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BIFURCATION OF CONTRACTING SINGULAR CYCLES *

BY RAFAEL LABARCA

Dedicated to the memory of Professor R. Chuaqui (R.I.P.)

ABSTRACT. - The aim of this work is to continue the analysis of a new mechanism, the singular cycles, through which a vector field, depending on parameter, may evolve when the parameter varies from a vector field exhibiting simple dynamics into one having non-trivial dynamics. Specifically, if we start with a Morse - Smale vector field and move through a generic one-parameter family of vector fields to a contracting singular cycle and beyond, we reach a region filled up mostly with hyperbolic flows. In fact, the Lebesgue measure of parameter values corresponding to non Axiom A flows is zero. Moreover we provide a complete description of the bifurcation set that appear in these families.

1. Introduction

The aim of this work is to continue the analysis of a new mechanism, the singular cycles, introduced in [3] and [1] through which a vector field, depending on parameters, may evolve when the parameter varies from a vector field exhibiting simple dynamics into one having non-trivial dynamics.

Let $M$ be a $C^\infty$, $m$-dimensional, compact, connected, boundaryless, riemannian manifold. Let $X \in X^r(M)$ be a $C^r$-vector field on $M$.

**Definition 1.** - A cycle for the vector field $X$ is a compact, invariant set $\Gamma \subset M$ formed by:

1. a finite number of singularities and periodic orbits $\Gamma_0 = \{\sigma_0, \ldots, \sigma_n\}$;
2. the complement $\Gamma_1 = (\Gamma \setminus \Gamma_0)$ is a set of non-periodic regular trajectories of the vector field $X$ that satisfies:
   
   (CC)$_1$ for any trajectory $\gamma \subset \Gamma_1$, there exists $0 \leq i \leq n$ such that $\omega(\gamma) \subset \sigma_{(i+1)\mod(n+1)}$ and $\alpha(\gamma) \subset \sigma_i$;

   (CC)$_2$ given $0 \leq i \leq n$ there exists a trajectory $\gamma \subset \Gamma_1$ such that $\omega(\gamma) \subset \sigma_{(i+1)\mod(n+1)}$ and $\alpha(\gamma) \subset \sigma_i$.

Here $\omega(\gamma)$ (respectively $\alpha(\gamma)$) denotes the $\omega$-limit set (respectively the $\alpha$-limit set) of the trajectory $\gamma$.

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A cycle will be called *singular* if it contains a singularity; *hyperbolic* if all the critical elements in $\Gamma$ are hyperbolic.

In this article we will deal with a 3-dimensional, hyperbolic, singular cycle, $\Gamma \subset M^3$, that contains a unique singularity, $\sigma_0(X)$, and periodic orbits $\sigma_1(X), \cdots, \sigma_n(X), n \geq 1$ (Fig. 1).

![Fig. 1](image-url)

We will assume the following regularity conditions:

1. $\Gamma = \{\sigma_0(X), \gamma_0(X), \sigma_1(X), \gamma_1(X), \gamma_2(X), \cdots, \sigma_n(X), \gamma_n(X), \gamma_{n+1}(X)\}$, where $W^u_i = W^u_{\sigma_i(X)}$ intersects transversally $W^s_{(i+1) \mod(n+1)}$ along the orbits $\gamma_1^1(X) \cup \gamma_2^2(X)$, $i = 1, \cdots, n$.

We let $\sigma_0(Y), \sigma_1(Y), \cdots, \sigma_n(Y)$ denote, respectively, the analytic continuation of $\sigma_0(X), \sigma_1(X), \cdots, \sigma_n(X)$; for any $Y \in \mathcal{U}_X$. Here $\mathcal{U}_X$ denotes a small neighborhood of $X$ in $X^r(M^3)$ with the usual $C^r$-topology, $r \geq 3$.

2. For any $Y \in \mathcal{U}_X$, the eigenvalues of $D_{\sigma_0(Y)}(Y) : T_{\sigma_0(Y)}(M^3) \to T_{\sigma_0(Y)}(M^3)$ are real numbers $-\lambda_3(Y) < -\lambda_1(Y) < 0 < \lambda_2(Y)$ and satisfy a $k$-Sternberg condition, $k$ big enough to guarantee that we have $C^2$-linearizing coordinates which depend $C^2$ on $Y \in \mathcal{U}_X$ in a neighborhood of $\sigma_0(Y)$.

3. For every $p \in \gamma_0(X)$ and every invariant manifold of $X$, passing through $\sigma_0(X)$ and $p, W(\sigma_0(X))$, and tangent (at $\sigma_0(X)$) to the space spanned by the eigenvectors associated to $-\lambda_1(X)$ and $\lambda_2(X)$, we have $T_p(W(\sigma_0(X))) + T_p(W^s_{\sigma_i(X)}) = T_p(M^3)$.

4. $\Gamma$ is isolated: that is, there exists an open set $U \supset \Gamma$ such that $\cap_t X_t(U) = \Gamma$; here $X_t$ denotes the flow defined by the vector field $X$.

5. Let $Q_i \subset M^3, 1 \leq i \leq n$, be a transversal section at $q_i(Y) \in \sigma_i(Y)$. We let $P_i(Y) : V_i \subset Q_i \to Q_i$ denote the first return map defined in a neighborhood of $q_i(Y)$, any $Y \in \mathcal{U}_X$. We assume the eigenvalues of $D_{q_i}P_i : T_{q_i}(V_i) \to T_{q_i}(Q_i)$ are real numbers.
and satisfy a $k$-Sternberg condition, $k$ big enough to guarantee that we have $C^3$-linearizing coordinates which depend $C^2$ on $Y \in U_X$ in a neighborhood of $q_i(Y)$;

(6) The number $\alpha(Y) = \frac{\lambda_1(Y)}{\lambda_2(Y)}$ is greater than one and

$$\beta(Y) = \frac{\lambda_3(Y)}{\lambda_2(Y)} > \alpha(Y) + 2.$$ 

A cycle $\Gamma$ as above is called a contracting singular cycle.

We let $\Gamma(Y, U) \subset M$ denote the set $\cap_i Y_i(U)$, for $Y \in U_X$ (that is, the maximal invariant set in the neighborhood $U$ for the vector field $Y$).

We let $\gamma_0(Y), \gamma_1(Y), \gamma_2(Y), \cdots; \gamma_1^n(Y), \gamma_2^n(Y)$ denote, respectively, the analytic continuation of the trajectories $\gamma_0(X), \cdots, \gamma_2^n(X)$ for any $Y \in U_X$. These trajectories are included in the unstable manifolds $W^u(\sigma_0(Y)), \cdots, W^u(\sigma_n(Y))$ respectively.

Comment: It is easy to see that there exists a codimension-one submanifold, $\mathcal{N} \subset X^r(M)$, containing $X$ such that:

(i) $Y \in \mathcal{N}$ implies $\Gamma(Y, U) = \{\sigma_0(Y), \gamma_0(Y), \cdots, \gamma_2^n(Y)\}$;

(ii) $(U_X \setminus \mathcal{N})$ has two connected components and one of them, which is denoted $U^-$, is such that $Y \in U^-$ implies $\Gamma(Y, U) = \{\sigma_0(Y), \sigma_1(Y), \gamma_1(Y), \gamma_2(Y), \cdots, \sigma_n(Y), \gamma_1^n(Y), \gamma_2^n(Y)\}$; and

(iii) Bifurcations for the maximal invariant set $\Gamma(Y, U)$ may appear only for $Y \in U^+ = (U_X \setminus (\mathcal{N} \cup U^-))$.

$U^+_H$ is defined to be the set of $Y \in U^+$ such that $\Gamma(Y, U)$ consists of $\Gamma_0$, a transitive hyperbolic set and a denumerable number of isolated hyperbolic periodic orbit, and $U^+_A$ as the set of $Y \in U^+$ such that $\Gamma(Y, U)$ consists of $\sigma_0(Y)$, a transitive hyperbolic set, a hyperbolic attracting periodic orbit (which is contained in the closure of the trajectory $\gamma_0(Y)$), and a denumerable number of isolated hyperbolic periodic orbit.

Under the above conditions we have the following:

**Theorem 1.** a) $U^+ \setminus (U^+_H \cup U^+_A)$ is laminated by codimension-one $C^1$-submanifolds of the following type:

$a_1)$ those laminas that present a saddle-node or a flip bifurcation for periodic orbits;

$a_2)$ those laminas that present a contracting singular cycle;

$a_3)$ those laminas that present a homoclinic behavior for the singularity; and

$a_4)$ those laminas that present a recurrent behavior for the analytic continuation of the trajectory $\gamma_0(X)$.

Moreover all elements in the same lamina have the same dynamics in the neighborhood $U$ (that is, given a lamina $L \subset U^+ \setminus (U^+_H \cup U^+_A)$ and $Y_1, Y_2 \in L$, there exists a homeomorphism $h : U \to U$ that is a topological equivalence between $Y_1$ and $Y_2$).

b) Any $Y \in U^+_H \cup U^+_A$ is structurally stable.

c) For any $Y \in (U^+ \setminus (U^+_H \cup U^+_A))$, $\Gamma(Y, U)$ decomposed into a chain recurrent expansive set, a denumerable number of isolated hyperbolic periodic orbits plus the closure of the trajectory $\gamma_0(Y)$.

Now let $\{X_\mu\} \subset U_X$ be a one-parameter family of vector fields such that $X_{\mu=0} \in \mathcal{N}$ and $\{X_\mu\}$ is transversal to $\mathcal{N}$ at $\mu = 0$. 

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THEOREM 2. - There exists $v = \nu(X_\mu) > 0$ such that:

$$m(\{\mu; 0 \leq \mu \leq v, X_\mu \notin (U^+_H \cup U^+_A)\}) = 0$$

(here $m(A)$ denotes the Lebesgue measure of the set $A \subset \mathbb{R}$).

Following [3] we may now state a corollary for Theorem 1.

COROLLARY. - Let $\{Y_\mu\}$ be another one-parameter family transversal to $N$ at $\mu = 0$. There exists a reparametrization $\rho : [0, \nu(X_\mu)] \to [0, \nu(Y_\mu)]$ and, for each $\mu \in [0, \nu(X_\mu)]$, a homeomorphism $h_\mu : U \to U$ that is a topological equivalence between $X_\mu|_U$ and $Y_\rho(\mu)|_U$.

Remark. - a) A particular case of Theorem 2 was proven by Pacifico and Rovella in [2]. In their case, $\Gamma$ is given by $\{\sigma_0(X), \gamma_0(X), \sigma_1(X), \gamma_1(X)\}$ and the associated first return map preserves orientation. A more general case of the Pacifico-Rovella result was proven by San Martin in [8].

The techniques they use to prove their result do not apply in our case.

b) For the case $\alpha(X) < 1$ (an expanding singular cycle), theorems 1 and 2 and the above Corollary 1 were proven by Bamon, Labarca, Mañé and Pacífico in [1].

c) The main difference between the unfolding of expanding and contracting singular cycles is the following: the unfolding of contracting singular cycles must have saddle-node and flip bifurcations whereas the unfolding of the expanding singular cycles does not.

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2. Proof of Theorem 1

This Chapter is organized in the following way: In section 2.1 we make the necessary change of coordinates to obtain a simpler form of the First Return Map. Section 2.2 is devoted to give a characterization of the elements in $U^+_H \cup U^+_A$. Sections 2.3 - 2.11 are devoted to the study of the one dimensional dynamics associated to a contracting singular cycle. In particular we obtain the proof of Theorem 1.

2.1. Change of coordinates and the First Return Map

Let $X \in \mathcal{X}^{r}(M^3)$ be a vector field having a contracting singular cycle, $\Gamma$, with isolated neighborhood $U \subset M$. For the sake of simplicity we will assume $\Gamma$ contains a unique periodic orbit, and later on in Section III.5 we will make comments on the general case. Here $\Gamma$ is the union of a singularity $\sigma_0 = \sigma_0(X)$, a periodic orbit $\sigma_1 = \sigma_1(X)$, an orbit $\gamma_0 = \gamma_0(X) \subset W^u_{\sigma_0}$ of nontransversal intersection between $W^u_{\sigma_0}$ and $W^s_{\sigma_1}$, and two orbits of transversal intersection between $W^u_{\sigma_1}$ and $W^s_{\sigma_1}, \gamma_1 = \gamma_1(X)$ and $\gamma_2 = \gamma_2(X)$. 

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Let $Q$ be a cross section to the flow $X$ at $q \in \sigma_1$ parametrized by \{$(x, y)/|x|, |y| \leq 1$\} and satisfying $W^a_{\sigma_1} \supset \{(x, 0); |x| \leq 1\}$ and $W^\alpha_{\sigma_1} \supset \{(0, y); |y| \leq 1\}$.

