, t)| \leq C (h_{\lambda}^{-1} h'_{\lambda})^{1/2} (m \|x\|)^{|\lambda|}.$$

Proof. – Put $|\lambda| = d$ and write

$$P_{\lambda} = \sum_{|\mu|=d} a_{\mu} p_{\mu}$$

so that

$$\langle P_\lambda, P_\lambda \rangle = \sum_{|\mu|=d} a_\mu^2 z_\mu(q, t).$$

By Cauchy's inequality we have

$$|P_\lambda|^2 \leq \left(\sum_{|\mu|=d} a_\mu^2 z_\mu(q, t) \right) \left(\sum_{|\mu|=d} \frac{|p_\mu|^2}{z_\mu(q, t)} \right).$$

It follows from (3.9) that

$$\sum_{|\mu|=d} a_\mu^2 z_\mu(q, t) = h_\lambda^{-1} h'_\lambda.$$

Put

$$C_1 = \max_{\ell(\lambda) \leq m} \frac{1 - t^{\lambda_i}}{1 - q^{\lambda_i}}$$

so that

$$\sum_{|\mu|=d} \frac{|p_\mu|^2}{z_\mu(q, t)} \leq C_1 \sum_{|\mu|=d} z_\mu^{-1} p_\mu^2(|x_1|, \dots, |x_m|).$$

Since $\sum_{|\mu|=d} z_\mu^{-1} p_\mu = \sum_{|\mu|=d} m_\mu$ ([Ma2, p. 17]), we obtain

$$\sum_{|\mu|=d} \frac{|p_\mu|^2}{z_\mu(q, t)} \leq C_1 m^d \binom{m+d-1}{d} \|x\|^{2d}.$$

Note that

$$\binom{m+d-1}{d} = \frac{(d+1)(d+2) \cdots (d+m-1)}{(m-1)!} \leq d^{m-1} \frac{m!}{(m-1)!}.$$

Put $C_2 = \max_d m^{-d} d^{m-1}$. This gives

$$\sum_{|\mu|=d} \frac{|p_\mu|^2}{z_\mu(q, t)} \leq C_1 C_2 m^{2d+1} \|x\|^{2d}.$$

Hence by setting $C = (m C_1 C_2)^{1/2}$, we arrive at the desired inequality.

THEOREM 3.8. — We have

- (1) If $r \leq s$, then the series (3.10) converges absolutely for all $x \in \mathbb{C}^m$.
- (2) If $r = s + 1$, then the series (3.10) converges absolutely for $\|x\| < m^{-1}$.
- (3) If $r > s + 1$, then the series (3.10) does not converge absolutely for $x \neq (0, \dots, 0)$ unless it terminates.

Proof. – We compare the series (3.10) with the series

$${}_r\phi_s(a_1, \dots, a_r; b_1, \dots, b_s; z) = \sum_{d=0}^{\infty} \frac{\prod_{k=1}^r (a_k; q)_d \left\{ (-1)^d q^{d(d-1)/2} \right\}^{s+1-r}}{\prod_{l=1}^s (b_l; q)_d} \frac{z^d}{(q; q)_d}$$

which is known to have radius of convergence $\rho = \infty$ if $r \leq s$, $\rho = 1$ if $r = s + 1$, $\rho = 0$ if $r > s + 1$ unless it terminates [GR, p. 5]. Put

$$a_{ki} = a_k t^{-(i-1)}, \quad b_{li} = b_l t^{-(i-1)}.$$

Note first that

$$(h_\lambda h'_\lambda)^{-1} \left(\prod_{i=1}^m (q; q)_{\lambda_i} \right)^2 \leq (1-t)^{-m}$$

and

$$t^{(r-s)b(\lambda)} \leq R^{|\lambda|}$$

where we put $R = 1$ if $r \geq s$, and $R = t^{(r-s)m}$ if $r < s$. Then by Lemma 3.7 we have

$$\begin{aligned} & \sum_{\lambda} \left| \frac{\prod_{i=1}^r (a_i)_{\lambda}^{(q,t)} \left\{ (-1)^{|\lambda|} q^{b(\lambda')} \right\}^{s+1-r}}{\prod_{i=1}^s (b_i)_{\lambda}^{(q,t)} h'_{\lambda}(q, t)} P_{\lambda} \right| \\ & \leq (1-t)^{-m/2} \sum_{\lambda} q^{(s+1-r)b(\lambda')} t^{(r-s)b(\lambda)} \left| \frac{\prod_{k=1}^r \prod_{i=1}^m (a_{ki}; q)_{\lambda_i} \left(\frac{h_{\lambda}}{h'_{\lambda}} \right)^{1/2} \frac{P_{\lambda}}{\prod_{i=1}^m (q; q)_{\lambda_i}}}{\prod_{l=1}^s \prod_{i=1}^m (b_{li}; q)_{\lambda_i}} \right| \\ & \leq (1-t)^{-m/2} C \sum_{\lambda} t^{(r-s)b(\lambda)} \left| \frac{\prod_{k=1}^r \prod_{i=1}^m (a_{ki}; q)_{\lambda_i} q^{(s+1-r)b(\lambda')} \frac{(m||x||)^{|\lambda|}}{\prod_{i=1}^m (q; q)_{\lambda_i}}}{\prod_{l=1}^s \prod_{i=1}^m (b_{li}; q)_{\lambda_i}} \right| \\ & \leq (1-t)^{-m/2} C \prod_{i=1}^m \sum_{\lambda_i=0}^{\infty} \left| \frac{\prod_{k=1}^r (a_{ki}; q)_{\lambda_i} q^{(s+1-r)\lambda_i(\lambda_i-1)/2} \frac{(Rm||x||)^{\lambda_i}}{\prod_{i=1}^m (q; q)_{\lambda_i}}}{\prod_{l=1}^s (b_{li}; q)_{\lambda_i}} \right|. \end{aligned}$$

This completes the proof of the assertions (1) and (2).

For the proof of (3), suppose $x_i \neq 0$, $1 \leq i \leq a$, and $x_i = 0$ otherwise. Note that

$$h'_{\lambda}{}^{-1} \prod_{i=1}^m (q; q)_{\lambda_i} \geq ((1-q)(1-qt^m)^{-1})^{|\lambda|}.$$

Put $\theta = (1 - q)(1 - qt^m)^{-1}$. Picking up the terms with $\lambda = (d, d, \dots, d) = (d^a)$ (so that $P_\lambda = x_1 \cdots x_a$ by (3.1)), we obtain

$$\begin{aligned} & \sum_{\lambda} \left| \frac{\prod_{i=1}^r (a_i)_{\lambda}^{(q,t)} \left\{ (-1)^{|\lambda|} q^{b(\lambda')} \right\}^{s+1-r}}{\prod_{i=1}^s (b_i)_{\lambda}^{(q,t)} h'_{\lambda}(q, t)} P_{\lambda} \right| \\ & \geq \sum_{\lambda} t^{(r-s)b(\lambda)} \left| \frac{\prod_{k=1}^r \prod_{i=1}^m (a_{ki}; q)_{\lambda_i} q^{(s+1-r)b(\lambda')}}{\prod_{l=1}^s \prod_{i=1}^m (b_{li}; q)_{\lambda_i}} \frac{\theta^{|\lambda|} P_{\lambda}}{\prod_{i=1}^m (q; q)_{\lambda_i}} \right| \\ & \geq \sum_{d=0}^{\infty} \left| \frac{\prod_{k=1}^r \prod_{i=1}^a (a_{ki}; q)_d q^{a(s+1-r)d(d-1)/2}}{\prod_{l=1}^s \prod_{i=1}^a (b_{li}; q)_d} \frac{|t^{(r-s)a(a-1)/2} \theta^a x_1 \cdots x_a|^d}{((q; q)_d)^a} \right|. \end{aligned}$$

This last series is easily shown to be divergent, thereby completing the proof of (3).

4. q -Difference system of q -hypergeometric function

4.1. SUMMED-UP EQUATION. — As in the case of $q = 1$ [Kan2], we shall consider the q -difference equation formed by summing the q -difference equations, multiplied by $A_i(x; t)$ each, in the system (2.26). First we introduce auxiliary operators:

$$D_2^{(q,t)} = \frac{1-q}{1-t} \sum_{1 \leq i < j \leq m} A_{ij}(x; t) x_i x_j \frac{\partial^2}{\partial_q x_i \partial_q x_j} - \frac{1}{1-t} \sum_{1 \leq i \neq j \leq m} A_{ij}(x; t) x_i \frac{\partial}{\partial_q x_i}$$

where

$$A_{ij}(x; t) = t \prod_{k=1, k \neq i, j}^m \frac{(tx_i - x_k)(tx_j - x_k)}{(x_i - x_k)(x_j - x_k)},$$

and

$$\varepsilon = \varepsilon_m = \sum_{i=1}^m A_i(x; t) \frac{\partial}{\partial_q x_i}.$$

It is known that [Ma3]

$$(4.1) \quad D_2^{(q,t)} P_{\lambda} = f_{\lambda}(q, t) P_{\lambda}$$

where

$$\begin{aligned} f_{\lambda} &= \frac{1}{(1-q)(1-t)} \sum_{1 \leq i < j \leq m} t^{2m-i-j} (q^{\lambda_i + \lambda_j} - 1) \\ &= \sum_{1 \leq i < j \leq m} \left\{ \frac{(1 - t^{m-i} q^{\lambda_i})(1 - t^{m-j} q^{\lambda_j})}{(1-q)(1-t)} - \frac{(1 - t^{m-i})(1 - t^{m-j})}{(1-q)(1-t)} \right\} + \frac{1-m}{1-t} e_{\lambda}. \end{aligned}$$

Denote by Λ_m^r the vector space of symmetric homogeneous polynomials of degree r in $x_1, \dots, x_m$ with coefficients in $\mathbb{Q}(q, t)$.

LEMMA 4.1. – ε defines a linear mapping from Λ_m^r to Λ_m^{r-1} for every r .

Proof. – Since the $p_\lambda(x_1, \dots, x_m)$'s with $|\lambda| = r$ and $\ell(\lambda) \leq m$ form a basis of Λ_m^r , it suffices to show $\varepsilon p_\lambda \in \Lambda_m^{r-1}$. But this easily boils down to prove the case of $\lambda = (r)$. We see

$$\varepsilon p_r = \frac{1 - q^r}{1 - q} \sum_{i=1}^m A_i x_i^{r-1} = \frac{1 - q^r}{1 - q^{r-1}} D_1 p_{r-1}.$$

Hence the lemma follows immediately from (3.2) and the fact that the $P_\lambda(x_1, \dots, x_m)$'s with $|\lambda| = r - 1$ and $\ell(\lambda) \leq m$ form a basis of Λ_m^{r-1} .

Let us denote by $L_m = L_m^{(q,t)}$ the q -difference operator formed by summing the q -difference operators, multiplied by $A_i = A_i(x; t)$ each, in the system (2.26):

$$\begin{aligned} (4.2) \quad L_m &= \sum_{i=1}^m x_i (c - abqx_i) A_i T_i(A_i) \frac{\partial^2}{\partial_q x_i^2} + (1 - t) \\ &\quad \times \sum_{1 \leq i \neq j \leq m} \frac{x_i x_j (c - abx_j)}{qx_i - tx_j} A_i T_i(A_j) \frac{\partial^2}{\partial_q x_i \partial_q x_j} \\ &\quad + \sum_{i=1}^m \left\{ \frac{t^{m-1} - c}{1 - q} + \frac{1}{1 - q} ((1 - a)(1 - b)t^{m-1} - (t^{m-1} - abq)) x_i \right\} A_i \frac{\partial}{\partial_q x_i} \\ &\quad + \frac{1 - t}{1 - q} \left\{ \sum_{i=1}^m \frac{1 - T_i(A_i)}{1 - t} (c - abqx_i) A_i \frac{\partial}{\partial_q x_i} \right. \\ &\quad \left. - \sum_{1 \leq i \neq j \leq m} \frac{x_j (c - abx_j)}{qx_i - tx_j} A_i T_i(A_j) \frac{\partial}{\partial_q x_j} \right\} \\ &\quad - \frac{(1 - a)(1 - b)t^{m-1}}{(1 - q)^2} \frac{1 - t^m}{1 - t}. \end{aligned}$$

Now we can state the following crucial lemma.

LEMMA 4.2. – *We have*

$$\begin{aligned} (4.3) \quad L_m &= \frac{c}{1 - q} (D_1 \varepsilon - \varepsilon D_1) - ab \left(D_1^2 - \frac{1 - t^2}{t(1 - q)} D_2 \right) + \frac{t^{m-1}}{1 - q} \varepsilon \\ &\quad + \frac{1}{1 - q} \left\{ \frac{2ab(1 - t^m)}{1 - t} - (a + b)t^{m-1} \right\} D_1 - \frac{(1 - a)(1 - b)t^{m-1}}{(1 - q)^2} \frac{1 - t^m}{1 - t}. \end{aligned}$$

Proof. – We show that

$$\begin{aligned}
 (4.4) \quad & \frac{1}{1-q}(D_1\varepsilon - \varepsilon D_1) \\
 &= \sum_{i=1}^m x_i A_i T_i(A_i) \frac{\partial^2}{\partial_q x_i^2} + (1-t) \sum_{1 \leq i < j \leq m} x_i x_j \left\{ \frac{A_i T_i(A_j)}{q x_i - t x_j} + \frac{A_j T_j(A_i)}{q x_j - t x_i} \right\} \frac{\partial^2}{\partial_q x_i \partial_q x_j} \\
 &\quad - \frac{1}{1-q} \sum_{i=1}^m A_i T_i(A_i) \frac{\partial}{\partial_q x_i} - \frac{1-t}{1-q} \sum_{1 \leq i \neq j \leq m} \frac{x_j}{q x_i - t x_j} A_i T_i(A_j) \frac{\partial}{\partial_q x_j}
 \end{aligned}$$

$$\begin{aligned}
 (4.5) \quad & D_1^2 - \frac{1-t^2}{t(1-q)} D_2 \\
 &= \sum_{i=1}^m q x_i^2 A_i T_i(A_i) \frac{\partial^2}{\partial_q x_i^2} \\
 &\quad + (1-t) \sum_{1 \leq i < j \leq m} x_i x_j \left\{ \frac{x_j A_i T_i(A_j)}{q x_i - t x_j} + \frac{x_i A_j T_j(A_i)}{q x_j - t x_i} \right\} \frac{\partial^2}{\partial_q x_i \partial_q x_j} \\
 &\quad + \frac{1}{1-q} \sum_{i=1}^m \left\{ -q x_i A_i T_i(A_i) - (1-t) \sum_{1 \leq j \leq m, j \neq i} \frac{x_i^2}{q x_j - t x_i} A_j T_j(A_i) \right. \\
 &\quad \left. + \frac{2 - (t^{m-1} + t^m)}{1-t} x_i A_i \right\} \frac{\partial}{\partial_q x_i}.
 \end{aligned}$$

