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$q$-Selberg integrals and Macdonald polynomials


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q-SELBerg INTEGRALS AND MACDONALD POLYNOMIALS

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ABSTRACT. - We consider a Jackson integral with special integrand (g-Selberg integral) and give an explicit formula of a system of q-difference equations satisfied by it. We also define a kind of hypergeometric function having series expansions in terms of Macdonald polynomials and show that this function satisfies a q-difference equation formed by summing up equations of the q-difference system above after multiplying each by a suitable factor. We can thus conclude the q-Selberg integral to be the hypergeometric function in our sense. This implies, in particular, the q-integration formula of Macdonald polynomials due to Kadell [Kad2]. These results reproduce our previous ones [Kan2] if we put q = t^a and let t \to 1.

1. Introduction

The purpose of this paper is to give q-analogues of our previous results in [Kan2]. Fix q with 0 < q < 1 and set (x)_{\infty} = (x; q)_{\infty} = \prod_{i=0}^{\infty}(1 - xq^i) and (x)_{a} = (x; q)_{a} = (x)_{\infty}/(xq^a)_{\infty}. For x = (x_1, \ldots, x_m) \in \mathbb{C}^m and t = (t_1, \ldots, t_n) \in (\mathbb{C}^*)^n put

\begin{align}
\Phi(t) &= \prod_{j=1}^{n} t_j^{(j-1)(1-q)} \frac{(qt_j)_{\infty}}{(q^a t_j)_{\infty}} \prod_{1 \leq i < j \leq n} \frac{(q^{1-i}t_j/t_i)_{\infty}}{(q^{i-j}t_j/t_i)_{\infty}} f(x, t) \\
f(x, t) &= \prod_{1 \leq i \leq m, 1 \leq j \leq n} \frac{(x_it_j)_{\infty}}{(q^a x_i t_j)_{\infty}} \\
\Phi_0(t) &= \Phi(t)D(t)
\end{align}

where D(t) = \prod_{1 \leq i < j \leq n} (t_i - t_j).

For \xi \in (\mathbb{C}^*)^n we put [0, \xi_{\infty})_q = \{(q^n \xi_1, \ldots, q^n \xi_n) | (s_1, \ldots, s_n) \in \mathbb{Z}^n\}. The Jackson integral of a function f on (\mathbb{C}^*)^n over [0, \xi_{\infty})_q is defined by

\begin{align}
\int_{[0, \xi_{\infty})_q} f(t_1, \ldots, t_n)\mathbf{\tilde{w}} = (1 - q)^n \sum_{s \in \mathbb{Z}} f(q^s \xi_1, \ldots, q^s \xi_n) \\
\mathbf{\tilde{w}} = \frac{d_1 t_1}{t_1} \wedge \cdots \wedge \frac{d_n t_n}{t_n}
\end{align}
provided it exists. Similarly, the integral over $[0, \xi] = [0, \xi_1] \times \cdots \times [0, \xi_n]$ is defined by
\[
\int_{[0,\xi]} f(t_1, \ldots, t_n) \, d\xi = (1 - q)^n \sum_{s_i \in I_{\geq 0}} f(\xi_1 q^{s_1}, \ldots, \xi_n q^{s_n}).
\]

We consider the integral
\[
(1.4) \quad qS_{n,m}(\alpha, \beta, \gamma; \mu; x_1, \ldots, x_m; \xi) \quad (qS_{n,m}(x) \text{ for short}) = \int_{[0,\xi_\infty]} \Phi_0(t) \, d\xi.
\]

Write $q^k = \{q^k; k \in \mathbb{Z}\}$. We assume the following condition which assures that $\Phi(t)$ has no poles on $[0,\xi_\infty]_q$.

\[\begin{align*}
&\left\{ \begin{array}{l}
q^\gamma \xi_j / \xi_i \notin q^k \quad \text{for } 1 \leq i \leq j \leq n \text{ or } 2\gamma - 1 \in \mathbb{Z}_{\geq 0}; \\
q^\beta \xi_j \notin q^k \quad \text{for } 1 \leq j \leq n \text{ or } \beta - 1 \in \mathbb{Z}_{\geq 0}; \\
q^n x_i \xi_j \notin q^k \quad \text{for } 1 \leq i \leq m, 1 \leq j \leq n.
\end{array} \right.
\]

For convergence of the integral we assume also

\[\begin{align*}
&\text{(C1)} \\
&\left\{ \begin{array}{l}
\text{Re } \alpha + n - 1 > 4(n-1)\max\{\text{Re } \gamma, 0\}, \\
\text{Re } \alpha + n - 1 + \text{Re } \beta - 1 + m\text{Re } \mu < -2(n-1)|\text{Re } \gamma|.
\end{array} \right.
\]

For the proof of convergence under the conditions (C1), (C2), see the Appendix A.

One of our main results (Theorem 4.11) states that if $\mu = 1$ or $-\gamma$, then $qS_{n,m}(x)$ has an explicit series expansion in terms of A-type Macdonald polynomials [Ma2] (in the case $\mu = -\gamma$, we need to choose $\xi = \xi_F =: (1, q^\gamma, \ldots, q^{(n-1)\gamma})$). This precisely corresponds to our previous result that $S_{n,m}(x) := \lim_{q \to 0, \epsilon \to 1} qS_{n,m}(x)$ has an explicit series expansion in terms of Jack polynomials [Kan2, Theorem 5, p. 1106] (see also [Ko]).

In the case $f(x, t) \equiv 1 (m = 0)$, $qS_{n,0}(\alpha, \beta, \gamma; \xi)$ has been evaluated by K. Aomoto in [Ao2] (cf. also [Ao3]):

\[
(1.5) \quad qS_{n,0}(\alpha, \beta, \gamma; \xi) = q^{\frac{n(n-1)^2}{2}} \prod_{j=1}^n \xi_j^{\alpha-2(j-1)\gamma} \frac{\vartheta(\xi_j q^{\alpha+\beta-(n-1)\gamma}) \vartheta(q^{\beta+j-1}\gamma) \vartheta(q^{\gamma})}{\vartheta(q^{\alpha+\beta-(n-j)\gamma}) \vartheta(q^{\beta})(q^{\gamma})} \times \prod_{1 \leq i < j \leq n} \frac{\vartheta(\xi_j / \xi_i)}{\vartheta(q^\gamma \xi_j / \xi_i)} \times \prod_{j=1}^n \frac{\Gamma_q(\alpha + n - 1 - (n + j - 2)\gamma) \Gamma_q(\beta + (j - 1)\gamma) \Gamma_q(j\gamma)}{\Gamma_q(\alpha + \beta + n - 1 - (n - j)\gamma) \Gamma_q(j\gamma)}.
\]

Here $\vartheta(x)$ denotes the Jacobi elliptic theta function $(x)_\infty (q/x)_\infty (q)_\infty$ and $\Gamma_q(x)$ denotes the $q$-gamma function $(1 - q)^{1-x}(q)_\infty / (q^x)_\infty$. We notice that when $n = 1$, this integral is
nothing but the Ramanujan's $1 \psi_1$ sum [As1]. In Appendix B we shall give a self-contained proof of (1.5) based on $q$-difference equation (the $m = 1$ case of Theorem 4.11). If $\xi = \xi_F$, one can simplify the formula (1.5) to get

$$qS_{n,0}(\alpha, \beta, \gamma; \xi_F) = q^{A_n} \prod_{j=1}^{n} \frac{\Gamma_q(\alpha + n - 1 - (n + j - 2)\gamma)\Gamma_q(\beta + (j - 1)\gamma)\Gamma_q(j\gamma)}{\Gamma_q(\alpha + \beta + n - 1 - (n - j)\gamma)\Gamma_q(\gamma)},$$

where $A_n = \sum_{j=1}^{n} (\alpha - 2(j-1)\gamma + n - 1)(j-1)\gamma$. If $\gamma$ is equal to a positive integer $k$, (1.6) reduces to the Askey-Habsieger-Kadell's formula [As, H, Kad1] (see Proposition 5.2):

$$\int_{[0,1]^n} \prod_{j=1}^{n} \frac{q^{t_j}}{q^{t_j}} \prod_{1 \leq i < j \leq n} q^{2k} \frac{q^{1-k}t_j}{t_i} \frac{q^{2k}}{2k} \prod_{i=1}^{n} \frac{\Gamma_q(x + (n - j)k)\Gamma_q(y + (n - j)k)\Gamma_q(1 + jk)}{\Gamma_q(x + y + (2n - j - 1)k)\Gamma_q(1 + k)},$$

where $\alpha$ and $\beta$ are identified with $x + (n - 1)(2k - 1)$ and $y$ respectively.

In Section 2 we show that, when $\mu = 1$ or $-\gamma$, $qS_{n,m}(x)$ satisfies a system of $q$-difference equations (Theorem 2.3, (2.26)). This system tends to the holonomic system of differential equations in [Kan2, (9), p. 1088] when $q \to 1$. In Section 3 we define a kind of $q$-hypergeometric function $\Phi^{(\alpha)}_{\mu}(a_1, \ldots, a_r; b_1, \ldots, b_s; x_1, \ldots, x_m)$ using $A$-type Macdonald polynomials. By setting $q = t^\alpha$ and $t \to 1$, $\Phi^{(\alpha)}_{\mu}(x)$ reduces to the hypergeometric function $F^{(\alpha)}(x)$ defined by using Jack polynomials [Kan2, Ko]. We notice that $2F^{(\alpha)}_1$ is a special case of $BC$-type hypergeometric function of Heckman-Opdam [BO]. In Section 4 we prove that $2F^{(\alpha)}_1(a, b; c; x)$ satisfies a $q$-difference equation formed by summing up equations of (2.26) multiplied each by a suitable factor. Therefore the uniqueness properties of solutions of this summed-up equation assures us that $qS_{n,m}(x)$ with $\mu = 1$ or $-\gamma$ is nothing but $2F^{(\alpha)}_1(a, b; c; x)$ if we adjust $(q, t)$ and $a, b, c$ suitably (Theorem 4.11). As a consequence we obtain an $q$-integration formula of Macdonald polynomials (Theorem 5.1). In the special case that $\xi = \xi_F$ and $\gamma = k$, a positive integer, this was conjectured and proved by Kadell [Kad2] in a different way from ours (though some details of the proof have been omitted in our copy of [Kad2]). This integration formula in turn gives explicit formulae of the values of $qS_{n,m}(a, b; c; x)$ at special points (Proposition 5.4). In a separate paper [Kan3] we shall show that Theorem 4.11 implies the constant term identities due to Forrester, Zeilberger and Cooper [F, Z, C].

In a recent preprint [BC], Barsky and Carpentier have given a different proof of our previous result [Kan2, Theorem 5] by employing a new method of G. Anderson. It would be interesting to know whether their argument has $q$-analogous counterpart.

Part of the results of this paper were announced in [Kan1]. The author thanks Prof. K. Aomoto for inspiring discussions and the referee for helpful remarks and suggestions.
2. \( q \)-difference system

2.1. Let \( T_{q,t_i} = T_i \) denote the \( q \)-shift operator on the \( i \)-th coordinate: \( T_i \varphi(t) = \varphi(t_1, \ldots, qt_i, \ldots, t_n) \) and set

\[
\frac{\partial \varphi}{\partial q t_i} = \frac{(T_i - 1) \varphi}{(q - 1) t_i}.
\]

Note that

\[
\frac{\partial (\varphi \psi)}{\partial q t_i} = \frac{\partial \varphi}{\partial q t_i} \psi + T_i \varphi \frac{\partial \psi}{\partial q t_i}
\]

which will be of frequent use. Put \( b_i(t) = T_i \Phi(t)/\Phi(t) \). In particular

\[
b_1(t) = q^\alpha \frac{1 - q^\beta t_1}{1 - q t_1} \prod_{j=2}^{m} \frac{t_1 - q^{-1} t_j}{t_1 - q^{-1} t_j} \prod_{k=1}^{n} \frac{1 - q^\mu x_k t_1}{1 - x_k t_1}.
\]

Define the covariant \( q \)-difference operator \( \nabla_i \) by

\[
\nabla_i \varphi(t) = \varphi(t) - b_i(t) T_i \varphi(t).
\]

Clearly

\[
\int_{[0, \xi \infty]} \Phi(t) \varphi(t) \tilde{\omega} = \int_{[0, \xi \infty]} T_i (\Phi(t) \varphi(t)) \tilde{\omega},
\]

provided the integral is convergent. Hence we have

\[
\int_{[0, \xi \infty]} \Phi(t) \nabla_i \varphi(t) \tilde{\omega} = 0.
\]

Let \( \mathfrak{S}_n \) denote the symmetric group of degree \( n \) and for \( \sigma \in \mathfrak{S}_n \) define

\[
(\sigma \varphi)(t) = \varphi(t_{\sigma(1)}, \ldots, t_{\sigma(n)}).
\]

Put

\[
U_\sigma(t) = \sigma \Phi(t)/\Phi(t).
\]

Then we have

\[
U_\sigma(t) = \prod_{1 \leq i < j \leq n, \sigma^{-1}(i) > \sigma^{-1}(j)} \left( \frac{t_j}{t_i} \right)^{2\gamma - 1} \frac{\vartheta(q^\gamma t_j/t_i)}{\vartheta(q^{-\gamma} t_j/t_i)}.
\]

By using \( \vartheta(qx) = -1/x \vartheta(x) \) one can easily verify that \( T_i U_\sigma(t) = U_\sigma(t) \) for every \( i \).

We assert that

\[
\int_{[0, \xi \infty]} \Phi(t) \sigma(\nabla_i \varphi(t)) \tilde{\omega} = 0
\]
provided the integral is convergent. In fact
\[
0 = \int_{[0, \xi \infty]} \sigma(\Phi(t))\sigma(\nabla_1 \varphi(t)) \tilde{\omega} = U_\sigma(\xi) \int_{[0, \xi \infty]} \Phi(t)\sigma(\nabla_1 \varphi(t)) \tilde{\omega}.
\]

Hence for the alternation \( A \varphi = \sum_{\sigma \in S_n} \text{sgn} \, \sigma \cdot (\sigma \varphi) \), we obtain the fundamental
\[
(2.1) \quad \int_{[0, \xi \infty]} \Phi(t) A(\nabla_1 \varphi(t)) \tilde{\omega} = 0.
\]

For complex \( Q \) we shall denote by \( Q D_n(t) = Q D(t) \) the product \( \prod_{1 \leq i < j \leq n} (t_i - Q t_j) \).

The following lemma ([Kad1, (4.10), p. 976], cf. also [Ma1, chapter 3, (1.3)]) is crucial to our calculations.

**Lemma 2.1.** – Let \( M \subset \{1, \ldots, n\} \). Then
\[
(2.2) \quad A(\prod_{j \in M} t_j Q D(t)) = Q^e(M) \frac{(Q; Q)^{|M|}(Q; Q)_{n-|M|} e_{|M|}(t) D(t)}{(1 - Q)^n}
\]

where \( e(M) = |\{(i, j) | 1 \leq i < j \leq n, i \notin M, j \in M\}| \) and \( e_r(t) \) denotes the elementary symmetric function of degree \( r \).

This lemma implies
\[
(2.3) \quad \sum_{M \in \{1, \ldots, n\}, |M| = r} Q^e(M) = \frac{(Q; Q)_n}{(Q; Q)_r (Q; Q)_{n-r}}
\]
\[
(2.4) \quad \sum_{M \in \{2, \ldots, n\}, |M| = r} Q^e(M) = Q^r \frac{(Q; Q)_{n-1}}{(Q; Q)_r (Q; Q)_{n-r-1}}.
\]

**Lemma 2.2.**

\[
(2.5) \quad A(Q D(t) \prod_{k=2}^n (1 - x t_k))
= \left\{ (Q; Q)_{n-1} (1 - x)^d \left( \prod_{k=1}^n (1 - x t_k) \right) \frac{d}{dQx} \left( \prod_{k=1}^n (1 - x t_k) \right) + (Q; Q)_n \prod_{k=1}^n (1 - x t_k) \right\} D(t)
\]
\[ (2.6) \quad A \left( \prod_{k=2}^{n} (t_1 - Q^{-1} t_k) \prod_{2 \leq h < k \leq n} (t_h - Qt_k) \prod_{k=2}^{n} (1 - xt_k) \right) \]

\[ = Q^{-(n-1)} \left( \frac{Q; Q}{(1 - Q)^n} D(t) \right) \left\{ \prod_{k=1}^{n} (1 - xt_k) - \prod_{k=1}^{n} (Q - xt_k) \right\}. \]

**Proof.** From (2.2) and (2.4) we have

\[ A(QD(t)) \sum_{M \in \{2, \ldots, n\}, |M| = r} \prod_{j \in M} t_j = Q^r (1 - Q^{n-r}) \left( \frac{Q; Q}{(1 - Q)^n} \right) c_r(t) D(t), \]

from which follows (2.5). For the proof of (2.6), observe that

\[ \text{LHS of (2.6) } = Q^{-(n-1)} A(QD(t) \prod_{k=1}^{n} (1 - xt_k)). \]

Then (2.6) follows at once since (2.2) and (2.3) imply

\[ A(QD(t)) \sum_{M \in \{1, \ldots, n-1\}, |M| = r} \prod_{j \in M} t_j = (1 - Q^{n-r}) \left( \frac{Q; Q}{(1 - Q)^n} \right) c_r(t) D(t). \]

Put

\[ A_i(x_1, \ldots, x_m; t) = \prod_{j=1, j \neq i}^{m} \frac{tx_i - x_j}{x_i - x_j}, \]

Expansion in partial fractions gives

\[ (2.7) \quad \prod_{j=1}^{m} \frac{x_j - tz}{x_j - z} = (1 - t) \sum_{j=1}^{m} \frac{x_j A_j(x; t)}{x_j - z} + t^m. \]

Replacing \( z \) by \( 1/z \) and \( m \) by \( m - 1 \), we have also

\[ (2.8) \quad \prod_{j=1, j \neq i}^{m} \frac{1 - x_j z/t}{1 - x_j z} = (t - 1) t^{1-m} \sum_{j=1, j \neq i}^{m} \frac{A_j(x; t)(x_j - x_i)}{(1 - x_j z)(tx_j - x_i)} + t^{1-m}. \]

Specializing \( z \) suitably in (2.7) and (2.8), we obtain

\[ (2.9) \quad \sum_{i=1}^{m} A_i(x; t) = \frac{1 - t^m}{1 - t}. \]
Multiplying both sides of (2.8) by \((1 - x_i z)^{-1}\) and using

\[ \frac{x_j - x_i}{(1 - x_i z)(1 - x_j z)} = \frac{x_j}{1 - x_j z} - \frac{x_i}{1 - x_i z} \]

and (2.12), we get

\[ \frac{1}{1 - x_i z} \prod_{j=1, j \neq i} A_j(x; t) = \frac{1}{1 - x_i z} \left( (t - 1) t^{1-m} \right) \]

\[ \times \sum_{j=1, j \neq i} \frac{A_j(x; t) x_j}{(1 - x_i z)(1 - x_j z)} + t^{1-m} A_i(x; t) \frac{1}{1 - x_i z}. \]

In what follows \(Q\) stands for \(q^7\).