Let $p = p(X)$ be the first intersection between $\gamma_0$ and $Q$. Then $p = (x_0, 0) = (x_0(X), 0)$ and we assume $x_0 > 0$. It is clear that a first return map, $F = F(X)$, is defined on a subset of $Q$. Moreover if $q_1 = (0, y_1) = (0, y_1(X))$ and $q_2 = (0, y_2) = (0, y_2(X))$ are such that their $u$-limit set is $\sigma_0$, then there are horizontal strips $R_1 = R_1(X)$ and $R_2 = R_2(X)$ such that $F$ is defined on $R_1 \cup R_2$. Here a horizontal strip is a closed set $C \subset Q$ bounded (in $Q$) by two disjoint continuous curves connecting the vertical sides of $Q$, \{$(1, y)/|y| \leq 1$, and \{$(1, y)/|y| \leq 1$\}.

Since $\Gamma$ is isolated, we have that $\Gamma \cap Q \subset \{(x, y)/y \geq 0\}$ and that:

$$F(R_1 \cup R_2) \subset \{(x, y)/y \leq 0\}$$

(See Fig. 2).

If $Y \in \mathcal{X}$ is near $X$, then $W^r(\sigma_1(Y))$ intersects $Q$ at a curve $c(Y)$, and the first intersection of $W^u(\sigma_0(Y))$ with $Q$ is a point $p(Y)$. Note that both $c(Y)$ and $p(Y)$ vary smoothly with $Y$. The implicit function theorem on Banach spaces implies that the condition $p(Y) \in c(Y)$ defines a $C^2$-codimension one submanifold, $\mathcal{N}$, in a neighborhood of $X, U \subset \mathcal{X}$, such that $(U \setminus \mathcal{N})$ has two connected components: one of them, which we denote by $U^-$, is characterized by $p(Y) \in Q$ and lies below $c(Y)$; we let $U^+$ denote the other component.

Clearly, $Y \in U^-$ implies $\Gamma(Y, U) = \{\sigma_0(Y), \sigma_1(Y), \gamma^1_1(Y), \gamma^2_1(Y)\}$ and hence the dynamics of the vector field $Y$ in $U$ is simple.

If $Y \in U^+$, then $\sigma_1(Y)$ has transversal homoclinic orbits and therefore $Y$ does not have simple dynamics in $U$. As before we note that there exists a first return map $F_Y$ defined on a subset of $Q$, every $Y \in U^+$.

Since $\Gamma(Y, U)$ is the saturation of $\Gamma(Y, U) \cap Q$ by the flow $Y_t$, and $\Gamma(Y, U) \cap Q$ is the maximal invariant set of $F_Y$, it is necessary to describe the dynamics of $F_Y$ to understand...
the dynamics of $Y$ on $\Gamma(Y, U)$. For this we choose coordinates $(x, y)$ on $Q$, that depend $C^2$ on $Y$, such that:

(i) $\{(x, 0)/|x| \leq 1\} \subset W^s(\sigma_1(Y))$;
(ii) $\{(0, y)/|y| \leq 1\} \subset W^u(\sigma_2(Y))$;
(iii) $\Gamma(Y, U) \cap Q \subset Q^+ = \{(x, y)/x \geq 0, y \geq 0\}$; and
(iv) the analytic continuation of the point $p = p(X) = \gamma_0(X) \cap Q$ is a point $p(Y) = (x(Y), y(Y))$, with $0 < x(Y) < 1$.

Note that $Y \in U^+$ if and only if $y(Y) > 0$.

Moreover $\Gamma(Y, U) \not\subset \{\sigma_0(Y), \sigma_1(Y), \gamma_1(Y), \gamma_2(Y)\}$ if and only if $y(Y) \geq 0$.

For $Y \in U$ such that $y(Y) > 0$, let $q_1(Y) = (0, y_1(Y))$ (resp., $q_2(Y) = (0, y_2(Y))$) be the analytic continuation of the point $q_1$ (resp., $q_2$). Since $\omega(q_i(Y)) = \sigma_0(Y)$ and $\alpha(q_i(Y)) = \sigma_1(Y)$, $i = 1, 2$, there are horizontal strips $R^i_Y \ni q_i(Y)$ such that the positive orbits of points at $R^i_Y$ first pass near $\sigma_0(Y)$ and afterwards return to $Q$. On the other hand, the positive orbits of points at a horizontal strip $R_Y$ containing $W^s(\sigma_1(Y)) \cap Q$ goes around the closed orbit $\sigma_1(Y)$ and then return to $Q$ (see Fig. 3).

Fig. 3

Therefore $F_Y$ is defined on $R_Y \cup R^1_Y \cup R^2_Y$, and the restriction of $F_Y$ to $R_Y$ coincides with the Poincaré map, $P_Y$, associated to $\sigma_1(Y)$. We further assume $P_Y$ is linear on $R_Y$.

Let $\xi_Y > 1$ and $\tau_Y < 1$ be the eigenvalues of $DP_Y(0, 0)$. We have $R^1_Y = \{(x, y)/x \geq 0, \Theta^1_Y(x) \leq y \leq \Theta^1(x)\}, R^2_Y = \{(x, y)/x \geq 0, \Theta^2 \leq y \leq \Theta^2_Y(x)\}$, where $\Theta^i_Y(x) = \Theta^i(Y, x)$ is a smooth real function satisfying $\{(x, \Theta^i_Y(x)), 0 \leq x \leq 1\} \subset W^s(\sigma_0(Y))$ and $(0, \Theta^i_Y(0)) = q_i(Y), i = 1, 2$. Moreover if $\delta^i_Y(x) = \delta^i(Y, x)$ is such that $\{(x, \Theta^i_Y(x) + (-1)^{i+1}\delta^i_Y(x)), 0 \leq x \leq 1\} \subset F^{-1}_Y(\{(x, 0), 0 \leq x \leq 1\}) \subset F^{-1}_Y(W^s(\sigma_1(Y)))$ $i = 1, 2$, then there is $\varepsilon > 0$ such that $\Theta^1 - \varepsilon > \Theta^1_Y(x) + \delta^1_Y(x)$ and $\Theta^2 + \varepsilon < \Theta^2_Y(x) - \delta^2_Y(x)$, every $x$.

Making a linear change of coordinates we may also assume that

(v) $|(\Theta^i_Y(x))| < 100$ and that $\delta_Y$ goes to zero uniformly in the $C^2$-topology when $Y$ approaches $\mathcal{N}$.
Clearly $R_Y = \{(x,y)/x \geq 0, 0 \leq y \leq \xi_Y^{-1} \Theta_Y(x)\}$ and $F_Y(x,y) = (\tau_Y x, \xi_Y y)$, for $(x,y) \in R_Y$.

To obtain the expressions of $F_Y(x,y)$, for $(x,y) \in R^1_Y \cup R^2_Y$, we proceed as follows:

Let $-\lambda_3(Y) < -\lambda_1(Y) < 0 < \lambda_2(Y)$ be the eigenvalues of $DY(\sigma_0(Y))$. We set $\alpha(Y) = \frac{\lambda_1(Y)}{\lambda_2(Y)}$ and $\beta(Y) = \frac{\lambda_3(Y)}{\lambda_2(Y)}$.

For $Y \in \mathcal{U}$, let $(x_1,x_2,x_3)$ be $C^3$-linearizing coordinates, in a neighborhood $U_0 \ni \sigma_0(Y)$, that depend $C^2$ on $Y$. We let $L$ and $\tilde{L}$ denote the planes $x_1 = 1$ and $x_2 = 1$, respectively.

For $(x,y) \in R^1_Y$, we have $F_Y(x,y) = \pi_3 \circ \pi_2 \circ \pi_1(x,y) = (f_1(x,y), g_1(x,y))$ where:

(a) $\pi_1 : V \subset Q^+ \to L$ is a diffeomorphism such that $\pi_1(x,\Theta_Y(x)) = (x_3,0)$, for $0 \leq x \leq 1$, and $D\pi_1(x,y) = \begin{bmatrix} a_i(x,y) & b_i(x,y) \\ c_i(x,y) & d_i(x,y) \end{bmatrix}$ where $|a_i(x,y)|, |d_i(x,y)| \leq K_1$, and $k_1, K_1$ are positive real constants. Up to replacing $\{(x,\Theta_Y(x)), x \in [0,1]\}$ with some negative iterate of it (and shrinking $U$) if necessary; we may assume that there are $0 < \eta << 1$ such that $\frac{|c_i(x,y)|}{|d_i(x,y)|} \leq \eta$, every $(x,y) \in R^1_Y$ and $Y \in \mathcal{U}^+$;

(b) $\pi_2 : L \to \tilde{L}$ is given by $\pi_2(x_3,x_2) = (\tilde{x}_3 = x_3 x_2^{b_2}, \tilde{x}_1 = x_2^{a_2})$;

(c) $\pi_3 : \tilde{L} \to Q$ is a diffeomorphism such that

$$D\pi_3(\tilde{x}_3, \tilde{x}_1) = \begin{bmatrix} \tilde{a}(\tilde{x}_3, \tilde{x}_1) & \tilde{b}(\tilde{x}_3, \tilde{x}_1) \\ \tilde{c}(\tilde{x}_3, \tilde{x}_1) & \tilde{d}(\tilde{x}_3, \tilde{x}_1) \end{bmatrix}$$

with $k_2 \leq |\tilde{a}(\tilde{x}_3, \tilde{x}_1)|, |\tilde{d}(\tilde{x}_3, \tilde{x}_1)| \leq K_2$, some positive constants $k_2, K_2$. Moreover, by replacing $p(Y)$ with some positive iterate of it (also contained in $W^u(\sigma_0(Y)) \cap S$), if necessary, we may assume that the quotient $|\tilde{b}|/|\tilde{d}|$ is small enough, and hence that $|\tilde{b}|/|\tilde{d}| \leq \eta$, some small $\eta > 0$.

We now state a very useful lemma that establishes the existence of a $C^3$-invariant stable foliation for $F_Y$ that depends $C^2$ on $Y$. The proof follows from the techniques in [4]; e.g. as may be found in [1] and [5].

**LEMMA 1.** For every $Y \in \mathcal{U}$, there exists an invariant $C^3$ stable foliation for $F_Y$, $F^s_Y$, that depends $C^2$ on $Y$.

After a $C^3$ change of coordinates, this lemma implies that $\Theta_Y(x), \delta_Y(x)$ and $g^i_Y(x,y)$ are maps that do not depend on $x$.

For the sake of simplicity, we assume that $\Theta_Y(x) \equiv 1$ and that $\delta_Y(x) = 1 - \delta$. We also have $c_i(x,y) \equiv 0$. Since $\pi_1(x,y)$ is a diffeomorphism, we have that $a_i(x,y) \neq 0$ and that $d_i(x,y) \neq 0$, every $(x,y)$. Thus we conclude that there are real positive constants $C$ and $K$ such that:

(d) $0 \leq \left| \frac{\partial}{\partial x} f_1^i(x,y) \right| \leq K x_2^{b_2} + r_1^i(x,y)$,

$$\left| \frac{\partial}{\partial y} f_1^i(x,y) \right| = K x_2^{a_2 - 1} + r_1^i(x,y)$$

and

$$\left| \frac{\partial}{\partial y} g_1^i(x) \right| \leq C x_2^{a_2 - 1} + r_3^i(y)$$
where, respectively, $|r_1(x, y)| \leq (\text{constant}) \cdot x_2^\alpha y^{-1}, |r_2(x, y)| \leq (\text{constant}) \cdot x_2^\alpha y$, and $|r_3(y)| \leq (\text{constant}) \cdot x_2^\alpha y$. In the above inequalities we replace $x_2$ with $y - (1 - \delta)$ or $1 - y$, according that $i = 1$ or $2$.

Moreover,

$$(e) \quad f_Y^1(x, 1 - \delta) = xy = f_Y^2(x, 1), \text{ for } x \in [0, 1], \text{ and } g_Y^1(1 - \delta) = yY = g_Y^2(1);$$

$$(f) \quad f_Y^1(x, 1 - \delta + \delta_Y^1) \subset \{(x, 0); x \in [0, 1]\},$$

$$f_Y^2(x, 1 - \delta_Y^2) \subset \{(x, 0); x \in [0, 1]\}, \text{ any } x \in [0, 1],$$

and $g_Y^1(1 - \delta + \delta_Y^1) = 0 = g_Y^2(1 - \delta_Y^2)$.

Conditions (d), (e) and (f) imply $f_Y^1 = A_Y^1 y_1^{1/\alpha_y}$, where $A_Y^1$ is a positive constant for $i = 1, 2$.