One can check (4.3) without difficulty assuming (4.4) and (4.5). It is clear that the coefficient of $\partial^2/\partial_q x_i^2$ in $D_1\varepsilon - \varepsilon D_1$ is $(1-q)x_i A_i T_i(A_i)$. For the coefficient of $\partial^2/\partial_q x_i \partial_q x_j$, it suffices to observe

$$\begin{aligned}
 & \text{the coefficient of } \partial^2/\partial_q x_i \partial_q x_j \text{ in } D_1\varepsilon - \varepsilon D_1 \\
 &= x_i A_i T_i(A_j) + x_j A_j T_j(A_i) - x_j A_i T_i(A_j) - x_i A_j T_j(A_i) \\
 &= (x_i - x_j)(A_i T_i(A_j) - A_j T_j(A_i)) \\
 &= (x_i - x_j) \frac{A_{ij}}{t} \left\{ \frac{(t x_i - x_j)(t x_j - q x_i)}{(x_i - x_j)(x_j - q x_i)} - \frac{(t x_j - x_i)(t x_i - q x_j)}{(x_j - x_i)(x_i - q x_j)} \right\} \\
 &= \frac{A_{ij}}{t} \frac{(1-q)(1-t)(q-t)x_i x_j (x_i + x_j)}{(q x_i - x_j)(q x_j - x_i)}.
 \end{aligned}$$

and

$$\begin{aligned}
 & \frac{A_i T_i(A_j)}{q x_i - t x_j} + \frac{A_j T_j(A_i)}{q x_j - t x_i} \\
 &= \frac{A_{ij}}{t} \left\{ \frac{t x_i - x_j}{(x_i - x_j)(q x_i - x_j)} + \frac{t x_j - x_i}{(x_j - x_i)(q x_j - x_i)} \right\} \\
 &= \frac{A_{ij}}{t} \frac{(q-t)(x_i + x_j)}{(q x_i - x_j)(q x_j - x_i)}.
 \end{aligned}$$

We have also

$$\begin{aligned} & \text{the coefficient of } \partial/\partial_q x_i \text{ in } D_1 \varepsilon - \varepsilon D_1 \\ &= \sum_{j=1}^m A_j x_j \frac{\partial A_i}{\partial_q x_j} - \sum_{j=1}^m A_j \frac{\partial(A_i x_i)}{\partial_q x_j} \\ &= -A_i T_i(A_i) - \sum_{1 \leq j \leq m, j \neq i} (x_i - x_j) A_j \frac{\partial A_i}{\partial_q x_j}. \end{aligned}$$

Since

$$\begin{aligned} (x_i - x_j) \frac{\partial A_i}{\partial_q x_j} &= \prod_{k=1, k \neq i, j} \frac{tx_i - x_k}{x_i - x_k} \left(\frac{tx_i - qx_j}{x_i - qx_j} - \frac{tx_i - x_j}{x_i - x_j} \right) \frac{x_i - x_j}{(q-1)x_j} \\ &= \prod_{k=1, k \neq i, j} \frac{tx_i - x_k}{x_i - x_k} \frac{(1-t)x_i}{qx_j - x_i} \\ &= \frac{(1-t)x_i}{qx_j - tx_i} T_j(A_i), \end{aligned}$$

the proof of (4.4) is complete.

The coefficient of $\partial^2/\partial_q x_i^2$ in $D_1^2 - \frac{1-t^2}{t(1-q)} D_2$ is clearly as in (4.5). For the coefficient of $\partial^2/\partial_q x_i \partial_q x_j$, note that

$$\begin{aligned} & x_i x_j (A_i T_i(A_j) + A_j T_j(A_i)) \\ &= \frac{A_{ij}}{t} \frac{x_i x_j}{x_i - x_j} \frac{(qx_i - tx_j)(tx_i - x_j)(qx_j - x_i) - (qx_j - tx_i)(tx_j - x_i)(qx_i - x_j)}{(qx_i - x_j)(qx_j - x_i)} \end{aligned}$$

and

$$\begin{aligned} & (qx_i - tx_j)(tx_i - x_j)(qx_j - x_i) - (qx_j - tx_i)(tx_j - x_i)(qx_i - x_j) \\ &= (qx_i - x_j + (1-t)x_j)(tx_i - x_j)(qx_j - x_i) - (qx_j - x_i + (1-t)x_i)(tx_j - x_i)(qx_i - x_j) \\ &= (1+t)(x_i - x_j)(qx_i - x_j)(qx_j - x_i) \\ & \quad + (1-t)\{x_j(qx_j - x_i)(tx_i - x_j) - x_i(qx_i - x_j)(tx_j - x_i)\}. \end{aligned}$$

Hence

$$\begin{aligned} & \text{the coefficient of } \partial^2/\partial_q x_i \partial_q x_j \text{ in } D_1^2 - \frac{1-t^2}{t(1-q)} D_2 \\ &= (1-t) \frac{A_{ij}}{t} \frac{x_i x_j \{x_j(qx_j - x_i)(tx_i - x_j) - x_i(qx_i - x_j)(tx_j - x_i)\}}{(x_i - x_j)(qx_i - x_j)(qx_j - x_i)} \\ &= (1-t) x_i x_j \left\{ \frac{x_j A_i T_i(A_j)}{qx_i - tx_j} + \frac{x_i A_j T_j(A_i)}{qx_j - tx_i} \right\}. \end{aligned}$$

Since

$$\begin{aligned} & \text{the coefficient of } \partial/\partial_q x_i \text{ in } D_1^2 - \frac{1-t^2}{t(1-q)} D_2 \\ &= \frac{1}{1-q} \left\{ \frac{1-t^m}{1-t} x_i A_i - q x_i A_i T_i(A_i) - \sum_{1 \leq j \leq m, j \neq i} x_i A_j T_j(A_i) + \frac{1+t}{t} \sum_{1 \leq j \leq m, j \neq i} x_i A_{ij} \right\}, \end{aligned}$$

we conclude the proof of (4.5) from

$$\begin{aligned} & \sum_{1 \leq j \leq m, j \neq i} \left\{ A_j T_j(A_i) - \frac{1+t}{t} A_{ij} - (1-t) \frac{x_i}{q x_j - t x_i} A_j T_j(A_i) \right\} \\ &= \sum_{1 \leq j \leq m, j \neq i} A_j \prod_{k=1, k \neq i, j}^m \frac{t x_i - x_k}{x_i - x_k} \left\{ \frac{t x_i - q x_j}{x_i - q x_j} - (1+t) \frac{x_j - x_i}{t x_j - x_i} - (1-t) \frac{x_i}{q x_j - x_i} \right\} \\ &= \sum_{1 \leq j \leq m, j \neq i} A_j \prod_{k=1, k \neq i, j}^m \frac{t x_i - x_k}{x_i - x_k} \frac{t x_i - x_j}{t x_j - x_i} \\ &= -A_i \sum_{1 \leq j \leq m, j \neq i} A_j \frac{x_j - x_i}{t x_j - x_i} \\ &= -\frac{1-t^{m-1}}{1-t} A_i \end{aligned}$$

where the last equality follows from (2.10).

4.2. GENERALIZED BINOMIAL COEFFICIENTS.

DEFINITION 4.3. – For any partitions λ and μ of length $\leq m$, the generalized binomial coefficient $\binom{\lambda}{\mu}_m$ is defined by

$$\varepsilon \left(\frac{t^{b(\lambda)}}{\epsilon_{t^m, t}(P_\lambda)} P_\lambda(x_1, \dots, x_m) \right) = \sum_{\mu} \binom{\lambda}{\mu}_m \frac{t^{b(\mu)}}{\epsilon_{t^m, t}(P_\mu)} P_\mu(x_1, \dots, x_m).$$

Remark. – If we put $q = t^\alpha$ and let $t \rightarrow 1$, then it is readily seen that $\binom{\lambda}{\mu}_m$ reduces to the generalized binomial coefficient defined by using Jack polynomials [Kan2, p. 1096]. In this case the following theorem has been announced by [L] and proved by [Kan2].

Denote $\mu \subset \lambda$ if $\mu_i \leq \lambda_i$ for every i .

THEOREM 4.4. – (1) $\binom{\lambda}{\mu}_m \neq 0$ if and only if $\mu \subset \lambda$ and $|\mu| = |\lambda| - 1$.

(2) the $\binom{\lambda}{\mu}_m$'s are independent of the dimension m in the sense that $\binom{\lambda}{\mu}_r = \binom{\lambda}{\mu}_s$ provided $r, s \leq \ell(\lambda)$.

We leave the proof to Section 6. We write $\binom{\lambda}{\mu}$ dropping the subscript m . For each partition λ , we put $\lambda_{(i)} = (\lambda_1, \dots, \lambda_{i-1}, \lambda_i + 1, \lambda_{i+1}, \dots)$ and $\lambda^{(i)} = (\lambda_1, \dots, \lambda_{i-1}, \lambda_i - 1, \lambda_{i+1}, \dots)$ and call them admissible if the parts are in nonincreasing order. We shall write

$\lambda_{(i,j)} = (\lambda_{(i)})_{(j)}$, $\lambda^{(i,j)} = (\lambda^{(i)})_{(j)}$. By Theorem 4.4 (1), $\binom{\lambda}{\mu} = 0$ unless $\lambda = \mu_{(i)}$ (or $\mu = \lambda^{(i)}$) for some i . Hence, in view of $b(\lambda) - b(\lambda^{(i)}) = i - 1$, we have

$$(4.6) \quad \varepsilon P_\lambda(x_1, \dots, x_m) = c_\lambda h'_\lambda \sum_{i=1}^m \binom{\lambda}{\lambda^{(i)}} \frac{t^{1-i}}{c_{\lambda^{(i)}} h'_{\lambda^{(i)}}} P_{\lambda^{(i)}}(x_1, \dots, x_m)$$

where we have put

$$c_\lambda = c_\lambda(q, t, m) = \frac{\varepsilon_{t^m, t}(P_\lambda)}{h'_\lambda} = \frac{(t^m)_\lambda^{(q, t)}}{h_\lambda} h'_\lambda.$$

The summation in (4.6) is over all i such that $\lambda^{(i)}$ is admissible. This convention will be used in all future summations involving $\lambda_{(i)}$ or $\lambda^{(i)}$.

PROPOSITION 4.5. – *The formal series*

$$S(x_1, \dots, x_m) = \sum_\lambda \gamma_\lambda \frac{P_\lambda(x_1, \dots, x_m)}{h'_\lambda}$$

satisfies the summed-up equation $L_m(S) = 0$ if and only if the coefficients γ_λ satisfy the following recurrence relations

$$(4.7) \quad \frac{1}{(1-q)c_\lambda} \sum_{i=1}^m c_{\lambda_{(i)}} \binom{\lambda_{(i)}}{\lambda} t^{m+1-2i} (t^{i-1} - q^{\lambda_i} c) \gamma_{\lambda_{(i)}} \\ + \left\{ -ab \left(e_\lambda^2 - \frac{1-t^2}{t(1-q)} f_\lambda \right) + \left(\frac{2ab(1-t^m)}{(1-q)(1-t)} - \frac{(a+b)t^{m-1}}{1-q} \right) e_\lambda \right. \\ \left. - \frac{(1-a)(1-b)t^{m-1}(1-t^m)}{(1-q)^2(1-t)} \right\} \gamma_\lambda = 0.$$

Proof. – One can easily verify that, using (3.2), (4.1) and (4.6), the left-hand side of (4.7) is the coefficient of P_λ/h'_λ in $L_m(S)$.

Note that for $r \leq m$ we have

$$(4.8) \quad S(x_1, \dots, x_r) := S(x_1, \dots, x_r, 0, \dots, 0) = \sum_\lambda \gamma_\lambda \frac{P_\lambda(x_1, \dots, x_m)}{h'_\lambda}.$$

The recurrence relation (4.7) implies the following uniqueness property.

COROLLARY 4.6. – *Assume that $(c)_\lambda^{(q, t)} \neq 0$ for any partition λ of length $\leq m$. If the formal series (4.8) satisfies $L_r(S(x_1, \dots, x_r)) = 0$ for every $r \leq m$, and $S(0, \dots, 0) = 0$, then $S(x_1, \dots, x_m) \equiv 0$.*

Proof. – Note that the coefficient of $\gamma_{(i)}$ of (4.7) is not zero because of the assumption $(c)_\lambda^{(q,t)} \neq 0$. We prove $S(x_1, \dots, x_r) = 0$ by induction on r , the case $r = 1$ being immediate from (4.7). Clearly it suffices to show $S(x_1, \dots, x_m) = 0$ assuming $S(x_1, \dots, x_r) = 0$ for $r \leq m-1$, i.e. $\gamma_\lambda = 0$ if $\ell(\lambda) \leq m-1$. For γ_κ with $\kappa_m = 1$, put $\lambda = \kappa^{(m)}$ or $\lambda_{(m)} = \kappa$. Substituting this λ into (4.7) immediately shows that $\gamma_{\lambda_{(m)}}$ is a linear combination of γ_λ and $\gamma_{\lambda_{(i)}}$, $i < m$. Thus we find $\gamma_{\lambda_{(m)}} = 0$. The general case follows by induction on κ_m . Let us denote by $(2.26)_m$ the system (2.26) to express its dimensional dependence.

LEMMA 4.7. – *If $S(x_1, \dots, x_m)$ is a solution of $(2.26)_m$, then $S(x_1, \dots, x_r)$, $1 \leq r \leq m$, is a solution of $(2.26)_r$.*

Proof. – Clearly it suffices to prove the case of $r = m-1$. Substitute $S(x_1, \dots, x_m)$ into $(2.26)_m$ with $i \neq m$ and put $x_m = 0$. Then one finds that $S(x_1, \dots, x_{m-1})$ satisfies that the system $(2.26)_{m-1}$ multiplied by t .

This lemma implies that if $S(x_1, \dots, x_m)$ is a solution of $(2.26)_m$, then $L_r(S(x_1, \dots, x_r)) = 0$ for $r \leq m$. Hence by Corollary 4.6 we obtain

LEMMA 4.8. – *Assume that $(c)_\lambda^{(q,t)} \neq 0$ for any partition λ of length $\leq m$. If $S(x_1, \dots, x_m)$ is a solution of $(2.26)_m$ and $S(0, \dots, 0) = 0$, then $S(x_1, \dots, x_m) \equiv 0$.*

We next provide some formulas for the $\binom{\lambda}{\mu}$'s.