2.2. Case of \(\mu = 1\). - In this case we see

\[ f(x, t) = \prod_{1 \leq j \leq m, 1 \leq k \leq n} (1 - x_j t_k). \]

Put

\[ \varphi_i(x, t) = \frac{1 - t_1}{1 - x_i t_1} \prod_{j=1, j \neq i} x_j \frac{1 - q^{-1} x_j t_1}{1 - x_j t_1} Q D(t), \]

so that

\[ b_1(t) T_1 \varphi_i(x, t) = q^{a+n-1} \prod_{k=2}^n (t_1 - Q^{-1} t_k) \prod_{2 \leq h < k \leq n} (t_h - Q t_k). \]

We want to calculate \(A(\varphi_i)\) and \(A(b_1 T_1 \varphi_i)\). Since

\[ 1 - t_1 = \frac{1 - x_i t_1}{x_i} + 1 - \frac{1}{x_i} \]

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we see

\begin{equation}
A(\varphi_i) = \frac{1}{x_i} \mathcal{A} \left( \prod_{j=1, j \neq i}^m \frac{1 - q^{-1}x_j t_1}{1 - x_j t_1} qD(t) \right) \\
+ \left( 1 - \frac{1}{x_i} \right) \mathcal{A} \left( \frac{1}{1 - x_i t_1} \prod_{j=1, j \neq i}^m \frac{1 - q^{-1}x_j t_1}{1 - x_j t_1} qD(t) \right). \tag{2.14}
\end{equation}

Substituting \( t = q \) and \( z = t_1 \) in (2.8) and (2.13) gives

\begin{equation}
\prod_{j=1, j \neq i}^m \frac{1 - q^{-1}x_j t_1}{1 - x_j t_1} = (q - 1)q^{1-m} \sum_{j=1, j \neq i}^m \frac{A_j(x; q)(x_j - x_i)}{(1 - x_j t_1)(qx_j - x_i)} + q^{1-m} \tag{2.15}
\end{equation}

\begin{equation}
\frac{1}{1 - x_i t_1} \prod_{j=1, j \neq i}^m \frac{1 - q^{-1}x_j t_1}{1 - x_j t_1} = (q - 1)q^{1-m} \sum_{j=1, j \neq i}^m \frac{A_j(x; q)x_j}{(1 - x_j t_1)(qx_j - x_i)} \\
+ q^{1-m} \frac{A_i(x; q)}{1 - x_i t_1}. \tag{2.16}
\end{equation}

Substituting (2.15) and (2.16) into (2.14) and applying (2.5), (2.10) and (2.11), we have

\[
\frac{f(x, t)}{D(t)} A(\varphi_i) = \frac{(Q; Q)_n}{(1 - Q)^n} f(x, t) \\
+ q^{1-m}(1 - q) \frac{(Q; Q)_{n-1}}{(1 - Q)^{n-1}} \sum_{j=1, j \neq i}^m \frac{x_j(x_j - 1)}{qx_j - x_i} A_j(x; q) \frac{\partial f(x, t)}{\partial x_j} \\
+ q^{1-m} \frac{(Q; Q)_{n-1}}{(1 - Q)^{n-1}} (1 - x_i) A_i(x; q) \frac{\partial f(x, t)}{\partial x_i}.
\]

Hence

\begin{equation}
T_{Q, x_i} \left( \frac{f(x, t)}{D(t)} A(\varphi_i) \right) \tag{2.17}
= \frac{(Q; Q)_{n-1}}{(1 - Q)^{n-1}} \left\{ \frac{(1 - Q^n)}{1 - Q} \left( f(x, t) + (Q - 1)x_i \frac{\partial f}{\partial Q x_i} \right) \\
+ q^{1-m}(1 - q) \sum_{j=1, j \neq i}^m \frac{x_j(x_j - 1)}{qx_j - Q x_i} T_{Q, x_i}(A_j(x; q)) \\
\left( \frac{\partial f}{\partial Q x_j} + (Q - 1)x_i \frac{\partial^2 f}{\partial Q x_i \partial Q x_j} \right) \\
+ q^{1-m}(1 - Q x_i) T_{Q, x_i}(A_i(x; q)) \left( \frac{\partial f}{\partial Q x_i} + (Q - 1)x_i \frac{\partial^2 f}{\partial Q x_i^2} \right) \right\}.
\end{equation}
Next we calculate $A(b_1 T_1 \varphi_i)$. Since
\[
1 - q^a t_1 = q^a \frac{1 - x_i t_1}{x_i} + 1 - \frac{q^a}{x_i},
\]
we see
\[
A(b_1 T_1 \varphi_i) = q^{a+\beta} n^{-1} \frac{1}{x_i} A\left(\prod_{k=2}^{n} (t_1 - Q^{-1} t_k) \prod_{2 \leq h < k \leq n} (t_h - t_k)\right) + q^{a+n-1} \left(1 - \frac{q^a}{x_i}\right) A\left((1 - x_i t_1)^{-1} \prod_{k=2}^{n} (t_1 - Q^{-1} t_k) \prod_{2 \leq h < k \leq n} (t_h - t_k)\right).
\]
Applying (2.6), we have
\[
f(x, t) A(b_1 T_1 \varphi_i) = q^{a+\beta+n-1} Q^{-n+1} (Q; Q)_{n-1} \frac{f(x, t)}{(1 - Q)^n} x_i + q^{a+n-1} Q^{-n+1} (Q; Q)_{n-1} \frac{(1 - Q)^n}{(1 - Q)^n} \times \left(1 - \frac{q^a}{x_i}\right) \prod_{1 \leq k \leq n} (1 - x_i t_k) \left\{ \prod_{k=1}^{n} (1 - x_i t_k) - \prod_{k=1}^{n} (Q - x_i t_k) \right\}.
\]
Hence
\[
(2.18) \quad T_{Q, x_i} \left(\frac{f(x, t)}{D(t)} A(b_1 T_1 \varphi_i)\right)
\]
\[
= q^{a+\beta+n-1} Q^{-n} (Q; Q)_{n-1} \frac{1}{(1 - Q)^n} x_i \left( f(x, t) + (Q - 1) x_i \frac{\partial f}{\partial Q x_i} \right) + q^{a+n-1} Q^{-n} (Q; Q)_{n-1} \frac{(1 - Q)^n}{(1 - Q)^n} \left( (Q - 1) x_i \frac{\partial f}{\partial Q x_i} + (Q - 1)^n f(x, t) \right)
\]
\[
= q^{a+n-1} Q^{-n} (Q; Q)_{n-1} \frac{1}{(1 - Q)^n} \left\{ \frac{1 - Q^n}{1 - Q} f(x, t) + (Q^n - x_i) \frac{\partial f}{\partial Q x_i} \right\}.
\]
It is clear from (2.1) that
\[
(2.19) \quad \int_{[0, \xi \infty]} T_{Q, x_i} (\Phi(t) A(\nabla_1 \varphi(t))) \omega = 0.
\]
Substituting (2.17) and (2.18) into (2.19), we arrive at \( S = q^{S_{n,m}(x)} \)

\[
(2.20) \quad 0 = q^{-(\alpha+n-1)}Q^{-1}x_i(1 - Qx_i)T_{Q,x_i}(A(x; q)) \frac{\partial^2 S}{\partial Qx_i^2} + q^{-(\alpha+n-1)}(1 - q) \sum_{j=1, j \neq i}^{m} \frac{Q^{-1}x_i x_j (1 - x_j)}{Qx_i - qx_j} T_{Q,x_i}(A_j(x; q)) \frac{\partial^2 S}{\partial Qx_i \partial Qx_j} \\
+ \left\{ \frac{q^{m-1} - c}{1 - Q} + \frac{1}{1 - Q} \left((1 - a)(1 - b)q^{m-1} - (q^{m-1} - abQ)q^{m-1} - (1 - a)(1 - b)q^{m-1}ight) \right\} q^\beta Q^{-1} \frac{\partial S}{\partial Qx_i} \\
+ 1 - q \frac{1 - Q}{1 - q} \left( \frac{1 - T_{Q,x_i}(A(x; q))}{1 - q} (c - abQ \beta Qx_i)q^\beta Q^{-1} \frac{\partial S}{\partial Qx_i} \right) \\
- \sum_{j=1, j \neq i}^{m} \frac{q^{-(\alpha+n-1)}Q^{-1}x_i (1 - x_j)}{Qx_i - qx_j} T_{Q,x_i}(A_j(x; q)) \frac{\partial S}{\partial Qx_j} \\
- \frac{(1 - a)(1 - b)q^{m-1}}{(1 - Q)^2} S
\]

where

\[ a = Q^{-n}, \quad b = q^{-(\alpha+n-1)}Q^{-n-1}, \quad c = q^{-(\alpha + \beta + n - 1)}. \]

Now we change the variables:

\[ x_i = q^\beta Q^{-1}y_i, \quad i = 1, \ldots, m. \]

Note that

\[ T_{Q,x_i} = T_{Q,y_i}, \quad \frac{\partial}{\partial Qx_i} = q^\beta Q \frac{\partial}{\partial Qy_i}, \]

and

\[ A(x; q) = A(y; q). \]

(2.20) is transformed into

\[
(2.21) \quad y_i (c - abQy_i)T_{Q,y_i}(A(y; q)) \frac{\partial^2 S}{\partial Qy_i^2} + (1 - q) \times \sum_{j=1, j \neq i}^{m} \frac{y_i y_j (c - aby_j)}{Qy_i - qy_j} T_{Q,y_i}(A_j(y; q)) \frac{\partial^2 S}{\partial Qy_i \partial Qy_j} \\
+ \left\{ \frac{q^{m-1} - c}{1 - Q} + \frac{1}{1 - Q} \left((1 - a)(1 - b)q^{m-1} - (q^{m-1} - abQ)q^{m-1} - (1 - a)(1 - b)q^{m-1}\right) \right\} \frac{\partial S}{\partial Qy_i}
\]
\[ + \frac{1 - q^m}{1 - Q} \left\{ \frac{1 - T_{Q,y_i}(A_i(y;q))}{1 - q} (c - abQy_i) \frac{\partial S}{\partial Qy_i} \right\} \]
\[ - \sum_{j=1, j \neq i}^{m} \frac{y_j(c - aby_j)}{Qy_i - qy_j} T_{Q,y_j}(A_j(y;q)) \frac{\partial S}{\partial Qy_j} \]
\[ - \frac{(1 - a)(1 - b)q^{m-1}}{(1 - Q)^2} S = 0. \]

2.3. Case of \( \mu = -\gamma \). In this case we have
\[ f(x,t) = \prod_{1 \leq j \leq m, 1 \leq k \leq n} \frac{(x_j t_k)_{\infty}}{(Q^{-1}x_j t_k)_{\infty}}. \]

Put
\[ \varphi_i(x,t) = \frac{1 - t_1}{1 - (qQ)^{-1}x_j t_1} QD(t), \]
so that
\[ b_1(t)T_1 \varphi_i(x,t) = q^{\alpha+n-1} \frac{1 - q^3 t_1}{1 - Q^{-1}x_j t_1} \prod_{k=2}^{n} \frac{t_1 - q^{-\gamma} t_k}{t_1 - q^{\gamma} t_k} \prod_{j=1}^{m} \frac{1 - Q^{-1}x_j t_1}{1 - x_j t_1} QD(t). \]

Then one can proceed in a similar way as in the case of \( \mu = 1 \). We have (we omit the details of calculation)
\[ \frac{f(x,t)}{D(t)} A(\varphi_i) = \frac{(Q;Q)_n}{(1 - Q)^n} \frac{qQ}{x_i} f(x,t) + \frac{(Q;Q)_{n-1}}{(1 - Q)^n} \left( \frac{1 - qQ}{x_i} \right) (T_{q^{-1},x_i} f(x,t) - Q^n f(x,t)), \]
so that
\[ (2.22) \]
\[ T_{q,x_i} \left( \frac{f(x,t)}{D(t)} A(\varphi_i) \right) = \frac{(Q;Q)_{n-1}}{(1 - Q)^n} \left\{ (1 - Q^n)f(x,t) + (q - 1)Q(1 - Q^{n-1} x_i) \frac{\partial f}{\partial x_i} \right\}. \]

We have also
\[ \frac{f(x,t)}{D(t)} A(b_1T_1 \varphi_i) = q^{\alpha+n-1} Q^{2-m-n} \left\{ \frac{(Q;Q)_n}{(1 - Q)^n} f(x,t) \right\} \]
\[ - (1 - q)Q^n \left( \frac{Q;Q)_{n-1}}{(1 - Q)^n} (q^3 - x_i) A_i(x;Q) \frac{\partial f}{\partial q x_i} \right) \]
\[ - (1 - q)Q^n \left( \frac{(Q;Q)_{n-1}}{(1 - Q)^n} \sum_{j=1, j \neq i}^{m} \frac{x_j (q^3 - x_j)}{x_i - Q x_j} A_j(x;Q) \frac{\partial f}{\partial q x_j} \right). \]
Hence

\begin{align}
(2.23) \quad & T_{q,x} \left( \frac{f(x,t)}{D(t)} A(b_{1}, T_{1} \varphi_{i}) \right) \\
& = q^{\alpha+n-1}Q^{2-m-n}(Q; Q)^{n-1} \left( (1 - Q^{n})Q^{m-1}\left( f(x,t) + (q - 1)x_{i} \frac{\partial f}{\partial x_{i}} \right) \\
& - q(1 - q)Q^{n}(q^{\beta-1} - x_{i})T_{q,x_{i}}(A_{i}(x; Q)) \left( \frac{\partial f}{\partial q x_{i}} + (q - 1)x_{i} \frac{\partial^{2} f}{\partial q x_{i}^{2}} \right) \\
& - (1 - q)Q^{n}(1 - Q) \sum_{j=1,j \neq i}^{m} \frac{x_{j}(q^{\beta-1} - x_{j})}{qx_{i} - Qx_{j}} T_{q,x_{i}}(A_{j}(x; Q)) \\
& \times \left( \frac{\partial f}{\partial q x_{j}} + (q - 1)x_{j} \frac{\partial^{2} f}{\partial q x_{i} \partial q x_{j}} \right) \right). 
\end{align}

Substituting (2.22) and (2.23) into (2.19) gives the equation corresponding to (2.20):

\begin{align}
(2.24) \quad & 0 = q^{\alpha+n}Qx_{i}(q^{\beta-1} - x_{i})T_{q,x_{i}}(A_{i}(x; Q)) \frac{\partial^{2} S}{\partial q x_{i}^{2}} \\
& + q^{\alpha+n-1}(1 - Q) \sum_{j=1,j \neq i}^{m} \frac{Qx_{i}x_{j}(q^{\beta-1} - x_{j})}{qx_{i} - Qx_{j}} T_{q,x_{i}}(A_{j}(x; Q)) \frac{\partial^{2} S}{\partial q x_{i} \partial q x_{j}} \\
& + \left\{ \frac{Q^{m}}{1 - q} + \frac{Q^{m-1}}{1 - q} (-Q^{n} - q^{\alpha+n-1}Q^{-(n-1)} + q^{\alpha+n-1}Q_{x_{i}}) \right\} \frac{\partial S}{\partial q x_{i}} \\
& - \frac{q^{\alpha+n}Q}{1 - q}(q^{\beta-1} - x_{i})T_{q,x_{i}}(A_{i}(x; Q)) \frac{\partial S}{\partial q x_{i}} \\
& - \frac{1 - Q}{1 - q} Q^{\alpha+n-1} \sum_{j=1,j \neq i}^{m} \frac{x_{j}(q^{\beta-1} - x_{j})}{qx_{i} - Qx_{j}} T_{q,x_{i}}(A_{j}(x; Q)) \frac{\partial S}{\partial q x_{j}} \\
& - \frac{(1 - Q^{n})(1 - q^{\alpha+n-1}Q^{-(n-1)})Q^{m-1}}{(1 - q)^{2}} S.
\end{align}

Hence changing variables as

\[ x_{i} = Qy_{i}, \quad i = 1, \ldots, m \]

yields

\begin{align}
(2.25) \quad & y_{i}(c - aby_{i})T_{q,y_{i}}(A_{i}(y; Q)) \frac{\partial^{2} S}{\partial y_{i}^{2}} \\
& + (1 - Q) \sum_{j=1,j \neq i}^{m} \frac{y_{i}y_{j}(c - aby_{j})}{qy_{i} - Qy_{j}} T_{q,y_{i}}(A_{j}(y; Q)) \frac{\partial^{2} S}{\partial y_{i} \partial y_{j}} 
\end{align}
where

\[ a = Q^n, \quad b = q^{a+n-1}Q^{-(n-1)}, \quad c = q^{a+b+n-1}. \]

We have thus proved

**Theorem 2.3**. Assume \( \mu = 1 \) or \(-\gamma\). Then \( qS_{n,m(\alpha,\beta,\gamma,\mu;\xi)} \) satisfies the following system of \( q \)-difference equations (\( T_i = T_{q,x_i} \)).

\[
(2.26) \quad x_i(c-abqx_i)T_i(A_i(x;t)) \frac{\partial^2 S}{\partial x_i^2} + (1-q) \sum_{j=1, j\neq i}^m \frac{x_ix_j(c-abx_j)}{qx_i-tx_j} T_i(A_j(x;t)) \frac{\partial^2 S}{\partial x_i \partial x_j}
\]

\[
+ \left\{ \frac{t^{m-1} - c}{1-q} + \frac{1}{1-q} \left[ (1-a)(1-b)t^{m-1} - (t^{m-1} - abq) \right] x_i \right\} \frac{\partial S}{\partial x_i}
\]

\[
+ \frac{1-t}{1-q} \left\{ \frac{1-T_i(A_i(x;t))}{1-t} (c-abqx_i) \frac{\partial S}{\partial x_i} \right\}
\]

\[
- \sum_{j=1, j\neq i}^m \frac{x_j(c-abx_j)}{qx_i-tx_j} T_i(A_j(x;t)) \frac{\partial S}{\partial x_j}
\]

\[
- \frac{(1-a)(1-b)t^{m-1}}{(1-q)^2} S = 0, \quad i = 1, \ldots, m,
\]

where if \( \mu = 1 \), then put \( (q,t) = (Q,q) \), \( a = Q^{-n}, b = q^{-(\alpha+n-1)}Q^{n-1}, c = q^{-(\alpha+\beta+n-1)} \)
and change \( x_i \) with \( Q^{-\alpha}Qx_i, 1 \leq i \leq m \). If \( \mu = -\gamma \), then put \( (q,t) = (q,Q) \), \( a = Q^n, b = q^{\alpha+\beta+n-1}Q^{-(n-1)}, c = q^{\alpha+\beta+n-1} \) and change \( x_i \) with \( Q^{-1}x_i, 1 \leq i \leq m \).

**Remark**. In the theorem above the change of variables means that we change only the variables of the \( q \)-difference equations. We do not change the variables of the unknown function \( S \). The same remark applies also to the following Theorem 2.4 and 2.5.
2.4. VARIANTS. – One can calculate a system of $q$-difference equations satisfied by the integral $\int qS_{n,m}(\alpha, \beta, \gamma; \mu; x_1^{-1}, \ldots, x_m^{-1}; \xi)$ provided $\mu = 1$ or $-\gamma$ in the same way as in the case of Theorem 2.3. If $\mu = 1$, then put

$$\varphi_i(x, t) = \frac{1 - t_1}{1 - t_1/x_i} Q D(t).$$

We have

$$\frac{f(x, t)}{D(t)} A(\varphi_i) = \frac{(Q; Q)_{n-1}}{(1 - Q)^n} \left\{ (1 - Q^n)x_i f(x, t) + (1 - x_i) \left( T_{Q^{-1}, x_i} f(x, t) - Q^n f(x, t) \right) \right\}$$

and

$$\frac{f(x, t)}{D(t)} A(b_l T_l \varphi_i) = q^{\alpha+n-1} Q^{1-n} (Q; Q)_{n-1} \left\{ (1 - Q^n)f(x, t) + (1 - Q)Q^n x_i (1 - q^a x_i) A_i(x; q) \frac{\partial f}{\partial Q x_i} \right. \right.$$}

$$+ (1 - q)(1 - Q) Q^n \sum_{j=1, j \neq i}^{m} \frac{x_i x_j (1 - q^a x_j)}{x_i - q x_j} A_j(x; q) \frac{\partial f}{\partial Q x_j} \}.$$

Substituting these into (2.19) gives

$$0 = q^{\alpha+n-1} Q^2 x_i^2 (1 - q^a Q x_i) T_{Q, x_i} (A_i(x; q)) \frac{\partial^2 S}{\partial Q x_i^2}$$

$$+ q^{\alpha+n-1}(1 - q) Q^2 \sum_{j=1, j \neq i}^{m} \frac{x_i^2 x_j (1 - q^a x_j)}{Q x_i - q x_j} T_{Q, x_i} (A_j(x; q)) \frac{\partial^2 S}{\partial Q x_i \partial Q x_j}$$

$$+ \frac{x_i}{1 - Q} \left\{ Q(Q^{n-1} - x_i) + q^{\alpha+n-1} Q^{-(n-1)} (1 - Q^n) \right\} \frac{\partial S}{\partial Q x_i}$$

$$- q^{\alpha+n-1} Q^2 x_i (1 - q^a Q x_i) T_{Q, x_i} (A_i(x; q)) \frac{\partial S}{\partial Q x_i}$$

$$- \frac{(1 - q) Q^{\alpha+n-1} Q^2}{1 - Q} \sum_{j=1, j \neq i}^{m} \frac{x_i x_j (1 - q^a x_j)}{Q x_i - q x_j} T_{Q, x_i} (A_j(x; q)) \frac{\partial S}{\partial Q x_j}$$

$$+ \frac{(1 - Q^n)(1 - q^{\alpha+n-1} Q^{-(n-1)})}{(1 - Q)^2} S.$$

If $\mu = -\gamma$, then put

$$\varphi_i(x, t) = \frac{1 - t_1}{1 - (qQ)^{-1}t_1/x_i} \prod_{j=1, j \neq i}^{m} \frac{1 - q^{-1}t_1/x_j}{1 - (qQ)^{-1}t_1/x_j} Q D(t).$$
We have

\[
\frac{f(x, t)}{D(t)} \mathcal{A}(\varphi_i) = \frac{(Q; Q)^{n-1}}{(1 - Q)^n} \left\{ (1 - Q^n) f(x, t) - (1 - q)x_i(1 - qQx_i) A_i(x; Q) \frac{\partial f}{\partial qx_i} \\
- (1 - q)(1 - Q) \sum_{j=1, j \neq i}^{m} \frac{x_i x_j (1 - qQx_j)}{x_i - Qx_j} A_j(x; Q) \frac{\partial f}{\partial qx_j} \right\}
\]

and

\[
\frac{f(x, t)}{D(t)} \mathcal{A}(b_i T_1 \varphi_i) = q^{a+n-1} Q^{-(n-1)} \frac{(Q; Q)^{n-1}}{(1 - Q)^n} \left\{ (1 - Q^n) q^3 x_i f(x, t) \\
+ (1 - q^3 x_i) \prod_{k=1}^{n} \frac{f(x, t)}{1 - t_k / x_i} \left( \prod_{k=1}^{n} (1 - t_k / x_i) - \prod_{k=1}^{n} (Q - t_k / x_i) \right) \right\}.
\]