Finally, by making another $C^3$-change of coordinates, we obtain $F_Y(x, y) = (f_Y(x, y), g_Y(y))$, with

$$g_Y(y) = \begin{cases} 
\xi_Y y, & \text{for } y \in [0, \xi_Y^{-1}] \\
y_Y - J(Y, y)(y - (1 - \delta))^{\alpha_y}, & \text{for } y \in [1 - \delta, 1 - \delta + \delta_Y^1] \\
y_Y - K(Y, y)(1 - y)^{\alpha_y}, & \text{for } y \in [1 - \delta_Y^2, 1]. 
\end{cases}$$

Here $J(Y, y)$ and $K(Y, y)$ are $C^2$-maps on $Y$, whereas $C^3$-maps on $y$ for $y \neq 1, 1 - \delta$.

Furthermore using (d), (e) and (f), we obtain:

$$(g) \quad \frac{\partial}{\partial y} g_Y(x) \leq C|1 - y|^{\alpha_y - 1} \quad \text{or} \quad \frac{\partial}{\partial y} g_Y(y) \leq C|y - (1 - \delta)|^{\alpha_y - 1} \quad \text{according,}$$

respectively, that $y \in [1 - \delta_Y^2, 1]$ or that $y \in [1 - \delta, 1 - \delta + \delta_Y^1]$.

Also

(i) $\frac{\partial}{\partial y} J(Y, y) \leq K_0$ and $\left\| \frac{\partial}{\partial Y} K(Y, y) \right\|$ is small;

(ii) $\frac{\partial}{\partial y} J(Y, y) \leq K_0$ and $\left\| \frac{\partial}{\partial Y} J(Y, y) \right\|$ is small;

(iii) $J(X, 1 - \delta) > 0$ and $K(X, 1) > 0$.

$$(h) \quad 0 \leq \frac{\partial}{\partial x} f_Y(x, y) \leq K|1 - y|^{\beta_y} \quad \text{or} \quad 0 \leq \frac{\partial}{\partial x} f_Y(x, y) \leq K|y - (1 - \delta)|^{\beta_y}, \quad \text{and}$$

$$\frac{\partial}{\partial y} f_Y(x, y) \leq K|1 - y|^{\alpha_y - 1} \quad \text{or} \quad \frac{\partial}{\partial y} f_Y(x, y) \leq K|y - (1 - \delta)|^{\alpha_y - 1}; \quad \text{according, respectively,}$$

that $y \in [1 - \delta_Y^2, 1]$ or that $y \in [1 - \delta, 1 - \delta + \delta_Y^1]$.

We do not lose generality if, in the sequel, we assume that, for $Y \in \mathcal{U} : \alpha(Y) = \alpha, \beta(Y) = \beta, \xi_Y = \xi$ and $\gamma_Y = \gamma$.

Furthermore since the map $Y \rightarrow y_Y$ is a $C^2$-submersion, we can find $C^2$-coordinates $(v, \mu)$ in the neighborhood $\mathcal{U}(\mu \in \mathbb{R})$ such that:

(i) $\{(v, \mu)/\mu = 0\} \subset \mathcal{N} \cap \mathcal{U}$;

(ii) $F_{(v, \mu)}(x, y) = (\tau x, \xi y)$ if $0 \leq y \leq \xi^{-1}$,

$$F_{(v, \mu)}(x, y) = \left( x(\mu, v) + f^2(v, \mu; x, y), \mu - K(v, \mu; y)(1 - y)^\alpha \right)$$

for $1 - \delta^2(v, \mu) \leq y \leq 1$,

$$F_{(v, \mu)}(x, y) = \left( x(v, \mu) + f^1(v, \mu; x, y), \mu - J(v, \mu; y)(y - (1 - \delta))^{\alpha_y} \right), \text{for}$$

$1 - \delta \leq y \leq 1 - \delta + \delta^1(v, \mu)$. 

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Under these conditions we obtain \( \delta^i(v, \mu) = A^i(v)\mu^{1/\alpha} \), with \( \left\| \frac{\partial A^i}{\partial v} \right\| \) small numbers, for \( i = 1, 2 \).

We will use the notations \( a(v, \mu) = 1 - \delta^2(v, \mu) \) and \( b(v, \mu) = 1 - \delta + \delta^4(v, \mu) \).

2.2.

For a proof of Theorem 1 we first give a characterization of the elements in \( \mathcal{U}_H^+ \cup \mathcal{U}_A^+ \).

Choose \( \mu_1 > 0 \) and \( n_0 \in \mathbb{N} \) such that \( \xi^{-n_0} \mu_1 = 1, 1 > 1 \).

**Lemma 2.** - For \( (v, \mu) \in \mathcal{U} \) such that \( \xi^{-n_0} < \mu \leq \mu_1 \), we have that

\[
\Lambda(v, \mu) = \{(x, y)/F_n^{\mu}(x, y) \in R(v, \mu) \cup R_1(v, \mu) \cup R_2(v, \mu), n \in \mathbb{Z}\}
\]

is a hyperbolic transitive set.

**Proof.** - See Lemma 2 in [1].

We next assume \( 0 < \mu \leq \xi^{-n_0} = \mu_0 \).

Set \( I_0(v, \mu) = [0, \xi^{-1}], I_{01}(v, \mu) = \xi^{-1}, 1 - \delta, \)

\[
I_1(v, \mu) = [1 - \delta, b(v, \mu)], I_{12}(v, \mu) = \delta, a(v, \mu), \quad \text{and} \quad I_2(v, \mu) = [a(v, \mu), 1].
\]

For \( (v, \mu) \in \mathcal{U} \), let \( L(v, \mu, \cdot) : \cup_{i=0}^2 I_i(v, \mu) \to [0, 1] \) be the map \( L(v, \mu; y) = \pi_y \circ F(v, \mu)(x, y) \) second component of the first return map \( F(v, \mu)(x, y) \).

Define \( L_1(v, \mu; y) = L(v, \mu; y) \) and \( L_{n+1}(v, \mu; y) = L(v, \mu; L_n(v, \mu; y)) \) for \( n \geq 1 \).

Let

\[
\Lambda(v, \mu) = \{y \in [0, 1]/L_n(v, \mu; y) \in \cup_{i=0}^2 I_i(v, \mu), n \geq 0\}
\]

\( \Gamma_0 = \{(v, \mu) \in \mathcal{U} : 1 \notin \Lambda(v, \mu)\} \)

and

\( \Gamma_1 = \{(v, \mu) \in \mathcal{U} : 1 \in \Lambda(v, \mu) \text{ and there exists a hyperbolic attracting periodic orbit for the map } L(v, \mu; \cdot)\} \)

**Lemma 3.** - For \( (v, \mu) \in \Gamma_0 \) we have that \( \Lambda(v, \mu) \) is a hyperbolic set for the map \( L(v, \mu; \cdot) \).

**Proof.** - Let \( (v, \mu) \in \Gamma_0 \) and \( n = n(v, \mu) \) be the integer such that \( L_n(v, \mu; 1) \in I_{01}(v, \mu) \cup I_{12}(v, \mu) \). Due to the continuity of the map \( (v, \mu; y) \to L_n(v, \mu; y) \) we can find neighborhoods \( U_1 \subset I_1(v, \mu), U_2 \subset I_2(v, \mu) \) of the points \( 1 - \delta \) and \( 1 \), respectively, such that \( y \in U_1 \cup U_2 \) implies \( L_n(v, \mu; y) \in I_{01}(v, \mu) \cup I_{12}(v, \mu) \). This, in turn, implies that \( \Lambda(v, \mu) \) is a compact invariant set with all its periodic points hyperbolic repelling and without critical points. Hence, by applying a result proved by Mañé [6] to the restriction map

\[
L(v, \mu; \cdot) / (I_{01}(v, \mu) \cup I_{12}(v, \mu) \setminus U_1 \cup U_2)
\]

the result now follows. ■

**Definition 2.** - Let \( I \subset J \) be two intervals. We will say \( f \in C^k(I, J), k \geq 1 \), satisfies Axiom A if:

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(i) $f$ has a finite number of hyperbolic, attracting periodic orbits and no other attractors.
(ii) Let $B(f)$ denote the basin of attraction of the attracting periodic orbits for $f$. The set $\sum(f) = I \setminus B(f)$ is a hyperbolic set for $f$.

**Lemma 4.** For $(u, \mu) \in \Gamma_1$ we have that $L(v, \mu; \cdot)$ satisfies Axiom A.

**Proof.** We note that $L(v, \mu; \cdot)|_{I_1(v, \mu) \cup I_2(v, \mu)}$ has negative Schwarzian derivative. By Singer's theorem we obtain that the attracting periodic orbit attracts all the critical points (since that all critical points eventually have the same orbit).

Since $L(v, \mu; \cdot)$ has a hyperbolic attracting periodic orbit, we have that it does not have saddle-node or attracting flip bifurcations. Since these are the only non-hyperbolic periodic orbits that appear in our family (see sections 2.3 through 2.14), we conclude that $\Lambda(v, \mu)$ does not contain non-hyperbolic periodic orbits. In particular, all the periodic points in $(\Lambda(v, \mu) \setminus B(L(v, \mu; \cdot)))$ are hyperbolic. This implies that $(\Lambda(v, \mu) \setminus B(L(v, \mu; \cdot)))$ is a hyperbolic set (see [dM, pg. 128]).

Using the techniques of [3] or [1], it is easy to see that $(v, \mu) \in \Gamma_0$ if and only if $(v, \mu) \in \mathcal{U}_H^+$ and $(v, \mu) \in \Gamma_1$ if and only if $(v, \mu) \in \mathcal{U}_A^+$. Part b) of Theorem 1 now follows.

2.3. Since $X \in \mathcal{U}_X$ we have $X = (v_0, 0)$ some $v_0$.

In the sequel we will deal with $(v, \mu) \in \mathcal{U}_X$ such that $-\xi^{-(n_0 - 1)} \leq \mu \leq \xi^{-(n_0 - 1)}$;

$\|v - v_0\| \leq r_0$, some $r_0 > 0$ small, and $n_0 \in \mathbb{N}$ choosen such that the number:

$$Q_0 = \inf\{\alpha(A^1(v))^{-1}\xi^\frac{n_0}{\alpha} (1 - \delta - \xi^{-1}), \alpha(A^2(v))^{-1}\xi^\frac{n_0}{\alpha} (1 - \delta - \xi^{-1}); v \in V\}$$

satisfies $Q_0 > 2, \frac{2}{Q_0(1 - \xi^{-1})} < 1$ and, $\xi^{-1/\alpha}Q_0 > 1$.

Throughout, we will consider $k_0 \in \mathbb{N}$ such that $k_0 \geq n_0$.

Let $B(k_0)$ be the set $\{(v, \mu) \in \mathcal{U}/1 - \delta \leq \xi^{k_0 - 1}\mu \leq 1; \|v - v_0\| \leq r_0\}$.

For $(v, \mu) \in B(k_0)$ denote by $D\left(\begin{array}{c}1 \\ j(j)\end{array}\right)(v, \mu) \subset I_1(v, \mu)$ $D\left(\begin{array}{c}2 \\ j(j)\end{array}\right)(v, \mu) \subset I_2(v, \mu))$

the interval satisfying:

$$L\left(v, \mu; D\left(\begin{array}{c}i \\ j(j)\end{array}\right)(v, \mu)\right) = \xi^{-(k_0 - 1)} \xi^{-j} \left[1 - \delta, 1\right], \text{ for } j \geq 1, \quad i = 1, 2.$$

$D\left(\begin{array}{c}i \\ 0(0)\end{array}\right)(v, \mu) \subset I_i(v, \mu)$ will denote, the interval satisfying:

$$L\left(v, \mu; D\left(\begin{array}{c}i \\ 0(0)\end{array}\right)(v, \mu)\right) = \xi^{-(k_0 - 1)} \left[1 - \delta, \xi^{k_0 - 1}\mu\right], \quad i = 1, 2.$$

Note that

$$D\left(\begin{array}{c}1 \\ 0(0)\end{array}\right)(v, \xi^{-(k_0 - 1)}(1 - \delta)) = \{1 - \delta\} \text{ and that } D\left(\begin{array}{c}2 \\ 0(0)\end{array}\right)(v, \xi^{-(k_0 - 1)}(1 - \delta)) = \{1\}.$$
For \( j \geq 1 \), we let \( \left\{ z\left(\frac{i}{j}\right)(v, \mu), y\left(\frac{i}{j}\right)(v, \mu) \right\} \) denote the boundary points of the interval \( D\left(\frac{i}{j}\right)(v, \mu) \). These two points are defined by the equations

\[
L\left(v, \mu; z\left(\frac{i}{j}\right)(v, \mu)\right) = \xi^{-(k_0-1)}\xi^{-j(1-\delta)} \text{ and } \\
L\left(v, \mu; y\left(\frac{i}{j}\right)(v, \mu)\right) = \xi^{-(k_0-1)}\xi^{-j} .
\]

For \( j = 0 \), we have that \( D\left(\frac{1}{0}\right)(v, \mu) = \left[ 1 - \delta, z\left(\frac{1}{0}\right)(v, \mu) \right] \) and that \( D\left(\frac{2}{0}\right)(v, \mu) = \left[ z\left(\frac{2}{0}\right)(v, \mu), 1 \right] \) where \( L\left(v, \mu; z\left(\frac{i}{j}\right)(v, \mu)\right) = \xi^{-(k_0-1)}(1-\delta), i = 1, 2. \)

We note that:

\[
\lim_{\mu \to \xi^{-(k_0-1)(1-\delta)}} \frac{\partial z\left(\frac{1}{0}\right)}{\partial \mu}(v, \mu) = +\infty \text{ and } \lim_{\mu \to \xi^{-(k_0-1)(1-\delta)}} \frac{\partial z\left(\frac{2}{0}\right)}{\partial \mu}(v, \mu) = -\infty .
\]

The proof of the following lemma is easy and left to the reader.