LEMMA 4.9. – *We have*

$$(4.9) \quad \sum_{i=1}^m c_{\lambda_{(i)}} \binom{\lambda_{(i)}}{\lambda} = \frac{1-t^m}{(1-q)(1-t)} c_\lambda$$

$$(4.10) \quad \sum_{i=1}^m q^{\lambda_i} t^{m-i} c_{\lambda_{(i)}} \binom{\lambda_{(i)}}{\lambda} = \left\{ \frac{1-t^m}{(1-q)(1-t)} - e_\lambda \right\} t^{m-1} c_\lambda$$

$$(4.11) \quad \sum_{i=1}^m (q_i^\lambda t^{m-i})^2 c_{\lambda_{(i)}} \binom{\lambda_{(i)}}{\lambda} = \left\{ \frac{t^{2m-2}(1-t^m)}{(1-q)(1-t)} - \frac{2t^{m-1}(1-t^m)}{1-te_\lambda} \right. \\ \left. + (1-q)t^{m-1}e_\lambda^2 + (t-1)(t^{m-1} + t^{m-2})f_\lambda \right\} c_\lambda.$$

Proof. – We first show

$$(4.12) \quad \sum_{i=1}^m A_i x_i = t^{m-1} m_1$$

$$(4.13) \quad \sum_{i=1}^m A_i x_i^2 = t^{m-1} m_{(2)} + t^{m-2}(t-1)m_{(1,1)}$$

$$(4.14) \quad \sum_{1 \leq i \neq j \leq m} A_{ij}(x; t) x_i = \frac{t^{m-1}(1-t^{m-1})}{1-t} m_1$$

$$(4.15) \quad \sum_{1 \leq i < j \leq m} A_{ij}(x; t) x_i x_j = t^{2m-3} m_{(1,1)}$$

where $m_1 = m_{(1)} = x_1 + \cdots + x_m$. Replacing z with $1/z$ in (2.7) yields

$$\prod_{i=1}^m \frac{t - zx_i}{1 - zx_i} = (t-1) \sum_{i=1}^m \frac{zx_i A_i(x; t)}{1 - zx_i} + t^m.$$

Differentiating both sides once (resp. twice) with respect to z and then setting $z = 0$ gives (4.12) (resp. (4.13)). By (2.10) and (2.7) with $z = tx_i$ we have

$$\begin{aligned} \sum_{j=1, j \neq i}^m \prod_{k=1, k \neq i, j}^m \frac{tx_j - x_k}{x_j - x_k} \frac{x_i - x_j}{tx_i - x_j} &= \sum_{j=1, j \neq i}^m \prod_{k=1, k \neq i, j}^m \frac{tx_j - x_k}{x_j - x_k} \left\{ \frac{1}{t} + \frac{t-1}{t} \frac{x_j}{x_j - tx_i} \right\} \\ &= \frac{1}{t} \frac{1-t^{m-1}}{1-t} + \frac{t-1}{t} \frac{1}{1-t} \left\{ -t^{m-1} + \prod_{k=1, k \neq i}^m \frac{x_k - t^2 x_i}{x_k - tx_i} \right\} \\ &= \frac{1}{t} \left\{ \frac{1-t^m}{1-t} - \prod_{k=1, k \neq i}^m \frac{x_k - t^2 x_i}{x_k - tx_i} \right\}. \end{aligned}$$

Hence we obtain

$$\begin{aligned} &\sum_{1 \leq i \neq j \leq m} A_{ij}(x; t) x_i \\ &= t \sum_{i=1}^m A_i x_i \left\{ \sum_{j=1, j \neq i}^m \prod_{k=1, k \neq i, j}^m \frac{tx_j - x_k}{x_j - x_k} \frac{x_i - x_j}{tx_i - x_j} \right\} \\ &= \frac{1-t^m}{1-t} \sum_{i=1}^m A_i x_i - \sum_{i=1}^m A_i(x; t^2) x_i \\ &= \frac{1-t^m}{1-t} t^{m-1} m_1 - t^{2m-2} m_1 \\ &= \frac{t^{m-1}(1-t^{m-1})}{1-t} m_1. \end{aligned}$$

The proof of (4.15) is similar to that of (4.14) and we omit it.

Setting $a = 0$ in (3.12) yields

$$(4.16) \quad \sum_{\lambda} \frac{t^{b(\lambda)}}{h'_{\lambda}} P_{\lambda}(x_1, \dots, x_m) = \prod_{i=1}^m \frac{1}{(x_i; q)_{\infty}}.$$

It follows from (2.9) that

$$\begin{aligned}\varepsilon\left(\prod_{i=1}^m \frac{1}{(x_i; q)_\infty}\right) &= \frac{1}{1-q} \left(\sum_{i=1}^m A_i\right) \prod_{i=1}^m \frac{1}{(x_i; q)_\infty} \\ &= \frac{1-t^m}{(1-q)(1-t)} \prod_{i=1}^m \frac{1}{(x_i; q)_\infty}.\end{aligned}$$

Hence applying ε to both sides of (4.16), we have

$$\frac{1-t^m}{(1-q)(1-t)} \sum_{\lambda} \frac{t^{b(\lambda)}}{h'_{\lambda}} P_{\lambda}(x_1, \dots, x_m) = \sum_{\lambda} c_{\lambda} \left(\sum_{i=1}^m \binom{\lambda}{\lambda^{(i)}} \frac{t^{b(\lambda^{(i)})}}{c_{\lambda^{(i)}} h'_{\lambda^{(i)}}} P_{\lambda^{(i)}}\right).$$

Equating coefficients of P_{λ} in both sides yields (4.9).

For the proof of (4.10), note that by (4.12) and (2.9) we see

$$\begin{aligned}D_1\left(\frac{1}{\prod_{i=1}^m (x_i; q)_\infty}\right) &= \frac{1}{1-q} \left(\left(\sum_{i=1}^m A_i x_i\right) \prod_{i=1}^m \frac{1}{(x_i; q)_\infty}\right) \\ &= \frac{t^{m-1}}{1-q} \frac{m_1}{\prod_{i=1}^m (x_i; q)_\infty}\end{aligned}$$

and

$$\begin{aligned}\varepsilon\left(\frac{m_1}{\prod_{i=1}^m (x_i; q)_\infty}\right) &= \left\{\sum_{i=1}^m A_i + \frac{1}{1-q} \sum_{i=1}^m T_i(m_1) A_i\right\} \frac{1}{\prod_{i=1}^m (x_i; q)_\infty} \\ &= \left\{\frac{1-t^m}{1-t} + \left(\frac{1-t^m}{(1-q)(1-t)} - t^{m-1}\right) m_1\right\} \frac{1}{\prod_{i=1}^m (x_i; q)_\infty},\end{aligned}$$

so that we obtain

$$\varepsilon D_1\left(\frac{1}{\prod_{i=1}^m (x_i; q)_\infty}\right) = \frac{t^{m-1}}{1-q} \left\{\frac{1-t^m}{1-t} + \left(\frac{1-t^m}{(1-q)(1-t)} - t^{m-1}\right) m_1\right\} \prod_{i=1}^m \frac{1}{(x_i; q)_\infty}.$$

Hence applying εD_1 to both sides of (4.16) gives

$$\begin{aligned}\sum_{\lambda} \frac{t^{b(\lambda)} e_{\lambda}}{h'_{\lambda}} \varepsilon P_{\lambda}(x_1, \dots, x_m) &= \frac{t^{m-1}(1-t^m)}{(1-q)(1-t)} \sum_{\lambda} \frac{t^{b(\lambda)}}{h'_{\lambda}} P_{\lambda}(x_1, \dots, x_m) \\ &\quad + \left(\frac{1-t^m}{(1-q)(1-t)} - t^{m-1}\right) \sum_{\lambda} \frac{t^{b(\lambda)} e_{\lambda}}{h'_{\lambda}} P_{\lambda}(x_1, \dots, x_m).\end{aligned}$$

Substituting (4.6) in the left-hand side and equating coefficients of P_{λ} of both sides gives (4.10) (use $e_{\lambda^{(i)}} = e_{\lambda} + t^{m-i} q^{\lambda_i}$ and then (4.9)).

By virtue of (4.14) and (4.15) we can derive

$$D_2\left(\prod_{i=1}^m \frac{1}{(x_i; q)_\infty}\right) = \frac{t^{m-1}}{(1-q)(1-t)} \left\{ t^{m-2} m_{(1,1)} - \frac{1-t^{m-1}}{1-t} m_1 \right\} \prod_{i=1}^m \frac{1}{(x_i; q)_\infty}.$$

Also by (4.12) and (4.13) we have

$$\begin{aligned} \varepsilon\left(\frac{m_{(1,1)}}{\prod_{i=1}^m (x_i; q)_\infty}\right) &= \left\{ \frac{1-t^{m-1}}{1-t} m_1 + \left(\frac{1-t^m}{(1-q)(1-t)} - (t^{m-1} + t^{m-2}) \right) m_{(1,1)} \right\} \\ &\quad \times \prod_{i=1}^m \frac{1}{(x_i; q)_\infty}. \end{aligned}$$

Hence applying εD_2 to both sides of (4.16), we get

$$\begin{aligned} &\sum_{\lambda} \frac{t^{b(\lambda)} f_{\lambda}}{h'_{\lambda}} \varepsilon P_{\lambda}(x) \\ &= \frac{t^{m-1}}{(1-q)(1-t)} \left\{ t^{m-2} \varepsilon\left(\frac{m_{(1,1)}}{\prod_{i=1}^m (x_i; q)_\infty}\right) - \frac{1-t^{m-1}}{1-t} \varepsilon\left(\frac{m_1}{\prod_{i=1}^m (x_i; q)_\infty}\right) \right\} \\ &= \frac{t^{m-1}}{(1-q)(1-t)} \left\{ t^{m-2} \left(\frac{1-t^m}{(1-q)(1-t)} - (t^{m-1} + t^{m-2}) \right) \frac{m_{(1,1)}}{\prod_{i=1}^m (x_i; q)_\infty} \right. \\ &\quad \left. + \frac{1-t^{m-1}}{1-t} \left(t^{m-2} + t^{m-1} - \frac{1-t^m}{(1-q)(1-t)} \right) \frac{m_1}{\prod_{i=1}^m (x_i; q)_\infty} \right. \\ &\quad \left. - \frac{(1-t^{m-1})(1-t^m)}{(1-t)^2} \frac{1}{\prod_{i=1}^m (x_i; q)_\infty} \right\} \\ &= \left(\frac{1-t^m}{(1-q)(1-t)} - (t^{m-1} + t^{m-2}) \right) \sum_{\lambda} \frac{t^{b(\lambda)} f_{\lambda}}{h'_{\lambda}} P_{\lambda}(x) \\ &\quad - \frac{t^{m-1}(1-t^{m-1})(1-t^m)}{(1-q)(1-t)^3} \sum_{\lambda} \frac{t^{b(\lambda)}}{h'_{\lambda}} P_{\lambda}(x). \end{aligned}$$

Substituting (4.6) into the left-hand side and equating coefficients of P_{λ} of both sides yields

$$\sum_{\lambda} f_{\lambda(i)} c_{\lambda(i)} \binom{\lambda(i)}{\lambda} = \left\{ \left(\frac{1-t^m}{(1-q)(1-t)} - (t^{m-1} + t^{m-2}) \right) f_{\lambda} - \frac{t^{m-1}(1-t^{m-1})(1-t^m)}{(1-q)(1-t)^3} \right\} c_{\lambda}.$$

Since

$$f_{\lambda(i)} = f_{\lambda} + \left\{ (1-q)e_{\lambda} - \frac{1-t^m}{1-t} \right\} \frac{t^{m-i} q^{\lambda_i}}{1-t} + \frac{(t^{m-i} q^{\lambda_i})^2}{1-t},$$

one can simplify the left-hand side by using (4.9) and (4.10) and this completes the proof of (4.11).

4.3. q -DIFFERENCE SYSTEM. — We now state one of the main results of this paper.

THEOREM 4.10. — *The hypergeometric function ${}_2\Phi_1^{(q,t)}(a, b; c; x_1, \dots, x_m)$ is the unique solution of the summed-up equation $L_m(S) = 0$ subject to the following condition:*

- (a) $S(x)$ is a symmetric function in $x_1, \dots, x_m$.
- (b) $S(x)$ is analytic at the origin with $S(0) = 1$.
- (c) $S(x_1, \dots, x_r, 0, \dots, 0)$ is a solution of $L_r(S) = 0$ for every $r \leq m$.

Proof. — The uniqueness is immediate from Corollary 4.6. Put $\gamma_\lambda = (a)_\lambda^{(q,t)}(b)_\lambda^{(q,t)}/(c)_\lambda^{(q,t)}$. Then we see

$$\gamma_{\lambda(i)} = \gamma_\lambda (t^{i-1} - q^{\lambda_i} a)(t^{i-1} - q^{\lambda_i} b)(t^{i-1} - q^{\lambda_i} c)^{-1}.$$

By virtue of (4.7), the proof that $L_m({}_2\Phi_1^{(q,t)}) = 0$ (and also (c)) boils down to show

$$\begin{aligned} & \frac{1}{(1-q)c_\lambda} \sum_{i=1}^m c_{\lambda(i)} \binom{\lambda(i)}{\lambda} t^{m+1-2i} (t^{i-1} - q^{\lambda_i} a)(t^{i-1} - q^{\lambda_i} b) - ab \left(e_\lambda^2 - \frac{1-t^2}{t(1-q)} f_\lambda \right) \\ & + \left(\frac{2ab(1-t^m)}{(1-q)(1-t)} - \frac{(a+b)t^{m-1}}{1-q} \right) e_\lambda - \frac{(1-a)(1-b)t^{m-1}(1-t^m)}{(1-q)^2(1-t)} = 0. \end{aligned}$$

But this is an immediate consequence of Lemma 4.9.