Substituting these into (2.19) yields

\[
0 = qx_i^2 (1 - q^2 Qx_i) T_{q, x_i}(A_i(x; Q)) \frac{\partial^2 S}{\partial x_i^2}
+ (1 - Q) \sum_{j=1, j \neq i}^{m} \frac{qx_i^2 x_j (1 - qQx_j)}{qx_i - Qx_j} T_{q, x_i}(A_j(x; Q))
\times \frac{\partial^2 S}{\partial qx_i \partial qx_j} + \frac{x_i}{1 - q} \left\{ q^{a+n-1} Q^{-(n-1)} (1 - q^3 + 1) Q^n x_i \\
- (1 - Q^n + (1 - q^3 Qx_i) q) \right\} \frac{\partial S}{\partial qx_i}
+ \frac{x_i (1 - q^2 Qx_i)}{1 - q} \left( 1 - T_{q, x_i}(A_i(x; Q)) \right) \frac{\partial S}{\partial qx_i}
- \frac{1}{1 - q} \sum_{j=1, j \neq i}^{m} \frac{qx_i x_j (1 - qQx_j)}{qx_i - Qx_j} T_{q, x_i}(A_j(x; Q)) \frac{\partial S}{\partial qx_j}
\times \frac{\partial^2 S}{\partial qx_i \partial qx_j} + \frac{1 - Q^n}{(1 - q)^2} S.
\]

From (2.27) and (2.28) we can conclude

**Theorem 2.4.** Assume \( \mu = 1 \) or \( -\gamma \). Then \( q S_{n, m}(\alpha, \beta, \gamma, \mu; x_1^{-1}, \ldots, x_m^{-1}; \xi) \) satisfies the following system of q-difference equations.

\[
x_i(c - abqx_i) T_i(A_i(x; t)) \frac{\partial^2 S}{\partial qx_i^2} + (1 - q) \sum_{j=1, j \neq i}^{m} \frac{x_i x_j (c - abx_j)}{qx_i - tx_j} T_i(A_j(x; t)) \frac{\partial^2 S}{\partial qx_i \partial qx_j}
\]
\[ \begin{align*}
&+ \frac{a t^{m-1} - c/q + ac/q - c}{1 - q} + \frac{1}{1 - q} (abq - a^2 t^{m-1}) x_i \frac{\partial S}{\partial q x_i} \\
&+ \frac{1 - t}{1 - q} \left( 1 - T(A_i(x; t)) \right) (c - abq x_i) \frac{\partial S}{\partial q x_i} - \sum_{j=1, j \neq i}^m x_j (c - abx_j) \frac{\partial S}{\partial q x_j} \\
- \frac{(1 - a)(a t^{m-1} - c/q - c)}{(1 - q)^2} x_i S = 0, \quad i = 1, \ldots, m,
\end{align*} \]

where if \( \mu = 1 \), then put \((q, t) = (Q, q), a = Q^{-n}, b = q^{-(\alpha + n - 1)} Q^{n-1}, c = q^{-(\alpha + \beta + n - 1)} \) and change \( x_i \) with \( q^{-\beta} Q x_i, 1 \leq i \leq m \). If \( \mu = -\gamma \), then put \((q, t) = (q, Q), a = Q^n, b = q^{\alpha + n - 1} Q^{-(n-1)}, c = q^{\alpha + \beta + n - 1} \) and change \( x_i \) with \( Q^{-1} x_i, 1 \leq i \leq m \).

From this theorem one can derive the following theorem by straightforward calculation.

**Theorem 2.5.** Assume \( \mu = 1 \), or \(-\gamma\). Then \((x_1 \cdots x_m)_{n}^\alpha S_{m,n}(\alpha, \beta, \gamma, \mu; x_1^{-1}, \ldots, x_m^{-1}; \xi)\) satisfies the system (2.26) in which if \( \mu = 1 \), then put \((q, t) = (Q, q), a = Q^{-n}, b = q^{-(\alpha + n - 1)} Q^{n-1}, c = q^{-(\alpha + \beta + n - 1)} \) and change \( x_i \) with \( q^{-\beta} Q x_i, 1 \leq i \leq m \). If \( \mu = -\gamma \), then put \((q, t) = (q, Q), a = Q^n, b = q^{\alpha + n - 1} Q^{-(n-1)}, c = q^{\alpha + \beta + n - 1} \) and change \( x_i \) with \( Q^{-1} x_i, 1 \leq i \leq m \).

### 3. \textit{q}-Hypergeometric functions

#### 3.1. Macdonald polynomials

Let \( \lambda = (\lambda_1, \lambda_2, \ldots) \) be a partition and \( \lambda' = (\lambda'_1, \lambda'_2, \ldots) \) the conjugate partition. The number \( \lambda_1 \) of parts of \( \lambda \) is denoted by \( \ell(\lambda) \), called the length of \( \lambda \). If \( \lambda \) has \( m_1 \) parts equal to 1, \( m_2 \) parts equal to 2, and so on, we write \( \lambda = (m_1 \ 2^{m_2} \cdots) \) and denote \( \prod_{r \geq 1} (m_r m_r !) = z_\lambda \). We write \( |\lambda| = \sum \lambda_i \). If \( \mu \) is another partition, then write \( \mu \leq \lambda \) when \( |\mu| = |\lambda| \) and \( \mu_1 + \cdots + \mu_i \leq \lambda_1 + \cdots + \lambda_i \) for all \( i \). Given a partition \( \lambda = (\lambda_1, \ldots, \lambda_m) \) of length \( \leq m \), the monomial symmetric polynomial \( m_\lambda(x_1, \ldots, x_m) \) is defined by

\[
m_\lambda(x_1, \ldots, x_m) = \sum x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_m^{\lambda_m},
\]

where the sum is over all distinct monomials obtainable from \( x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_m^{\lambda_m} \) by permutations of the \( x \)'s. In particular when \( \lambda = (r) \) we have the \( r \)-th power sum:

\[
m_{(r)} = p_r(x_1, \ldots, x_m) = \sum_{i=1}^m x_i^r.
\]

For each partition \( \lambda \), we set \( p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots \).

Let \( q, t \) be independent indeterminates and \( Q(q, t) \) be the field of rational functions in \( q \) and \( t \). We have the fundamental ([Ma2, (2.8)]).

**Theorem 3.1.** For each partition \( \lambda \) of length \( \leq m \) there exists a unique symmetric polynomial \( P_\lambda(x_1, \ldots, x_m; q, t) \) with coefficients in \( Q(q, t) \) satisfying

\[
P_\lambda = \sum_{\mu \leq \lambda} u_{\lambda \mu} m_\mu,
\]

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where $u_{\lambda\mu} \in \mathbb{Q}(q, t)$ and $u_{\lambda\lambda} = 1$; and

\begin{equation}
(3.2)\quad D_1^{(q, t)} P_\lambda = e_\lambda P_\lambda
\end{equation}

where $D_1^{(q, t)}$ and $e_\lambda(q, t)$ are defined by

\begin{equation}
(3.3)\quad D_1^{(q, t)} = \sum_{i=1}^{m} A_i(x; t) x_i \frac{\partial}{\partial q x_i}, \quad e_\lambda(q, t) = \sum_{i=1}^{m} \frac{1 - q^{\lambda_i}}{1 - q} t^{m - i}.
\end{equation}

We understand that $P_\lambda(x_1, \ldots, x_m) = 0$ if $\ell(\lambda) \geq m + 1$. One can readily verify that $P_\lambda(x_1, \ldots, x_r, 0, \ldots, 0) = P_\lambda(x_1, \ldots, x_r)$.

Denote the ring of symmetric polynomials in $x_1, \ldots, x_m$ over the field $F = \mathbb{Q}(q, t)$ by $\Lambda_{m, F}$. Define a scalar product $\langle , \rangle$ on $\Lambda_{m, F}$ by

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda(q, t)$$

where

$$z_\lambda(q, t) = z_\lambda \prod_{i=1}^{\ell(\lambda)} \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}}.$$

Note that $D_1^{(q, t)}$ has the following properties ([Ma2, (2.7.1)-(2.7.3)]):

$$D_1^{(q, t)} m_\lambda = \sum_{\mu \subseteq \lambda} c_{\lambda\mu} m_\mu$$

for each partition $\lambda$ of length $\leq m$;

$$\langle D_1^{(q, t)} f, g \rangle = \langle g, D_1^{(q, t)} g \rangle$$

for all $f, g \in \Lambda_{m, F}$;

$$\lambda \neq \mu \Rightarrow c_{\lambda\lambda} \neq c_{\mu\mu}.$$

From these properties one can deduce that the condition (3.2) in the Theorem 3.1 can be replaced by

$$\langle P_\lambda, P_\mu \rangle = 0 \quad if \quad \lambda \neq \mu.$$

We shall need a kind of specialization formula of $P_\lambda$. Let $u$ be a new indeterminate and define a homomorphism

$$\epsilon_{u, t} : \Lambda_{m, F} \rightarrow F[u]$$

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by

\[ \epsilon_{r}(p_{r}) = \frac{1 - u^{r}}{1 - t^{r}} \]

for each \( r \geq 1 \). Then we see

\[ \epsilon_{t^{m}, t}(f) = f(1, t, \ldots, t^{m-1}). \]

Consider the diagram of \( \lambda \) in which the rows and columns are arranged as in a matrix, with the \( i \)th row consisting of \( \lambda_{i} \) boxes. For each square \( s = (i, j) \) in the diagram of \( \lambda \), let

\[ a(s) = \lambda_{i} - j, \quad a'(s) = j - 1, \]

\[ l(s) = \lambda_{j'} - i, \quad l'(s) = i - 1, \]

and put

\[ h_{\lambda}(q, t) = \prod_{s \in \lambda} (1 - q^{a(s)} t^{l(s)+1}), \quad h'_{\lambda}(q, t) = \prod_{s \in \lambda} (1 - q^{a(s)+1} t^{l(s)}) \]

and

\[ b_{\lambda} = b_{\lambda}(q, t) = \frac{h_{\lambda}(q, t)}{h'_{\lambda}(q, t)}. \]

One has different expressions of \( h_{\lambda} \) and \( h'_{\lambda} \).

**Proposition 3.2.** - Let \( \lambda \) be a partition of length \( \leq n \). Then

\[ h'_{\lambda}(q, t) = (q)^{n} \prod_{i=1}^{n} \left( q^{\lambda_{i}+1} t^{n-i} \right)^{-1} \prod_{1 \leq i < j \leq n} \frac{(q^{\lambda_{i}-\lambda_{j}+1} t^{j-i-1})}{(q^{\lambda_{i}-\lambda_{j}+1} t^{j-i-1})}. \]

\[ h_{\lambda}(q, t) = (t)^{n} \prod_{i=1}^{n} \left( q^{\lambda_{i}} t^{n-i+1} \right)^{-1} \prod_{1 \leq i < j \leq n} \frac{(q^{\lambda_{i}-\lambda_{j}+1} t^{j-i})}{(q^{\lambda_{i}-\lambda_{j}+1} t^{j-i})}. \]

**Proof.** - We prove (3.5). One can prove (3.6) in the same way. Put

\[ C = \{ i \mid \lambda_{i+1} < \lambda_{i} \}, \]

so that

\[ \prod_{s \in \lambda} (1 - q^{a(s)+1} t^{l(s)}) = \prod_{i \in C} \prod_{r=0}^{i-1} \prod_{j=\lambda_{i+1}+1}^{\lambda_{i}} (1 - q^{\lambda_{i}-r-j+1} t^{r}). \]
Observe that
\[
\prod_{1 \leq i < j \leq n} (q^{\lambda_i - \lambda_j + 1} t^{j-i}) = \prod_{r=1}^{n-1} \prod_{i=r+1}^{n} (q^{\lambda_i - \lambda_r + 1} t^{r}) = (q)^{-n} \prod_{r=0}^{n-1} \prod_{i=r+1}^{n} (q^{\lambda_i - \lambda_r + 1} t^{r})
\]
\[
\prod_{1 \leq i < j \leq n} (q^{\lambda_i - \lambda_j + 1} t^{j-i-1}) = \prod_{r=0}^{n-1} \prod_{i=r+1}^{n} (q^{\lambda_i - \lambda_r + 1} t^{r}) = \left( \prod_{r=0}^{n-1} (q^{\lambda_r - \lambda_{r+1} + 1} t^{r}) \right)^{n-1} \prod_{r=0}^{n-1} \prod_{i=r+1}^{n} (q^{\lambda_i - \lambda_{i+1} + 1} t^{r}),
\]

Hence we get
\[
\text{RHS of (3.5)} = \prod_{r=0}^{n-1} \prod_{i=r+1}^{n} \frac{(q^{\lambda_i - \lambda_r + 1} t^{r})}{(q^{\lambda_i - \lambda_{i+1} + 1} t^{r})} = \prod_{i \in \mathbb{C}} \prod_{r=0}^{i-1} \frac{(q^{\lambda_i - \lambda_r + 1} t^{r})}{(q^{\lambda_i - \lambda_{i+1} + 1} t^{r})} = \prod_{i \in \mathbb{C}} \prod_{r=0}^{i-1} \prod_{j=\lambda_i+1}^{\lambda_{i+1}} (1 - q^{\lambda_i - \lambda_{j+1} + 1} t^{r}),
\]
as desired.

We define the generalized factorial \((a)^{(q,t)}_{\lambda}\) by
\[
(a)^{(q,t)}_{\lambda} = \prod_{s \in \lambda} (t^{\lambda(s)} - q^{\alpha(s)} a).
\]

The following explicit formula ([Ma2, (5.3)]) is essential:

**Theorem 3.3.** We have
\[
\epsilon_{\alpha}(P_{\lambda}(q,t)) = \frac{(u)^{(q,t)}_{\lambda}}{h_{\lambda}(q,t)}.
\]

**Remark.** This formula is equivalent to the \(q\)-binomial theorem for the \(q\)-hypergeometric functions defined in the next subsection (see Theorem 3.5). We shall treat this problem in a separate paper.

We shall also need ([Ma2, (3.11)]):
\[
(3.7) \quad \sum_{\lambda} b_{\lambda}(q,t) P_{\lambda}(x;q,t) P_{\lambda}(y;q,t) = \prod_{i,j} \frac{(tx,ty; q)_{\infty}}{(x_{i},y_{j}; q)_{\infty}},
\]

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or its dual [Ma2, (3.12)]

\[(3.8) \quad \sum_{\lambda} P_\lambda(x; q, t) P_\lambda(y; t, q) = \prod_{i,j}(1 + x_i y_j).\]

This is a consequence of [Ma2, Sect. 5, p. 159]:

\[(3.9) \quad \langle P_\lambda, P_\lambda \rangle = b_\lambda(q, t)^{-1}.\]

3.2. \(q\)-HYPERGEOMETRIC FUNCTION. — For a partition \(\lambda\), denote \(b(\lambda) = \sum (i - 1) \lambda_i = \sum \lambda_i^2/2\).

**Definition 3.4.** — Let \(a_1, \ldots, a_r\) and \(b_1, \ldots, b_s\) be complex numbers such that \((b_j)_{(a,r)} \neq 0, 1 \leq j \leq m\) for any partition of length \(\leq m\). The \(q\)-hypergeometric function \(\Phi_s^{(q,t)}(a_1, \ldots, a_r; b_1, \ldots, b_s; x_1, \ldots, x_m)\) is defined by

\[(3.10) \quad \Phi_s^{(q,t)}(a_1, \ldots, a_r; b_1, \ldots, b_s; x) = \sum_{\lambda} \frac{\prod_{i=1}^r (a_i)_{(q,t)} \prod_{j=1}^s (b_j)_{(q,t)}^{\lambda_i^2/2} \prod_{i=1}^r (x_i)_{(q,t)}}{\prod_{i=1}^s (b_j)_{(q,t)}^{\lambda_i^2/2} \prod_{i=1}^r (x_i)_{(q,t)}} P_\lambda(x; q, t).\]

As a consequence of the factor \(\{(-1)^{|\lambda|} q^{b(\lambda')}\}^{s+1-r}\), it follows that

\[(3.11) \quad \lim_{a \to \infty} \Phi_s^{(q,t)}(a_1, \ldots, a_r, a; b_1, \ldots, b_s; x_1/a, \ldots, x_m/a) = \Phi_s^{(q,t)}(a_1, \ldots, a_r; b_1, \ldots, b_s; x).\]

When \(m = 1\), \(\Phi_s^{(q,t)}(x)\) reduces to the ordinary \(q\)-hypergeometric function \(\phi_s(x)\) (cf. [An, GR]), being independent of \(t\).

**Theorem 3.5.** — We have

\[(3.12) \quad \Phi_s^{(q,t)}(a_1; \ldots, x_m) = \prod_{i=1}^m (ax_i; q)_\infty (x_i; q)_\infty.\]

**Proof.** — Note first that [Ma2, (2.6)]:

\[(3.13) \quad \prod_{i,j} (tx_i y_j; q)_\infty (x_i y_j; q)_\infty = \sum_{\lambda} z_\lambda(q, t)^{-1} p_\lambda(x) p_\lambda(y).\]

In particular we have

\[\prod_{i=1}^m (ax_i; q)_\infty (x_i; q)_\infty = \sum_{\lambda} z_\lambda(q, a)^{-1} p_\lambda(x).\]
It follows from (3.7) and (3.13) that
\[
\begin{equation}
\sum_{\lambda} b_{\lambda}(q, t) P_{\lambda}(x; q, t) P_{\lambda}(y; q, t) = \sum_{\lambda} z_{\lambda}(q, t)^{-1} p_{\lambda}(x) p_{\lambda}(y)
\end{equation}
\]
so
\[
\begin{equation}
\sum_{|\lambda| \leq l} b_{\lambda}(q, t) P_{\lambda}(x; q, t) P_{\lambda}(y; q, t) = \sum_{|\lambda| \leq l} z_{\lambda}(q, t)^{-1} p_{\lambda}(x) p_{\lambda}(y)
\end{equation}
\]
for arbitrary \( l \). In view of Theorem 3.3 and
\[
\epsilon_{q, t}(p_{\lambda}(y)) = \prod_{i=1}^{\nu(\lambda)} \frac{1 - a_{\lambda_i}}{1 - t^{\lambda_i}},
\]
applying \( \epsilon_{q, t} \) to both sides of (3.15) considered as polynomials in \( y \) yields
\[
\begin{equation}
\sum_{|\lambda| \leq l} \sum_{\mu} \frac{(a)^{(q, t)}}{h_{\lambda}^{(q, t)}} P_{\lambda}(x) = \sum_{|\lambda| \leq l} \sum_{\mu} \frac{(a)^{(q, t)}}{h_{\lambda}^{(q, t)}} p_{\lambda}(x).
\end{equation}
\]
Since \( l \) is arbitrary, this clearly gives (3.12).

**Corollary 3.6**

(3.17) \( 0 \Phi_{0}^{(q, t)}(x_1, \ldots, x_m) = \prod_{i=1}^{m} (x_i; q)_\infty. \)

**Proof.** – This follows at once from (3.11) because
\[
\lim_{a \to \infty} \prod_{i=1}^{m} \frac{(x_i; q)_\infty}{(x_i/a; q)_\infty} = \prod_{i=1}^{m} (x_i; q)_\infty.
\]

Next we consider the convergence of the series. We assume \( 0 < t < 1 \).