**Lemma 5.** – Given \( \varepsilon > 0 \) we can find \( j_0 \in \mathbb{N} \) such that

\[
\max \left\{ \sup \left\{ \left| b(v, \mu) - z\left(\frac{1}{j}\right)(v, \mu) \right|, \left| \frac{\partial b}{\partial v}(v, \mu) - \frac{\partial z\left(\frac{1}{j}\right)}{\partial \mu}(v, \mu) \right|, \right. \right. \\
\left. \left. \left| \frac{\partial b}{\partial \mu}(v, \mu) - \frac{\partial z\left(\frac{1}{j}\right)}{\partial \mu}(v, \mu) \right| \right\}, \right.
\]

\[
\sup \left\{ \left| b(v, \mu) - y\left(\frac{1}{j}\right)(v, \mu) \right|, \left| \frac{\partial b}{\partial v}(v, \mu) - \frac{\partial y\left(\frac{1}{j}\right)}{\partial \mu}(v, \mu) \right|, \right. \\
\left. \left. \left| \frac{\partial b}{\partial \mu}(v, \mu) - \frac{\partial y\left(\frac{1}{j}\right)}{\partial \mu}(v, \mu) \right| \right\}, \right.
\]

\[
\sup \left\{ \left| a(v, \mu) - z\left(\frac{2}{j}\right)(v, \mu) \right|, \left| \frac{\partial a}{\partial v}(v, \mu) - \frac{\partial z\left(\frac{2}{j}\right)}{\partial \mu}(v, \mu) \right|, \right. \\
\left. \left. \left| \frac{\partial a}{\partial \mu}(v, \mu) - \frac{\partial z\left(\frac{2}{j}\right)}{\partial \mu}(v, \mu) \right| \right\}, \right.
\]
\[
\sup \left\{ \left| a(v, \mu) - y \left( \frac{2}{j} \right) (v, \mu) \right|, \left| \frac{\partial a}{\partial v} (v, \mu) - \frac{\partial y}{\partial v} \left( \frac{2}{j} \right) (v, \mu) \right|, \left| \frac{\partial a}{\partial \mu} (v, \mu) - \frac{\partial y}{\partial \mu} \left( \frac{2}{j} \right) (v, \mu) \right| \right\}; (v, \mu) \in B(k_0) \right\} < \varepsilon,
\]

for any \( j \geq j_0 \) : that is, the sequences of maps \( \left( z \left( \frac{1}{j} \right) \right), \left( y \left( \frac{1}{j} \right) \right) \) (resp. \( \left( z \left( \frac{2}{j} \right) \right), \left( y \left( \frac{2}{j} \right) \right) \)) converge to \( b(v, \mu) \) (resp. \( a(v, \mu) \)) in the uniform \( C^1 \)-topology in \( B(k_0) \).

We also note the following fact: for any \( j \geq 1\), \( y \in D \left( \frac{i}{j} \right) (v, \mu) \) and \( y' \in D \left( \frac{i}{j+1} \right) (v, \mu) \) we have

\[
\left| \frac{\partial L}{\partial y} (v, \mu, y') \right| \geq \lambda_j > 1,
\]

where the sequence \( \left( \lambda_j \right) \) satisfies \( \lim_{j \to \infty} \lambda_j = 1 \).

We now have the following result for \( (v, \mu) \in B(k_0) \).

**Lemma 6.**

\[
\min \left\{ \left| \frac{\partial L_{k_0}}{\partial y} \left( v, \mu; y \left( \frac{1}{1} \right) \right) \right|, \left| \frac{\partial L_{k_0}}{\partial y} \left( v, \mu; y \left( \frac{2}{1} \right) \right) \right| \right\} \geq \xi^{\frac{k_0-n-1}{\alpha}} Q_0
\]

**Proof.** Since \( L_{k_0}(v, \mu; y) = \xi^{k_0-1} L(v, \mu; y) \), for \( (v, \mu) \in B(k_0) \), \( y \left( \frac{1}{1} \right) (v, \mu) \leq y \leq b(v, \mu) \) or \( a(v, \mu) \leq y \leq y \left( \frac{2}{1} \right) (v, \mu) \) we have

\[
\frac{\partial L_{k_0}}{\partial y} (v, \mu, y \left( \frac{1}{1} \right) (v, \mu)) = -\xi^{k_0-1} \alpha J \left( v, \mu; y \left( \frac{1}{1} \right) (v, \mu) \right) \left( y \left( \frac{1}{1} \right) (v, \mu) - (1 - \delta) \right)^{-1}
\]

\[
\left( \ast \right) \left[ y \left( \frac{1}{1} \right) (v, \mu) - (1 - \delta) \right] \frac{\partial J}{\partial y} (v, \mu, y \left( \frac{1}{1} \right))
\]

\[
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For $y \left( \frac{1}{1} \right)(v, \mu)$ we have:

$$\mu - J \left( v, \mu, y \left( \frac{1}{1} \right) \right) \left( y \left( \frac{1}{1} \right) - (1 - \delta) \right)^\alpha = \xi^{-k_0}$$

and

$$1 - \delta < y \left( \frac{1}{1} \right)(v, \mu) < 1 - \delta + A^1(v) \mu^{1/\alpha}.$$

Since $\xi^{-(k_0-1)}(1 - \delta) \leq \mu \leq \xi^{-(k_0-1)}$, we obtain

$$\xi^{-(k_0-1)}(1 - \delta)^{1/\alpha} \leq \mu^{1/\alpha} \leq \xi^{-(k_0-1)}$$

and hence $(\mu^{1/\alpha})^{-1} \geq \xi^{k_0-1}$.

Therefore

$$\left| \alpha \xi^{k_0-1} J \left( v, \mu, y \left( \frac{1}{1} \right) \right) \left( y \left( \frac{1}{1} \right) - (1 - \delta) \right)^{\alpha-1} \right| > \alpha (A^1(v))^{-1} \xi^{k_0-1} (1 - \delta - \xi^{-1}).$$

Using this fact in equation (*) the result follows for $y \left( \frac{1}{1} \right)(v, \mu)$. The proof for

$$\left| \frac{\partial L_{k_0}}{\partial y} \left( v, \mu, y \left( \frac{2}{1} \right) \right)(v, \mu) \right|$$

is analogous. 

**Corollary 1.** For $(v, \mu) \in B(k_0)$ and $y \in D \left( \frac{i}{j} \right)(v, \mu), j \geq 1$, we have that

$$\left| \frac{\partial L_{k_0}}{\partial y} (v, \mu, y) \right| \geq \xi^{\frac{k_0-n_0-1}{\alpha}} Q_0, \text{ for } y \in D \left( \frac{i}{1} \right)(v, \mu),$$

and that

$$\left| \frac{\partial L_{k_0}}{\partial y} (v, \mu; y) \right| \geq \lambda_1 \cdots \lambda_{j-1} \xi^{\frac{k_0-n_0-1}{\alpha}} Q_0, \text{ for } y \in D \left( \frac{i}{j} \right)(v, \mu)$$

and any $j \geq 2$.

2.4. Associated to $\left( \frac{i}{j} \right)$ we next define the one-dimensional map

$$g \left( \frac{i}{j} \right)(v, \mu, \cdot) : D \left( \frac{i}{j} \right)(v, \mu) \to [1 - \delta, 1] \text{ by } g \left( \frac{i}{j} \right)(v, \mu; y) = L_{k_0+j}(v, \mu; y).$$

Applying Corollary 1 we have that

$$\left| \frac{\partial g \left( \frac{i}{1} \right)}{\partial y} (v, \mu, y) \right| \geq \xi^{\frac{k_0-n_0-1}{\alpha}} Q_0 = P_1, \text{ for } y \in D \left( \frac{i}{1} \right)(v, \mu)$$

and that

$$\left| \frac{\partial g \left( \frac{i}{j} \right)}{\partial y} (v, \mu; y) \right| \geq \lambda_1 \cdots \lambda_{j-1} \xi^{\frac{k_0-n_0-1}{\alpha}} Q_0 = P_j, \text{ for } y \in D \left( \frac{i}{j} \right)(v, \mu)$$

any $j \geq 2$. 

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From these estimates we get that the maps $g_{i,j}^i(v,\mu; y), i = 1,2, j \geq 1$, are $C^\infty$-expanding diffeomorphisms onto their images (that are $[1 - \delta,1]$). Moreover, for $i = 1$ all the maps $g_{1,j}^1(v,\mu)$ reverse orientation, and for $i = 2$ all the maps $g_{2,j}^2(v,\mu)$ preserve orientation.

Now given any sequence of two symbols, $\left(\left(\begin{array}{c} i_0 \\
 j_0 \end{array}\right), \left(\begin{array}{c} i_1 \\
 j_1 \end{array}\right), \ldots \right)$, let us define a sequence of nested sets and maps:

$D_{i_0,j_0}^i(v,\mu) \supset D_{i_0,j_0}^i(v,\mu) \supset \cdots \supset D_{i_0,j_0}^i(v,\mu) \supset \cdots$

and

$g\left(\left(\begin{array}{c} i_0 \\
 j_0 \end{array}\right), \left(\begin{array}{c} i_1 \\
 j_1 \end{array}\right)\right)(v,\mu), \ldots, g\left(\left(\begin{array}{c} i_0 \\
 j_0 \end{array}\right), \left(\begin{array}{c} i_1 \\
 j_1 \end{array}\right), \ldots, \left(\begin{array}{c} 1_r \\
 j_r \end{array}\right)\right)(v,\mu)$,

as follows:

$D_{i_0,j_0}^i(v,\mu) = \{ y \in D_{i_0,j_0}^i(v,\mu) : g\left(\left(\begin{array}{c} i_0 \\
 j_0 \end{array}\right), \left(\begin{array}{c} i_1 \\
 j_1 \end{array}\right), \ldots, \left(\begin{array}{c} 1_r \\
 j_r \end{array}\right)\right)(v,\mu; y) \in D_{i_0,j_0}^i(v,\mu) \}.$

For $D_{i_0,j_0}^i(v,\mu) \neq \emptyset$ we associate a map

$g\left(\left(\begin{array}{c} i_0 \\
 j_0 \end{array}\right), \left(\begin{array}{c} i_1 \\
 j_1 \end{array}\right)\right)(v,\mu; y) : D_{i_0,j_0}^i(v,\mu) \rightarrow [1 - \delta,1]$ defined by

$g\left(\left(\begin{array}{c} i_0 \\
 j_0 \end{array}\right), \left(\begin{array}{c} i_1 \\
 j_1 \end{array}\right)\right)(v,\mu; y) = g\left(\left(\begin{array}{c} i_0 \\
 j_0 \end{array}\right), \left(\begin{array}{c} i_1 \\
 j_1 \end{array}\right), \ldots, \left(\begin{array}{c} 1_r \\
 j_r \end{array}\right)\right)(v,\mu; y).$

For $r \geq 2$ and $D_{i_0,j_0}^i, \ldots, \left(\begin{array}{c} i_{r-1} \\
 j_{r-1} \end{array}\right)\right)(v,\mu) \neq \emptyset$, we define

$D_{i_0,j_0}^i(v,\mu) = \{ y \in D_{i_0,j_0}^i(v,\mu) / \}

g\left(\left(\begin{array}{c} i_0 \\
 j_0 \end{array}\right), \left(\begin{array}{c} i_1 \\
 j_1 \end{array}\right), \ldots, \left(\begin{array}{c} i_{r-1} \\
 j_{r-1} \end{array}\right)\right)(v,\mu; y) \in D_{i_0,j_0}^i(v,\mu) \}.$

Associated to those $D_{i_0,j_0}^i(v,\mu)$ that are non-empty define the map

$g\left(\left(\begin{array}{c} i_0 \\
 j_0 \end{array}\right), \ldots, \left(\begin{array}{c} 1_r \\
 j_r \end{array}\right)\right)(v,\mu; y) : D_{i_0,j_0}^i(v,\mu) \rightarrow [1 - \delta,1]$ by