Next we compare the hypergeometric series ${}_2\Phi_1^{(q,t)}(a, b; c; x_1, \dots, x_m)$ with the q -Selberg integral ${}_qS_{n,m}(\alpha, \beta, \gamma, \mu; x; \xi)$ with $\mu = 1$ or $-\gamma$. If $\mu = 1$, then ${}_qS_{n,m}(x)$, being a polynomial, is analytic at the origin. But if $\mu = -\gamma$, then in general ${}_qS_{n,m}(x)$ has poles at $x_i = \xi_j^{-1} q^{\gamma+k}$, $k \in \mathbb{Z}$, so that the origin is an essential singularity. In this case we choose $\xi = \xi_F = (1, Q, \dots, Q^{n-1})$. Then the integral ${}_qS_{n,m}(x; \xi_F)$ over $[0, \xi_F]_q$ is analytic at the origin because the integral is done only over the set $< \xi_F >$ consisting of the points such that $t_1 = q^{\nu_1}, t_2/t_1 = q^{\gamma+\nu_2}, \dots, t_n/t_{n-1} = q^{\gamma+\nu_n}$ for each $\nu_j \in \mathbb{Z}_{\geq 0}$ (this is the so called “ α -stable cycle” in [Ao1]). Combining Theorem 2.3, Lemma 4.8 and Theorem 4.10, we now obtain

THEOREM 4.11. — *We have $(q^\beta Q^{-1}x = (q^\beta Q^{-1}x_1, \dots, q^\beta Q^{-1}x_m)$ etc.)*

$$(4.17) \quad {}_qS_{n,m}(\alpha, \beta, \gamma, 1; q^\beta Q^{-1}x; \xi) = C \cdot {}_2\Phi_1^{(Q,q)}(Q^{-n}, q^{-(\alpha+n-1)}Q^{n-1}; q^{-(\alpha+\beta+n-1)}; x)$$

$$(4.18) \quad {}_qS_{n,m}(\alpha, \beta, \gamma, -\gamma; Qx; \xi_F) = C_F \cdot {}_2\Phi_1^{(q,Q)}(Q^n, q^{\alpha+n-1}Q^{-(n-1)}; q^{\alpha+\beta+n-1}; x)$$

where $C = {}_qS_{n,0}(\alpha, \beta, \gamma; \xi)$, $C_F = {}_qS_{n,0}(\alpha, \beta, \gamma; \xi_F)$.

The condition that ${}_2\Phi_1^{(q,t)}(a, b; c; x_1, \dots, x_m)$ satisfies the system (2.26) is equivalent to an infinite system of polynomial equations in q, t, a, b, c . The formula (4.18) implies that these equations hold when $t = q^\gamma$, $a = q^{n\gamma}$, $b = q^{\alpha+n-1-(n-1)\gamma}$, $c = q^{\alpha+\beta+n-1}$. Since n, α, β, γ are arbitrary, these equations hold for any q, t, a, b, c . Thus we arrive at

THEOREM 4.12. – *The hypergeometric series ${}_2\Phi_1^{(q,t)}(a, b; c; x_1, \dots, x_m)$ is the unique solution of the system (2.26) subject to the following conditions:*

- (a) $S(x)$ is a symmetric function in $x_1, \dots, x_m$.
- (b) $S(x)$ is analytic at the origin with $S(0) = 1$.

5. Consequences

5.1. INTEGRATION FORMULA OF MACDONALD POLYNOMIALS.

Put

$${}_qD(\alpha, \beta, \gamma; t) = \prod_{j=1}^n t_j^{\alpha+(j-1)(1-2\gamma)} \frac{(qt_j)_\infty}{(q^\beta t_j)_\infty} \prod_{1 \leq i < j \leq n} \frac{(q^{1-\gamma} t_j / t_i)_\infty}{(q^\gamma t_j / t_i)_\infty} D(t).$$

Theorem 4.11 implies the following integration formula.

THEOREM 5.1.

$$\begin{aligned} & \int_{[0, \xi \infty]_q} P_\lambda(t; q, Q) {}_qD(\alpha, \beta, \gamma; t) \tilde{\omega} \\ &= {}_qS_{n,0}(\alpha, \beta, \gamma; \xi) \frac{(Q^n)_\lambda^{(q,Q)} (q^{\alpha+n-1} Q^{-(n-1)})_\lambda^{(q,Q)}}{h_\lambda(q, Q) (q^{\alpha+\beta+n-1})_\lambda^{(q,Q)}} \\ &= {}_qS_{n,0}(\alpha, \beta, \gamma; \xi) P_\lambda(1, Q, \dots, Q^{n-1}; q, Q) \frac{(q^{\alpha+n-1} Q^{-(n-1)})_\lambda^{(q,Q)}}{(q^{\alpha+\beta+n-1})_\lambda^{(q,Q)}}. \end{aligned}$$

Proof. – Replacing x by $-q^\beta Q^{-1}x$, λ by λ' and setting $y = t = (t_1, \dots, t_n)$, $(q, t) = (Q, q)$ in (3.8), we have

$$\prod_{1 \leq i \leq m, 1 \leq j \leq n} (1 - q^\beta Q^{-1} x_i t_j) = \sum_{\lambda} (-q^\beta Q^{-1})^{|\lambda'|} P_{\lambda'}(x; Q, q) P_\lambda(t; q, Q).$$

Substituting this into (4.17) yields

$$\begin{aligned} (5.1) \quad & \sum_{\lambda} (-q^\beta Q^{-1})^{|\lambda'|} P_{\lambda'}(x; Q, q) \int_{[0, \xi \infty]_q} P_\lambda(t; q, Q) {}_qD(\alpha, \beta, \gamma; t) \tilde{\omega} \\ &= C \cdot {}_2\Phi_1^{(Q,q)}(Q^{-n}, q^{-(\alpha+n-1)} Q^{n-1}; q^{-(\alpha+\beta+n-1)}; x) \\ &= C \sum_{\lambda'} \frac{(Q^{-n})_{\lambda'}^{(Q,q)} (q^{-(\alpha+n-1)} Q^{n-1})_{\lambda'}^{(Q,q)}}{(q^{-(\alpha+\beta+n-1)})_{\lambda'}^{(Q,q)} h'_{\lambda'}(Q, q)} P_{\lambda'}(x; Q, q). \end{aligned}$$

Note that in general we have

$$(a)_\lambda^{(q,t)} = (-a)^{|\lambda|} (a^{-1})_{\lambda'}^{(t,q)}, \quad h'_\lambda(q, t) = h_{\lambda'}(t, q).$$

Hence equating the coefficients of $P_\lambda(x; Q, q)$ in (5.1) immediately gives the desired first equality. The second equality is a direct consequence of Theorem 3.3.

We next show Theorem 5.1 implies the integration formula of Kadell [Kad2]. Assume $\gamma = k$, a positive integer. Put $x = \alpha + (n-1)(1-2k)$, $y = \beta$.

PROPOSITION 5.2. – Assume $\operatorname{Re}(x) > 0$, $y \neq 0, -1, -2, \dots$. We have

$$\begin{aligned}
 (5.2) \quad & \int_{[0,1]^n} P_\lambda(t; q, q^k) \prod_{j=1}^n t_j^x \frac{(qt_j)_\infty}{(q^y t_j)_\infty} \prod_{1 \leq i < j \leq n} t_i^{2k} (q^{1-k} t_j / t_i)_{2k} \tilde{\omega} \\
 &= q^{kx \binom{n}{2} + 2k^2 \binom{n}{3}} \frac{(q^{nk})_\lambda^{(q, q^k)}}{h_\lambda(q, q^k)} \\
 &\quad \times \prod_{i=1}^n \frac{\Gamma_q(ik+1) \Gamma_q(x + (n-i)k + \lambda_i) \Gamma_q(y + (n-i)k)}{\Gamma_q(k+1) \Gamma_q(x+y + (2n-i-1)k + \lambda_i)} \\
 &= q^{kx \binom{n}{2} + 2k^2 \binom{n}{3}} P_\lambda(1, q^k, \dots, q^{(n-1)k}; q, q^k) \\
 &\quad \cdot \prod_{i=1}^n \frac{\Gamma_q(ik+1) \Gamma_q(x + (n-i)k + \lambda_i) \Gamma_q(y + (n-i)k)}{\Gamma_q(k+1) \Gamma_q(x+y + (2n-i-1)k + \lambda_i)}.
 \end{aligned}$$

Proof. – Observe that $\prod_{1 \leq i < j \leq n} t_i^{2k-1} (q^{1-k} t_j / t_i)_{2k-1}$ is antisymmetric. Using Lemma 2.1, (4.17) and Theorem 5.1, we have

LHS of (5.2)

$$\begin{aligned}
 &= \int_{\langle \xi_F \rangle} P_\lambda(t; q, q^k) \prod_{j=1}^n t_j^x \frac{(qt_j)_\infty}{(q^y t_j)_\infty} \mathcal{A} \left(\prod_{1 \leq i < j \leq n} (t_i - q^k t_j) \right) t_i^{2k-1} (q^{1-k} t_j / t_i)_{2k-1} \tilde{\omega} \\
 &= \frac{(q^k; q^k)_n}{(1 - q^k)^n} \int_{\langle \xi_F \rangle} P_\lambda(t; q, q^k)_q D(x + (n-1)(2k-1), y, k; t) \tilde{\omega} \\
 &= \frac{(q^k; q^k)_n}{(1 - q^k)^n} q^{A_n} \frac{(q^{nk})_\lambda}{h_\lambda(q, q^k)} \frac{\prod_{(i,j) \in \lambda} (1 - q^{x+(n-i)k+j-1})}{\prod_{(i,j) \in \lambda} (1 - q^{x+y+(2n-i-1)k+j-1})} \\
 &\quad \cdot \prod_{i=1}^n \frac{\Gamma_q(ik) \Gamma_q(x + (n-i)k) \Gamma_q(y + (n-i)k)}{\Gamma_q(k) \Gamma_q(x+y + (2n-i-1)k)}.
 \end{aligned}$$

Hence (5.2) is immediate from the formulas

$$\Gamma_q(x+1) = \frac{1-q^x}{1-q} \Gamma_q(x)$$

and

$$A_n = \sum_{j=1}^n (\alpha - 2(j-1)k + n-1)(j-1)k = kx \binom{n}{2} + 2k^2 \binom{n}{3}.$$

Kadell [Kad2] has given a different proof of Proposition 5.2 in a slightly different expression:

$$\begin{aligned} \text{LHS of (5.2)} &= q^{B_n} \frac{(q^k; q^k)_n}{(1 - q^k)^n} \prod_{1 \leq i < j \leq n} \frac{(\lambda_i - \lambda_j + (j - i)k)_k}{(1 - q)^k} \\ &\quad \times \prod_{i=1}^n \frac{\Gamma_q(x + (n - i)k + \lambda_i) \Gamma_q(y + (n - i)k)}{\Gamma_q(x + y + (2n - i - 1)k + \lambda_i)}, \end{aligned}$$

where $B_n = k \sum_{i=1}^n (i - 1) \lambda_i + kx \binom{n}{2} + 2k^2 \binom{n}{3}$. This is checked by utilizing (3.6):

$$h_\lambda(q, q^k) = (q^k)_\infty^n \prod_{i=1}^n (q^{\lambda_i + (n-i+1)k})_\infty^{-1} \prod_{1 \leq i < j \leq n} (q^{\lambda_i - \lambda_j + (j-i)k})_k^{-1}$$

and observing

$$(q^{nk})_\lambda^{(q, q^k)} (q^{\lambda_i + (n-i+1)k})_\infty = q^{k \sum_{i=1}^n (i-1) \lambda_i} (q^k)_\infty^n.$$

5.2. INTEGRAL REPRESENTATION OF ${}_r\Phi_s^{(q,t)}(a, b; c; x)$.

Let $a_1, \dots, a_r$ and $b_1, \dots, b_s$ be such that $(b_j)_\lambda^{(q,t)} \neq 0$ for any j and any partition λ of length $\leq m$. Assume $m \leq n$, and put

$$\begin{aligned} (5.3) \quad & {}_r\Phi_s^{(q,t)}(a_1, \dots, a_r; b_1, \dots, b_s; x_1, \dots, x_m; y_1, \dots, y_n) \\ &= \sum_\lambda \frac{\prod_{i=1}^r (a_i)_\lambda^{(q,t)} \left\{ (-1)^{|\lambda|} q^{b(\lambda')} \right\}^{s+1-r}}{\prod_{i=1}^s (b_i)_\lambda^{(q,t)} h'_\lambda(q, t) P_\lambda(1, t, \dots, t^{n-1})} P_\lambda(x; q, t) P_\lambda(y; q, t). \end{aligned}$$

This series converges in a neighborhood of origin only if $r \leq s + 1$ and its proof is similar to that of Theorem 3.8.

PROPOSITION 5.3. — *Let $a_{r+1} = q^\varepsilon$ and $b_{s+1} = q^\eta$ and put $\alpha = \varepsilon + (n - 1)(\gamma - 1)$, $\beta = \eta - \varepsilon - (n - 1)\gamma$. We have*

$$\begin{aligned} (5.4) \quad & {}_{r+1}\Phi_{s+1}^{(q,t)}(a_1, \dots, a_{r+1}; b_1, \dots, b_{s+1}; x_1, \dots, x_m) \\ &= {}_qS_{n,0}(\alpha, \beta, \gamma; \xi_F)^{-1} \\ &\quad \times \int_{[0, \xi_F \infty]_q} {}_r\Phi_s^{(q,t)}(a_1, \dots, a_r; b_1, \dots, b_s; x_1, \dots, x_m; y_1, \dots, y_n) \\ &\quad \times {}_qD(\alpha, \beta, \gamma; y) \tilde{\omega} \end{aligned}$$

provided the right-hand side is convergent.

Proof. – This is an immediate consequence of Theorem 5.1 because

$$\frac{(q^{\alpha+n-1}Q^{-(n-1)})_{\lambda}^{(q,Q)}}{(q^{\alpha+\beta+n-1})_{\lambda}^{(q,Q)}} = \frac{(q^{\varepsilon})_{\lambda}^{(q,Q)}}{(q^{\eta})_{\lambda}^{(q,Q)}}.$$

PROPOSITION 5.4. – Let $a = q^{\delta}, b = q^{\varepsilon}, c = q^{\eta}$ and $\alpha = \varepsilon + (n-1)(\gamma-1)$, $\beta = \eta - \varepsilon - (n-1)\gamma$. We have

$$(5.5) \quad {}_2\Phi_1^{(q,Q)}(q^{-N}, b; c; q, qQ, \dots, qQ^{n-1}) = q^{Nn(\alpha+(n-1)(1-\gamma))} \prod_{i=1}^n \frac{(q^{\beta+(i-1)\gamma})_N}{(q^{\alpha+\beta+n-1-(i-1)\gamma})_N},$$

where $\delta = -N, N \in \mathbb{Z}_{\geq 0}$ and

$$(5.6) \quad {}_2\Phi_1^{(q,Q)}(a, b; c; c/ab, cQ/ab, \dots, cQ^{n-1}/ab) \\ = \prod_{i=1}^n \frac{\Gamma_q(\beta - \delta + (i-1)\gamma) \Gamma_q(\alpha + \beta + n - 1 - (i-1)\gamma)}{\Gamma_q(\beta + (i-1)\gamma) \Gamma_q(\alpha + \beta - \delta + n - 1 - (i-1)\gamma)}$$

provided the left-hand side is convergent.