**Lemma 3.7.** – Let \( ||x|| = \max \{ |x_1|, \ldots, |x_m| \} \). There exists a positive constant \( C \) depending only on \( q, t \) and \( m \) such that
\[
|P_{\lambda}(x; q, t)| \leq C(h_{\lambda}^{-1}h_{\lambda}')^{1/2}(m ||x||)^{|\lambda|}.
\]

**Proof.** – Put \( |\lambda| = d \) and write
\[
P_{\lambda} = \sum_{|\mu| = d} a_{\mu} p_{\mu}
\]
so that
\[ (P_\lambda, P_\lambda) = \sum_{|\mu|=d} a_\mu^2 z_\mu(q,t). \]

By Cauchy's inequality we have
\[ |P_\lambda|^2 \leq \left( \sum_{|\mu|=d} a_\mu^2 z_\mu(q,t) \right) \left( \sum_{|\mu|=d} |p_\mu|^2 z_\mu(q,t) \right). \]

It follows from (3.9) that
\[ \sum_{|\mu|=d} a_\mu^2 z_\mu(q,t) = h_\lambda^{-1} h_\lambda'. \]

Put
\[ C_1 = \max_{\ell(\lambda) \leq m} \frac{1 - t^{\lambda_i}}{1 - q^{\lambda_i}} \]
so that
\[ \sum_{|\mu|=d} \frac{|p_\mu|^2}{z_\mu(q,t)} \leq C_1 \sum_{|\mu|=d} z_\mu^{-1} p_\mu^2 (|x_1|, \ldots, |x_m|). \]

Since \( \sum_{|\mu|=d} z_\mu^{-1} p_\mu = \sum_{|\mu|=d} m_\mu \) ([Ma2, p. 17]), we obtain
\[ \sum_{|\mu|=d} \frac{|p_\mu|^2}{z_\mu(q,t)} \leq C_1 m^d \binom{m+d-1}{d} ||x||^{2d}. \]

Note that
\[ \binom{m+d-1}{d} = \frac{(d+1)(d+2) \cdots (d+m-1)}{(m-1)!} \leq d^{m-1} \frac{m!}{(m-1)!}. \]

Put \( C_2 = \max_d m^{-d} d^{m-1} \). This gives
\[ \sum_{|\mu|=d} \frac{|p_\mu|^2}{z_\mu(q,t)} \leq C_1 C_2 m^{2d+1} ||x||^{2d}. \]

Hence by setting \( C = (mC_1 C_2)^{1/2} \), we arrive at the desired inequality.

**Theorem 3.8.** - We have

1. If \( r \leq s \), then the series (3.10) converges absolutely for all \( x \in \mathbb{C}^m \).
2. If \( r = s + 1 \), then the series (3.10) converges absolutely for \( ||x|| < m^{-1} \).
3. If \( r > s + 1 \), then the series (3.10) does not converge absolutely for \( x \neq (0, \ldots, 0) \) unless it terminates.

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Proof. – We compare the series (3.10) with the series

\[
\sum_{d=0}^{\infty} \prod_{i=1}^r (a_k; q) d \left\{ (-1)^d q^{d(d-1)/2} \right\}^{s+1-r} \frac{z^d}{(q; q)_d}
\]

which is known to have radius of convergence \( \rho = \infty \) if \( r \leq s \), \( \rho = 1 \) if \( r = s + 1 \), \( \rho = 0 \) if \( r > s + 1 \) unless it terminates [GR, p. 5]. Put

\[
a_{ki} = a_k t^{-(i-1)}, \quad b_{ki} = b_k t^{-(i-1)}.
\]

Note first that

\[
(h_\lambda h'_\lambda)^{-1} \left( \prod_{i=1}^m (q; q)_{\lambda_i} \right)^2 \leq (1 - t)^{-m}
\]

and

\[
t^{(r-s)b(\lambda)} \leq R^{m(\lambda)}
\]

where we put \( R = 1 \) if \( r \geq s \), and \( R = t^{(r-s)m} \) if \( r < s \). Then by Lemma 3.7 we have

\[
\sum_{\lambda} \left| \frac{\prod_{i=1}^r (a_i)}{\prod_{i=1}^r (b_i)} \right| \left\{ (-1)^{m(\lambda)} q^{b(\lambda)} \right\}^{s+1-r} \frac{P_{\lambda}}{P_{\lambda}^{(h_\lambda h'_\lambda)^{-1} \left( \prod_{i=1}^m (q; q)_{\lambda_i} \right)^2 \leq (1 - t)^{-m}}
\]

This completes the proof of the assertions (1) and (2).

For the proof of (3), suppose \( x_i \neq 0, 1 \leq i \leq a, \) and \( x_i = 0 \) otherwise. Note that

\[
h'_\lambda^{-1} \prod_{i=1}^m (q; q)_{\lambda_i} \geq ((1 - q)(1 - qt^m)^{-1})^{m(\lambda)}
\]
Put $\theta = (1 - q)(1 - qt^m)^{-1}$. Picking up the terms with $\lambda = (d, d, \ldots , d) = (d^n)$ (so that $P_\lambda = x_1 \cdots x_n$ by (3.1)), we obtain

$$\sum_{\lambda} \left| \frac{\prod_{i=1}^{r}(a_{i\lambda}^{(q,t)}) \{-1\}^{\lambda} \lambda^{b(\lambda')} \}}{\prod_{i=1}^{r}(b_{i\lambda}^{(q,t)}) \lambda^{r}} \right| P_\lambda$$

$$\geq \sum_{\lambda} \left| \prod_{i=1}^{r}(\theta^{(r-s)b(\lambda)} \frac{\prod_{i=1}^{m} (a_{ki}; q)_{\lambda_i} \lambda{\lambda}}{\prod_{i=1}^{m} (b_{ki}; q)_{\lambda_i} \lambda^{r}} \right| \prod_{i=1}^{m} (q; q)_{\lambda_i} \right|$$

$$\geq \sum_{d=0}^{\infty} \left| \prod_{i=1}^{r} \prod_{i=1}^{m} (a_{ki}; q)_{d} q^{d(s+1-r)(d-1)/2} \frac{\prod_{i=1}^{m} (q; q)_{\lambda_i} \lambda^{r}}{(q; q)_{d}^{d}} \right|$$

This last series is easily shown to be divergent, thereby completing the proof of (3).

4. $q$-Difference system of $q$-hypergeometric function

4.1. Summed-up Equation. As in the case of $q = 1$ [Kan2], we shall consider the $q$-difference equation formed by summing the $q$-difference equations, multiplied by $A_i(x; t)$ each, in the system (2.26). First we introduce auxiliary operators:

$$D_{2}^{(q,t)} = \frac{1 - q}{1 - t} \sum_{1 \leq i < j \leq m} A_{ij}(x; t) x_i x_j \frac{\partial^2}{\partial q x_i \partial q x_j} - \frac{1}{1 - t} \sum_{1 \leq i \neq j \leq m} A_{ij}(x; t) x_i \frac{\partial}{\partial q x_i}$$

where

$$A_{ij}(x; t) = t \prod_{k=1, k \neq i,j}^{m} \frac{(tx_i - x_k)(tx_j - x_k)}{(x_i - x_k)(x_j - x_k)}$$

and

$$\varepsilon = \varepsilon_m = \sum_{i=1}^{m} A_i(x; t) \frac{\partial}{\partial q x_i}$$

It is known that [Ma3]

$$D_{2}^{(q,t)} P_\lambda = f_\lambda(q, t) P_\lambda$$

where

$$f_\lambda = \frac{1}{(1 - q)(1 - t)} \sum_{1 \leq i < j \leq m} t^{2m-i-j}(q^{\lambda_i + \lambda_j} - 1)$$

$$= \sum_{1 \leq i < j \leq m} \left\{ \frac{(1 - t^{m-i}q^{\lambda_i}) (1 - t^{m-j}q^{\lambda_j})}{(1 - q)(1 - t)} - \frac{(1 - t^{m-i})(1 - t^{m-j})}{(1 - q)(1 - t)} \right\} + \frac{1 - m}{1 - t} \varepsilon_\lambda.$$
Denote by \( \Lambda^r_m \) the vector space of symmetric homogeneous polynomials of degree \( r \) in \( x_1, \ldots, x_m \) with coefficients in \( \mathbb{Q}(q, t) \).

**Lemma 4.1.** \( \varepsilon \) defines a linear mapping from \( \Lambda^r_m \) to \( \Lambda^{r-1}_m \) for every \( r \).

**Proof.** Since the \( p_{\lambda}(x_1, \ldots, x_m)'s \) with \( |\lambda| = r \) and \( \ell(\lambda) \leq m \) form a basis of \( \Lambda^r_m \), it suffices to show \( \varepsilon p_{\lambda} \in \Lambda^{r-1}_m \). But this easily boils down to prove the case of \( \lambda = (r) \). We see

\[
\varepsilon p_r = \frac{1 - q^r}{1 - q} \sum_{i=1}^{m} A_i x_i^{r-1} - \frac{1 - q^r}{1 - q} D_1 p_{r-1}.
\]

Hence the lemma follows immediately from (3.2) and the fact that the \( P_{\lambda}(x_1, \ldots, x_m)'s \) with \( |\lambda| = r - 1 \) and \( \ell(\lambda) \leq m \) form a basis of \( \Lambda^{r-1}_m \).

Let us denote by \( L_m = L_m(q,t) \) the \( q \)-difference operator formed by summing the \( q \)-difference operators, multiplied by \( A_i = A_i(x,t) \) each, in the system (2.26):

\[
(4.2) \quad L_m = \sum_{i=1}^{m} x_i(c - abq x_i) A_i T_i(A_i) \frac{\partial^2}{\partial q x_i^2} + (1 - t)
\]

\[
\times \sum_{1 \leq i < j \leq m} \frac{x_i x_j (c - abq x_j)}{q x_i - t x_j} A_i T_i(A_j) \frac{\partial^2}{\partial q x_i \partial q x_j}
\]

\[
+ \sum_{i=1}^{m} \left\{ \frac{t^{m-1} - c}{1 - q} + \frac{1}{1 - q} \left( (1 - a)(1 - b)t^{m-1} - (t^{m-1} - abq) \right) x_i \right\} A_i \frac{\partial}{\partial q x_i}
\]

\[
+ \frac{1 - t}{1 - q} \left\{ \sum_{i=1}^{m} \frac{1 - T_i(A_i)}{1 - t} (c - abq x_i) A_i \frac{\partial}{\partial q x_i}
\]

\[
- \sum_{1 \leq i \neq j \leq m} \frac{x_j (c - abq)}{q x_i - t x_j} A_j T_j(A_i) \frac{\partial}{\partial q x_j}
\]

\[
- \frac{(1 - a)(1 - b)t^{m-1}}{(1 - q)^2} \frac{1 - t^m}{1 - t}.
\]

Now we can state the following crucial lemma.

**Lemma 4.2.** We have

\[
(4.3) \quad L_m = \frac{c}{1 - q} (D_1 \varepsilon - \varepsilon D_1) - ab \left( D_1^2 - \frac{1 - t^2}{t(1 - q)} D_2 \right) + \frac{t^{m-1}}{1 - q} \varepsilon
\]

\[
+ \frac{1}{1 - q} \left\{ \frac{2ab(1 - t^m)}{1 - t} - (a + b)t^{m-1} \right\} D_1 - \frac{(1 - a)(1 - b)t^{m-1}}{(1 - q)^2} \frac{1 - t^m}{1 - t}.
\]
Proof. We show that

\[ \frac{1}{1-q}(D_1 \varepsilon - \varepsilon D_1) = \sum_{i=1}^{m} x_i A_i T_i(A_i) \frac{\partial^2}{\partial q x_i^2} + (1-t) \sum_{1 \leq i < j \leq m} x_i x_j \left( A_i T_i(A_j) + A_j T_j(A_i) \right) \frac{\partial^2}{\partial q x_i \partial q x_j} \]

\[ - \frac{1}{1-q} \sum_{i=1}^{m} A_i T_i(A_i) \frac{\partial}{\partial q x_i} - \frac{1-t}{1-q} \sum_{1 \leq i < j \leq m} x_j A_i T_i(A_j) \frac{\partial}{\partial q x_j} \]

(4.4) \quad \frac{1}{1-q} \sum_{i=1}^{m} x_i A_i T_i(A_i) \frac{\partial^2}{\partial q x_i^2} + (1-t) \sum_{1 \leq i < j \leq m} x_i x_j \left( \frac{A_i T_i(A_j)}{q x_i - t x_j} + \frac{A_j T_j(A_i)}{q x_j - t x_i} \right) \frac{\partial^2}{\partial q x_i \partial q x_j} \]

(4.5) \quad \frac{1}{1-q} \sum_{i=1}^{m} q x_i^2 A_i T_i(A_i) \frac{\partial^2}{\partial q x_i^2} + (1-t) \sum_{1 \leq i < j \leq m} x_i x_j \left( \frac{x_i A_i T_i(A_j)}{q x_i - t x_j} + \frac{x_j A_j T_j(A_i)}{q x_j - t x_i} \right) \frac{\partial^2}{\partial q x_i \partial q x_j} \]

One can check (4.3) without difficulty assuming (4.4) and (4.5). It is clear that the coefficient of \( \frac{\partial^2}{\partial q x_i^2} \) in \( D_1 \varepsilon - \varepsilon D_1 \) is \( (1-q)x_i A_i T_i(A_i) \). For the coefficient of \( \frac{\partial^2}{\partial q x_i \partial q x_j} \), it suffices to observe

the coefficient of \( \frac{\partial^2}{\partial q x_i \partial q x_j} \) in \( D_1 \varepsilon - \varepsilon D_1 \)

\[ = x_i A_i T_i(A_j) + x_j A_j T_j(A_i) - x_j A_i T_i(A_j) - x_i A_j T_j(A_i) \]

\[ = (x_i - x_j)(A_i T_i(A_j) - A_j T_j(A_i)) \]

\[ = (x_i - x_j) A_{ij} \left\{ \frac{(tx_i - x_j)(tx_j - qx_i)}{(x_i - x_j)(x_j - qx_i)} - \frac{(tx_j - x_i)(tx_i - qx_j)}{(x_j - x_i)(x_i - qx_j)} \right\} \]

\[ = \frac{A_{ij} (1-q)(1-t)(q-t)x_i x_j(x_i + x_j)}{(q x_i - x_j)(q x_j - x_i)}. \]

and

\[ \frac{A_i T_i(A_j)}{q x_i - t x_j} + \frac{A_j T_j(A_i)}{q x_j - t x_i} \]

\[ = \frac{A_{ij} \left\{ \frac{tx_i - x_j}{(x_i - x_j)(q x_i - x_j)} + \frac{tx_j - x_i}{(x_j - x_i)(q x_j - x_i)} \right\}}{t} \]

\[ = \frac{A_{ij} \left\{ (q-t)(x_i + x_j) \right\}}{t \left( q x_i - x_j \right) \left( q x_j - x_i \right)} \].
We have also

\[ \frac{\partial}{\partial q x_i} (A_i x_j A_j \frac{\partial A_i}{\partial q x_j} - A_i T_i(A_i) - \sum_{1 \leq j \leq m, j \neq i} (x_i - x_j) A_j \frac{\partial A_j}{\partial q x_j} \].

Since

\[ (x_i - x_j) \frac{\partial A_i}{\partial q x_i} = \prod_{k=1, k \neq i,j} \frac{t x_i - x_k}{x_i - x_k} \left( \frac{t x_i - q x_j}{x_i - q x_j} - \frac{t x_i - x_j}{x_i - x_j} \right) \frac{x_i - x_j}{(q - 1)x_j} \]

\[ = \prod_{k=1, k \neq i,j} \frac{t x_i - x_k}{x_i - x_k} \frac{(1 - t)x_i}{x_i - x_i} \frac{q x_j - x_i}{q x_j - x_i} \]

\[ = \frac{(1 - t)x_i}{q x_j - t x_i} T_j(A_i), \]

the proof of (4.4) is complete.

The coefficient of \( \frac{\partial^2}{\partial q x_i \partial q x_j} \) in \( D_1 - \frac{1 - t^2}{t(1 - q)} D_2 \) is clearly as in (4.5). For the coefficient of \( \frac{\partial^2}{\partial q x_i \partial q x_j} \), note that

\[ x_i x_j (A_i T_i(A_j) + A_j T_j(A_i)) \]

\[ = \frac{A_{ij}}{t} \frac{x_i x_j}{x_i - x_j} (q x_j - x_i) (q x_j - x_i) - (q x_j - x_i) (t x_j - x_i) (q x_i - x_j) \]

and

\[ (q x_i - t x_j) (t x_i - x_j) (q x_j - x_i) - (q x_j - t x_i) (t x_j - x_i) (q x_i - x_j) \]

\[ = (q x_i - x_j) + (1 - t)x_j) (t x_i - x_j) (q x_j - x_i) - (q x_j - x_i) (t x_j - x_i) (q x_i - x_j) \]

\[ = (1 + t)(x_i - x_j) (q x_i - x_j) (q x_j - x_i) \]

\[ + (1 - t) \{x_j (q x_j - x_i) (t x_j - x_i) - x_i (q x_i - x_j) (t x_j - x_i) \}. \]

Hence

the coefficient of \( \frac{\partial^2}{\partial q x_i} \partial q x_j \) in \( D_1 - \frac{1 - t^2}{t(1 - q)} D_2 \)

\[ = (1 - t) \frac{A_{ij}}{t} \frac{x_i x_j \{x_j (q x_j - x_i) (t x_j - x_i) - x_i (q x_i - x_j) (t x_j - x_i) \}}{(x_i - x_j) (q x_i - x_j) (q x_j - x_i)} \]

\[ = (1 - t) x_i x_j \left\{ \frac{x_j A_i T_i(A_j)}{q x_j - t x_i} + \frac{x_i A_j T_j(A_i)}{q x_i - t x_j} \right\}. \]
Since

the coefficient of $\partial/\partial_q x_i$ in $D_1^2 = \frac{1 - t^2}{t(1 - q)} D_2$

\[
\frac{1}{1 - q}\left\{\frac{1 - t^m}{1 - t} x_i A_i - qx_i A_i T_i(A_i) - \sum_{1 \leq j \leq m, j \neq i} x_i A_i T_j(A_i) + \frac{1 + t}{t} \sum_{1 \leq j \leq m, j \neq i} x_i A_{ij}\right\},
\]

we conclude the proof of (4.5) from

\[
\sum_{1 \leq j \leq m, j \neq i} \left\{ A_j T_j(A_i) - \frac{1 + t}{t} A_{ij} - \frac{x_i}{qx_j - tx_i} A_j T_j(A_i)\right\}
\]

\[
= \sum_{1 \leq j \leq m, j \neq i} A_j \prod_{k=1, k \neq i,j}^m \frac{tx_i - x_k}{x_i - x_k} \left\{ \frac{tx_i - qx_j}{x_i - qx_j} - \frac{x_j - x_i}{tx_j - x_i} - \frac{1 + t}{t} \frac{x_i}{qx_j - x_i} \right\}
\]

\[
= \sum_{1 \leq j \leq m, j \neq i} A_j \prod_{k=1, k \neq i,j}^m \frac{tx_i - x_k}{x_i - x_k} \frac{tx_j - x_i}{tx_j - x_i}
\]

\[
= - A_i \sum_{1 \leq j \leq m, j \neq i} A_j \frac{x_j - x_i}{tx_j - x_i}
\]

\[
= - \frac{1 - t^{m-1}}{1 - t} A_i
\]

where the last equality follows from (2.10).

4.2. GENERALIZED BINOMIAL COEFFICIENTS.

DEFINITION 4.3. – For any partitions $\lambda$ and $\mu$ of length $\leq m$, the generalized binomial coefficient $\binom{\lambda}{\mu}$ is defined by

\[
\binom{\lambda}{\mu} = \sum_{\nu \in \Lambda_m} \frac{\nu!}{\nu^\lambda \nu^\mu} \binom{\lambda}{\mu_m} \binom{\mu}{\nu}
\]

Remark. – If we put $q = t^\alpha$ and let $t \to 1$, then it is readily seen that $\binom{\lambda}{\mu}_m$ reduces to the generalized binomial coefficient defined by using Jack polynomials [Kan2, p. 1096]. In this case the following theorem has been announced by [L] and proved by [Kan2].

Denote $\mu \subset \lambda$ if $\mu_i \leq \lambda_i$ for every $i$.

THEOREM 4.4. – (1) $\binom{\lambda}{\mu}_m \neq 0$ if and only if $\mu \subset \lambda$ and $|\mu| = |\lambda| - 1$.

(2) the $\binom{\lambda}{\mu}_m$'s are independent of the dimension $m$ in the sense that $\binom{\lambda}{\mu}_r = \binom{\lambda}{\mu}_s$ provided $r, s \leq \ell(\lambda)$.