$g\left(\left(\begin{array}{c} i_0 \\
 j_0 \end{array}\right), \ldots, \left(\begin{array}{c} 1_r \\
 j_r \end{array}\right)\right)(v,\mu; y) = g\left(\left(\begin{array}{c} 1_r \\
 j_r \end{array}\right)\right)(v,\mu; y).$
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Remark 1. – Given any finite set of two symbols, \( \left\{ \left( \frac{i_0}{j_0}, \ldots, \frac{i_r}{j_r} \right) \right\} \), such that \( j_k \geq 1 \), for \( k = 0, 1, \ldots, r \), by Corollary 1 we have that:

\[
\left| \frac{\partial}{\partial \mu} \left(g \left( \left( \frac{i_0}{j_0}, \ldots, \frac{i_r}{j_r} \right) \right) \right)(v, \mu; y) \right| \geq P_{j_0} \cdots P_{j_r},
\]

any \( y \in D \left( \left( \frac{i_0}{j_0}, \ldots, \frac{1}{j_1} \right) \right) (v, \mu) \). From this inequality we conclude

\[
\left| D \left( \left( \frac{i_0}{j_0}, \ldots, \frac{i_r}{j_r} \right) \right) (v, \mu) \right| \leq (P_{j_0} \cdots P_{j_{r-1}})^{-1} \left| D \left( \frac{i_r}{j_r} \right) (v, \mu) \right|
\]

and hence

\[
\sum_{(i_0, j_0) \geq 1} \left( \sum_{(i_1, j_1) \geq 1} \left( \sum_{(i_r, j_r) \geq 1} \left| D \left( \left( \frac{i_0}{j_0}, \ldots, \frac{i_r}{j_r} \right) \right) (v, \mu) \right| \right) \right) \leq \delta \cdot \left( \frac{2}{P_1(1 - \xi^{-1})} \right)^r;
\]

that is, for any \( (v, \mu) \in B(k_0) \) we have :

Corollary 2. – The set of points

\[
y \in \left( I_1(v, \mu) \setminus D \left( \frac{1}{0} \right) (v, \mu) \right) \cup \left( I_2(v, \mu) \setminus D \left( \frac{2}{0} \right) (v, \mu) \right)
\]

that satisfy

(i) \( L_i(v, \mu; y) \) is defined, all \( i \geq 1 \), and

(ii) there is no \( i_0 \in \mathbb{N} \) such that \( L_{i_0}(v, \mu; y) \in D \left( \frac{1}{0} \right) (v, \mu) \cup D \left( \frac{2}{0} \right) (v, \mu) \).

is a hyperbolic set of zero Lebesgue measure.

Remark 2. – Let denote the set above by \( C \left( \left( \frac{1}{1}, \frac{2}{1} \right) \right)(v, \mu) \). As a consequence we obtain that its closure is a Cantor set of zero Lebesgue measure.

2.5. Let us now consider any sequence of two symbols \( \left( \left( \frac{i_0}{j_0}, \frac{i_1}{j_1}, \ldots \right) \right) \), where \( i_k = 1, 2 \) and \( j_k \geq 1 \), all \( k \in \mathbb{N} \).

Let

\[
z_r(v, \mu) = z \left( \left( \frac{i_0}{j_0}, \frac{i_r}{j_r} \right) \right) (v, \mu), \quad y_r(v, \mu) = y \left( \left( \frac{i_0}{j_0}, \frac{i_r}{j_r} \right) \right) (v, \mu)
\]

denote the boundary points of the interval \( D \left( \left( \frac{i_0}{j_0}, \ldots, \frac{i_r}{j_r} \right) \right)(v, \mu) \) defined, respectively, by the conditions

\[
\Delta_r(v, \mu, z_r(v, \mu)) = g \left( \left( \frac{i_0}{j_0}, \ldots, \frac{i_r}{j_r} \right) \right) (v, \mu; z \left( \left( \frac{i_0}{j_0}, \ldots, \frac{i_r}{j_r} \right) \right) (v, \mu) = 1 - \delta
\]
and
\[ \Delta_r(v, \mu, y_r(v, \mu)) = g\left(\left(\frac{i_0}{j_0}, \ldots, \frac{i_r}{j_r}\right)\right)(v, \mu; y\left(\left(\frac{i_0}{j_0}, \ldots, \frac{i_r}{j_r}\right)\right)(v, \mu)) = 1 \]

From these relations we obtain
\[
\frac{\partial z_r}{\partial v}(v, \mu) = -\frac{\partial \Delta_r}{\partial v}(v, \mu, z_r(v, \mu)) \\
\frac{\partial z_r}{\partial \mu}(v, \mu) = -\frac{\partial \Delta_r}{\partial \mu}(v, \mu, z_r(v, \mu))
\]

Let us compute inductively the derivatives in the right-hand side.
Since \( \Delta_r(v, \mu; y) = g\left(\frac{i_r}{j_r}\right)(v, \mu; \Delta_{r-1}(v, \mu; y)) \), we have
\[
\frac{\partial \Delta_r}{\partial v}(v, \mu; y) = -\frac{\partial g}{\partial v}\left(\frac{i_r}{j_r}\right)(v, \mu; \Delta_{r-1}(v, \mu; y)) \\
+ \frac{\partial g}{\partial y}\left(\frac{i_r}{j_r}\right)(v, \mu; \Delta_{r-1}(v, \mu; y)) \cdot \frac{\partial \Delta_{r-1}}{\partial v}(v, \mu; y) = \\
\frac{\partial g}{\partial v}\left(\frac{i_r}{j_r}\right)(y, \mu; \Delta_{r-1}(v, \mu; y)) \\
+ \frac{\partial g}{\partial v}\left(\frac{i_r}{j_r}\right)(v, \mu; \Delta_{r-1}(v, \mu; y)) \cdot \frac{\partial \Delta_{r-1}}{\partial v}(v, \mu; y) \\
+ \frac{\partial g}{\partial y}\left(\frac{i_r}{j_r}\right)(v, \mu; \Delta_{r-1}(v, \mu; y)) \cdot \frac{\partial \Delta_{r-1}}{\partial y}(v, \mu; \Delta_{r-2}) \\
+ \frac{\partial g}{\partial v}\left(\frac{i_{r-1}}{j_{r-1}}\right)(v, \mu; \Delta_{r-1}(v, \mu; y)) \\
+ \cdots + \frac{\partial g}{\partial y}\left(\frac{i_1}{j_1}\right)(v, \mu; \Delta_{r-1}(v, \mu; y)) \cdot \frac{\partial \Delta_0}{\partial v}(v, \mu; y)
\]

We have a similar relation for \( \frac{\partial \Delta_r}{\partial \mu}(v, \mu; y) \) by replacing \( \frac{\partial}{\partial \mu} \) for \( \frac{\partial}{\partial v} \) wherever it corresponds in the above formulas.
The other derivative yields

\[
\frac{\partial \Delta_r}{\partial y}(v, \mu; y) = \frac{\partial g}{\partial y}(v, \mu; \Delta_{r-1}(v, \mu; y)) \cdots \frac{\partial g}{\partial y}(v, \mu; y).
\]

Denoting by \( g_r \) the map \( g(j_r) \), we have:

\[
\frac{\partial z_r}{\partial v}(v, \mu) = \frac{\left\{ -\left[ \frac{\partial g_r}{\partial y}(v, \mu; \Delta_{r-1}(z_r)) + \cdots \right] \right.}{\left. + \frac{\partial g_r}{\partial y}(v, \mu; \Delta_{r-1}(z_r)) \cdots \frac{\partial g_0}{\partial y}(v, \mu; z_r) \right\}}
\]

and

\[
\frac{\partial z_r}{\partial \mu}(v, \mu) = \frac{\left\{ -\left[ \frac{\partial g_r}{\partial y}(v, \mu; \Delta_{r-1}(z_r)) + \cdots \right] \right.}{\left. + \frac{\partial g_r}{\partial y}(v, \mu; \Delta_{r-1}(z_r)) \cdots \frac{\partial g_0}{\partial y}(v, \mu; z_r) \right\}}
\]

Now, for any \( \binom{i_0}{j_0} \), we have

\[
\left| \begin{array}{c}
\frac{\partial g}{\partial v}(v, \mu; y) \\
\frac{\partial g}{\partial y}(v, \mu; y)
\end{array} \right| = \left| \begin{array}{c}
\frac{\partial L}{\partial v}(v, \mu; y) \\
\frac{\partial L}{\partial y}(v, \mu; y)
\end{array} \right|
\]

and

\[
\left| \begin{array}{c}
\frac{\partial g}{\partial \mu}(v, \mu; y) \\
\frac{\partial g}{\partial y}(v, \mu; y)
\end{array} \right| = \left| \begin{array}{c}
\frac{\partial L}{\partial \mu}(v, \mu; y) \\
\frac{\partial L}{\partial y}(v, \mu; y)
\end{array} \right|
\]
We note that the sequence \((z_r(v, \mu))\) converges uniformly in the \(C^0\)-topology to

\[
z_\infty(v, \mu) = z\left(\begin{pmatrix} i_0 \\ j_0 \end{pmatrix}, \begin{pmatrix} i_1 \\ j_1 \end{pmatrix}, \cdots\right)(v, \mu)
\]
i.e.,

\[
\lim_{r \to \infty} \sup\{|z_\infty(v, \mu) - z_r(v, \mu)|; (v, \mu) \in B(k_0)\} = 0.
\]

From this fact and the above computation for the derivatives of the maps \(z_r(v, \mu)\), and since all the \(g\left(\begin{pmatrix} i_r \\ j_r \end{pmatrix}\right)\), \(j_r \geq 1\) are \(C^\infty\)-diffeomorphisms, after a cumbersome computation, we obtain

**Lemma 7.** - The sequence \((z_r(v, \mu))\) satisfies the following property: Given \(\varepsilon > 0\) there is an \(r_0 \in \mathbb{N}\) such that

\[
\sup\{|z_{r+p}(v, \mu) - z_r(v, \mu)|, \left\| \frac{\partial z_{r+p}(v, \mu) - \partial z_r(v, \mu)}{\partial v} \right\|; (v, \mu) \in B(k_0)\} < \varepsilon \text{ for } r \geq r_0, \ p \in \mathbb{N};
\]

that is, the sequence \((z_r(v, \mu))\) is a Cauchy sequence of maps in the uniform \(C^1\)-topology.

In particular we have that the map \((v, \mu) \mapsto z_\infty(v, \mu)\) is a \(C^1\)-map on \(B(k_0)\).

Let us now denote by

\[
G(v, \mu, \cdot) : \bigcup_{i=1}^2 \bigcup_{j \geq 1} D\left(\begin{pmatrix} i \\ j \end{pmatrix}\right)(v, \mu) \to [1 - \delta, 1]
\]

the map defined by \(G(v, \mu, y) = g\left(\begin{pmatrix} i \\ j \end{pmatrix}\right)(v, \mu, y), \text{ for } y \in D\left(\begin{pmatrix} i \\ j \end{pmatrix}\right)(v, \mu)\).

Let

\[
C\left(\begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}\right)(v, \mu)
\]
denote the set of points \(y \in \left[y\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)(v, \mu), y\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right)(v, \mu)\right]\) such that it is defined \(G_k(v, \mu, y)(G_{k+1}(v, \mu, y) = G(v, \mu, G_k(v, \mu, y)), G_1(v, \mu, y) = G(v, \mu, y))\) for all \(k \in \mathbb{N}\) and \(G_k(v, \mu, y) \in \left[y\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)(v, \mu), y\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right)(v, \mu)\right]\).

Associated with any point \(y \in C\left(\begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}\right)(v, \mu)\) we may define a sequence \(\Gamma(v, \mu) : \mathbb{N} \to \left\{\begin{pmatrix} i \\ j \end{pmatrix} \mid i = 1, 2; j \geq 1\right\}\) by

\[
\Gamma(v, \mu)(k) = \begin{pmatrix} i_s \\ j_s \end{pmatrix} \Leftrightarrow G_k(v, \mu)(y) \in D\left(\begin{pmatrix} i_s \\ j_s \end{pmatrix}\right)(v, \mu).
\]
This defines a map $C\left(\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}\right)(v, \mu) \Gamma(v, \mu) \rightarrow \Sigma_1$,

$$\Sigma_1 = \left\{ \varGamma : \mathbb{N} \rightarrow \left\{ \begin{pmatrix} i \\ j \end{pmatrix} ; i = 1, 2; j \geq 1 \right\} \right\}$$

which is, as usual, a homeomorphism and satisfies

$$\Gamma(v, \mu) \circ G(v, \mu) = \sigma_1 \circ \Gamma(v, \mu),$$

where $\Sigma_1 \sigma_1 \Sigma_1$ denotes the shift map $\sigma_1(\varGamma)(k) = \varGamma(k + 1)$.