Proof. – By Theorem 3.5 we have

$$(5.7) \quad {}_1\Phi_0^{(q,Q)}(q^{-N}; q, qQ, \dots, qQ^{n-1}; y_1, \dots, y_n) = \prod_{j=1}^n (q^{-N+1}y_j)_N,$$

$$(5.8) \quad {}_1\Phi_0^{(q,Q)}(a; c/ab, cQ/ab, \dots, cQ^{n-1}/ab; y_1, \dots, y_n) = \prod_{j=1}^n \frac{(q^{\beta}y_j)_{\infty}}{(q^{\beta-\gamma}y_j)_{\infty}}.$$

Substituting (5.7) (resp. (5.8)) in the right-hand side of (5.4) with $m = n, r = 1, s = 0$ and $a_1 = q^{-N}$ (resp. $a_1 = a$) gives

$$\begin{aligned} {}_2\Phi_1^{(q,Q)}(q^{-N}, b; c; q, qQ, \dots, qQ^{n-1}) &= q^{Nn(\alpha+(n-1)(1-\gamma))} \frac{{}_qS_{n,0}(\alpha, \beta + N, \gamma; q^{-N}\xi_F)}{{}_qS_{n,0}(\alpha, \beta, \gamma; \xi_F)} \\ &= q^{Nn(\alpha+(n-1)(1-\gamma))} \frac{{}_qS_{n,0}(\alpha, \beta + N, \gamma; \xi_F)}{{}_qS_{n,0}(\alpha, \beta, \gamma; \xi_F)}, \\ {}_2\Phi_1^{(q,Q)}(a, b; c; c/ab, cQ/ab, \dots, cQ^{n-1}/ab) &= \frac{{}_qS_{n,0}(\alpha, \beta - \delta, \gamma; \xi_F)}{{}_qS_{n,0}(\alpha, \beta, \gamma; \xi_F)}. \end{aligned}$$

Hence the proof follows from the explicit formula (4.19).

6. Proof of Theorem 4.4

6.1. SKEW MACDONALD POLYNOMIALS. – For any partition λ, μ, ν define rational functions $f_{\mu\nu}^\lambda(q, t)$ by

$$f_{\mu\nu}^\lambda = f_{\mu\nu}^\lambda(q, t) = \frac{\langle P_\lambda, P_\mu P_\nu \rangle}{\langle P_\lambda, P_\lambda \rangle}.$$

Equivalently,

$$(6.1) \quad P_\mu P_\nu = \sum_{\lambda} f_{\mu\nu}^\lambda(q, t) P_\lambda.$$

Clearly $f_{\mu\nu}^\lambda = 0$ unless $|\lambda| = |\mu| + |\nu|$. Moreover it holds that $f_{\mu\nu}^\lambda = 0$ unless $\lambda \supset \mu$ and $\lambda \supset \nu$ [Ma2, (4.2)].

If λ, μ are partitions, define *skew Macdonald polynomials* $P_{\lambda/\mu}$ by

$$(6.2) \quad P_{\lambda/\mu} = b_\lambda^{-1} b_\mu \sum_{\nu} b_\nu f_{\mu\nu}^\lambda(q, t) P_\nu.$$

Hence $P_{\lambda/\mu} = 0$ unless $\lambda \supset \mu$. Let $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$ be two sequences of independent indeterminates. Then we have [Ma2, (4.5)]:

$$(6.3) \quad P_\lambda(x, y) = \sum_{\mu} P_{\lambda/\mu}(x) P_\mu(y).$$

Put

$$\widetilde{f_{\mu\nu}^\lambda} = \frac{b_\mu b_\nu}{b_\lambda} f_{\mu\nu}^\lambda.$$

Setting $x = x_m, y = (x_1, \dots, x_{m-1})$ in (6.3), we get

$$(6.4) \quad P_\lambda(x_1, \dots, x_m) = \sum_{\mu} \left(\sum_r \widetilde{f_{\mu(r)}^\lambda} \right) x_m^r P_\mu(x_1, \dots, x_{m-1}).$$

If $\lambda \supset \mu$ then the skew shape λ/μ (regarded as a difference $\lambda - \mu$ of diagrams) is called a *horizontal r -strip* (resp. *vertical r -strip*) if $|\lambda/\mu| = r$ and no two distinct squares of λ/μ lie in the same column (resp. row). Then $\widetilde{f_{\mu(r)}^\lambda} \neq 0$ (resp. $f_{\mu(1^r)}^\lambda \neq 0$) if and only if $\lambda \supset \mu$ and λ/μ is a horizontal (resp. vertical) r -strip [Ma2, (4.8)]. Moreover they can be explicitly evaluated as follows. For each square s and each partition λ , define

$$(6.5) \quad b_\lambda(s) = b_\lambda(s; q, t) = \frac{1 - q^{a(s)} t^{l(s)+1}}{1 - q^{a(s)+1} t^{l(s)}}$$

if $s \in \lambda$, and $b_\lambda(s) = 1$ if $s \notin \lambda$. If S is any set of squares (contained in the diagram of λ or not), put

$$(6.6) \quad b_\lambda(S) = \prod_{s \in S} b_\lambda(s).$$

Now let λ, μ be partitions such that $\lambda \supset \mu$ and λ/μ is a horizontal r -strip. Let $C_{\lambda/\mu}$ (resp. $R_{\lambda/\mu}$) denote the union of the columns (resp. rows) that contain squares of λ/μ . Then [Ma2, (5.12)]:

$$(6.7) \quad f_{\mu(r)}^\lambda = b_{(r)}^{-1} b_\lambda(C_{\lambda/\mu}) / b_\mu(C_{\lambda/\mu}).$$

Observe that

$$(6.8) \quad \widetilde{f_{\mu(1)}^\lambda} = b_{(1)} \frac{b_\mu(R_{\lambda/\mu})}{b_\lambda(R_{\lambda/\mu})} = \frac{1-t}{1-q} \frac{b_\mu(R_{\lambda/\mu})}{b_\lambda(R_{\lambda/\mu})}.$$

If λ, μ be partitions such that $\lambda \supset \mu$ and λ/μ is a vertical r -strip, then applying the duality theorem [Ma2, (3.5)] (cf. [Ma3, Chap.6, (7.9)]) to (6.7), we obtain

$$(6.9) \quad f_{\mu(1r)}^\lambda = b_\lambda(\bar{R}_{\lambda/\mu}) / b_\mu(\bar{R}_{\lambda/\mu}),$$

where $\bar{R}_{\lambda/\mu}$ denotes the union of rows that do not contain squares of λ/μ .

6.2. LEMMAS. – Put

$$(6.10) \quad \left(\widetilde{\lambda} \right)_m = \prod_{s \in \lambda} (1 - q^{a'(s)} t^{m-l'(s)}) \left(\prod_{s \in \mu} (1 - q^{a'(s)} t^{m-l'(s)}) \right)^{-1} h_\mu h_\lambda^{-1} \left(\lambda \right)_m,$$

so that from Definition 4.3 we have

$$(6.11) \quad \varepsilon P_\lambda(x_1, \dots, x_m) = \sum_{|\mu|=|\lambda|-1} \left(\widetilde{\lambda} \right)_m P_\mu(x_1, \dots, x_m).$$

LEMMA 6.1. – Let λ and μ be partitions of $\ell(\lambda) \leq m$ and $\ell(\mu) \leq m-1$. We have

$$\begin{aligned} \left(\widetilde{\lambda} \right)_m &= t \left(\widetilde{\lambda} \right)_{m-1} + \widetilde{f_{\mu(1)}^\lambda}, \text{ if } \ell(\lambda) \leq m-1, \\ \left(\widetilde{\lambda} \right)_m &= \widetilde{f_{\mu(1)}^\lambda}, \text{ if } \ell(\lambda) = m. \end{aligned}$$

Proof. – Setting $x_m = 0$ in (6.11) yields

$$t\varepsilon P_\lambda(x_1, \dots, x_{m-1}) + \left. \frac{\partial P_\lambda(x_1, \dots, x_m)}{\partial_q x_m} \right|_{x_m=0} = \sum_{\mu} \widetilde{\binom{\lambda}{\mu}}_m P_\mu(x_1, \dots, x_{m-1}),$$

in which we see by (6.4) that

$$\left. \frac{\partial P_\lambda(x_1, \dots, x_m)}{\partial_q x_m} \right|_{x_m=0} = \sum_{|\mu|=|\lambda|-1} \widetilde{f_{\mu(1)}^\lambda} P_\mu(x_1, \dots, x_{m-1}).$$

Hence equating the coefficients of $P_\mu(x_1, \dots, x_{m-1})$ in both sides gives the desired formulas.

For each partition λ , define $\lambda_* = (\lambda_1 - 1, \lambda_2 - 1, \dots, \lambda_{\ell(\lambda)} - 1)$. One can readily derive from Theorem 3.1 that if $\ell(\lambda) = m$, then

$$(6.12) \quad P_\lambda(x_1, \dots, x_m) = \left(\prod_{i=1}^m x_i \right) P_{\lambda_*}(x_1, \dots, x_m).$$

LEMMA 6.2. – For partitions λ and μ of length $= m$, we have

$$\widetilde{\binom{\lambda}{\mu}}_m = q \widetilde{\binom{\lambda_*}{\mu_*}}_m + \widetilde{f_{\lambda_*(1^{m-1})}^\mu}.$$

Proof. – It follows from (6.12) that

$$(6.13) \quad \varepsilon P_\lambda(x_1, \dots, x_m) = \varepsilon \left(\prod_{i=1}^m x_i \right) P_{\lambda_*}(x_1, \dots, x_m) + q \left(\prod_{i=1}^m x_i \right) \varepsilon P_{\lambda_*}(x_1, \dots, x_m).$$

We assert in general that

$$(6.14) \quad \varepsilon e_r(x_1, \dots, x_m) = \frac{1 - t^{m-r+1}}{1 - t} e_{r-1}(x_1, \dots, x_m).$$

In fact

$$\begin{aligned} \sum_{i=1}^m A_i \frac{\partial}{\partial_q x_i} e_r(x_1, \dots, x_m) &= \sum_{i=1}^m A_i e_{r-1}(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_m) \\ &= \sum_{i=1}^m A_i (e_{r-1}(x_1, \dots, x_m) - x_i e_{r-2}(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_m)) \\ &= \frac{1 - t^m}{1 - t} e_{r-1}(x_1, \dots, x_m) - D_1 e_{r-1}(x_1, \dots, x_m) \\ &= \frac{1 - t^{m-r+1}}{1 - t} e_{r-1}(x_1, \dots, x_m). \end{aligned}$$

Substituting (6.14) with $r = m$ into (6.13) and equating the coefficients of $P_\mu(x_1, \dots, x_m)$ of both sides gives the desired formula.

LEMMA 6.3. — *We have*

$$\begin{aligned}
 (6.15) \quad & \sum_{\lambda, \mu} f_{\nu(1^r)}^\lambda \left(\widetilde{\binom{\lambda}{\mu}} \right)_m P_\mu(x_1, \dots, x_m) \\
 &= \left(\frac{1-t^m}{1-t} + qe_\nu \right) e_{r-1} P_\nu(x_1, \dots, x_m) \\
 &\quad - D_1 \left(e_{r-1} P_\nu(x_1, \dots, x_m) \right) + e_r \varepsilon P_\nu(x_1, \dots, x_m).
 \end{aligned}$$

Proof. — The left-hand side is nothing but $\varepsilon(e_r P_\nu(x_1, \dots, x_m))$. On the other hand using (6.14) and that

$$\begin{aligned}
 T_{q, x_i} e_r(x_1, \dots, x_m) &= e_r(x_1, \dots, x_m) + (q-1)x_i e_{r-1}(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_m) \\
 &= e_r(x_1, \dots, x_m) + qx_i e_{r-1}(x_1, \dots, x_m) - x_i T_{q, x_i} e_{r-1}(x_1, \dots, x_m),
 \end{aligned}$$

we have

$$\begin{aligned}
 & \varepsilon(e_r P_\nu(x_1, \dots, x_m)) \\
 &= (\varepsilon e_r) P_\nu(x_1, \dots, x_m) + \sum_{i=1}^m A_i T_{q, x_i}(e_r) \frac{\partial}{\partial_q x_i} P_\nu(x_1, \dots, x_m) \\
 &= \frac{1-t^{m-r+1}}{1-t} e_{r-1} P_\nu(x_1, \dots, x_m) + e_r \varepsilon P_\nu(x_1, \dots, x_m) + qe_{r-1} D_1 P_\nu(x_1, \dots, x_m) \\
 &\quad - \sum_{i=1}^m A_i x_i T_{q, x_i}(e_{r-1}) \frac{\partial}{\partial_q x_i} P_\nu(x_1, \dots, x_m).
 \end{aligned}$$

Hence the formula (6.15) is immediate from

$$\begin{aligned}
 D_1(e_{r-1} P_\nu(x_1, \dots, x_m)) &= \frac{t^{m-r+1} - t^m}{1-t} e_{r-1} P_\nu(x_1, \dots, x_m) \\
 &\quad + \sum_{i=1}^m A_i x_i T_{q, x_i}(e_{r-1}) \frac{\partial}{\partial_q x_i} P_\nu(x_1, \dots, x_m).
 \end{aligned}$$

LEMMA 6.4.

$$\begin{aligned}
 (6.17) \quad & \sum_{\lambda, \mu} f_{\nu(2)}^\lambda \left(\widetilde{\binom{\lambda}{\mu}} \right)_m P_\mu(x_1, \dots, x_m) \\
 &= \frac{1}{1-qt} \{ (1+q)(1-t^m) - (1-q^2)e_\nu \} e_1 P_\nu(x_1, \dots, x_m) \\
 &\quad + \frac{t(1-q^2)}{1-qt} D_1(e_1 P_\nu(x_1, \dots, x_m)) + P_{(2)} \varepsilon P_\nu(x_1, \dots, x_m).
 \end{aligned}$$

Proof. – One can readily deduce from Theorem 3.1 that

$$P_{(2)} = m_{(2)} + \frac{(1+q)(1-t)}{1-qt} m_{(1,1)},$$

so

$$(6.18) \quad \frac{\partial P_{(2)}}{\partial_q x_i} = \frac{t(1-q^2)}{1-qt} x_i + \frac{(1+q)(1-t)}{1-qt} e_1,$$

from which we get

$$(6.19) \quad \varepsilon P_{(2)} = \frac{(1+q)(1-qt^m)}{1-qt} e_1.$$

We have

$$(6.20) \quad \varepsilon(P_{(2)} P_\nu) = \varepsilon(P_{(2)}) P_\nu + P_{(2)} \varepsilon(P_\nu) + (q-1) \sum_{i=1}^m A_i x_i \frac{\partial P_{(2)}}{\partial_q x_i} \frac{\partial P_\nu}{\partial_q x_i}$$

and by (6.16) with $r = 2$ that

$$(6.21) \quad (q-1) \sum_{i=1}^m A_i x_i^2 \frac{\partial P_\nu}{\partial_q x_i} = D_1(e_1 P_\nu) - (t^{m-1} + e_\nu) e_1 P_\nu.$$

Substituting (6.18) and (6.19) into (6.20) and then applying (6.21) yields (6.17).