We leave the proof to Section 6. We write $\binom{\lambda}{\mu}$ dropping the subscript $m$. For each partition $\lambda$, we put $\lambda_{(i)} = (\lambda_1, \ldots, \lambda_{i-1}, \lambda_i+1, \lambda_{i+1}, \ldots)$ and $\lambda^{(i)} = (\lambda_1, \ldots, \lambda_{i-1}, \lambda_i-1, \lambda_{i+1}, \ldots)$ and call them admissible if the parts are in nonincreasing order. We shall write
\(\lambda_{(i,j)} = (\lambda_{(i)})_{(j)}, \lambda^{(i,j)} = (\lambda^{(i)})_{(j)}\). By Theorem 4.4 (1), \(\lambda^{(i)} = 0\) unless \(\lambda = \mu^{(i)}\) (or \(\mu = \lambda^{(i)}\)) for some \(i\). Hence, in view of \(b(\lambda) - b(\lambda^{(i)}) = i - 1\), we have

\[
\varepsilon P_\lambda(x_1, \ldots, x_m) = c_\lambda h_\lambda' \sum_{i=1}^{m} \left( \lambda_{(i)} \right) \frac{t^{1-i}}{c_{\lambda^{(i)}} h_{\lambda^{(i)}}'} P_{\lambda^{(i)}}(x_1, \ldots, x_m)
\]

where we have put

\[c_\lambda = c(\lambda, q, t, m) = \frac{\varepsilon_{m,t}(P_\lambda)}{h_\lambda'} = \frac{(t^m)_{(q,t)}}{h_\lambda'}.\]

The summation in (4.6) is over all \(i\) such that \(\lambda^{(i)}\) is admissible. This convention will be used in all future summations involving \(\lambda_{(i)}\) or \(\lambda^{(i)}\).

**Proposition 4.5.** - The formal series

\[S(x_1, \ldots, x_m) = \sum_\lambda \gamma_\lambda \frac{P_\lambda(x_1, \ldots, x_m)}{h_\gamma'}
\]

satisfies the summed-up equation \(L_m(S) = 0\) if and only if the coefficients \(\gamma_\lambda\) satisfy the following recurrence relations

\[
\frac{1}{1-q} \sum_{i=1}^{m} c_{\lambda^{(i)}} \left( \lambda_{(i)} \right) t^{m+1-2i} (t^{i-1} - q^{\lambda_{(i)}}) \gamma_{\lambda^{(i)}}
\]

\[+ \left\{ -ab \left( e_\lambda^2 \frac{1}{t} \right) + \frac{2ab(1-t^m)}{(1-q)(1-t)} - \frac{(a+b)t^{m-1}}{1-q} \right\} e_\lambda
\]

\[- \frac{(1-a)(1-b)t^{m-1}(1-t^m)}{(1-q)^2(1-t)} \right\} \gamma_\lambda = 0.
\]

**Proof.** - One can easily verify that, using (3.2), (4.1) and (4.6), the left-hand side of (4.7) is the coefficient of \(P_\lambda/h_\lambda'\) in \(L_m(S)\).

Note that for \(r \leq m\) we have

\[
S(x_1, \ldots, x_r) := S(x_1, \ldots, x_r, 0, \ldots, 0) = \sum_\lambda \gamma_\lambda \frac{P_\lambda(x_1, \ldots, x_m)}{h_\gamma'}.
\]

The recurrence relation (4.7) implies the following uniqueness property.

**Corollary 4.6.** - Assume that \((c_{(q,t)}^{(i)})_{\lambda} \neq 0\) for any partition \(\lambda\) of length \(\leq m\). If the formal series (4.8) satisfies \(L_r(S(x_1, \ldots, x_r)) = 0\) for every \(r \leq m\), and \(S(0, \ldots, 0) = 0\), then \(S(x_1, \ldots, x_m) = 0\).
Proof. — Note that the coefficient of \( \gamma(i) \) of (4.7) is not zero because of the assumption \((c)_{\lambda}^{(q,t)} \neq 0\). We prove \( S(x_1, \ldots, x_r) = 0 \) by induction on \( r \), the case \( r = 1 \) being immediate from (4.7). Clearly it suffices to show \( S(x_1, \ldots, x_m) = 0 \) assuming \( S(x_1, \ldots, x_r) = 0 \) for \( r \leq m - 1 \), i.e. \( \gamma_\lambda = 0 \) if \( \ell(\lambda) \leq m - 1 \). For \( \gamma_\lambda \) with \( \kappa_m = 1 \), put \( \lambda = \kappa^{(m)} \) or \( \lambda^{(m)} = \kappa \). Substituting this \( \lambda \) into (4.7) immediately shows that \( \gamma_{\lambda(m)} \) is a linear combination of \( \gamma_\lambda \) and \( \gamma_{\lambda(i)} \), \( i < m \). Thus we find \( \gamma_{\lambda(m)} = 0 \). The general case follows by induction on \( \kappa_m \).

Let us denote by \((2.26)_m \) the system (2.26) to express its dimensional dependence.

**Lemma 4.7.** — If \( S(x_1, \ldots, x_m) \) is a solution of \((2.26)_m \), then \( S(x_1, \ldots, x_r), 1 \leq r \leq m \), is a solution of \((2.26)_r \).

Proof. — Clearly it suffices to prove the case of \( r = m - 1 \). Substitute \( S(x_1, \ldots, x_m) \) into \((2.26)_m \) with \( i \neq m \) and put \( x_m = 0 \). Then one finds that \( S(x_1, \ldots, x_{m-1}) \) satisfies that the system \((2.26)_{m-1} \) multiplied by \( t \).

This lemma implies that if \( S(x_1, \ldots, x_m) \) is a solution of \((2.26)_m \), then \( L_r(S(x_1, \ldots, x_r)) = 0 \) for \( r < m \). Hence by Corollary 4.6 we obtain

**Lemma 4.8.** — Assume that \((c)_{\lambda}^{(q,t)} \neq 0\) for any partition \( \lambda \) of length \( \leq m \). If \( S(x_1, \ldots, x_m) \) is a solution of \((2.26)_m \) and \( S(0, \ldots, 0) = 0 \), then \( S(x_1, \ldots, x_m) = 0 \).

We next provide some formulas for the \( \binom{\lambda}{\mu} \)'s.

**Lemma 4.9.** — We have

\[
\sum_{i=1}^{m} c_{\lambda(i)} \left( \frac{\lambda_i}{\lambda} \right) = \frac{1 - t^m}{(1 - q)(1 - t)} c_\lambda
\]

\[
\sum_{i=1}^{m} q^{|\lambda|} t^{m-i} c_{\lambda(i)} \left( \frac{\lambda_i}{\lambda} \right) = \left\{ \frac{1 - t^m}{(1 - q)(1 - t)} - e_\lambda \right\} t^{m-1} c_\lambda
\]

\[
\sum_{i=1}^{m} (q^{|\lambda|} t^{m-i})^2 c_{\lambda(i)} \left( \frac{\lambda_i}{\lambda} \right) = \left\{ \frac{t^{2m-2}(1 - t^m)}{(1 - q)(1 - t)} - \frac{2t^{m-1}(1 - t^m)}{1 - te_\lambda} \right. \\
\left. + (1 - q)t^{m-1} e_\lambda^2 + (t - 1)(t^{m-1} + t^{m-2})f_\lambda \right\} c_\lambda
\]

Proof. — We first show

\[
\sum_{i=1}^{m} A_i x_i = t^{m-1} m_1
\]

\[
\sum_{i=1}^{m} A_i x_i^2 = t^{m-1} m_{(2)} + t^{m-2}(t - 1)m_{(1,1)}
\]
\(\sum_{1 \leq i \neq j \leq m} A_{ij}(x; t)x_i = \frac{t^{m-1}(1 - t^{m-1})}{1 - t} m_1\) \\
(4.15) \ \ \ \ \ \sum_{1 \leq i < j \leq m} A_{ij}(x; t)x_i x_j = t^{2m-3} m_{(1, 1)}

where \(m_1 = m_{(1)} = x_1 + \cdots + x_m\). Replacing \(z\) with \(1/z\) in (2.7) yields

\[
\prod_{i=1}^{m} \frac{t - zx_i}{1 - zx_i} = (t - 1) \sum_{i=1}^{m} \frac{x_i A_i(x; t)}{1 - zx_i} + t^m.
\]

Differentiating both sides once (resp. twice) with respect to \(z\) and then setting \(z = 0\) gives (4.12) (resp. (4.13)). By (2.10) and (2.7) with \(z = tx_i\), we have

\[
\sum_{j=1, j \neq i}^{m} \prod_{k=1, k \neq i, j}^{m} \frac{tx_j - x_k x_i - x_j}{x_j - x_k} \frac{tx_i - x_j}{tx_i} = \sum_{j=1, j \neq i}^{m} \prod_{k=1, k \neq i, j}^{m} \frac{tx_j - x_k x_i - x_j}{x_j - x_k} \left\{ \frac{1 + t - 1}{t} \frac{x_j}{x_j - tx_i} \right\}
\]

\[
= \frac{1 - t^{m-1}}{t - 1} + \frac{1 - t}{t - 1} \left\{ -t^{m-1} + \prod_{k=1, k \neq i}^{m} \frac{x_k - t^2 x_i}{x_k - tx_i} \right\}
\]

\[
= \frac{1}{t} \left\{ \frac{1 - t^{m}}{1 - t} - \prod_{k=1, k \neq i}^{m} \frac{x_k - t^2 x_i}{x_k - tx_i} \right\}.
\]

Hence we obtain

\[
\sum_{1 \leq i \neq j \leq m} A_{ij}(x; t)x_i = t \sum_{i=1}^{m} A_i x_i \left\{ \sum_{j=1, j \neq i}^{m} \prod_{k=1, k \neq i, j}^{m} \frac{tx_j - x_k x_i - x_j}{x_j - x_k} t x_i - x_j \right\}
\]

\[
= \frac{1 - t^{m}}{t - 1} \sum_{i=1}^{m} A_i x_i - \sum_{i=1}^{m} A_i(x; t^2) x_i - \prod_{i=1}^{m} \frac{x_i}{x_i; q_{\infty}}.
\]

The proof of (4.15) is similar to that of (4.14) and we omit it.

Setting \(a = 0\) in (3.12) yields

\[
\sum_{\lambda} \frac{t^{b(\lambda)}}{h_{\lambda}} P_{\lambda}(x_1, \ldots, x_m) = \prod_{i=1}^{m} \frac{1}{(x_i; q_{\infty})}.
\]
It follows from (2.9) that
\[ \varepsilon \left( \prod_{i=1}^{m} \frac{1}{(x_i; q)_\infty} \right) = \frac{1}{1 - q} \left( \sum_{i=1}^{m} A_i \right) \prod_{i=1}^{m} \frac{1}{(x_i; q)_\infty} = \frac{1 - t^m}{(1 - q)(1 - t)} \prod_{i=1}^{m} \frac{1}{(x_i; q)_\infty}. \]

Hence applying \( \varepsilon \) to both sides of (4.16), we have
\[
\frac{1 - t^m}{(1 - q)(1 - t)} \sum_{\lambda} \frac{t^{b(\lambda)}}{h^*_\lambda} P_\lambda(x_1, \ldots, x_m) = \sum_{\lambda} c_\lambda \left( \sum_{\lambda^{(i)}} \frac{\lambda^{(i)}}{\lambda^{(i)} h^{\lambda^{(i)}*}} \right) \frac{t^{b(\lambda^{(i)})}}{c_{\lambda^{(i)}} h^{\lambda^{(i)}*}} P_{\lambda^{(i)}}.
\]

Equating coefficients of \( P_\lambda \) in both sides yields (4.9).

For the proof of (4.10), note that by (4.12) and (2.9) we see
\[
D_1 \left( \prod_{i=1}^{m} \frac{1}{(x_i; q)_\infty} \right) = \frac{1}{1 - q} \left( \sum_{i=1}^{m} A_i x_i \right) \prod_{i=1}^{m} \frac{1}{(x_i; q)_\infty} = t^{m-1} \frac{m_1}{1 - q \prod_{i=1}^{m} (x_i; q)_\infty}
\]
and
\[ \varepsilon \left( \prod_{i=1}^{m} \frac{1}{(x_i; q)_\infty} \right) = \left\{ \sum_{i=1}^{m} A_i + \frac{1}{1 - q} \sum_{i=1}^{m} T_i(m_1) A_i \right\} \prod_{i=1}^{m} \frac{1}{(x_i; q)_\infty} = \left\{ \frac{1 - t^m}{1 - t} + \frac{1 - t^m}{(1 - q)(1 - t)} - t^{m-1} \right\} m_1 \prod_{i=1}^{m} \frac{1}{(x_i; q)_\infty}, \]
so that we obtain
\[ \varepsilon D_1 \left( \prod_{i=1}^{m} \frac{1}{(x_i; q)_\infty} \right) = \frac{t^{m-1}}{1 - q} \left\{ \frac{1 - t^m}{1 - t} + \frac{1 - t^m}{(1 - q)(1 - t)} - t^{m-1} \right\} m_1 \prod_{i=1}^{m} \frac{1}{(x_i; q)_\infty}. \]

Hence applying \( \varepsilon D_1 \) to both sides of (4.16) gives
\[
\sum_{\lambda} \frac{t^{b(\lambda)}}{h^*_\lambda} \varepsilon P_\lambda(x_1, \ldots, x_m) = \frac{t^{m-1}(1 - t^m)}{(1 - q)(1 - t)} \sum_{\lambda} \frac{t^{b(\lambda)}}{h^*_\lambda} P_\lambda(x_1, \ldots, x_m)
+ \left( \frac{1 - t^m}{(1 - q)(1 - t)} - t^{m-1} \right) \sum_{\lambda} \frac{t^{b(\lambda)}}{h^*_\lambda} P_{\lambda^{(i)}}(x_1, \ldots, x_m).
\]

Substituting (4.6) in the left-hand side and equating coefficients of \( P_\lambda \) of both sides gives (4.10) (use \( e_{\lambda^{(i)}} = e_\lambda + t^{m-1} q^\lambda \) and then (4.9)).
By virtue of (4.14) and (4.15) we can derive

\[ D_2 \left( \prod_{i=1}^{m} \frac{1}{(x_i; q)_{\infty}} \right) = \frac{t^{m-1}}{(1 - q)(1 - t)} \left\{ t^{m-2} m_{(1,1)} - \frac{1 - t^{m-1}}{1 - t} m_1 \right\} \prod_{i=1}^{m} \frac{1}{(x_i; q)_{\infty}}. \]

Also by (4.12) and (4.13) we have

\[ \varepsilon \left( \prod_{i=1}^{m} \frac{m_{(1,1)}}{(x_i; q)_{\infty}} \right) = \left\{ \frac{1 - t^{m-1}}{1 - t} m_1 + \left( \frac{1 - t^m}{(1 - q)(1 - t)} - (t^{m-1} + t^{m-2}) \right) m_{(1,1)} \right\} \]

\[ \times \prod_{i=1}^{m} \frac{1}{(x_i; q)_{\infty}}. \]

Hence applying \( \varepsilon D_2 \) to both sides of (4.16), we get

\[
\sum_{\lambda} \frac{q^{b(\lambda)} f_{\lambda}}{h'_{\lambda}} \varepsilon P_{\lambda}(x) = \frac{t^{m-1}}{(1 - q)(1 - t)} \left\{ t^{m-2} \varepsilon \left( \frac{m_{(1,1)}}{\prod_{i=1}^{m} (x_i; q)_{\infty}} \right) - \frac{1 - t^{m-1}}{1 - t} \varepsilon \left( \frac{m_1}{\prod_{i=1}^{m} (x_i; q)_{\infty}} \right) \right\}
\]

\[
= \frac{t^{m-1}}{(1 - q)(1 - t)} \left\{ t^{m-2} \left( \frac{1 - t^m}{(1 - q)(1 - t)} - (t^{m-1} + t^{m-2}) \right) \prod_{i=1}^{m} \frac{m_{(1,1)}}{(x_i; q)_{\infty}} \right\}
\]

\[
+ \frac{1 - t^{m-1}}{1 - t} \left( t^{m-2} + t^{m-1} - \frac{1 - t^m}{(1 - q)(1 - t)} \right) \prod_{i=1}^{m} \frac{m_1}{(x_i; q)_{\infty}} - \frac{(1 - t^{m-1})(1 - t^m)}{(1 - q)(1 - t)^2} \prod_{i=1}^{m} \frac{1}{(x_i; q)_{\infty}} \right\}
\]

\[
= \left( \frac{1 - t^m}{(1 - q)(1 - t)} - (t^{m-1} + t^{m-2}) \right) \sum_{\lambda} \frac{q^{b(\lambda)} f_{\lambda}}{h'_{\lambda}} P_{\lambda}(x)
\]

\[- \frac{t^{m-1}(1 - t^{m-1})(1 - t^m)}{(1 - q)(1 - t)^3} \sum_{\lambda} \frac{q^{b(\lambda)}}{h'_{\lambda}} P_{\lambda}(x). \]

Substituting (4.6) into the left-hand side and equating coefficients of \( P_{\lambda} \) of both sides yields

\[
\sum_{\lambda} f_{\lambda(i)} c_{\lambda(i)} \left( \frac{\lambda(i)}{\lambda} \right) = \left\{ \left( \frac{1 - t^m}{(1 - q)(1 - t)} - (t^{m-1} + t^{m-2}) \right) f_{\lambda} - \frac{t^{m-1}(1 - t^{m-1})(1 - t^m)}{(1 - q)(1 - t)^3} \right\} c_{\lambda}. \]

Since

\[
f_{\lambda(i)} = f_{\lambda} + \left( (1 - q) c_{\lambda} - \frac{1 - t^m}{1 - t} \right) \frac{t^{m-i} q^{\lambda_i}}{1 - t} + \frac{(t^{m-i} q^{\lambda_i})^2}{1 - t}, \]

one can simplify the left-hand side by using (4.9) and (4.10) and this completes the proof of (4.11).
4.3. q-Difference System. – We now state one of the main results of this paper.

**Theorem 4.10.** The hypergeometric function \( \Phi_1^{(a,t)}(x) \) is the unique solution of the summed-up equation \( L_m(S) = 0 \) subject to the following condition:

(a) \( S(x) \) is a symmetric function in \( x_1, \ldots, x_m \).

(b) \( S(x) \) is analytic at the origin with \( S(0) = 1 \).

(c) \( S(x_1, \ldots, x_r, 0, \ldots, 0) \) is a solution of \( L_r(S) = 0 \) for every \( r \leq m \).

**Proof.** The uniqueness is immediate from Corollary 4.6. Put \( \gamma = (a, b, c) \). Then we see

\[
\gamma_{\text{sym}} = \gamma(\xi) = (a - 1 - q^\gamma a)(t^i - 1 - q^\gamma b)(t^{i-1} - 1 - q^\gamma c)^{-1}.
\]

By virtue of (4.7), the proof that \( L_m(\Phi_1^{(a,t)}) = 0 \) (and also (c)) boils down to show

\[
\frac{1}{1 - q} \sum_{\ell = 1}^{m} e_{\gamma}(\ell - 1 - q^\gamma a)(t^i - 1 - q^\gamma b)(t^{i-1} - 1 - q^\gamma c)^{-1}.
\]

But this is an immediate consequence of Lemma 4.9.

Next we compare the hypergeometric series \( \Phi_1^{(a,t)}(x) \) with the q-Selberg integral \( S_n,m(\alpha, \beta, \gamma, \mu; x; \xi) \) with \( \mu = 1 \) or \( -\gamma \). If \( \mu = 1 \), then \( S_n,m(x) \), being a polynomial, is analytic at the origin. But if \( \mu = -\gamma \), then in general \( q S_n,m(x) \) has poles at \( x_i = \xi_i q^\gamma k, k \in \mathbb{Z} \), so that the origin is an essential singularity. In this case we choose \( \xi = \xi_F = (1, Q, \ldots, Q^{n-1}) \). Then the integral \( q S_n,m(x; \xi_F) \) over \([0, \xi_F]\) is analytic because the integral is done only over the set \( < \xi_F > \) consisting of the points such that \( t_1 = q^\gamma, t_2/t_1 = q^\gamma k_2, \ldots, t_n/t_{n-1} = q^\gamma k_n \) for each \( k_j \in \mathbb{Z}_{\geq 0} \) (this is the so called “\( \alpha \)-stable cycle” in [Ao1]). Combining Theorem 2.3, Lemma 4.8 and Theorem 4.10, we now obtain

**Theorem 4.11.** We have \( q \Phi_1^{(a,t)}(x) = (q^\gamma Q^{-1}x_1, \ldots, q^\gamma Q^{-1}x_m) \) etc.)

(4.17) \( q S_n,m(\alpha, \beta, \gamma; 1; Q^\gamma Q^{-1}x; \xi) = C \cdot 2 \Phi_1^{(Q^\gamma, Q)}(Q^{-n}, Q^{-1}a, Q^{-1}b, Q^{-1}c, x) \)

(4.18) \( q S_n,m(\alpha, \beta, \gamma; 0; Q^\gamma Q^{-1}x; \xi_F) = C_F \cdot 2 \Phi_1^{(Q, Q^\gamma)}(Q^n, Q^{\alpha+n-1}Q^{-1}, Q^{\alpha+\beta+n-1}, x) \)

where \( C = q S_n(\alpha, \beta, \gamma; \xi) \), \( C_F = q S_n(\alpha, \beta, \gamma; \xi_F) \).