For $\varGamma \in \Sigma_1$ we denote $p_\varGamma(v, \mu) = (\Gamma(v, \mu))^{-1}(\varGamma)$. As in Lemma 7 we may prove the following:

**Corollary 3.** – The map $B(k_0)^{p_\varGamma}[1 - \delta, 1], (v, \mu) \mapsto p_\varGamma(v, \mu)$ is $C^1$. 

We observe that the closure of the set $C\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right)(v, \mu)$ contains the points $b(v, \mu), a(v, \mu)$ and all their preimages under the map $G(v, \mu, \cdot)$ which are contained in the interval $\left[ y\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right)(v, \mu), y\left(\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}\right)(v, \mu) \right]$.

Denoting by $s(v, \mu)$ any of these preimages it is clear that the map $B(k_0) \longrightarrow [1 - \delta, 1], (v, \mu) \mapsto s(v, \mu)$ is a $C^1$ map and can be approximated, in the $C^1$-uniform topology, by a sequence of maps $z\left(\begin{pmatrix} i_0 \\ j_0 \end{pmatrix}, ..., \begin{pmatrix} i_r \\ j_r \end{pmatrix}\right)(v, \mu)$ (or $y\left(\begin{pmatrix} i_0 \\ j_0 \end{pmatrix}, ..., \begin{pmatrix} i_r \\ j_r \end{pmatrix}\right)(v, \mu)$) as in lemma 5.

In this sense we will say that the closure of the set $C\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right)(v, \mu)$ is a $C^1$-Cantor set of Lebesgue measure zero for any $(v, \mu) \in B(k_0)$.

2.6. Let us now consider the surface $S_0 = \{(v, \mu; \xi^{k_0-1}\mu); (v, \mu) \in B(k_0)\} \subset \mathcal{U} \times [1 - \delta, 1]$.

Since $S_0$ is transversal to $Y\left(\begin{pmatrix} i \\ j \end{pmatrix}\right) = \left\{(v, \mu; y\left(\begin{pmatrix} i \\ j \end{pmatrix}\right)(v, \mu); (v, \mu) \in B(k_0)\right\}$, we have that the intersection $S_0 \cap Y\left(\begin{pmatrix} i \\ j \end{pmatrix}\right)$ defines a $C^1$-surface, $\tilde{Y}\left(\begin{pmatrix} i \\ j \end{pmatrix}\right)$, parametrized by

$$\left\{(v, C\left(\begin{pmatrix} i \\ j \end{pmatrix}\right)(v), \xi^{k_0-1}C\left(\begin{pmatrix} i \\ j \end{pmatrix}\right)(v)) ; \|v - v_0\| \leq r_0 \right\}.$$

This defines a $C^1$-map $C\left(\begin{pmatrix} i \\ j \end{pmatrix}\right) : V \rightarrow [0, \mu_0], v \mapsto C\left(\begin{pmatrix} i \\ j \end{pmatrix}\right)(v)$ that satisfies

$$G_0\left(v, C\left(\begin{pmatrix} i \\ j \end{pmatrix}\right)(v)\right)\left(y\left(\begin{pmatrix} i \\ j \end{pmatrix}\right)(v, C\left(\begin{pmatrix} i \\ j \end{pmatrix}\right)(v))\right) = 1.$$
This implies that the vector field \( X^{(i)}(v) \), associated to the point \( (v, C^{(i)}(v)) \in B(k_0) \subset U \), will satisfy the homoclinic condition

\[
\gamma_0(\sigma_0 X^{(i)}(v)) \subset W^s(\sigma_1 X^{(i)}(v)).
\]

The same will apply to the intersection \( S_0 \cap Z^{(i)} \) where

\[
Z^{(i)} = \left\{ (v, \mu; Z^{(i)}(v, \mu)) ; (v, \mu) \in B(k_0) \right\}.
\]

Next we consider

\[
C\left( \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix} \right) = \left\{ (v, \mu; C\left( \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix} \right)(v, \mu)) ; (v, \mu) \in B(k_0) \right\}
\]

\[
= \{ (v, \mu; pr(v, \mu)); (v, \mu) \in B(k_0), \Gamma \in \Sigma_1 \}.
\]

For any given \( C^1 \)-surface \( \{(v, \mu; pr(v, \mu)); (v, \mu) \in B(k_0)\} = P_0 \), we have that \( P_0 \) is transversal to \( S_0 \) and hence the intersection \( S_0 \cap P_0 \) will define a \( C^1 \)-surface, \( C_0 \), parametrized by \( \{ (v, C_0(v)); P_0(v, C_0(v)) ; v \in V \} \). We denote by \( X_0(v) \) the vector field associated to \( (v, C_0(v)) \in B(k_0) \subset U \). This vector field must satisfy one of the following conditions:

(i) the point \( pr(v, C_0(v)) \) represents a periodic point of the map \( G(v, C_0(v)) \). In this case denote by \( \sigma(pr(v, C_0(v))) \) the hyperbolic periodic orbit of the vector field \( X_0(v) \) associated to \( pr(v, C_0(v)) \). Under these conditions we must have \( \gamma_0(\sigma_0 X_0(v)) \subset W^s(\sigma(pr(v, C_0(v)))) \), that is, the vector field \( X_0(v) \) presents a contracting singular cycle or

(ii) the point \( pr(v, C_0(v)) \) has recurrent behavior with respect to the set \( C\left( \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix} \right)(v, C_0(v)) \) under the map \( G(v, C_0(v)) \). In this case the trajectory \( \gamma_0(\sigma_0 X_0(v)) \) has recurrent behavior in the neighborhood \( U \); or

(iii) the point \( pr(v, C_0(v)) \) is eventually periodic under the map \( G(v, C_0(v)) \) (that is there is \( s \in \mathbb{N} \) such that \( G_{s_0}(v, C_0(v), pr(v, C_0(v))) \) is a periodic point of the map \( G(v, C_0(v)) \)). In this case the situation for the vector field \( X_0(v) \) is analogous to (i) above.

Now take any preimage, \( s(v, \mu) \), of the points \( b(v, \mu) \) or \( a(v, \mu) \), in the closure of the set \( C\left( \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix} \right)(v, \mu) \). Since the \( C^1 \) surface \( S = \{(v, \mu, s(v, \mu)); (v, \mu) \in B(k_0)\} \) is transversal to \( S_0 \) then the intersection \( S \cap S_0 \) define a \( C^1 \) surface \( S_b \) (resp \( S_a \)) parametrized by \( \{ (v, b(v), s(v, b(v)) ; v \in V \} \) (resp. \( \{ (v, a(v), s(v, a(v)) ; v \in V \} \)). Let denote by \( X_b(v) \) (resp. \( X_a(v) \)) the vector field associated to \( (v, b) \in B(k_0) \) (resp. \( (v, a) \in B(k_0) \)). This vector field satisfies that:

\[
\gamma_0(\sigma_0 X_b(v)) \subset W^s(\sigma_1 X_b(v))\). (resp. \( \gamma_0(\sigma_0 X_a(v)) \subset W^s(\sigma_1 X_a(v)))\).
\]
2.7.
In general let us consider the set of bisequences
\[ \Sigma_0 = \left\{ \Gamma : \mathbb{N} \to \left\{ \left( \begin{array}{c} i \\ j \end{array} \right), i = 1, 2; j \geq 0 \right\} \right\} \]
and the map
\[ G(v, \mu, \cdot) : \bigcup_{i=1}^{2} \left( \bigcup_{j \geq 0} D\left( \begin{array}{c} i \\ j \end{array} \right)(v, \mu) \right) \to [1 - \delta, 1] \]
given by
\[ G(v, \mu, y) = g\left( \begin{array}{c} i \\ j \end{array} \right)(v, \mu; y), y \in D\left( \begin{array}{c} i \\ j \end{array} \right)(v, \mu) \]
and \((v, \mu) \in B(k_0)\).

Denote by \( M(v, \mu) \) the set of points \( y \in [1 - \delta, 1] \) such that it is defined \( G_k(v, \mu, y) \)
for all \( k \in \mathbb{N} \).

Associated with any \( y \in M(v, \mu) \) we can define a bisequence \( \Gamma(v, \mu)(y) \in \Sigma_0 \) by:
\[ (\Gamma(v, \mu)(y))(k) = \left( \begin{array}{c} i \\ j \end{array} \right) \iff G_k(v, \mu, y) \in D\left( \begin{array}{c} i \\ j \end{array} \right)(v, \mu) \]
Clearly \( \Gamma(v, \mu) : M(v, \mu) \to \Sigma_1 \) is continuous and satisfies \( \Gamma(v, \mu) \circ G(v, \mu) = \sigma_1 \circ \Gamma(v, \mu) \). Here \( \sigma_0 : \Sigma_0 \to \Sigma_0 \) is the shift map \( \sigma_0(\Gamma)(k) = \Gamma(k + 1) \).

**Definition 3.** - We will say that the bisequence \( \Gamma \in \Sigma_0 \) is admissible at the level \((v, \mu)\) if \( \Gamma(v, \mu)^{-1}(\Gamma) \neq \emptyset \).

**Remark 3.** - 1) We note that \( \Gamma(v, \xi^{-(k_0-1)}) \) is a surjective map, for any \((v, \xi^{-(k_0-1)}) \in B(k_0)\).
2) From 1) we conclude that, given \( \Gamma \in \Sigma_0 \), we can find a first parameter value \( \mu_\Gamma(v); \xi^{-(k_0-1)}(1 - \delta) \leq \mu_\Gamma(v) \leq \xi^{-(k_0-1)} \) such that \( \Gamma \) is admissible at the level \((v, \mu)\), any \( \mu \geq \mu_\Gamma(v) \) [for instance \( \mu_\Gamma(v) = \xi^{-(k_0-1)}(1 - \delta) \), any \( \Gamma \in \Sigma_1 \)].

**Definition 4.** - Assume \((v, \mu) \in B(k_0)\) is a parameter value that satisfies \( \{1 - \delta, 1\} \subset M(v, \mu) \). In this case we will call the bisequence \( \sigma_0(\Gamma(v, \mu))(1) = \sigma_0(\Gamma(v, \mu))(1 - \delta) \) the itinerary of the map \( G(v, \mu, \cdot) \), and we will denote it by \( \Theta(v, \mu) \). We will say a bisequence \( \Gamma \in \Sigma_0 \) is realizable if there is a parameter value \((v, \mu) \in B(k_0)\) such that \( \Theta(v, \mu) = \Gamma \). We will denote the bisequence \( \Gamma(v, \mu)(1) \) (resp. \( \Gamma(v, \mu)(1 - \delta) \)) by \( \Gamma_1(v, \mu)(\text{resp.} \Gamma_{1-\delta}(v, \mu)) \).
Remark 4. — The only bisequence that satisfies \( \Gamma = \left( \begin{array}{c} 1 \\ 0 \\ \end{array} \right) \) and is realizable is the bisequence \( \left( \begin{array}{c} 1 \\ 0 \\ \end{array} \right), \left( \begin{array}{c} 1 \\ 0 \\ \end{array} \right), \cdots \). From here we conclude that there are bisequences which are not realizable.

Denote by \( \text{Per}(\sigma_0) \subset \Sigma_0 \) the set of all periodic bisequences \( \Gamma \in \Sigma_0 \). It is clear that \( \text{Per}(\sigma_0) \) is a dense subset of \( \Sigma_0 \). Let \( \Sigma_2 \subset \text{Per}(\sigma_0) \) be the set of all periodic bisequences \( \Gamma \in (\text{Per}(\sigma_0) \setminus \Sigma_1) \) such that \( \Gamma = \left( \begin{array}{c} 1 \\ 0 \\ \end{array} \right) \) or \( \Gamma = \left( \begin{array}{c} 2 \\ 0 \\ \end{array} \right) \).