6.3. Now we turn to the proof of Theorem 4.4. We shall prove a stronger assertion: For any partitions λ and μ of length $\leq m$, we have

$$(6.22) \quad \left(\begin{smallmatrix} \lambda \\ \mu \end{smallmatrix} \right)_m = t^{b(\lambda)-b(\mu)} \frac{h'_\lambda}{h'_\mu} f_{\mu(1)}^\lambda,$$

which is, in view of (6.8) and (6.10), equivalent to

$$(6.23) \quad \widetilde{\left(\begin{smallmatrix} \lambda \\ \mu \end{smallmatrix} \right)}_m = \begin{cases} \frac{1-q^{\lambda_i-1} t^{m-i+1}}{1-q} \frac{b_\mu(R_{\lambda/\mu})}{b_\lambda(R_{\lambda/\mu})}, & \text{if } \mu = \lambda^{(i)}, \\ 0, & \text{if } \mu \not\subset \lambda. \end{cases}$$

Here note that $R_{\lambda/\mu}$ is the i -th row of λ . We shall denote the i -row (resp. j -th column) of λ by $R_{\lambda,i}$ (resp. $C_{\lambda,j}$) and write $b_\lambda(R_i)$ (resp. $b_\lambda(C_i)$) for $b_\lambda(R_{\lambda,i})$ (resp. $b_\lambda(C_{\lambda,i})$). Note that $b_\lambda(R_{\kappa,i}) = b_\lambda(R_i)$ provided $\kappa \supset \lambda$. (6.23) is rewritten as

$$(6.24) \quad \widetilde{\left(\begin{smallmatrix} \lambda \\ \mu \end{smallmatrix} \right)}_m = \begin{cases} \frac{1-q^{\lambda_i-1} t^{m-i+1}}{1-q} \frac{b_\mu(R_i)}{b_\lambda(R_i)}, & \text{if } \mu = \lambda^{(i)}, \\ 0, & \text{if } \mu \not\subset \lambda. \end{cases}$$

We prove (6.24) by induction on the dimension m . The case $m = 1$ is easy to check: Put $\lambda = (r)$, $\mu = (r-1)$. Then clearly $\widetilde{\left(\begin{smallmatrix} \lambda \\ \mu \end{smallmatrix}\right)}_m = (1-q^r)/(1-q)$ holds. On the other hand we see

$$b_\lambda(R_1) = \frac{(t; q)_r}{(q; q)_r}, b_\mu(R_1) = \frac{(t; q)_{r-1}}{(q; q)_{r-1}},$$

and therefore (6.24) follows at once. We assume that (6.24) holds in the dimensions $\leq m-1$. This implies that (6.24) holds in the case $\ell(\lambda) \leq m-1$. In fact by Lemma 6.1 we see that $\widetilde{\left(\begin{smallmatrix} \lambda \\ \mu \end{smallmatrix}\right)}_m = 0$ unless $\mu \subset \lambda$. Moreover if $\mu = \lambda^{(i)}$ and $\ell(\lambda) \leq m-1$, then by means of (6.8) we have

$$\begin{aligned} \widetilde{\left(\begin{smallmatrix} \lambda \\ \mu \end{smallmatrix}\right)}_m &= t \frac{1 - q^{\lambda_i-1} t^{m-i}}{1-q} \frac{b_\mu(R_i)}{b_\lambda(R_i)} + \frac{1-t}{1-q} \frac{b_\mu(R_i)}{b_\lambda(R_i)} \\ &= \frac{1 - q^{\lambda_i-1} t^{m-i+1}}{1-q} \frac{b_\mu(R_i)}{b_\lambda(R_i)}. \end{aligned}$$

If $\ell(\lambda) = m$, then $\mu = \lambda^{(m)}$ and $\lambda_m = 1$. Hence (6.24) is immediate from Lemma 6.1 also in this case.

Next suppose $\ell(\mu) = m$ and $\ell(\lambda) = m$ and that (6.24) holds for λ_* and μ_* . Then Lemma 6.2 implies that $\widetilde{\left(\begin{smallmatrix} \lambda \\ \mu \end{smallmatrix}\right)}_m = 0$ unless $\mu \subset \lambda$. If $\mu = \lambda^{(i)}$, then by (6.9) we have

$$(6.25) \quad \widetilde{\left(\begin{smallmatrix} \lambda \\ \mu \end{smallmatrix}\right)}_m = q \frac{1 - q^{\lambda_i-2} t^{m-i+1}}{1-q} \frac{b_{\mu_*}(R_i)}{b_{\lambda_*}(R_i)} + \frac{b_\mu(\bar{R}_{\mu/\lambda_*})}{b_{\lambda_*}(\bar{R}_{\mu/\lambda_*})}.$$

Observe that

$$\begin{aligned} b_{\mu_*}(R_i) &= b_\mu(R_i) b_\mu((i, 1))^{-1} = \frac{1 - q^{\lambda_i-1} t^{m-i}}{1 - q^{\lambda_i-2} t^{m-i+1}} b_\mu(R_i) \\ b_{\lambda_*}(R_i) &= b_\lambda(R_i) b_\lambda((i, 1))^{-1} = \frac{1 - q^{\lambda_i} t^{m-i}}{1 - q^{\lambda_i-1} t^{m-i+1}} b_\lambda(R_i) \end{aligned}$$

and

$$b_\mu(\bar{R}_{\mu/\lambda_*}) = b_\mu(R_i), \quad b_{\lambda_*}(\bar{R}_{\mu/\lambda_*}) = b_{\lambda_*}(R_i).$$

Substituting these into (6.25) gives (6.24). Iterating this argument, we see that the case $\ell(\lambda) = \ell(\mu) = m$ reduces to the case $\ell(\mu) \leq m-1$ (which we have just proved) or the case $\ell(\mu) = m$ and $\ell(\lambda) \leq m-1$.

It now remains to show that $\widetilde{\left(\begin{smallmatrix} \lambda \\ \mu \end{smallmatrix}\right)}_m = 0$ when $\ell(\lambda) \leq m-1$ and $\ell(\mu) = m$. We prove this by induction on $\ell(\lambda)$ and for fixed $\ell(\lambda)$ on $\lambda_{\ell(\lambda)}$, the case $\lambda = (1)$ being obvious.

Note first that $\widetilde{\left(\frac{\lambda}{\mu}\right)}_m = 0$ for λ of $\ell(\lambda) \leq p-1, p \leq m-1$ implies $\widetilde{\left(\frac{\lambda}{\mu}\right)}_m = 0$ for λ of $\ell(\lambda) \leq p$ and $\lambda_p = 1$. This follows from (6.15) by setting $r = 1, \nu = \lambda^{(p)}$ (so that $\ell(\nu) = p-1$) and equating the coefficients of P_μ of both sides. So we have reduced the proof to showing $\widetilde{\left(\frac{\lambda^{(p)}}{\mu}\right)}_m = 0$ for any partition λ of $|\lambda| = |\mu|$ and $\ell(\lambda) = p \leq m-1$ provided that $\widetilde{\left(\frac{\kappa}{\mu}\right)}_m = 0$ when $\ell(\kappa) \leq p-1$ or $\ell(\kappa) = p$ and $\kappa_p \leq \lambda_p$.

We divide the proof into several parts treating different cases. First we assume $p \leq m-2$ and derive necessary equalities from Lemma 6.3 and Lemma 6.4. We have

$$(6.26) \quad \widetilde{\left(\frac{\lambda_{(p+1,i)}^{(p)}}{\mu}\right)}_m = 0$$

provided $i \leq p-1$. This follows immediately from (6.15) if we set $r = 1, \nu = \lambda_{(i)}^{(p)}$ and compare the coefficients of P_μ using induction hypothesis. Setting $r = 1$ (resp. $r = 2$) and $\nu = \lambda$ or $\lambda_{(p+1)}^{(p)}$ (resp. $\nu = \lambda^{(p)}$) in (6.15) and comparing coefficients of P_μ using induction hypothesis and (6.26), we find that

$$(6.27) \quad f_{\lambda^{(p)}(1)}^{\lambda^{(p)}} \left(\frac{\lambda^{(p)}}{\mu} \right)_m + f_{\lambda^{(p)}(1)}^{\lambda_{(p+1)}^{(p)}} \left(\frac{\lambda_{(p+1)}^{(p)}}{\mu} \right)_m = \sum_i f_{\lambda^{(i)}(1)}^\mu \left(\frac{\lambda}{\lambda^{(i)}} \right)_m$$

$$(6.28) \quad f_{\lambda_{(p+1)}^{(p)}(1)}^{\lambda_{(p+1)}^{(p)}} \left(\frac{\lambda_{(p+1)}^{(p)}}{\mu} \right)_m + f_{\lambda_{(p+1)}^{(p)}(1)}^{\lambda_{(p+1,p+2)}^{(p)}} \left(\frac{\lambda_{(p+1,p+2)}^{(p)}}{\mu} \right)_m \\ + f_{\lambda_{(p+1)}^{(p)}(1)}^{\lambda_{(p+1,p+1)}^{(p)}} \left(\frac{\lambda_{(p+1,p+1)}^{(p)}}{\mu} \right)_m = \sum_i f_{\lambda_{(p+1)}^{(p,i)}(1)}^\mu \left(\frac{\lambda_{(p+1)}^{(p)}}{\lambda_{(p,i)}^{(p+1)}} \right)_m$$

$$(6.29) \quad f_{\lambda^{(p)}(1^2)}^{\lambda_{(p+1)}^{(p)}} \left(\frac{\lambda_{(p+1)}^{(p)}}{\mu} \right)_m + f_{\lambda^{(p)}(1^2)}^{\lambda_{(p+1,p+2)}^{(p)}} \left(\frac{\lambda_{(p+1,p+2)}^{(p)}}{\mu} \right)_m \\ = \left(\frac{1-t^m}{1-t} + qe_{\lambda^{(p)}} - e_\mu \right) f_{\lambda^{(p)}(1)}^\mu + \sum_i f_{\lambda^{(p,i)}(1^2)}^\mu \left(\frac{\lambda_{(p+1)}^{(p)}}{\lambda_{(p,i)}^{(p+1)}} \right)_m.$$

Similarly, setting $\nu = \lambda^{(p)}$ in (6.17) gives

$$(6.30) \quad f_{\lambda^{(p)}(2)}^{\lambda^{(p)}} \left(\frac{\lambda^{(p)}}{\mu} \right)_m + f_{\lambda^{(p)}(2)}^{\lambda_{(p+1)}^{(p)}} \left(\frac{\lambda_{(p+1)}^{(p)}}{\mu} \right)_m + f_{\lambda^{(p)}(2)}^{\lambda_{(p+1,p+1)}^{(p)}} \left(\frac{\lambda_{(p+1,p+1)}^{(p)}}{\mu} \right)_m \\ = \frac{1}{1-qt} \{ (1-q^2)(te_\mu - e_{\lambda^{(p)}} + (1+q)(1-t^m)) \} f_{\lambda^{(p)}(1)}^\mu \\ + \sum_i f_{\lambda^{(p,i)}(2)}^\mu \left(\frac{\lambda^{(p)}}{\lambda_{(p,i)}^{(p+1)}} \right)_m.$$

Here note that, because of induction hypothesis, the generalized binomial coefficients appearing in the right-hand sides of (6.27)–(6.30) are given by (6.24). So we regard these equalities as equations of unknowns $\left(\widetilde{\lambda_{(p)}^{(p)}}_{\mu}\right)_m$, $\left(\widetilde{\lambda_{(p+1)}^{(p+1)}}_{\mu}\right)_m$, $\left(\widetilde{\lambda_{(p+1,p+1)}^{(p)}}_{\mu}\right)_m$, and $\left(\widetilde{\lambda_{(p+1,p+2)}^{(p)}}_{\mu}\right)_m$.

Remark. – If $\lambda_p = 1$ (resp. $\lambda_p = 2$), we understand $\left(\widetilde{\lambda_{(p+1,p+1)}^{(p)}}_{\mu}\right)_m$, and $\left(\widetilde{\lambda_{(p+1,p+2)}^{(p)}}_{\mu}\right)_m$ (resp. $\left(\widetilde{\lambda_{(p+1,p+1)}^{(p)}}_{\mu}\right)_m$) to be zero and the following argument should be modified accordingly. We leave this task to the reader and assume henceforth that $\lambda_p \geq 3$.

Case 1. – $p \leq m - 2$ and $\mu \not\subset \lambda_{(p+1,p+2)}^{(p)}$. Observe that the right-hand sides of (6.27)–(6.30) are all vanishing. Hence, for the proof of $\left(\widetilde{\lambda_{(p)}^{(p)}}_{\mu}\right)_m = 0$, it suffices to show that the determinant of coefficient matrix of equations is not identically zero. For this purpose, set $t = 1$, then we have in general

$$b_{\lambda}(s) = \frac{1 - q^{a(s)}}{1 - q^{a(s)+1}}.$$

Hence by (6.8) and (6.9) we find that the coefficients appearing in the equations (6.27)–(6.29) are all equal to one. Also by (6.7) we have

$$b_{(1)}^{-1} f_{\lambda_{(p)}^{(p)}(2)} \Big|_{t=1} = b_{(1)}^{-1} f_{\lambda_{(p+1,p+1)}^{(p)}(2)} \Big|_{t=1} = 1$$

and

$$\begin{aligned} b_{(1)}^{-1} f_{\lambda_{(p)}^{(p)}(2)} \Big|_{t=1} &= (b_{(1)} b_{(2)})^{-1} \frac{b_{\lambda_{(p+1)}}(C_{\lambda_{(p+1)},1} \cup C_{\lambda_{(p)},\lambda_p})}{b_{\lambda_{(p)}}(C_{\lambda_{(p+1)},1} \cup C_{\lambda_{(p)},\lambda_p})} \Big|_{t=1} \\ &= (b_{(1)} b_{(2)}^{-1}) \Big|_{t=1} \frac{1 - q^{\lambda_p - 1}}{1 - q^{\lambda_p}} \frac{1 - q^{\lambda_p - 1}}{1 - q^{\lambda_p - 2}} \\ &= ((1 - q)(1 - q^{\lambda_p})(1 - q^{\lambda_p - 2}))^{-1} (1 - q^2)(1 - q^{\lambda_p - 1})^2, \end{aligned}$$

which we denote by $C(q)$. Therefore the determinant of the coefficient matrix (we multiply the equation (6.30) by $b_{(1)}^{-1}$) at $t = 1$ is

$$\begin{vmatrix} f_{\lambda_{(1)}^{(p)}} & f_{\lambda_{(1)}^{(p+1)}} & 0 & 0 \\ 0 & f_{\lambda_{(p+1)}^{(p)}(1)} & f_{\lambda_{(p+1,p+1)}^{(p)}(1)} & f_{\lambda_{(p+1,p+2)}^{(p)}(1)} \\ 0 & f_{\lambda_{(p)}^{(p)}(1^2)} & 0 & f_{\lambda_{(p+1,p+2)}^{(p)}(1^2)} \\ b_{(1)}^{-1} f_{\lambda_{(p)}^{(p)}(2)} & b_{(1)}^{-1} f_{\lambda_{(p+1)}^{(p+1)}(2)} & b_{(1)}^{-1} f_{\lambda_{(p+1,p+1)}^{(p)}(2)} & 0 \end{vmatrix}_{t=1}$$

$$= \begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & C(q) & 1 & 0 \end{vmatrix} = C(q) - 1 \neq 0,$$

as desired.