The condition that \( \Phi_1^{(a,t)}(x) \) satisfies the system (2.26) is equivalent to an infinite system of polynomial equations in \( q, t, a, b, c \). The formula (4.18) implies that these equations hold when \( t = q^\gamma, a = q^\gamma, b = q^\gamma a, c = q^\gamma b \). Since \( n, \alpha, \beta, \gamma \) are arbitrary, these equations hold for any \( q, t, a, b, c \). Thus we arrive at
THEOREM 4.12. – The hypergeometric series \( \Phi_1^{(q,t)}(a, b; c; x_1, \ldots, x_m) \) is the unique solution of the system (2.26) subject to the following conditions:

(a) \( S(x) \) is a symmetric function in \( x_1, \ldots, x_m \).
(b) \( S(x) \) is analytic at the origin with \( S(0) = 1 \).

5. Consequences

5.1. Integration Formula of Macdonald Polynomials.

Put

\[
q D(\alpha, \beta, \gamma; t) = \prod_{j=1}^{n} t_j^{\alpha+(j-1)(1-2\gamma)} \frac{(qt_j)_{\infty}}{(q^{\beta}t_j)_{\infty}} \prod_{1 \leq i < j \leq n} \frac{(q^{1-\gamma}t_j/t_i)_{\infty}}{(q^{\beta}t_j/t_i)_{\infty}} D(t).
\]

Theorem 4.11 implies the following integration formula.

THEOREM 5.1.

\[
\int_{[0, \infty]} P_{\lambda}(t; q, Q) q D(\alpha, \beta, \gamma; t) \, dt = \frac{q S_{n,0}(\alpha, \beta, \gamma; \xi)(Q^n)_{\lambda}^{(q,Q)}(q^{a+n-1}Q^{-(n-1)})_{\lambda}^{(q,Q)}}{h_{\lambda}(q, Q)(q^{a+\beta+n-1})_{\lambda}^{(q,Q)}}.
\]

Proof. – Replacing \( x \) by \(-q^\beta Q^{-1}x\), \( \lambda \) by \( \lambda' \) and setting \( y = t = (t_1, \ldots, t_n) \), \( (q, t) = (Q, q) \) in (3.8), we have

\[
\prod_{1 \leq i \leq m, 1 \leq j \leq n} (1 - q^\beta Q^{-1}x_it_j) = \sum_{\lambda} (-q^\beta Q^{-1})^{\lambda'} P_{\lambda'}(x; Q, q) P_{\lambda}(t; q, Q).
\]

Substituting this into (4.17) yields

\[
\sum_{\lambda} (-q^\beta Q^{-1})^{\lambda'} P_{\lambda'}(x; Q, q) \int_{[0, \infty]} P_{\lambda}(t; q, Q) q D(\alpha, \beta, \gamma; t) \, dt
\]

\[
= C \cdot 2 \Phi_1^{(Q,q)}(Q^{-n}, q^{-(a+n-1)}Q^{n-1}; q^{-(\alpha+\beta+n-1)}; x)
= C \sum_{\lambda'} \frac{(Q^{-n})^{\lambda'}_{\lambda}^{(Q,q)}(q^{-(\alpha+n-1)}Q^{n-1})_{\lambda}^{(Q,q)}}{(q^{-\alpha+\beta+n-1})_{\lambda}^{(Q,q)} h_{\lambda'}(Q, q)} P_{\lambda'}(x; Q, q).
\]

Note that in general we have

\[
(a)^{(q,t)}_{\lambda} = (-a)^{\lambda}(a^{-1})^{(t,q)}_{\lambda'}, \quad h_{\lambda'}(q, t) = h_{\lambda'}(t, q).
\]
Hence equating the coefficients of $P_\gamma(x; Q, q)$ in (5.1) immediately gives the desired first equality. The second equality is a direct consequence of Theorem 3.3.

We next show Theorem 5.1 implies the integration formula of Kadell [Kad2]. Assume $\gamma = k$, a positive integer. Put $x = \alpha + (n - 1)(1 - 2k), y = \beta$.

**Proposition 5.2.** – Assume $\text{Re}(x) > 0, y \neq 0, -1, -2, \ldots$. We have

$$
\int_{[0,1]^n} P_\lambda(t; q, q^k) \prod_{j=1}^{n} t_j^{\gamma}(q^{s_j})_\infty \prod_{1 \leq i < j \leq n} t_i^{2k}(q^{-1}t_j/t_i)_{2k} \tilde{\omega} = q^{\gamma x + \gamma y + k} \prod_{i=1}^{n} \frac{\Gamma_q(y + (n - i)k + \lambda_i)\Gamma_q(y + (n - i)k)}{\Gamma_q(k + 1)\Gamma_q(x + y + (2n - i - 1)k + \lambda_i)}.
$$

*Proof.* – Observe that $\prod_{1 \leq i < j \leq n} t_i^{2k-1}(q^{-1}t_j/t_i)_{2k-1}$ is antisymmetric. Using Lemma 2.1, (4.17) and Theorem 5.1, we have

$$
\text{LHS of (5.2)} = \int_{\xi \in \mathbb{R}} P_\lambda(t; q, q^k) \prod_{j=1}^{n} t_j^{\gamma}(q^{s_j})_\infty A(\prod_{1 \leq i < j \leq n} (t_i - q^{s_j})_t^{2k-1}(q^{-1}t_j/t_i)_{2k-1} \tilde{\omega} = \frac{(q^k; q^k)_n}{(1 - q^k)^n} \int_{\xi \in \mathbb{R}} P_\lambda(t; q, q^k)_q D(x + (n - 1)(2k - 1), y, k; t)\tilde{\omega} = \frac{(q^k; q^k)_n}{(1 - q^k)^n} \frac{h_\lambda(q^k, q^k)}{\prod_{i,j} (1 - q^{x+y+(2n-i-1)k+j-1})} \prod_{i=1}^{n} \frac{\Gamma_q(y + (n - i)k)\Gamma_q(x + y + (2n - i - 1)k)}{\Gamma_q(k + 1)\Gamma_q(x + y + (2n - i - 1)k)}.
$$

Hence (5.2) is immediate from the formulas

$$
\Gamma_q(x + 1) = \frac{1 - q^x}{1 - q} \Gamma_q(x)
$$

and

$$
A_n = \sum_{j=1}^{n} (\alpha - 2(j - 1)k + n - 1)(j - 1)k = kx \binom{n}{2} + 2k^2 \binom{n}{3}.
$$
Kadell [Kad2] has given a different proof of Proposition 5.2 in a slightly different expression:

\[
\text{LHS of (5.2)} = q^B_n \left( \frac{q^k; q^k}{1 - q^k} \right)_n \prod_{1 \leq i < j \leq n} \frac{(\lambda_i - \lambda_j + (j - i)k)_k}{(1 - q)^k} \\
\times \prod_{i=1}^n \frac{\Gamma_q(x + (n - i)k + \lambda_i) \Gamma_q(y + (n - i)k)}{\Gamma_q(x + y + (2n - i - 1)k + \lambda_i)},
\]

where \( B_n = k \sum_{i=1}^n (i - 1)\lambda_i + kx \binom{n}{2} + 2k^2 \binom{n}{3} \). This is checked by utilizing (3.6):

\[
h_\lambda(q, q^k) = (q^k)_\infty \prod_{i=1}^n (q^\lambda_i + (n - i + 1)k)_\infty^{-1} \prod_{1 \leq i < j \leq n} (q^{\lambda_i - \lambda_j + (j - i)k})_k^{-1}
\]

and observing

\[
(q^n q^k)_\lambda (q^\lambda_i + (n - i + 1)k)_\infty = q^k \sum_{i=1}^n (i - 1)\lambda_i (q^k)_\infty^n.
\]

5.2. Integral representation of \( r \Phi_s^{(q,t)}(a, b; c; x) \).

Let \( a_1, \ldots, a_r \) and \( b_1, \ldots, b_s \) be such that \( (b_j)_\lambda^{(q,t)} \neq 0 \) for any \( j \) and any partition \( \lambda \) of length \( \leq m \). Assume \( m \leq n \), and put

\[
(5.3) \quad r \Phi_s^{(q,t)}(a_1, \ldots, a_r; b_1, \ldots, b_s; x_1, \ldots, x_m; y_1, \ldots, y_n)
\]

\[
= \sum_{\lambda} \prod_{i=1}^s \frac{\prod_{i=1}^r (a_i)_\lambda^{(q,t)}}{\prod_{i=1}^s (b_j)_\lambda^{(q,t)}} \frac{((-1)^{\lambda_i} q^\lambda \lambda_i)^{s+1-r}}{P_\lambda(x; q, t) P_\lambda(1, t, \ldots, t^{n-1})}.
\]

This series converges in a neighborhood of origin only if \( r \leq s + 1 \) and its proof is similar to that of Theorem 3.8.

**Proposition 5.3.** - Let \( a_{r+1} = q^\varepsilon \) and \( b_{s+1} = q^\eta \) and put \( \alpha = \varepsilon + (n - 1)(\gamma - 1), \beta = \eta - \varepsilon - (n - 1)\gamma \). We have

\[
(5.4) \quad r_{s+1} s_n^{(q,t)}(a_1, \ldots, a_{r+1}; b_1, \ldots, b_{s+1}; x_1, \ldots, x_m)
\]

\[
= q S_{n,0}(\alpha, \beta, \gamma; \xi_F)^{-1}
\]

\[
\times \int_{[0, \xi_F]^{\infty}} q \Phi_s^{(q,t)}(a_1, \ldots, a_r; b_1, \ldots, b_s; x_1, \ldots, x_m; y_1, \ldots, y_n)
\]

\[
\times q D(\alpha, \beta, \gamma; y) \omega
\]

provided the right-hand side is convergent.
Proof. - This is an immediate consequence of Theorem 5.1 because
\[
\frac{(q^{\alpha+n-1}Q^{-(n-1)})_{i}^{(q,Q)}}{(q^{\alpha+n-1})_{i}^{(q,Q)}} = \frac{(q^{\gamma})_{i}^{(q,Q)}}{(q^{\gamma})_{i}^{(q,Q)}}.
\]

Proposition 5.4. - Let \( a = q^\delta, b = q^\varepsilon, c = q^n \) and \( \alpha = \varepsilon + (n - 1)(\gamma - 1), \beta = \eta - \varepsilon - (n - 1)\gamma. \) We have

\[
\text{(5.5)} \quad 2\Phi_1^{(q,Q)}(q^{-N}, b; c; q, qQ, \ldots, q^{n-1}) = q^{n(n-(n-1)(1-\gamma))} \prod_{i=1}^{n} \frac{(q^{\beta+(i-1)\gamma})_{N}}{(q^{\alpha+\beta+n-1-(i-1)\gamma})_{N}},
\]

where \( \delta = -N, N \in \mathbb{Z}_{\geq 0} \) and

\[
\text{(5.6)} \quad 2\Phi_1^{(q,Q)}(a, b; c; q, qQ, \ldots, q^{n-1}/ab)
= \prod_{i=1}^{n} \frac{\Gamma_q(\beta - \delta + (i - 1)\gamma)\Gamma_q(\alpha + \beta + n - 1 - (i - 1)\gamma)}{\Gamma_q(\beta + (i - 1)\gamma)\Gamma_q(\alpha + \beta - \delta + n - 1 - (i - 1)\gamma)}
\]

provided the left-hand side is convergent.

Proof. - By Theorem 3.5 we have

\[
\text{(5.7)} \quad 1\Phi_0^{(q,Q)}(q^{-N}; q, qQ, \ldots, q^{n-1}; y_1 \ldots, y_n) = \prod_{j=1}^{n} (q^{-N+1}y_j)_{N},
\]

\[
\text{(5.8)} \quad 1\Phi_0^{(q,Q)}(a; c/ab, cQ/ab, \ldots, cQ^{n-1}/ab; y_1 \ldots, y_n) = \prod_{j=1}^{n} \frac{(q^{\beta}y_j)_{\infty}}{(q^{\beta-\gamma}y_j)_{\infty}}.
\]

Substituting (5.7) (resp. (5.8)) in the right-hand side of (5.4) with \( m = n, r = 1, s = 0 \) and \( a_1 = q^{-N} \) (resp. \( a_1 = a \)) gives

\[
2\Phi_1^{(q,Q)}(q^{-N}, b; c; q, qQ, \ldots, q^{n-1}) = q^{n(n-(n-1)(1-\gamma))} \frac{q^{S_n,0}(\alpha, \beta + N, \gamma; q^{-N}\xi_F)}{q^{S_n,0}(\alpha, \beta, \gamma; \xi_F)}
\]

\[
= q^{n(n-(n-1)(1-\gamma))} \frac{q^{S_n,0}(\alpha, \beta + N, \gamma; \xi_F)}{q^{S_n,0}(\alpha, \beta, \gamma; \xi_F)},
\]

\[
2\Phi_1^{(q,Q)}(a, b; c; q, qQ, \ldots, q^{n-1}/ab) = q^{S_n,0}(\alpha, \beta - \delta, \gamma; \xi_F). \]

Hence the proof follows from the explicit formula (4.19).
6. Proof of Theorem 4.4

6.1. SKewed Macdonald Polynomials. – For any partition $\lambda, \mu, \nu$ define rational functions $f_{\mu^\nu}^\lambda(q, t)$ by

$$f_{\mu^\nu}^\lambda = f_{\mu^\nu}(q, t) = \frac{< P_\lambda, P_\mu P_\nu >}{< P_\lambda, P_\lambda >}.$$  

Equivalently,

$$P_\mu P_\nu = \sum_\lambda f_{\mu^\nu}^\lambda(q, t) P_\lambda.$$  

Clearly $f_{\mu^\nu}^\lambda = 0$ unless $|\lambda| = |\mu| + |\nu|$. Moreover it holds that $f_{\mu^\nu}^\lambda = 0$ unless $\lambda \supset \mu$ and $\lambda \supset \nu$ [Ma2, (4.2)].

If $\lambda, \mu$ are partitions, define skew Macdonald polynomials $P_{\lambda/\mu}$ by

$$P_{\lambda/\mu} = b_{\lambda}^{-1} b_{\mu} \sum_\nu b_{\nu} f_{\mu^\nu}^\lambda(q, t) P_\nu.$$  

Hence $P_{\lambda/\mu} = 0$ unless $\lambda \supset \mu$. Let $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ be two sequences of independent indeterminates. Then we have [Ma2, (4.5)]:

$$P_{\lambda}(x, y) = \sum_\mu P_{\lambda/\mu}(x) P_\mu(y).$$

Put

$$\tilde{f}_{\mu\nu}^\lambda = \frac{b_{\mu} b_{\nu}}{b_{\lambda}} f_{\mu^\nu}^\lambda.$$  

Setting $x = x_m, y = (x_1, \ldots, x_{m-1})$ in (6.3), we get

$$P_{\lambda}(x_1, \ldots, x_m) = \sum_\mu \left( \sum_r \tilde{f}_{\mu^\nu}^\lambda \right) x_m^r P_\mu(x_1, \ldots, x_{m-1}).$$

If $\lambda \supset \mu$ then the skew shape $\lambda/\mu$ (regarded as a difference $\lambda - \mu$ of diagrams) is called a horizontal $r$-strip (resp. vertical $r$-strip) if $|\lambda/\mu| = r$ and no two distinct squares of $\lambda/\mu$ lie in the same column (resp. row). Then $f_{\mu^\nu}^\lambda \neq 0$ (resp. $f_{\mu^\nu}^\lambda \neq 0$) if and only if $\lambda \supset \mu$ and $\lambda/\mu$ is a horizontal (resp. vertical) $r$-strip [Ma2, (4.8)]. Moreover they can be explicitly evaluated as follows. For each square $s$ and each partition $\lambda$, define

$$b_{\lambda}(s) = b_{\lambda}(s; q, t) = \frac{1 - q^{a(s)l(s) + 1}}{1 - q^{a(s)+1l(s)}}$$
if \( s \in \lambda \), and \( b_\lambda(s) = 1 \) if \( s \not\in \lambda \). If \( S \) is any set of squares (contained in the diagram of \( \lambda \) or not), put

\[
b_\lambda(S) = \prod_{s \in S} b_\lambda(s).
\]

(6.6)

Now let \( \lambda, \mu \) be partitions such that \( \lambda \supset \mu \) and \( \lambda/\mu \) is a horizontal \( r \)-strip. Let \( C_{\lambda/\mu} \) (resp. \( R_{\lambda/\mu} \)) denote the union of the columns (resp. rows) that contain squares of \( \lambda/\mu \). Then [Ma2, (5.12)]:

(6.7)

\[
f^\lambda_\mu = b^{-1}_r b_\lambda(C_{\lambda/\mu})/b_\mu(C_{\lambda/\mu}).
\]

Observe that

(6.8)

\[
f^\lambda_{\mu(1)} = b(1) b_\mu(R_{\lambda/\mu}) = 1 - t b_\mu(R_{\lambda/\mu}).
\]

If \( \lambda, \mu \) be partitions such that \( \lambda \supset \mu \) and \( \lambda/\mu \) is a vertical \( r \)-strip, then applying the duality theorem [Ma2, (3.5)] (cf. [Ma3, Chap.6, (7.9)]) to (6.7), we obtain

(6.9)

\[
f^\lambda_{\mu(1')} = b_\lambda(\bar{R}_{\lambda/\mu})/b_\mu(\bar{R}_{\lambda/\mu}),
\]

where \( \bar{R}_{\lambda/\mu} \) denotes the union of rows that do not contain squares of \( \lambda/\mu \).

6.2. LEMMAS. - Put

(6.10)

\[
\binom{\lambda}{\mu}_m = \prod_{s \in \lambda} (1 - q^{a(s) t m - t'(s)}) \left( \prod_{s \in \mu} (1 - q^{a(s) t m - t'(s)}) \right)^{-1} h_\mu h^{-1}_\lambda \binom{\lambda}{\mu}_m,
\]

so that from Definition 4.3 we have

(6.11)

\[
e P_\lambda(x_1, \ldots, x_m) = \sum_{|\mu|=|\lambda|-1} \binom{\lambda}{\mu}_m P_\mu(x_1, \ldots, x_m).
\]

LEMMA 6.1. - Let \( \lambda \) and \( \mu \) be partitions of \( \ell(\lambda) \leq m \) and \( \ell(\mu) \leq m - 1 \). We have

\[
\binom{\lambda}{\mu}_m = t \binom{\lambda}{\mu}_{m-1} + f^\lambda_{\mu(1)}, \text{ if } \ell(\lambda) \leq m - 1,
\]

\[
\binom{\lambda}{\mu}_m = f^\lambda_{\mu(1)}, \text{ if } \ell(\lambda) = m.
\]
Proof. Setting $x_m = 0$ in (6.11) yields

$$t e^x(x_1, \ldots, x_{m-1}) + \frac{\partial P_\lambda(x_1, \ldots, x_m)}{\partial x_m} \bigg|_{x_m = 0} = \sum_{\mu \in [\lambda]} \binom{\lambda}{\mu}(\lambda)_{m} \mu \cdot P_\mu(x_1, \ldots, x_{m-1}),$$

in which we see by (6.4) that

$$\frac{\partial P_\lambda(x_1, \ldots, x_m)}{\partial x_m} \bigg|_{x_m = 0} = \sum_{[\mu] = [\lambda] - 1} f^{\lambda}_{\mu_1} P_\mu(x_1, \ldots, x_{m-1}).$$

Hence equating the coefficients of $P_\mu(x_1, \ldots, x_{m-1})$ in both sides gives the desired formulas.