Given \( \Gamma \in \Sigma_2 \) we let \( \Gamma_0 \) denote its period (i.e., \( \Gamma = (\Gamma_0, \Gamma_0, \Gamma_0, \cdots) \)). We have the following proposition:

**Proposition 1.** — For those \( \Gamma \in \Sigma_2 \) which satisfy that \( \sigma_0(\Gamma) \) is realizable and the number of \( \left( \begin{array}{c} 1 \\ 0 \\ \end{array} \right) \) that appears in \( \Gamma_0 \) is odd, we can find values of the parameter \( \mu_{\Gamma_0}(v) < \mu_{\Gamma_0}(v) < \mu_{2\Gamma_0}(v) \) such that:

i) for any \( (v, \mu) \in B(\Gamma_0), \mu_{\Gamma_0}(v) < \mu < \mu_{\Gamma_0}(v) \), the associated one-dimensional map \( G(v, \mu, \cdot) \) has an attracting, hyperbolic, periodic orbit whose period is \( \#(\Gamma_0) \). Moreover, one point of this orbit is contained in \( D(\sigma_0^k(\Gamma_0))(v, \mu), \) any \( 0 \leq k \leq \#(\Gamma_0) - 1 \).

ii) for any \( (v, \mu) \in B(\Gamma_0), \mu_{\Gamma_0}(v) < \mu < \mu_{2\Gamma_0}(v) \), the associated one-dimensional map \( G(v, \mu, \cdot) \) has an attracting, hyperbolic, periodic orbit whose period is \( 2\#(\Gamma_0) \). Moreover, two points of this orbit are contained in \( D(\sigma_0^k(\Gamma_0))(v, \mu), \) any \( 0 \leq k \leq \#(\Gamma_0) - 1 \).

iii) for \( (v, \mu_{\Gamma_0}(v)) \in B(\Gamma_0) \) we have that \( D(\sigma_0^k(\Gamma_0))(v, \mu) \) is a single point, and the associated one-dimensional map \( G(v, \mu, \cdot) \) satisfies

\[
G_1(\Gamma_0)(v, \mu)(D(\sigma_0^k(\Gamma_0))(v, \mu)) = D(\sigma_0^k(\Gamma_0))(v, \mu),
\]

any \( 0 \leq k \leq \#(\Gamma_0) - 1 \).

iv) for \( (v, \mu_{\Gamma_0}(v)) \in B(\Gamma_0) \) the associated one-dimensional map \( G(v, \mu, \cdot) \) has a flip bifurcation of the attracting periodic orbit. Moreover, one point of this orbit is contained in the interior of \( D(\sigma_0^k(\Gamma_0))(v, \mu_{\Gamma_0}(v)), \) any \( 0 \leq k \leq \#(\Gamma_0) - 1 \).

v) for \( (v, \mu_{2\Gamma_0}(v)) \in B(\Gamma_0) \) the associated one-dimensional map \( G(v, \mu_{2\Gamma_0}(v), \cdot) \) satisfies

\[
G_1(\Gamma_0)(\partial D(\sigma_0^k(\Gamma_0))(v, \mu_{2\Gamma_0}(v))) = \partial D(\sigma_0^k(\Gamma_0))(v, \mu_{2\Gamma_0}(v))
\]

and interchanges the points in \( \partial D(\sigma_0^k(\Gamma_0))(v, \mu_{2\Gamma_0}(v)), \) any \( 0 \leq k \leq \#(\Gamma_0) - 1 \).

In particular for \( \Gamma_0 = \left( \begin{array}{c} 1 \\ 0 \\ \end{array} \right) \) (resp. \( \Gamma_0 = \left( \begin{array}{c} 2 \\ 0 \\ \end{array} \right) \)) we have that \( G_2(\Gamma_0)(v, \mu, 1 - \delta) = 1 - \delta \) (resp. \( G_2(\Gamma_0)(v, \mu, 1 - \delta) = 1 \), \( \mu = \mu_{2\Gamma_0} \)).

vi) for \( \mu_{\Gamma_0}(v) \leq \mu \leq \mu_{2\Gamma_0}(v) \), the pre-image \( \Gamma(v, \mu)^{-1}(\sigma_0^k(\Gamma)) \) is the interval \( D(\sigma_0^k(\Gamma))(v, \mu) \).

vii) for any \( (v, \mu) \in B(\Gamma_0) \) such that \( \mu > \mu_{2\Gamma_0}(v) \), we have that \( \Gamma(v, \mu)^{-1}(\sigma_0^k(\Gamma)) \) is a hyperbolic repelling fixed point of the map \( G_2(\Gamma_0)(v, \mu, \cdot) \). Moreover \( D(\sigma_0^k(\Gamma))(v, \mu) \) is exactly this repelling fixed point and

viii) all the maps \( v \to \mu_{\Gamma_0}(v), v \to \mu_{\Gamma_0}(v), \) and \( v \to \mu_{2\Gamma_0}(v) \) are \( C^1 \).
Proof. Without loss assume \( \Gamma = (\Gamma_0, \Gamma_0, \ldots) \) where \( \Gamma_0 = \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \). Later we will make some comments on the general case.

In this situation \( \mu_{\Gamma_0} = \xi^{-(k_0-1)}(1-\delta) \). For \( (v, \mu) \in B(k_0) \) and \( y \in D \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \) define:

\[
E(v, \mu; y) = G(v, \mu, y) - y.
\]

We have:

\[
\frac{\partial E}{\partial y}(v, \mu; y) \bigg|_{y=1-\delta, \mu=\xi^{-(k_0-1)}(1-\delta)} = -1
\]

By applying the implicit function theorem we can find a \( C^2 \)-map \( y = y(v, \mu) \in D \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \) such that \( E(v, \mu; y(v, \mu)) = 0 \).

That is, \( G(v, \mu, y(v, \mu)) = y(v, \mu) \).

For fixed \( v \) such that \( (v, \mu) \in B(k_0) \) we have:

\[
\frac{\partial y}{\partial \mu}(v, \mu) = \frac{\frac{\partial G}{\partial \mu}(v, \mu; y(v, \mu))}{1 - \frac{\partial G}{\partial y}(v, \mu; y(v, \mu))}.
\]

Since \( \frac{\partial G}{\partial \mu}(v, \mu; y) > 0; (v, \mu) \in B(k_0), y \in D \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \) and \( \frac{\partial G}{\partial y}(v, \mu; y) \leq 0, \) for \( (v, \mu) \in B(k_0), y \in D \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \), we conclude that \( \frac{\partial y}{\partial \mu}(v, \mu) > 0, (v, \mu) \in B(k_0) \) and

\[
\frac{\partial y}{\partial \mu}(v, \mu) \leq \frac{\partial y}{\partial \mu}(v, \mu) \bigg|_{\mu=\xi^{-(k_0-1)}(1-\delta)} = \xi^{k_0-1}.
\]

Since

\[
\frac{\partial G}{\partial y}(v, \mu, y(v, \mu)) \bigg|_{\mu=\xi^{-(k_0-1)}(1-\delta)} = 0
\]

we conclude, for \( \mu \) near \( \xi^{-(k_0-1)}(1-\delta) \) such that \( (v, \mu) \in B(k_0) \) that \( y = y(v, \mu) \) is an attracting fixed point for the map \( G(v, \mu, \cdot) \).

Now a cumbersome computation will show that

\[
\frac{\partial}{\partial \mu} \left( \frac{\partial}{\partial y} (G(v, \mu, y)) \right) \bigg|_{y=y(v, \mu)} \leq 0.
\]

Moreover, for \( \mu > \xi^{-(k_0-1)}(1-\delta) \) we have:

\[
\frac{\partial G}{\partial \mu}(v, \mu, y(v, \mu)) = -\frac{\xi^{k_0-1} \mu - y(v, \mu)}{J(v, \mu, y(v, \mu))} \left[ \frac{\partial J}{\partial y}(v, \mu, y) + \frac{\alpha J^{1+\frac{1}{\alpha}} \xi^{k_0-1} y}{\xi^{k_0-1} \mu - y^\alpha} \right].
\]
So, there exist a unique value \( \mu = \mu_{I_0}^f(v) \) such that
\[
\frac{\partial G}{\partial y}(v, \mu, y)\bigg|_{\mu = \mu_{I_0}^f(v)} = -1.
\]

Now it is not hard to see that:
\[
\frac{\partial^3}{\partial y^3}(G(v, \mu, y(v, \mu)))\bigg|_{\mu = \mu_{I_0}^f(v)} < 0.
\]

Under these circumstances we may consider the \( C^2 \)-map
\[
H(v, \mu; y) = \begin{cases} 
\frac{G_2(v, \mu, y) - y}{y - y(v, \mu)}, & y \neq y(v, \mu) \\
\frac{\partial}{\partial y}(G_2(v, \mu, y)) - 1, & y = y(v, \mu).
\end{cases}
\]

Clearly \( H(v, \mu_{I_0}^f(v), y(v, \mu_{I_0}^f(v))) = 0 \) and
\[
\frac{\partial H}{\partial \mu}(v, \mu; y)\bigg|_{\mu = \mu_{I_0}^f(v)} = \frac{\partial}{\partial y}\left(\frac{\partial G_2(v, \mu, y)}{y(v, \mu_{I_0}^f(v))}\right)\bigg|_{y = y(v, \mu_{I_0}^f(v))} \neq 0.
\]

In this case there is a smooth map \( \mu = \mu(v, y) \) such that \( H(v, \mu(v, y), y) = 0 \).
For \( y \neq y(v, \mu) \) we have \( G_2(v, \mu, y) = y \) which is a period two point for the map \( G(v, \mu, \cdot) \).

It is easy to see that
\[
\frac{\partial \mu}{\partial y}(v, y)\bigg|_{y = y(v, \mu)} = 0
\]
and that
\[
\frac{\partial^2 \mu}{\partial y^2}\bigg|_{y = y(v, \mu)} > 0.
\]

We note that, whenever defined, the interval \( \{(v, \mu)\} \times [0, 1] \) intersects the graph of the map \( \mu = \mu(v, y) \) into two points: \( (v, \mu; y_1), (v, \mu; y_2) \). These two points satisfy \( G(v, \mu(v, y_1), y_1) = y_2, G(v, \mu(v, y_2), y_2) = y_1, \) and \( y_1 \leq y(v, \mu) \leq y_2 \). Since
\[
\left| \frac{\partial G_2(v, \mu, y(v, \mu))}{\partial y} \right| > 1,
\]
for \( \mu \geq \mu_{I_0}^f(v) \), and since this absolute value is equal to one only for \( \mu = \mu_{I_0}^f(v) \), we have that
\[
\left| \frac{\partial G_2(v, \mu(v, y_2), y_2)}{\partial y} \right| < 1,
\]
any \( \mu > \mu_{I_0}^f(v) \) wherever \( y_2 \) is defined.
Since the graph of the map \( \mu = \mu(v, y) \) intersects transversally the graph of the map \( (v, \mu) \mapsto G(v, \mu 1 - \delta) \), their intersection defines a \( C^1 \)-map \( \mu = \mu_{2\Gamma_0}(v) \) and thus the proof of Proposition 1 is now complete in the case \( \Gamma_0 = \left( \begin{array}{c} 1 \\ 0 \\ \vdots \end{array} \right) \).

In the general case we can proceed as follows:

Let \( \Gamma_0 = \left( \begin{array}{c} 1 \\ 0 \\ \vdots \end{array} \right) \) here \( r = \sharp(\Gamma_0) - 1 \), and consider

\[
D\left( \begin{array}{c} 1 \\ 0 \\ \vdots \\ \vdots \\ 1 \\ 0 \end{array} \right) (v, \mu)
= D(\Gamma_0)(v, \mu) \subset D\left( \begin{array}{c} 1 \\ 0 \\ \vdots \end{array} \right) (v, \mu) \subset \cdots \subset D\left( \begin{array}{c} 1 \\ 0 \end{array} \right) (v, \mu).
\]

Clearly we have \( G_{\sharp(\Gamma_0)}(v, \mu, 1 - \delta) \in D(\Gamma_0)(v, \mu) \).

Let \( \mu_{\Gamma_0}(v) = \inf \{ \mu; (v, \mu) \in B(k_0), \Theta(v, \mu) = \sigma_0(\Gamma) \} \). For \( \mu = \mu_{\Gamma_0}(v) \) we must have \( G_{\sharp(\Gamma_0)}(v, \mu, 1 - \delta) = 1 - \delta \) (and therefore \( D(\Gamma_0)(v, \mu_{\Gamma_0}(v)) = 1 - \delta \)).

Now we define the map \( E(v, \mu, y), y \in D(\Gamma_0)(v, \mu), (v, \mu) \in B(k_0) \) such that \( \mu \geq \mu_{\Gamma_0}(v) \) by:

\[
E(v, \mu, y) = G_{\sharp(\Gamma_0)}(v, \mu, y) - y
\]

Now the proof of the proposition 1 follows as in the previous case.

2.8.

Let \( \Gamma \in \Sigma_2 \) and denote by \( \Gamma_0 \) its period.

**Proposition 2.** – For those \( \Gamma \in \Sigma_2 \) such that \( \sigma_0(\Gamma) \) is realizable and the number of \( \frac{1}{j} \) that appears in \( \Gamma_0 \) is even, we can find values of the parameter \( \mu_{\Gamma}(v) = \mu_{\Gamma_0}(v) < \mu_{\Gamma_0}(v) \) such that:

i) for \( (v, \mu_{\Gamma_0}(v)) \in B(k_0) \), the associated one-dimensional map \( G(v, \mu_{\Gamma_0}(v), \cdot) \) has a saddle-node bifurcation whose period is \( \sharp(\Gamma_0) \). Moreover, one point of this orbit is contained in the boundary of the interval \( D(\sigma_0^k(\Gamma))(v, \mu) \), any \( 0 \leq k \leq \sharp(\Gamma_0) - 1 \).

ii) for \( (v, \mu) \in B(k_0) ; \mu_{\Gamma_0}(v) < \mu < \mu_{\Gamma_0}(v) \), the associated one-dimensional map \( G(v, \mu, \cdot) \) has an attracting, hyperbolic, periodic orbit and a repelling, hyperbolic, periodic orbit contained in the interior of \( D(\Gamma)(v, \mu) \cup D(\sigma_0^k(\Gamma))(v, \mu) \cup \cdots \cup D(\sigma_0^k(\Gamma)(\Gamma_0 - 1))(v, \mu) \).