Case 2. $-p = m - 2$ and $\mu \subset \lambda_{(p+1,p+2)}^{(p)}$. It necessarily follows that $\mu = \lambda_{(p+1,p+2)}^{(p,r)}$ for some $r \leq p$, and therefore the right-hand side of (6.27) is clearly zero. We see also

$$\text{RHS of (6.29)} = f_{\lambda_{(p,r)}^{(p,r)}(1^2)}^{\lambda_{(p+1,p+2)}^{(p,r)}} \left(\widetilde{\lambda_{(p)}^{(p)}} \right).$$

Hence it suffices to show that

$$(6.31) \quad f_{\lambda_{(p)}^{(p)}(1^2)}^{\lambda_{(p+1,p+2)}^{(p)}} \left(\widetilde{\lambda_{(p+1,p+2)}^{(p)}} \right)_m = f_{\lambda_{(p,r)}^{(p,r)}(1^2)}^{\lambda_{(p+1,p+2)}^{(p,r)}} \left(\widetilde{\lambda_{(p,r)}^{(p,r)}} \right)_m.$$

By (6.24) we have

$$\begin{aligned} & \left(\widetilde{\lambda_{(p)}^{(p)}} \right)_m^{-1} \left(\widetilde{\lambda_{(p+1,p+2)}^{(p)}} \right)_m \\ &= b_{\lambda_{(p,r)}^{(p,r)}}(R_r)^{-1} b_{\lambda_{(p+1,p+2)}^{(p,r)}}(R_r) b_{\lambda_{(p+1,p+2)}^{(p)}}(R_r)^{-1} b_{\lambda_{(p)}^{(p)}}(R_r) \\ &= b_{\lambda_{(p,r)}^{(p,r)}}((r, 1))^{-1} b_{\lambda_{(p+1,p+2)}^{(p,r)}}((r, 1)) b_{\lambda_{(p+1,p+2)}^{(p)}}((r, 1))^{-1} b_{\lambda_{(p)}^{(p)}}((r, 1)). \end{aligned}$$

On the other hand by (6.9) we see also

$$\begin{aligned} f_{\lambda_{(p)}^{(p)}(1^2)}^{\lambda_{(p+1,p+2)}^{(p)}} &= b_{\lambda_{(p)}^{(p)}}(C_{\lambda_{(p)},1})^{-1} b_{\lambda_{(p+1,p+2)}^{(p)}}(C_{\lambda_{(p)},1}), \\ f_{\lambda_{(p,r)}^{(p,r)}(1^2)}^{\lambda_{(p+1,p+2)}^{(p,r)}} &= b_{\lambda_{(p,r)}^{(p,r)}}(C_{\lambda_{(p,r)},1})^{-1} b_{\lambda_{(p+1,p+2)}^{(p,r)}}(C_{\lambda_{(p,r)},1}), \end{aligned}$$

so that

$$\begin{aligned} & \left(f_{\lambda_{(p)}^{(p)}(1^2)}^{\lambda_{(p+1,p+2)}^{(p)}} \right)^{-1} f_{\lambda_{(p,r)}^{(p,r)}(1^2)}^{\lambda_{(p+1,p+2)}^{(p,r)}} \\ &= b_{\lambda_{(p)}^{(p)}}((r, 1)) b_{\lambda_{(p,r)}^{(p,r)}}((r, 1))^{-1} b_{\lambda_{(p+1,p+2)}^{(p,r)}}((r, 1)) b_{\lambda_{(p+1,p+2)}^{(p)}}((r, 1))^{-1}. \end{aligned}$$

This completes the proof of (6.31).

Case 3. $-p = m - 1$ and $\mu \not\subset \lambda_{(p+1)}$. In this case by (6.15) with $r = 1$ (resp. $r = 2$) and $\nu = \lambda$ (resp. $\nu = \lambda^{(p)}$) and induction hypothesis we have for some $s < p$

$$(6.32) \quad f_{\lambda(1)}^{\lambda_{(p)}} \left(\widetilde{\mu} \right)_m + f_{\lambda(1)}^{\lambda_{(p+1)}} \left(\widetilde{\mu} \right)_m = 0$$

$$(6.33) \quad f_{\lambda^{(p)}(1^2)}^{\lambda^{(p+1)}} \left(\widetilde{\lambda^{(p+1)}}_{\mu} \right)_m + f_{\lambda^{(p)}(1^2)}^{\lambda^{(p)}_{(p+1,s)}} \left(\widetilde{\lambda^{(p)}_{(p+1,s)}}_{\mu} \right)_m = \sum_i f_{\lambda^{(p,i)}(1^2)}^{\mu} \left(\widetilde{\lambda^{(p,i)}}_{\mu} \right)_m.$$

Observe that, as $\mu \not\subset \lambda_{(p+1)}$ is assumed, $f_{\lambda^{(p,i)}(1^2)}^{\mu}$ does not vanish only if μ is of the form

$$(6.34) \quad \mu = \lambda_{(p+1,s)}^{(p,r)}, \quad r < p, \quad r \neq s,$$

and $i = r$. If not, then one can readily conclude from Lemma 6.2 and induction hypothesis that $\left(\widetilde{\lambda^{(p)}_{(p+1,s)}}_{\mu} \right)_m = 0$. Hence $\left(\widetilde{\lambda^{(p+1)}}_{\mu} \right)_m = 0$ follows from (6.33), so that we obtain $\left(\widetilde{\lambda^{(p)}}_{\mu} \right)_m = 0$ from (6.32).

We now assume (6.34). It clearly suffices to show

$$(6.35) \quad f_{\lambda^{(p)}(1^2)}^{\lambda^{(p)}_{(p+1,s)}} \left(\widetilde{\lambda^{(p)}_{(p+1,s)}}_{\lambda^{(p,r)}_{(p+1,s)}} \right)_m = f_{\lambda^{(p,r)}(1^2)}^{\lambda^{(p,r)}_{(p+1,s)}} \left(\widetilde{\lambda^{(p,r)}}_{\lambda^{(p,r)}_{(p+1,s)}} \right)_m.$$

By (6.24) we have

$$(6.36) \quad \left(\widetilde{\lambda^{(p,r)}}_{\lambda^{(p,r)}_{(p+1,s)}} \right)_m^{-1} \left(\widetilde{\lambda^{(p)}_{(p+1,s)}}_{\lambda^{(p,r)}_{(p+1,s)}} \right)_m = b_{\lambda^{(p,r)}}(R_r)^{-1} b_{\lambda^{(p,r)}_{(p+1,s)}}(R_r) b_{\lambda^{(p)}_{(p+1,s)}}(R_r)^{-1} b_{\lambda^{(p)}}(R_r) \\ = b_{\lambda^{(p,r)}}(S)^{-1} b_{\lambda^{(p,r)}_{(p+1,s)}}(S) b_{\lambda^{(p)}_{(p+1,s)}}(S)^{-1} b_{\lambda^{(p)}}(S),$$

where $S = (r, 1) \cup (r, \lambda_s + 1)$ if $r < s$ and $= (r, 1)$ if $r > s$. On the other hand by (6.9) we have also

$$f_{\lambda^{(p)}(1^2)}^{\lambda^{(p)}_{(p+1,s)}} = b_{\lambda^{(p)}}(C'_1 \cup C_{\lambda_s+1})^{-1} b_{(1)}^{-2} b_{\lambda^{(p)}_{(p+1,s)}}(C'_1 \cup C_{\lambda_s+1}) \\ f_{\lambda^{(p,r)}(1^2)}^{\lambda^{(p,r)}_{(p+1,s)}} = b_{\lambda^{(p,r)}}(C'_1 \cup C_{\lambda_s+1})^{-1} b_{(1)}^{-2} b_{\lambda^{(p,r)}_{(p+1,s)}}(C'_1 \cup C_{\lambda_s+1}),$$

where $C'_1 = C_1 \setminus (s, 1)$. Observe that

$$b_{\lambda^{(p,r)}}(C'_1 \cup C_{\lambda_s+1})^{-1} b_{\lambda^{(p)}}(C'_1 \cup C_{\lambda_s+1}) = b_{\lambda^{(p,r)}}(S)^{-1} b_{\lambda^{(p)}}(S) \\ b_{\lambda^{(p)}_{(p+1,s)}}(C'_1 \cup C_{\lambda_s+1})^{-1} b_{\lambda^{(p,r)}_{(p+1,s)}}(C'_1 \cup C_{\lambda_s+1}) = b_{\lambda^{(p,r)}_{(p+1,s)}}(S)^{-1} b_{\lambda^{(p)}_{(p+1,s)}}(S),$$

to get

$$\left(f_{\lambda^{(p)}(1^2)}^{\lambda^{(p)}_{(p+1,s)}} \right)^{-1} f_{\lambda^{(p,r)}(1^2)}^{\lambda^{(p,r)}_{(p+1,s)}} = \text{RHS of (6.36)}.$$

This completes the proof of (6.35).

Case 4. — $p = m - 1$ and $\mu \subset \lambda_{(p+1)}$. So $\mu = \lambda_{(p+1)}^{(r)}$ for some $r < p + 1$ and $\mu_m = 1$. One can readily derive from (6.15) with $r = 1$ and $\nu = \lambda$ and induction hypothesis that

$$f_{\lambda(1)}^{\lambda_{(p)}} \left(\widetilde{\lambda_{(p)}}_{\mu} \right)_m + f_{\lambda(1)}^{\lambda_{(p+1)}} \left(\widetilde{\lambda_{(p+1)}}_{\mu} \right)_m = f_{\lambda^{(r)}(1)}^{\mu} \left(\widetilde{\lambda}_{\lambda^{(r)}} \right)_m.$$

So it remains only to show

$$f_{\lambda(1)}^{\lambda_{(p+1)}} \left(\widetilde{\lambda_{(p+1)}}_{\lambda_{(p+1)}^{(r)}} \right)_m = f_{\lambda^{(r)}(1)}^{\lambda_{(p+1)}} \left(\widetilde{\lambda}_{\lambda^{(r)}} \right)_m.$$

This is concluded, as in the previous cases, from (6.9) and (6.24): It holds that

$$\begin{aligned} \left(\widetilde{\lambda}_{\lambda^{(r)}} \right)_m^{-1} \left(\widetilde{\lambda_{(p+1)}}_{\lambda_{(p+1)}^{(r)}} \right)_m &= \left(f_{\lambda(1)}^{\lambda_{(p+1)}} \right)^{-1} f_{\lambda^{(r)}(1)}^{\lambda_{(p+1)}} \\ &= b_{\lambda_{(p+1)}}((r, 1))^{-1} b_{\lambda}((r, 1)) b_{\lambda^{(r)}}((r, 1))^{-1} b_{\lambda_{(p+1)}^{(r)}}((r, 1)). \end{aligned}$$

We have completed the proof of Theorem 4.4.

Appendix A. Convergence of the integral

We show that the integral ${}_qS_{n,m}(\alpha, \beta, \gamma, \mu; x_1, \dots, x_m; \xi)$ converges under the conditions $(C_1), (C_2)$. It is immediate that, if $(aq^s)_{\infty}/(bq^s)_{\infty}$ has no pole at any $s \in \mathbb{Z}$, then

$$\left| \frac{(aq^s)_{\infty}}{(bq^s)_{\infty}} \right| \leq \begin{cases} M_1, & s \geq 0, \\ M_2 |a/b|^{-s}, & s < 0, \end{cases}$$

where $M_1 = \max_{s \geq 0} |(aq^s)_{\infty}/(bq^s)_{\infty}|$, $M_2 = |(a)_{\infty}/(b)_{\infty}| \max_{s \geq 0} |(a^{-1})_s/(b^{-1})_s|$. Using this, for $t_j = \xi_j q^{s_j}$ one has

$$\begin{aligned} \left| \frac{(q^{1-\gamma} t_j / t_i)_{\infty}}{(q^{\gamma} t_j / t_i)_{\infty}} (1 - t_j / t_i) \right| &\leq \begin{cases} Cte., & s_j - s_i \geq 0, \\ Cte. |q^{2\gamma}|^{s_j - s_i}, & s_j - s_i < 0, \end{cases} \\ \left| \frac{(q t_j)_{\infty}}{(q^{\beta} t_j)_{\infty}} \right| &\leq \begin{cases} Cte., & s_j \geq 0, \\ Cte. |q^{\beta-1}|^{s_j}, & s_j < 0, \end{cases} \\ \left| \prod_{i=1}^m \frac{(x_i t_j)_{\infty}}{(q^{\mu} x_i t_j)_{\infty}} \right| &\leq \begin{cases} Cte., & s_j \geq 0, \\ Cte. |q^{m\mu}|^{s_j}, & s_j < 0. \end{cases} \end{aligned}$$

For $s \in \mathbb{Z}$, we put

$$a_s = \begin{cases} 1, & s \geq 0, \\ q^{\beta-1+m\mu}, & s < 0. \end{cases}$$

Case $\operatorname{Re} \gamma \geq 0$. — So $|q^{\gamma}| \leq 1$ and it follows from the inequality above that

$$\left| \frac{(q^{1-\gamma} t_j / t_i)_{\infty}}{(q^{\gamma} t_j / t_i)_{\infty}} (1 - t_j / t_i) \right| \leq Cte. |q^{2\gamma}|^{-|s_j| - |s_i|}.$$

Hence

$$|\Phi_0(\xi_1 q^{s_1}, \dots, \xi_n q^{s_n})| \leq Cte. \prod_{j=1}^n |q^{(\alpha+n-1-2(j-1)\gamma)s_j-2(n-1)\gamma|s_j|} a_{s_j}|.$$

The condition $(\mathcal{C}_2)$ in the case $\operatorname{Re} \gamma \geq 0$ is equivalent to

$$\sum_{s=0}^{\infty} |q^{(\alpha+n-1-4(n-1)\gamma)s}| + \sum_{s=-1}^{-\infty} |q^{(\alpha+n-1+\beta-1+m\mu+2(n-1)\gamma)s}| < \infty.$$

This clearly implies the convergence of the series

$$(A.1) \quad \int_{[0, \xi\infty]_q} \Phi_0(t) \tilde{\omega} = (1-q)^n \sum_{s_i \in \mathbb{Z}} \Phi_0(\xi_1 q^{s_1}, \dots, \xi_n q^{s_n}).$$

Case $\operatorname{Re} \gamma < 0$. – We see

$$\left| \frac{(q^{1-\gamma} t_j / t_i)_{\infty}}{(q^{\gamma} t_j / t_i)_{\infty}} (1 - t_j / t_i) \right| \leq Cte.,$$

so that

$$|\Phi_0(\xi_1 q^{s_1}, \dots, \xi_n q^{s_n})| \leq Cte. \prod_{j=1}^n |q^{(\alpha+n-1-2(j-1)\gamma)s_j} a_{s_j}|.$$

The condition $(\mathcal{C}_2)$ in the case $\operatorname{Re} \gamma < 0$ is equivalent to

$$\sum_{s=0}^{\infty} |q^{(\alpha+n-1)s}| + \sum_{s=-1}^{-\infty} |q^{(\alpha+n-1+\beta-1+m\mu-2(n-1)\gamma)s}| < \infty.$$

This implies the convergence of the series (A.1).