For each partition $\lambda$, define $\lambda^* = (\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_\ell(\lambda) - 1)$. One can readily derive from Theorem 3.1 that if $\ell(\lambda) = m$, then

$$P_\lambda(x_1, \ldots, x_m) = \left( \prod_{i=1}^{m} x_i \right) \mu_{\lambda^*}(x_1, \ldots, x_m).$$

**Lemma 6.2.** For partitions $\lambda$ and $\mu$ of length $m$, we have

$$\binom{\lambda}{\mu}_m = q \binom{\lambda}{\mu^*}_m + f^{\lambda}_{\mu_1(1^{m-1})}.$$

**Proof.** It follows from (6.12) that

$$\prod_{i=1}^{m} x_i \mu_{\lambda^*}(x_1, \ldots, x_m) = \sum_{\mu \in [\lambda]} \binom{\lambda}{\mu}_m \mu_{\lambda^*}(x_1, \ldots, x_m).$$

We assert in general that

$$\prod_{i=1}^{m} x_i \mu_{\lambda^*}(x_1, \ldots, x_{m-1}) = \sum_{\mu \in [\lambda]} \binom{\lambda}{\mu}_m \mu_{\lambda^*}(x_1, \ldots, x_{m-1}).$$

In fact

$$\sum_{i=1}^{m} A_i \frac{\partial}{\partial x_i} e_r(x_1, \ldots, x_m) = \sum_{i=1}^{m} A_i e_{r-1}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m)$$

$$= \sum_{i=1}^{m} A_i e_{r-1}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m) - x_i e_{r-2}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m)$$

$$= \frac{1 - t^{m-r+1}}{1 - t} e_{r-1}(x_1, \ldots, x_m),$$

$$= \frac{1 - t^{m-r+1}}{1 - t} e_{r-1}(x_1, \ldots, x_m).$$
Substituting (6.14) with \( r = m \) into (6.13) and equating the coefficients of \( P_\mu(x_1, \ldots, x_m) \) of both sides gives the desired formula.

**Lemma 6.3.** We have

\[
\sum_{\lambda, \mu} f^\lambda_{\nu(1^r)} \left( \frac{\lambda}{\mu} \right)_m P_\mu(x_1, \ldots, x_m) = \left( \frac{1 - t^m}{1 - t} + q e_\nu \right) e_{r-1} P_\nu(x_1, \ldots, x_m) - D_1 \left( e_{r-1} P_\nu(x_1, \ldots, x_m) \right) + e_r P_\nu(x_1, \ldots, x_m).
\]

**Proof.** The left-hand side is nothing but \( \varepsilon(e_r P_\nu(x_1, \ldots, x_m)) \). On the other hand using (6.14) and that

\[
T_{q,x_i} e_r(x_1, \ldots, x_m) = e_r(x_1, \ldots, x_m) + (q - 1)x_i e_{r-1}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m)
\]

we have

\[
\varepsilon(e_r P_\nu(x_1, \ldots, x_m)) = (\varepsilon e_r) P_\nu(x_1, \ldots, x_m) + \sum_{i=1}^m A_i T_{q,x_i}(e_r) \frac{\partial}{\partial q x_i} P_\nu(x_1, \ldots, x_m)
\]

\[
= \frac{1 - t^{m-r+1}}{1 - t} e_{r-1} P_\nu(x_1, \ldots, x_m) + e_r P_\nu(x_1, \ldots, x_m) + q e_{r-1} D_1 P_\nu(x_1, \ldots, x_m)
\]

\[
- \sum_{i=1}^m A_i x_i T_{q,x_i}(e_{r-1}) \frac{\partial}{\partial q x_i} P_\nu(x_1, \ldots, x_m).
\]

Hence the formula (6.15) is immediate from

\[
D_1 \left( e_{r-1} P_\nu(x_1, \ldots, x_m) \right) = \frac{t^{m-r+1} - t^m}{1 - t} e_{r-1} P_\nu(x_1, \ldots, x_m)
\]

\[
+ \sum_{i=1}^m A_i x_i T_{q,x_i}(e_{r-1}) \frac{\partial}{\partial q x_i} P_\nu(x_1, \ldots, x_m).
\]

**Lemma 6.4.**

\[
\sum_{\lambda, \mu} f^\lambda_{\nu(2)} \left( \frac{\lambda}{\mu} \right)_m P_\mu(x_1, \ldots, x_m) = \frac{1}{1 - qt} \left\{ (1 + q)(1 - t^m) - (1 - q^2)e_1 P_\nu(x_1, \ldots, x_m)
\right.
\]

\[
+ \frac{t(1 - q^2)}{1 - qt} D_1 \left( e_1 P_\nu(x_1, \ldots, x_m) \right) + P_{(2)} e P_\nu(x_1, \ldots, x_m).
\]
Proof. – One can readily deduce from Theorem 3.1 that
\[ P_2 = m_2 + \frac{(1 + q)(1 - t)}{1 - qt} m_{(1,1)}, \]
so
\[
\frac{\partial P_2}{\partial q x_i} = \frac{t(1 - q^2)}{1 - qt} x_i + \frac{(1 + q)(1 - t)}{1 - qt} e_1,
\]
from which we get
\[
\epsilon P_2 = \frac{(1 + q)(1 - qt^m)}{1 - qt} e_1.
\]
We have
\[
\epsilon(P_2P_\nu) = \epsilon(P_2)P_\nu + P_2\epsilon(P_\nu) + (q - 1) \sum_{i=1}^{m} A_i x_i \frac{\partial P_\nu}{\partial q} \frac{\partial P_2}{\partial q x_i}
\]
and by (6.16) with \( r = 2 \) that
\[
(q - 1) \sum_{i=1}^{m} A_i x_i^2 \frac{\partial P_\nu}{\partial q x_i} = D_1(e_1 P_\nu) - (t^{m-1} + e_\nu)e_1 P_\nu.
\]
Substituting (6.18) and (6.19) into (6.20) and then applying (6.21) yields (6.17).

6.3. Now we turn to the proof of Theorem 4.4. We shall prove a stronger assertion: For any partitions \( \lambda \) and \( \mu \) of length \( \leq m \), we have
\[
\lambda \mu \]
which is, in view of (6.8) and (6.10), equivalent to
\[
\lambda \mu \]
Here note that \( R_{\lambda/\mu} \) is the \( i \)-th row of \( \lambda \). We shall denote the \( i \)-row (resp. \( j \)-th column) of \( \lambda \) by \( R_{\lambda,i} \) (resp. \( C_{\lambda,j} \)) and write \( b_\lambda(R_i) \) (resp. \( b_\lambda(C_i) \)) for \( b_\lambda(R_{\lambda,i}) \) (resp. \( b_\lambda(C_{\lambda,i}) \)). Note that \( b_\lambda(R_{\kappa/i}) = b_\lambda(R_i) \) provided \( \kappa \supset \lambda \). (6.23) is rewritten as
\[
\lambda \mu \]
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We prove (6.24) by induction on the dimension \( m \). The case \( m = 1 \) is easy to check: Put \( \lambda = (r), \mu = (r-1) \). Then clearly \( \binom{\lambda}{\mu} = (1-q^r)/(1-q) \) holds. On the other hand we see

\[
b_\lambda(R_1) = \frac{(t;q)_r}{(q;q)_r}, \quad b_\mu(R_1) = \frac{(t;q)_{r-1}}{(q;q)_{r-1}},
\]

and therefore (6.24) follows at once. We assume that (6.24) holds in the dimensions \( \leq m - 1 \). This implies that (6.24) holds in the case \( \ell(\lambda) \leq m - 1 \). In fact by Lemma 6.1 we see that \( \binom{\lambda}{\mu} = 0 \) unless \( \mu \subset \lambda \). Moreover if \( \mu = \lambda^{(i)} \) and \( \ell(\lambda) \leq m - 1 \), then by means of (6.8) we have

\[
\binom{\lambda}{\mu} = t \frac{1 - q^{\lambda_i - 1} t^{m-i}}{1 - q} \frac{b_\mu(R_i)}{b_\lambda(R_i)} + \frac{1 - t b_\mu(R_i)}{1 - q b_\lambda(R_i)} = \frac{1 - q^{\lambda_i - 1} t^{m-i+1}}{1 - q} \frac{b_\mu(R_i)}{b_\lambda(R_i)}.
\]

If \( \ell(\lambda) = m \), then \( \mu = \lambda^{(m)} \) and \( \lambda_m = 1 \). Hence (6.24) is immediate from Lemma 6.1 also in this case.

Next suppose \( \ell(\mu) = m \) and \( \ell(\lambda) = m \) and that (6.24) holds for \( \lambda_* \) and \( \mu_* \). Then Lemma 6.2 implies that \( \binom{\lambda}{\mu} = 0 \) unless \( \mu \subset \lambda \). If \( \mu = \lambda^{(i)} \), then by (6.9) we have

\[
(6.25)
\binom{\lambda}{\mu} = q \frac{1 - q^{\lambda_i - 2} t^{m-i+1}}{1 - q} \frac{b_{\mu_*}(R_i)}{b_\lambda(R_i)} + \frac{b_\mu(R_{\mu*/\lambda_*})}{b_\lambda(R_{\mu*/\lambda_*})}.
\]

Observe that

\[
b_{\mu_*}(R_i) = b_\mu(R_i) b_{\mu}((i,1))^{-1} = \frac{1 - q^{\lambda_i - 1} t^{m-i}}{1 - q^{\lambda_i - 2} t^{m-i+1}} b_\mu(R_i)
\]

\[
b_{\lambda_*}(R_i) = b_\lambda(R_i) b_{\lambda}((i,1))^{-1} = \frac{1 - q^{\lambda_i} t^{m-i}}{1 - q^{\lambda_i - 1} t^{m-i+1}} b_\lambda(R_i)
\]

and

\[
b_\mu(R_{\mu*/\lambda_*}) = b_\mu(R_i), \quad b_\lambda(R_{\mu*/\lambda_*}) = b_\lambda(R_i).
\]

Substituting these into (6.25) gives (6.24). Iterating this argument, we see that the case \( \ell(\lambda) = \ell(\mu) = m \) reduces to the case \( \ell(\mu) \leq m - 1 \) (which we have just proved) or the case \( \ell(\mu) = m \) and \( \ell(\lambda) \leq m - 1 \).

It now remains to show that \( \binom{\lambda}{\mu} = 0 \) when \( \ell(\lambda) \leq m - 1 \) and \( \ell(\mu) = m \). We prove this by induction on \( \ell(\lambda) \) and for fixed \( \ell(\lambda) \) on \( \lambda_{\ell(\lambda)} \), the case \( \lambda = (1) \) being obvious.
Note first that \( \left( \frac{\lambda}{\mu} \right)_m = 0 \) for \( \lambda \) of \( \ell(\lambda) \leq p - 1, p \leq m - 1 \) implies \( \left( \frac{\lambda}{\mu} \right)_m = 0 \) for \( \lambda \) of \( \ell(\lambda) \leq p \) and \( \lambda_p = 1 \). This follows from (6.15) by setting \( r = 1, \nu = \lambda(p) \) (so that \( \ell(\nu) = p - 1 \)) and equating the coefficients of \( P_\mu \) of both sides. So we have reduced the proof to showing \( \left( \frac{\lambda(p)}{\mu} \right)_m = 0 \) for any partition \( \lambda \) of \( |\lambda| = |\mu| \) and \( \ell(\lambda) = p \leq m - 1 \) provided that \( \left( \frac{\kappa}{\mu} \right)_m = 0 \) when \( \ell(\kappa) \leq p - 1 \) or \( \ell(\kappa) = p \) and \( \kappa_p \leq \lambda_p \).

We divide the proof into several parts treating different cases. First we assume \( p \leq m - 2 \) and derive necessary equalities from Lemma 6.3 and Lemma 6.4. We have

\[
\left( \frac{\lambda(p)}{\mu} \right)_m = 0
\]

provided \( i \leq p - 1 \). This follows immediately from (6.15) if we set \( r = 1, \nu = \lambda(p) \) and compare the coefficients of \( P_\mu \) using induction hypothesis. Setting \( r = 1 \) (resp. \( r = 2 \)) and \( \nu = \lambda(p) \) (resp. \( \nu = \lambda(p+1) \)) in (6.15) and comparing coefficients of \( P_\mu \) using induction hypothesis and (6.26), we find that

\[
f_{\lambda(p)}^{\lambda(1)} \left( \frac{\lambda(p)}{\mu} \right)_m + f_{\lambda(p)}^{\lambda(p+1)} \left( \frac{\lambda(p+1)}{\mu} \right)_m = \sum_i f_{\lambda(i)}^{\lambda(i)} \left( \frac{\lambda}{\lambda(i)} \right)_m
\]

\[
f_{\lambda(p+1)}^{\lambda(p+1)} \left( \frac{\lambda(p+1)}{\mu} \right)_m + f_{\lambda(p+1)}^{\lambda(p+1,p+2)} \left( \frac{\lambda(p+2)}{\mu} \right)_m + f_{\lambda(p+1)}^{\lambda(p+1,p+1)} \left( \frac{\lambda(p+1)}{\mu} \right)_m = \sum_i f_{\lambda(p+1)}^{\lambda(p+1)} \left( \frac{\lambda(p+1)}{\lambda(p+1)} \right)_m
\]

\[
f_{\lambda(p+1)}^{\lambda(p+1)} \left( \frac{\lambda(p+1)}{\mu} \right)_m + f_{\lambda(p+1)}^{\lambda(p+1,p+2)} \left( \frac{\lambda(p+2)}{\mu} \right)_m = \left( \frac{1 - t^m}{1 - t} + q e_{\lambda(p)} - e_\mu \right) f_{\lambda(p)}^{\lambda(p)} \left( \frac{\lambda(p)}{\mu} \right)_m + \sum_i f_{\lambda(p)}^{\lambda(p)} \left( \frac{\lambda(p)}{\lambda(p)} \right)_m.
\]

Similarly, setting \( \nu = \lambda(p) \) in (6.17) gives

\[
f_{\lambda(p)}^{\lambda(p)} \left( \frac{\lambda(p)}{\mu} \right)_m + f_{\lambda(p)}^{\lambda(p+1)} \left( \frac{\lambda(p+1)}{\mu} \right)_m + f_{\lambda(p)}^{\lambda(p+1,p+1)} \left( \frac{\lambda(p+1,p+1)}{\mu} \right)_m
\]

\[
= \frac{1}{1 - qt} \left\{ (1 - q^2) (t e_\mu - e_{\lambda(p)}) + (1 + q) (1 - t^m) \right\} f_{\lambda(p)}^{\lambda(p)} \left( \frac{\lambda(p)}{\mu} \right)_m + \sum_i f_{\lambda(p)}^{\lambda(p)} \left( \frac{\lambda(p)}{\lambda(p)} \right)_m.
\]
Here note that, because of induction hypothesis, the generalized binomial coefficients appearing in the right-hand sides of (6.27)-(6.30) are given by (6.24). So we regard these equalities as equations of unknowns \( \binom{\lambda(p)}{\mu} \), \( \binom{\lambda(p+1)}{\mu} \), \( \binom{\lambda(p)}{\mu} \), and \( \binom{\lambda(p+1, p+2)}{\mu} \).

**Remark.** If \( \lambda_p = 1 \) (resp. \( \lambda_p = 2 \)), we understand \( \binom{\lambda(p)}{\mu} \), and \( \binom{\lambda(p)}{\mu} \) (resp. \( \binom{\lambda(p+1, p+1)}{\mu} \)) to be zero and the following argument should be modified accordingly. We leave this task to the reader and assume henceforth that \( \lambda_p \geq 3 \).

**Case 1.** \(-p \leq m-2 \) and \( \mu \not\in \lambda(p+1, p+2) \). Observe that the right-hand sides of (6.27)-(6.30) are all vanishing. Hence, for the proof of \( \binom{\lambda(p)}{\mu} = 0 \), it suffices to show that the determinant of coefficient matrix of equations is not identically zero. For this purpose, set \( t = 1 \), then we have in general

\[
b_{\lambda}(s) = \frac{1 - q^{a(s)}}{1 - q^{a(s)+1}}.
\]

Hence by (6.8) and (6.9) we find that the coefficients appearing in the equations (6.27)-(6.29) are all equal to one. Also by (6.7) we have

\[
b^{-1}_{(1)} f_{\lambda(p)(2)} |_{t=1} = b^{-1}_{(1)} f_{\lambda(p+1, p+1)} |_{t=1} = 1
\]

and

\[
b^{-1}_{(1)} f_{\lambda(p+1)(2)} |_{t=1} = (b_{(1)} b_{(2)})^{-1} b_{\lambda(p+1)} (C_{\lambda(p+1), 1} \cup C_{\lambda(p), \lambda_p}) |_{t=1}
\]

\[
= (b_{(1)} b_{(2)})^{-1} \frac{1 - q^{a_p-1} - 1 - q^{a_p-1}}{1 - q^{a_p-2}}
\]

\[
= ((1 - q)(1 - q^{a_p})(1 - q^{a_p-2}))^{-1}(1 - q^2)(1 - q^{a_p-1})^2,
\]

which we denote by \( C(q) \). Therefore the determinant of the coefficient matrix (we multiply the equation (6.30) by \( b_{(1)}^{-1} \)) at \( t = 1 \) is

\[
\begin{vmatrix}
  f_{\lambda(p)} & f_{\lambda(p+1)} & 0 & 0 \\
  0 & f_{\lambda(p)} & f_{\lambda(p+1, p+1)} & f_{\lambda(p+1, p+2)} \\
  0 & f_{\lambda(p+1)} & f_{\lambda(p+1, p+1)} & 0 \\
  b_{(1)}^{-1} f_{\lambda(p)} & b_{(1)}^{-1} f_{\lambda(p+1)} & b_{(1)}^{-1} f_{\lambda(p+1, p+1)} & 0 \\
\end{vmatrix}_{t=1}
\]
as desired.

**Case 2.** \( p = m - 2 \) and \( \mu \subseteq \lambda^{(p)}_{(p+1,p+2)}. \)** It necessarily follows that \( \mu = \lambda^{(p,r)}_{(p+1,p+2)} \) for some \( r \leq p \), and therefore the right-hand side of (6.27) is clearly zero. We see also

\[
\text{RHS of (6.29)} = f_{\lambda^{(p,r)}_{(p+1,p+2)}}(\frac{\lambda^{(p)}}{\lambda^{(p,r)}_{(p+1,p+2)}}).
\]

Hence it suffices to show that

\[
(6.31) \quad f_{\lambda^{(p)}_{(p+1,p+2)}}(\lambda^{(p)}_{(p+1,p+2)}) \mid_{m=1} = f_{\lambda^{(p)}_{(p+1,p+2)}}(\lambda^{(p)}_{(p+1,p+2)}).
\]

By (6.24) we have

\[
\left( \lambda^{(p)}_{(p+1,p+2)} \right)^{-1} \left( \lambda^{(p+1,p+2)}_{(p+1,p+2)} \right) = b_{\lambda^{(p)}_{(p+1,p+2)}}(R_r) b_{\lambda^{(p)}_{(p+1,p+2)}}(R_r) b_{\lambda^{(p)}_{(p+1,p+2)}}(R_r)^{-1} b_{\lambda^{(p)}_{(p+1,p+2)}}(R_r)
\]

\[= b_{\lambda^{(p+1,p+2)}_{(p+1,p+2)}}((r,1))^{-1} b_{\lambda^{(p+1,p+2)}_{(p+1,p+2)}}((r,1))^{-1} b_{\lambda^{(p+1,p+2)}_{(p+1,p+2)}}((r,1)).
\]

On the other hand by (6.9) we see also

\[
f_{\lambda^{(p)}_{(p+1,p+2)}} = b_{\lambda^{(p)}_{(p+1,p+2)}}(C_{\lambda^{(p)}_{(p+1,p+2)}}) b_{\lambda^{(p)}_{(p+1,p+2)}}(C_{\lambda^{(p)}_{(p+1,p+2)}}),
\]

\[
f_{\lambda^{(p)}_{(p+1,p+2)}} = b_{\lambda^{(p)}_{(p+1,p+2)}}(C_{\lambda^{(p+1,p+2)}}) b_{\lambda^{(p+1,p+2)}}(C_{\lambda^{(p+1,p+2)}}),
\]

so that

\[
\left( f_{\lambda^{(p)}_{(p+1,p+2)}} \right) \mid_{(p+1,p+2)}^{-1} f_{\lambda^{(p+1,p+2)}}(C_{\lambda^{(p+1,p+2)}}) = b_{\lambda^{(p+1,p+2)}}((r,1))^{-1} b_{\lambda^{(p+1,p+2)}}((r,1))^{-1} b_{\lambda^{(p+1,p+2)}}((r,1)).
\]

This completes the proof of (6.31).