When $\xi = \xi_F$, as the summation in (A.1) is only over $0 \leq s_1 \leq s_2 \leq \dots \leq s_n$, one can relax the condition $(\mathcal{C}_2)$ into

$$\operatorname{Re} \alpha + n - 1 > \max\{2(n-1)\operatorname{Re} \gamma, 0\}.$$

Finally we note that, as $|P_{\lambda}(x)| \leq Cte. (|x_1| + \dots + |x_m|)^{|\lambda|}$, the integral $\int_{[0, \xi\infty]_q} P_{\lambda}(t; q, Q)_q D(\alpha, \beta, \gamma; t) \tilde{\omega}$ converges provided that

$$\begin{aligned} \operatorname{Re} \alpha + n - 1 &> 4(n-1)\max\{\operatorname{Re} \gamma, 0\}, \\ \operatorname{Re} \alpha + n - 1 + \operatorname{Re} \beta - 1 + m\operatorname{Re} \mu + |\lambda| &< -2(n-1)|\operatorname{Re} \gamma|. \end{aligned}$$

Appendix B. Evaluation of ${}_qS_{n,0}(\alpha, \beta, \gamma; \xi)$

We begin by showing that

$$(B.1) \quad {}_qS_{n,0}(\alpha + 1, \beta, \gamma; \xi) = q^{n(n-1)\gamma/2} \prod_{j=1}^n \frac{1 - q^{\alpha+n-1-(n+j-2)\gamma}}{1 - q^{\alpha+\beta+n-1-(n-j)\gamma}} {}_qS_{n,0}(\alpha, \beta, \gamma; \xi).$$

Indeed this is a consequence of the case $m = 1$ of (4.17): Equating the coefficients of x^n of both sides gives

$$(-q^\beta Q^{-1})^n {}_qS_{n,0}(\alpha + 1, \beta, \gamma; \xi) = \frac{(Q^{-n}; Q)_n}{(Q; Q)_n} \frac{(q^{-(\alpha+n-1)} Q^{n-1}; Q)_n}{(q^{-(\alpha+\beta+n-1)}; Q)_n} {}_qS_{n,0}(\alpha, \beta, \gamma; \xi),$$

and this leads to (B.1) immediately. We first prove (1.6) by induction on n , the case $n = 1$ being nothing but the q -beta integral formula [As1], [GR, p. 19]. We proceed as in [Kad1]. Set

$$(B.2) \quad {}_qpr_n(\alpha, \beta, \gamma) = q^{\frac{n(n-1)}{2}\alpha\gamma} \prod_{j=1}^n \frac{\Gamma_q(\alpha + n - 1 - (n + j - 2)\gamma)}{\Gamma_q(\alpha + \beta + n - 1 - (n - j)\gamma)}$$

$$(B.3) \quad {}_qQ_n(\alpha, \beta, \gamma) = \frac{{}_qS_{n,0}(\alpha, \beta, \gamma; \xi_F)}{{}_qpr_n(\alpha, \beta, \gamma)}.$$

By (B.1) and the equation $\Gamma_q(\alpha + 1) = (1 - q^\alpha)/(1 - q)\Gamma_q(\alpha)$, we see

$$(B.4) \quad {}_qQ_n(\alpha + 1, \beta, \gamma) = {}_qQ_n(\alpha, \beta, \gamma).$$

We extend ${}_qQ_n(\alpha, \beta, \gamma)$ to all α by this equation.

We assume that γ is real and $\gamma > 0$, $\operatorname{Re} \alpha + (n - 1)(1 - 2\gamma) > 0$. We show that

$$(B.5) \quad {}_qQ_n(\alpha, \beta, \gamma) = q^{C_n} \prod_{j=1}^n \frac{\Gamma_q(\beta + (j - 1)\gamma)\Gamma_q(j\gamma)}{\Gamma_q(\gamma)}$$

where $C_n = \sum_{j=1}^n (-2(j - 1)\gamma + n - 1)(j - 1)\gamma$. Rewriting ${}_qS_{n,0}(\alpha, \beta, \gamma; \xi_F)$ as iterated integral, we have

$$\begin{aligned} {}_qS_{n,0}(\alpha, \beta, \gamma; \xi_F) &= \int_{[0, q^{(n-1)\gamma}]} t_n^{\alpha+(n-1)(1-2\gamma)} \frac{(qt_n)_\infty}{(q^\beta t_n)_\infty} \\ &\quad \left[\int_{[0, (1, \dots, q^{(n-2)\gamma})]} \prod_{j=1}^{n-1} t_j^{\alpha+(j-1)(1-2\gamma)} \frac{(qt_j)_\infty}{(q^\beta t_j)_\infty} \right. \\ &\quad \times \left. \prod_{1 \leq i < j \leq n} \frac{(q^{1-\gamma} t_j/t_i)_\infty}{(q^\gamma t_j/t_i)_\infty} D(t) \frac{d_q t_1}{t_1} \wedge \dots \wedge \frac{d_q t_{n-1}}{t_{n-1}} \right] \frac{d_q t_n}{t_n}. \end{aligned}$$

Set $\alpha_0 = (n-1)(2\gamma-1)$. Observe that

$$\lim_{\alpha \rightarrow \alpha_0} \frac{1 - q^{\alpha - \alpha_0}}{1 - q} \int_{[0, q^{(n-1)\gamma}]} t_n^{\alpha - \alpha_0} \frac{(qt_n)_\infty}{(q^\beta t_n)_\infty} \frac{d_q t_n}{t_n} = 1.$$

Hence we obtain

$$\begin{aligned} \text{(B.6)} \quad & \lim_{\alpha \rightarrow \alpha_0} \frac{1 - q^{\alpha - \alpha_0}}{1 - q} {}_q S_{n,0}(\alpha, \beta, \gamma; \xi_F) = {}_q S_{n-1,0}(\alpha_0 + 1, \beta, \gamma; \xi_F) \\ & = q^{A_{n-1}} \prod_{j=1}^{n-1} \frac{\Gamma_q((n-j+1)\gamma) \Gamma_q(\beta + (j-1)\gamma) \Gamma_q(j\gamma)}{\Gamma_q(\beta + (n+j-1)\gamma) \Gamma_q(\gamma)} \end{aligned}$$

where

$$A_{n-1} = \sum_{j=1}^{n-1} (\alpha_0 - 2(j-1)\gamma + n-1)(j-1)\gamma = C_n + \frac{n(n-1)}{2} \alpha_0 \gamma.$$

On the other hand by (B.2) and (B.3) we have

$$\begin{aligned} \text{(B.7)} \quad & \lim_{\alpha \rightarrow \alpha_0} \frac{1 - q^{\alpha - \alpha_0}}{1 - q} {}_q S_{n,0}(\alpha, \beta, \gamma; \xi_F) \\ & = \lim_{\alpha \rightarrow \alpha_0} \frac{1 - q^{\alpha - \alpha_0}}{1 - q} {}_q pr_n(\alpha, \beta, \gamma) {}_q Q_n(\alpha, \beta, \gamma) \\ & = q^{\frac{n(n-1)}{2} \alpha_0 \gamma} \frac{1}{\Gamma_q(\beta + (n-1)\gamma)} \prod_{j=1}^{n-1} \frac{\Gamma_q((n-j)\gamma)}{\Gamma_q(\beta + (n+j-1)\gamma)} {}_q Q_n(\alpha_0, \beta, \gamma). \end{aligned}$$

Equating (B.6) and (B.7) yields

$$\begin{aligned} \text{(B.8)} \quad & {}_q Q_n(\alpha_0, \beta, \gamma) = q^{-\frac{n(n-1)}{2} \alpha_0 \gamma + A_{n-1}} \Gamma_q(\beta + (n-1)\gamma) \prod_{j=1}^{n-1} \frac{\Gamma_q(\beta + (n+j-1)\gamma)}{\Gamma_q((n-j)\gamma)} \\ & \quad \times \prod_{j=1}^{n-1} \frac{\Gamma_q((n-j+1)\gamma) \Gamma_q(\beta + (j-1)\gamma) \Gamma_q(j\gamma)}{\Gamma_q(\beta + (n+j-1)\gamma) \Gamma_q(\gamma)} \\ & = q^{C_n} \prod_{j=1}^n \frac{\Gamma_q(\beta + (j-1)\gamma) \Gamma_q(j\gamma)}{\Gamma_q(\gamma)}. \end{aligned}$$

This establishes (B.5) when $\alpha = \alpha_0 + k, k \in \mathbb{Z}$. One can show that ${}_q Q_n(\alpha_0, \beta, \gamma)$ is bounded in the strip $\alpha_0 + 1 \leq \operatorname{Re} \alpha \leq \alpha_0 + 2$ in the exactly same way as [Kad1]. Hence it is bounded for all α by (B.4), and thus (B.5) follows from Liouville's theorem that a

bounded entire function is constant. The restriction that γ is real and positive is easily removed by analytic continuation.

Now we turn to the proof of (1.5). Set

$$(B.9) \quad {}_qS_{n,0}(\alpha, \beta, \gamma; \xi) = c(\xi) {}_qS_{n,0}(\alpha, \beta, \gamma; \xi_F).$$

By the definition of Jackson integral, we see for any j that

$$(B.10) \quad T_{q,\xi_j} c(\xi) = c(\xi).$$

Observe that $c(\xi) \prod_{j=1}^n \xi_j^{2(j-1)\gamma-\alpha}$ is meromorphic on $(\mathbb{C}^*)^n$ with simple poles in $\{\xi \mid \prod_{1 \leq i < j \leq n} \vartheta(q^\gamma \xi_j / \xi_i) \prod_{j=1}^n \vartheta(q^\beta \xi_j) = 0\}$. We assert that $c(\xi)$ is vanishing on $\{\xi \mid \xi_j = \xi_i q^k, k \in \mathbb{Z}\}$. By (B.10), it suffices to show that if $\xi_i = \xi_j$, $i < j$, then ${}_qS_{n,0}(\alpha, \beta, \gamma; \xi) = 0$. Let σ_{ij} be the transposition of i and j . We have

$$\begin{aligned} \int_{[0, \xi_\infty]_q} \Phi_0(t) \tilde{\omega} &= \int_{[0, \sigma_{ij}(\xi)_\infty]_q} \Phi_0(t) \tilde{\omega} \\ &= \int_{[0, \xi_\infty]_q} \sigma_{ij}(\Phi_0(t)) \tilde{\omega} \\ &= -U_{\sigma_{ij}}(\xi) \int_{[0, \xi_\infty]_q} \Phi_0(t) \tilde{\omega}. \end{aligned}$$

Hence our assertion follows from (use $\vartheta(q/x) = \vartheta(x)$)

$$\begin{aligned} U_{\sigma_{ij}}(\xi) &= \prod_{\substack{1 \leq k < l \leq n \\ \sigma_{ij}(k) > \sigma_{ij}(l)}} \left(\frac{\xi_l}{\xi_k} \right)^{2\gamma-1} \frac{\vartheta(q^\gamma \xi_l / \xi_k)}{\vartheta(q^{1-\gamma} \xi_l / \xi_k)} \\ &= \frac{\vartheta(q^\gamma)}{\vartheta(q^{1-\gamma})} \prod_{i < k < j} \left(\frac{\xi_j}{\xi_k} \right)^{2\gamma-1} \frac{\vartheta(q^\gamma \xi_j / \xi_k)}{\vartheta(q^{1-\gamma} \xi_j / \xi_k)} \prod_{i < l < j} \left(\frac{\xi_l}{\xi_i} \right)^{2\gamma-1} \frac{\vartheta(q^\gamma \xi_l / \xi_i)}{\vartheta(q^{1-\gamma} \xi_l / \xi_i)} \\ &= 1. \end{aligned}$$

We are now able to write

$$c(\xi) = \prod_{j=1}^n \xi_j^{\alpha-2(j-1)\gamma} \frac{1}{\vartheta(q^\beta \xi_j)} \prod_{1 \leq i < j \leq n} \frac{\vartheta(\xi_j / \xi_i)}{\vartheta(q^\gamma \xi_j / \xi_i)} f(\xi)$$

where $f(\xi)$ is holomorphic on $(\mathbb{C}^*)^n$. One can derive from (B.10) that

$$T_{q,\xi_j} f(\xi) = -\frac{1}{q^{\alpha+\beta-(n-1)\gamma} \xi_j} f(\xi).$$

Therefore we conclude that

$$f(\xi) = Cte. \prod_{j=1}^n \vartheta(q^{\alpha+\beta-(n-1)\gamma} \xi_j).$$

Since $c(\xi_F) = 1$, we arrive at

$$c(\xi) = q^{\sum_{j=1}^n (2(j-1)\gamma - \alpha)} \prod_{j=1}^n \xi_j^{\alpha - 2(j-1)\gamma} \frac{\vartheta(\xi_j q^{\alpha+\beta-(n-1)\gamma}) \vartheta(q^{\beta+(j-1)\gamma}) \vartheta(q^{j\gamma})}{\vartheta(q^{\alpha+\beta-(n-j)\gamma}) \vartheta(\xi_j q^{\beta}) \vartheta(q^{\gamma})} \\ \prod_{1 \leq i < j \leq n} \frac{\vartheta(\xi_j / \xi_i)}{\vartheta(q^{\gamma} \xi_j / \xi_i)}.$$

Combining this with (B.9) and (1.6) completes the proof of (1.5).

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