**Case 3.** \( p = m - 1 \) and \( \mu \not\subseteq \lambda^{(p+1)}_{(p+1)}. \)** In this case by (6.15) with \( r = 1 \) (resp. \( r = 2 \)) and \( \nu = \lambda \) (resp. \( \nu = \lambda^{(p)}_{(p+1)} \)) and induction hypothesis we have for some \( s < p \)

\[
(6.32) \quad f_{\lambda^{(p)}_{(p+1,p+2)}}(\lambda^{(p)}_{(p+1,p+2)}) = 0
\]
(6.33) \[ f_{\lambda(p+1)}^{\lambda(p+1)} \left( \lambda(p+1) \right)_{m} + f_{\lambda(p+1)}^{\lambda(p+1)} \left( \lambda(p+1) \right)_{m} = \sum_{i} f_{\lambda(p+1)}^{\lambda(p+1)} \left( \lambda(p+1) \right)_{m}. \]

Observe that, as \( \mu \not\in \lambda(p+1) \) is assumed, \( f_{\lambda(p+1)}^{\lambda(p+1)} \) does not vanish only if \( \mu \) is of the form

(6.34) \[ \mu = \lambda^{(p,r)} \], \( r < p, r \neq s \),

and \( i = r \). If not, then one can readily conclude from Lemma 6.2 and induction hypothesis that \( f_{\lambda(p+1)}^{\lambda(p+1)} \) \( m \) = 0. Hence \( f_{\lambda(p+1)}^{\lambda(p+1)} \) \( m \) = 0 follows from (6.33), so that we obtain \( \lambda(p+1) \) \( m \) = 0 from (6.32).

We now assume (6.34). It clearly suffices to show

(6.35) \[ f_{\lambda(p+1)}^{\lambda(p+1)} \left( \lambda(p+1) \right)_{m} = f_{\lambda(p+1)}^{\lambda(p+1)} \left( \lambda(p+1) \right)_{m}. \]

By (6.24) we have

(6.36) \[ \left( \lambda(p+1) \right)_{m} = \lambda(p+r) \left( \lambda(p+r) \right)_{m} = b_{\lambda(p+r)} \left( R \right)^{-1} b_{\lambda(p+r)} \left( R \right)^{-1} b_{\lambda(p)} \left( R \right) \]

\[ = b_{\lambda(p+r)} \left( S \right)^{-1} b_{\lambda(p)} \left( S \right)^{-1} b_{\lambda(p)}(S), \]

where \( S = (r, 1) \cup (r, \lambda_s + 1) \) if \( r < s \) and \( = (r, 1) \) if \( r > s \). On the other hand by (6.9) we have also

\[ f_{\lambda(p+1)}^{\lambda(p+1)} = b_{\lambda(p)} \left( C_1 \cup C_{\lambda_s+1} \right)^{-1} b_{\lambda(p)} \left( C_1 \cup C_{\lambda_s+1} \right) \]

\[ f_{\lambda(p+1)}^{\lambda(p+1)} = b_{\lambda(p,r)} \left( C_1 \cup C_{\lambda_s+1} \right)^{-1} b_{\lambda(p,r)} \left( C_1 \cup C_{\lambda_s+1} \right), \]

where \( C_1 = C_1 \setminus \{s, 1\} \). Observe that

\[ b_{\lambda(p,r)} \left( C_1 \cup C_{\lambda_s+1} \right)^{-1} b_{\lambda(p)} \left( C_1 \cup C_{\lambda_s+1} \right) = b_{\lambda(p,r)}(S)^{-1} b_{\lambda(p)}(S) \]

\[ b_{\lambda(p)} \left( C_1 \cup C_{\lambda_s+1} \right)^{-1} b_{\lambda(p,r)} \left( C_1 \cup C_{\lambda_s+1} \right) = b_{\lambda(p,r)}(S)^{-1} b_{\lambda(p)}(S), \]

to get

\[ \left( f_{\lambda(p+1)}^{\lambda(p+1)} \right)^{-1} \left( f_{\lambda(p+1)}^{\lambda(p+1)} \right) = \text{RHS of } (6.36). \]

This completes the proof of (6.35).
Case 4. \( p = m - 1 \) and \( \mu \subset \lambda_{(p+1)} \). So \( \mu = \lambda_{(p+1)} \) for some \( r < p + 1 \) and \( \mu_m = 1 \). One can readily derive from (6.15) with \( r = 1 \) and \( \nu = \lambda \) and induction hypothesis that

\[
 f_{\lambda (1)}^{\lambda (p)} \left( \frac{\lambda (p)}{\lambda (r)} \right)_m + f_{\lambda (1)}^{\lambda (p+1)} \left( \frac{\lambda (p+1)}{\lambda (r)} \right)_m = f_{\lambda (1)}^{\lambda (r)} \left( \frac{\lambda (r)}{\lambda (r)} \right)_m.
\]

So it remains only to show

\[
 f_{\lambda (1)}^{\lambda (p+1)} \left( \frac{\lambda (p+1)}{\lambda (r)} \right)_m = f_{\lambda (1)}^{\lambda (r)} \left( \frac{\lambda (r)}{\lambda (r)} \right)_m.
\]

This is concluded, as in the previous cases, from (6.9) and (6.24): It holds that

\[
 \left( \frac{\lambda (r)}{\lambda (r)} \right)_m^{-1} \left( \frac{\lambda (p+1)}{\lambda (p+1)} \right)_m = \left( f_{\lambda (1)}^{\lambda (p+1)} \right)^{-1} f_{\lambda (1)}^{\lambda (p+1)} \left( \frac{\lambda (r)}{\lambda (r)} \right)_m.
\]

\[
 = b_{\lambda (p+1)}((r,1))^{-1} b_{\lambda (r)}((r,1))^{-1} b_{\lambda (p+1)}((r,1)).
\]

We have completed the proof of Theorem 4.4.

Appendix A. Convergence of the integral

We show that the integral \( q_{S_m}^{\infty m} (\alpha, \beta, \gamma, \mu; x_1, \ldots, x_m; \xi) \) converges under the conditions (C_1), (C_2). It is immediate that, if \( (aq^s)_\infty \) has no pole at any \( s \in \mathbb{Z} \), then

\[
 \frac{\left| (aq^s)_\infty \right|}{\left| (bq^s)_\infty \right|} \leq \begin{cases} 
 M_1, & s \geq 0, \\
 M_2 |a/b|^{-s}, & s < 0,
\end{cases}
\]

where \( M_1 = \max_{s \geq 0} \left| (aq^s)_\infty \right| / \left| (bq^s)_\infty \right|, M_2 = |(a)_\infty / (b)_\infty| \max_{s \geq 0} |(a^{-1})_s / (b^{-1})_s| \). Using this, for \( t_j = \xi q^{s_j} \), one has

\[
 \left| \frac{(q^{1-}\gamma t_j/t_i)_\infty (1 - t_j/t_i)}{(q^\gamma t_j/t_i)_\infty} \right| \leq \begin{cases} 
 C\text{te.}, & s_j - s_i \geq 0, \\
 C\text{te.} |q^{2\gamma}|^{s_j - s_i}, & s_j - s_i < 0,
\end{cases}
\]

\[
 \left| \frac{(qt_j)_\infty}{(q^\gamma t_j)_\infty} \right| \leq \begin{cases} 
 C\text{te.}, & s_j \geq 0, \\
 C\text{te.} |q^{\beta - 1}|^{s_j}, & s_j < 0,
\end{cases}
\]

\[
 \prod_{i=1}^m \left| \frac{(x_i t_j)_\infty}{(q^\mu x_i t_j)_\infty} \right| \leq \begin{cases} 
 C\text{te.}, & s_j \geq 0, \\
 C\text{te.} |q^{\mu}|^{s_j}, & s_j < 0.
\end{cases}
\]

For \( s \in \mathbb{Z} \), we put

\[
 a_s = \begin{cases} 
 1, & s \geq 0, \\
 q^{\beta - 1 + m\mu}, & s < 0.
\end{cases}
\]

Case Re \( \gamma \geq 0 \). - So \( |q^s| \leq 1 \) and it follows from the inequality above that

\[
 \left| \frac{(q^{1-}\gamma t_j/t_i)_\infty (1 - t_j/t_i)}{(q^\gamma t_j/t_i)_\infty} \right| \leq C\text{te.} |q^{2\gamma}|^{-|s_j|} |s_i|.
\]
Hence

\[ |\Phi_0(\xi_1 q^{s_1}, \ldots, \xi_n q^{s_n})| \leq Cte. \prod_{j=1}^{n} |q^{(\alpha+n-1-2(j-1)\gamma)s_j-2(n-1)\gamma s_j} a_{s_j}|. \]

The condition (C₂) in the case \( \operatorname{Re} \gamma \geq 0 \) is equivalent to

\[ \sum_{s=0}^{\infty} |q^{(\alpha+n-1-4(n-1)\gamma)s} s^2 + \sum_{s=-1}^{\infty} |q^{(\alpha+n-1+\beta-1+m\mu+2(n-1)\gamma)s}| < \infty. \]

This clearly implies the convergence of the series

\[ (A.1) \int_{[0, \xi_{\infty}]} \Phi_0(t) \tilde{\omega} = (1 - q)^n \sum_{s_1, \ldots, s_n \in \mathbb{Z}} \Phi_0(\xi_1 q^{s_1}, \ldots, \xi_n q^{s_n}). \]

Case \( \operatorname{Re} \gamma < 0 \). We see

\[ \left| \frac{(q^{1-\gamma} t_1^{\infty} / t_i^{\infty})(1 - t_j / t_i)}{(q^{\gamma} t_j / t_i^{\infty})} \right| \leq Cte., \]

so that

\[ |\Phi_0(\xi_1 q^{s_1}, \ldots, \xi_n q^{s_n})| \leq Cte. \prod_{j=1}^{n} |q^{(\alpha+n-1-2(j-1)\gamma)s_j} a_{s_j}|. \]

The condition (C₂) in the case \( \operatorname{Re} \gamma < 0 \) is equivalent to

\[ \sum_{s=0}^{\infty} |q^{(\alpha+n-1)s}| + \sum_{s=-1}^{\infty} |q^{(\alpha+n-1+\beta-1+m\mu+2(n-1)\gamma)s}| < \infty. \]

This implies the convergence of the series (A.1).

When \( \xi = \xi_F \), as the summation in (A.1) is only over \( 0 \leq s_1 \leq s_2 \leq \ldots \leq s_n \), one can relax the condition (C₂) into

\[ \operatorname{Re} \alpha + n - 1 > \max\{2(n - 1)\operatorname{Re} \gamma, 0\}. \]

Finally we note that, as \( |P_\lambda(x)| \leq Cte.(|x_1| + \ldots + |x_m|)^{\lambda} \), the integral

\[ \int_{[0, \xi_{\infty}]} P_\lambda(t; q, Q) D(\alpha, \beta, \gamma; t) \tilde{\omega} \]

converges provided that

\[ \operatorname{Re} \alpha + n - 1 > 4(n - 1)\max\{\operatorname{Re} \gamma, 0\}, \]

\[ \operatorname{Re} \alpha + n - 1 + \operatorname{Re} \beta - 1 + m\operatorname{Re} \mu + |\lambda| < -2(n - 1)\operatorname{Re} \gamma. \]
Appendix B. Evaluation of \( qS_{n,0}(\alpha, \beta, \gamma; \xi) \)

We begin by showing that

\[
qS_{n,0}(\alpha + 1, \beta, \gamma; \xi) = q^{n(n-1)/2} \prod_{j=1}^{n} \frac{1 - q^{\alpha + n - 1 - (n+j-2)\gamma}}{1 - q^{\alpha + \beta + n - 1 - (n-j)\gamma}} qS_{n,0}(\alpha, \beta, \gamma; \xi).
\]

Indeed this is a consequence of the case \( m = 1 \) of (4.17): Equating the coefficients of \( x^n \) of both sides gives

\[
(-q^\beta Q^{-1})^n qS_{n,0}(\alpha + 1, \beta, \gamma; \xi) = \frac{(Q^{-n}; Q)_n}{(Q; Q)_n} \frac{(q^{-\alpha+n-1}Q^{n-1}; Q)_n}{(q^{-\alpha+\beta+n-1}; Q)_n} qS_{n,0}(\alpha, \beta, \gamma; \xi),
\]

and this leads to (B.1) immediately. We first prove (1.6) by induction on \( n \), the case \( n = 1 \) being nothing but the \( q \)-beta integral formula \([\text{Asl}], [\text{GR}, \text{p. 19}]\). We proceed as in \([\text{Kadi}]\). Set

\[
qPr_n(\alpha, \beta, \gamma) = q^{\frac{n(n-1)}{2}\alpha \gamma} \prod_{j=1}^{n} \frac{\Gamma_q(\alpha + n - 1 - (n+j-2)\gamma)}{\Gamma_q(\alpha + \beta + n - 1 - (n-j)\gamma)}
\]

By (B.1) and the equation \( \Gamma_q(\alpha + 1) = (1 - q^\alpha)/(1 - q) \Gamma_q(\alpha) \), we see

\[
qQ_n(\alpha + 1, \beta, \gamma) = qQ_n(\alpha, \beta, \gamma).
\]

We extend \( qQ_n(\alpha, \beta, \gamma) \) to all \( \alpha \) by this equation.

We assume that \( \gamma \) is real and \( \gamma > 0 \), \( \Re \alpha + (n - 1)(1 - 2\gamma) > 0 \). We show that

\[
qQ_n(\alpha, \beta, \gamma) = qC_n \prod_{j=1}^{n} \frac{\Gamma_q(\beta + (j-1)\gamma) \Gamma_q(j\gamma)}{\Gamma_q(\gamma)}
\]

where \( C_n = \sum_{j=1}^{n} (-2(j-1)\gamma + n - 1)(j-1)\gamma \). Rewriting \( qS_{n,0}(\alpha, \beta, \gamma; \xi_F) \) as iterated integral, we have

\[
qS_{n,0}(\alpha, \beta, \gamma; \xi_F) = \int_{[0, q^{(n-1)\gamma}]} t_1^{\alpha + (n-1)(1-2\gamma)} (qt_1)^{\infty} (qt_n)^{\infty} \left[ \prod_{j=1}^{n-1} t_j^{\alpha + (j-1)(1-2\gamma)} \frac{(qt_j)^{\infty}}{(qt_j)^{\infty}} \right] \times \prod_{1 \leq i < j \leq n} \frac{(q^{1-\gamma}t_j/t_i)^{\infty}}{(qt_j/t_i)^{\infty}} D(t) \frac{dqt_j}{t_j} \wedge \cdots \wedge \frac{dqt_{n-1}}{t_{n-1}} \frac{dqt_n}{t_n}.
\]

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Set $\alpha_0 = (n - 1)(2\gamma - 1)$. Observe that

$$\lim_{\alpha \to \alpha_0} \frac{1 - q^{\alpha - \alpha_0}}{1 - q} \int_{[0, q^{(n - 1)\gamma}]} t^{\alpha - \alpha_0} (q \beta \gamma_\infty) \frac{d q \beta \gamma}{t_\gamma} = 1.$$  

Hence we obtain

\begin{equation}
(B.6) \quad \lim_{\alpha \to \alpha_0} \frac{1 - q^{\alpha - \alpha_0}}{1 - q} q S_{n, 0}(\alpha, \beta, \gamma; \xi_F) = q S_{n-1, 0}(\alpha_0 + 1, \beta, \gamma; \xi_F)
\end{equation}

where

$$A_{n-1} = \sum_{j=1}^{n-1} (\alpha_0 - 2(j - 1)\gamma + n - 1)(j - 1)\gamma = C_n + \frac{n(n - 1)}{2} \alpha_0 \gamma.$$  

On the other hand by (B.2) and (B.3) we have

\begin{equation}
(B.7) \quad \lim_{\alpha \to \alpha_0} \frac{1 - q^{\alpha - \alpha_0}}{1 - q} q S_{n, 0}(\alpha, \beta, \gamma; \xi_F) = \lim_{\alpha \to \alpha_0} \frac{1 - q^{\alpha - \alpha_0}}{1 - q} q P_{n}(\alpha, \beta, \gamma) q Q_n(\alpha, \beta, \gamma)
\end{equation}

Equating (B.6) and (B.7) yields

\begin{equation}
(B.8) \quad q Q_n(\alpha_0, \beta, \gamma) = q^{-\frac{n(n - 1)}{2} \alpha_0 \gamma + A_{n-1}} \Gamma_q((n - j + 1)\gamma) \prod_{j=1}^{n-1} \frac{\Gamma_q((n - j)\gamma)}{\Gamma_q((n - (j - 1))\gamma)} q Q_n(\alpha_0 + 1, \beta, \gamma)
\end{equation}

This establishes (B.5) when $\alpha = \alpha_0 + k, k \in \mathbb{Z}$. One can show that $q Q_n(\alpha_0, \beta, \gamma)$ is bounded in the strip $\alpha_0 + 1 \leq \text{Re} \alpha \leq \alpha_0 + 2$ in the exactly same way as [Kad1]. Hence it is bounded for all $\alpha$ by (B.4), and thus (B.5) follows from Liouville’s theorem that a
bounded entire function is constant. The restriction that \( \gamma \) is real and positive is easily removed by analytic continuation.

Now we turn to the proof of (1.5). Set

\[(B.9) \quad qS_{n,0}(\alpha, \beta, \gamma; \xi) = c(\xi) qS_{n,0}(\alpha, \beta, \gamma; \xi_F).\]

By the definition of Jackson integral, we see for any \( j \) that

\[(B.10) \quad T_{q,\xi_j} c(\xi) = c(\xi).\]

Observe that \( c(\xi) \prod_{j=1}^{n} \xi_j^{2(j-1)\gamma-\alpha} \) is meromorphic on \((\mathbb{C}^*)^n\) with simple poles in \( \{ \xi \mid \prod_{1 \leq i < j \leq n} \vartheta(q^{\gamma} \xi_j/\xi_i) \prod_{j=1}^{n} \vartheta(q^{\gamma} \xi_j) = 0 \} \). We assert that \( c(\xi) \) is vanishing on \( \{ \xi \mid \xi_j = q^{k}, k \in \mathbb{Z} \} \). By (B.10), it suffices to show that if \( \xi_i = \xi_j, \ i < j \), then \( qS_{n,0}(\alpha, \beta, \gamma; \xi) = 0 \). Let \( \sigma_{ij} \) be the transposition of \( i \) and \( j \). We have

\[
U_{\sigma_{ij}}(\xi) = \prod_{1 \leq k < l \leq n} \frac{\xi_{\sigma_{ij}(k)}}{\xi_{\sigma_{ij}(l)}} \left( \frac{\xi_{i}}{\xi_{k}} \right)^{2\gamma-1} \frac{\vartheta(q^{\gamma} \xi_i/\xi_k)}{\vartheta(q^{1-\gamma} \xi_i/\xi_k)}
\]

\[
= \frac{\vartheta(q^{\gamma})}{\vartheta(q^{1-\gamma})} \prod_{i < k < j} \left( \frac{\xi_{i}}{\xi_{k}} \right)^{2\gamma-1} \frac{\vartheta(q^{\gamma} \xi_i/\xi_k)}{\vartheta(q^{1-\gamma} \xi_i/\xi_k)} \prod_{i < l < j} \left( \frac{\xi_{l}}{\xi_{i}} \right)^{2\gamma-1} \frac{\vartheta(q^{\gamma} \xi_l/\xi_i)}{\vartheta(q^{1-\gamma} \xi_l/\xi_i)}
\]

\[= 1.\]

We are now able to write

\[
c(\xi) = \prod_{j=1}^{n} \xi_{j}^{\alpha-2(j-1)\gamma} \frac{1}{\vartheta(q^{\gamma} \xi_j)} \prod_{1 \leq i < j \leq n} \frac{\vartheta(\xi_j/\xi_i)}{\vartheta(q^{\gamma} \xi_j/\xi_i)} f(\xi)
\]

where \( f(\xi) \) is holomorphic on \((\mathbb{C}^*)^n\). One can derive from (B.10) that

\[
T_{q,\xi_j} f(\xi) = -\frac{1}{q^{\alpha + \beta - (n-1)\gamma} \xi_j} f(\xi).
\]
Therefore we conclude that
\[ f(\xi) = Cte. \prod_{j=1}^{n} \vartheta(q^{\alpha + \beta - (n-1)\gamma} \xi_j). \]

Since \( c(\xi_F) = 1 \), we arrive at
\[ c(\xi) = q^{\sum_{j=1}^{n} (2(j-1)\gamma - \alpha)} \prod_{j=1}^{n} \xi_j^{\alpha - 2(j-1)\gamma} \frac{\vartheta(\xi_j q^{\alpha + \beta - (n-1)\gamma}) \vartheta(q^{\beta + (j-1)\gamma}) \vartheta(q^{\gamma})}{\vartheta(q^{\alpha + \beta - (n-j)\gamma}) \vartheta(q^{\xi_j q^{\beta}}) \vartheta(q^{\gamma})} \prod_{1 \leq i < j \leq n} \frac{\vartheta(\xi_j / \xi_i)}{\vartheta(q^{\gamma} \xi_j / \xi_i)}. \]

Combining this with (B.9) and (1.6) completes the proof of (1.5).

**REFERENCES**


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