Moreover one point, of any of the two periodic orbits, is contained in \( D(\sigma_0^k(\Gamma))(v, \mu) \), any \( 0 \leq k \leq (\sharp(\Gamma_0) - 1) \).

iii) for \( (v, \mu = \mu_{\Gamma_0}(v)) \in B(k_0) \), the associated one-dimensional map satisfies

\[
G_{\sharp(\Gamma_0)}(v, \mu, \partial D(\sigma_0^k(\Gamma))(v, \mu)) = \partial D(\sigma_0^k(\Gamma))(v, \mu).
\]

Under these circumstances the points in the boundary are fixed points for the map \( G_{\sharp(\Gamma_0)} \).

Note that the boundary \( \partial D(\Gamma)(v, \mu) \) contains \( 1 - \delta \) or \( 1 \) depending on \( \Gamma_0 = \left( \begin{array}{c} 1 \\ 0 \\ \vdots \end{array} \right) \) or \( \left( \begin{array}{c} 2 \\ 0 \\ \vdots \end{array} \right) \), respectively.
iv) for \((v, \mu) \in B(k_0); \mu_{T_0}^{\alpha}(v) \leq \mu \leq \mu_{2T_0}(v)\) the pre-image \(\Gamma(v, \mu)^{-1}(\sigma_0^v(\Gamma))\) is the interval \(D(\sigma_0^v(\Gamma))(v, \mu)\).

v) for any \((v, \mu) \in B(k_0)\) such that \(\mu > \mu_{T_0}(v)\) we have that \(\Gamma(v, \mu)^{-1}(\sigma_0^v(\Gamma))\) is a hyperbolic, repelling fixed point of the map \(G_0^v(\Gamma)(v, \mu)\). Moreover \(D(\sigma_0^v(\Gamma))(v, \mu)\) is exactly this repelling fixed point.

vi) The maps \(V \rightarrow [1 - \delta, 1]; v \mapsto \mu_{\Gamma_0}^{\alpha}(v)\), and \(v \mapsto \mu_{\Gamma_0}(v)\) are \(C^1\).

Proof. - Assume \(\Gamma_0 = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \cdots\). Later we will comments on the general case.

In this situation \(\mu_{\Gamma_0}(v) = \xi^{-(k_0 - 1)}\).

For \((v, \mu) \in B(k_0)\) and \(y \in D(\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix})(v, \mu)\) define the map: \(E(v, \mu; y) = G(v, \mu; y) + y\).

We have :

\[ E(v, \mu; y) = \xi^{k_0 - 1}[\mu - K(v, \mu; y)(1 - y)^\alpha] - y. \]

and, hence, \(\frac{\partial E}{\partial \mu}(v, \mu; y)|_{y=1} = \xi^{k_0 - 1} \neq 0\), for any \((v, \mu) \in B(k_0)\). Therefore, by the implicit function theorem we obtain a \(C^1\)-map, twice differentiable in the \(y\)-variable \(\mu = \mu(v, y)\) such that: We solve the equation \(E(v, \mu; y) = 0\) for \((v, \mu) \in B(k_0), y \in D(\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix})(v, \mu)\) if and only if \(\mu = \mu(v, y)\).

From the relation \(E(v, \mu(v, y); y) = 0\) we obtain

\[
\frac{\partial \mu}{\partial y}(v, y) = \frac{\xi^{k_0 - 1}[\frac{\partial K}{\partial y}(v, \mu; y)(1 - y)^\alpha - \alpha K(v, \mu; y)(1 - y)^{\alpha - 1}] - 1}{\xi^{k_0} - \frac{\partial K}{\partial \mu}(v, \mu; y)(1 - y)^{\alpha}},
\]

and from this relation we have that: \(\frac{\partial \mu}{\partial y}(v, y) = 0\) if and only if

\[ H(v, y) = -\frac{\partial K}{\partial y}(v, \mu(v, y); y)(1 - y)^\alpha + \alpha K(v, \mu(v, y); y)(1 - y)^{\alpha - 1} - \xi^{-(k_0 - 1)} = 0. \]

Since \(|1 - y|\) is small, \(K(v, \mu; y) \neq 0\) and

\[
\frac{\partial H}{\partial y}(v, y) = (1 - y)^{\alpha - 2}\left[\frac{\partial^2 K}{\partial y^2}(v, \mu(v, y); y)(1 - y)^2 + 2\alpha \frac{\partial K}{\partial \mu}(v, \mu(v, y); y)(1 - y) - \alpha(\alpha - 1)K(v, \mu(v, y), y)\right],
\]

we have \(\frac{\partial H}{\partial y}(v, y) \neq 0\), any \((v, y)\) such that \(H(v, y) = 0\).

Hence by the implicit function theorem we find a \(C^1\)-map, \(y = y(v)\), that simultaneously satisfies equations \(E(v, \mu(v, y(v)); y(v)) = 0\) and \(\frac{\partial \mu}{\partial y}(v, y(v)) = 0\).
Figure 4 shows the above relations obtained for the maps $\mu(v, y)$ and $y(v)$.

Denote by $\mu^{\infty}_\Gamma = \mu(v, y(v))$. For this map we have:

$$G(v, \mu^{\infty}_\Gamma, y(v)) = y(v); \quad \frac{\partial G}{\partial y} (v, \mu^{\infty}_\Gamma, y(v)) \equiv 1$$

and

$$\frac{\partial^2 G}{\partial y^2} (v, \mu^{\infty}_\Gamma, y) \neq 0$$

That is; the one dimensional map $G(v, \mu^{\infty}_\Gamma, \cdot)$, has a saddle-node at the point $y = y(v) \in D(\frac{\partial}{\partial y})(v, \mu^{\infty}_\Gamma)$.

Now assume $(v, \mu) \in B(\gamma_0)$ satisfies $\mu^{\infty}_\Gamma < \mu < \mu^{\infty}_\Gamma(v)$. In this case the interval $\{(v, \mu)\} \times [1 - \delta, 1]$ intersects the graph of the map $\mu(v, y)$ into two points $(v, \mu; y_1)$ and $(v, \mu; y_2)$. These two points satisfy $G(v, \mu; y_1) = y_1$ and $G(v, \mu; y_2) = y_2$ with $y_1 < y_2$.

Again, an easy computation shows $\frac{\partial G}{\partial y} (v, \mu; y_1) > 1 > \frac{\partial G}{\partial y} (v, \mu; y_2)$. That is the map $G(v, \mu; \cdot)$ has a hyperbolic, attracting periodic orbit whose period is $\gamma_0$, at $y = y_2$; and a hyperbolic repelling, fixed point at $y = y_1$.

Observe that, for $(v, \mu) \in B(\gamma_0), \mu \leq \mu^{\infty}_\Gamma$, the one dimensional map $G(v, \mu; \cdot)$ does not have fixed points in $D(\frac{\partial}{\partial y})(v, \mu)$.

In the general case we can proceed as follows:

Let $\Gamma_0 = \left( \begin{pmatrix} 2
0
\end{pmatrix}, \begin{pmatrix} i_1
j_1
\end{pmatrix}, \cdots, \begin{pmatrix} i_r
j_r
\end{pmatrix} \right)$, here $r = \#(\Gamma_0) - 1$. Let us consider

$$D\left( \begin{pmatrix} 2
0
\end{pmatrix}, \begin{pmatrix} i_1
j_1
\end{pmatrix}, \cdots, \begin{pmatrix} i_r
j_r
\end{pmatrix} \right)(v, \mu) \subset D\left( \begin{pmatrix} 2
0
\end{pmatrix}, \begin{pmatrix} i_1
j_1
\end{pmatrix}, \cdots, \begin{pmatrix} i_{r-1}
j_{r-1}
\end{pmatrix} \right)(v, \mu) \subset \cdots \subset D\left( \begin{pmatrix} 2
0
\end{pmatrix} \right)(v, \mu).$$

Clearly we have $G_{\#(\Gamma_0)}(v, \mu, 1) \in D(\Gamma)(v, \mu)$. Let $\mu_{\Gamma_0}(v) = \sup\{\mu; (v, \mu) \in B(\gamma_0), \Theta(v, \mu) = \sigma_0(\Gamma)\}$. For $\mu = \mu_{\Gamma_0}(v)$ we must have
Now we define the map $E(v, \mu; y)$, $y \in D(\Gamma_0)(v, \mu)$, $(v, \mu) \in B(k_0)$ such that $\mu \leq \mu \Gamma_0(v)$ by:

$$E(v, \mu; y) = G_\#(\Gamma_0)(v, \mu; y) - y.$$  

Now the proof follows as in the previous case.

As a consequence of proposition 1 and 2 we get the following:

**Remark 5.** Assume $\Gamma_1(v, \mu)$ or $\Gamma_{1-\delta}(v, \mu)$ is a periodic itinerary. In this situation the associated one dimensional map $G(v, \mu, \cdot)$ satisfies one of the following:

(i) $D(\Gamma_1(v, \mu))(v, \mu)$ (or $D(\Gamma_{1-\delta}(v, \mu))(v, \mu)$) is an interval which contains, in its interior, a hyperbolic, attracting periodic orbit or

(ii) $D(\Gamma_1(v, \mu))(v, \mu)$ (or $D(\Gamma_{1-\delta}(v, \mu))(v, \mu)$) is an interval which contains a flip or a saddle-node periodic orbit or

(iii) $D(\Gamma_1(v, \mu))(v, \mu)$ (or $D(\Gamma_{1-\delta}(v, \mu))(v, \mu)$) is an interval and $y = 1$ (or $y = 1 - \delta$) is an attracting periodic orbit or

(iv) $D(\Gamma_1(v, \mu))(v, \mu) = \{1\}$ (or $D(\Gamma_{1-\delta}(v, \mu))(v, \mu) = \{1 - \delta\}$).

Let us now define an order relation among the elements of $\Sigma_0$.

We initially define

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} < \begin{pmatrix} 1 \\ 1 \end{pmatrix} < \cdots < \begin{pmatrix} 1 \\ n \end{pmatrix} < \begin{pmatrix} 2 \\ n+1 \end{pmatrix} < \cdots < \begin{pmatrix} 2 \\ n \end{pmatrix} < \cdots < \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

Let $\Gamma_1 \neq \Gamma_2$ be any two bisequences. Assume that

$$\left(\begin{pmatrix} i^1_0 \\ j^1_0 \end{pmatrix}, \ldots, \begin{pmatrix} i^1_k \\ j^1_k \end{pmatrix}\right) = \left(\begin{pmatrix} i^2_0 \\ j^2_0 \end{pmatrix}, \ldots, \begin{pmatrix} i^2_k \\ j^2_k \end{pmatrix}\right)$$

and that $\begin{pmatrix} i^1_{k+1} \\ j^1_{k+1} \end{pmatrix} \neq \begin{pmatrix} i^2_{k+1} \\ j^2_{k+1} \end{pmatrix}$.

- If there is an even number of $\begin{pmatrix} 1 \\ j \end{pmatrix}$ among $\begin{pmatrix} i^1_0 \\ j^1_0 \end{pmatrix}, \ldots, \begin{pmatrix} i^1_k \\ j^1_k \end{pmatrix}$ and $\begin{pmatrix} i^1_{k+1} \\ j^1_{k+1} \end{pmatrix} > \begin{pmatrix} i^2_{k+1} \\ j^2_{k+1} \end{pmatrix}$, we will say $\Gamma_1$ is greater than $\Gamma_2$ and we will denote $\Gamma_1 > \Gamma_2$.

- If there is an odd number of $\begin{pmatrix} 1 \\ j \end{pmatrix}$ among $\begin{pmatrix} i^1_0 \\ j^1_0 \end{pmatrix}, \ldots, \begin{pmatrix} i^1_k \\ j^1_k \end{pmatrix}$ and $\begin{pmatrix} i^1_{k+1} \\ j^1_{k+1} \end{pmatrix} < \begin{pmatrix} i^2_{k+1} \\ j^2_{k+1} \end{pmatrix}$, we will say $\Gamma_1$ is greater than $\Gamma_2$ and we will denote $\Gamma_1 > \Gamma_2$.

**Lemma 8.** The map $\Gamma(v, \mu) : M(v, \mu) \to \Sigma_0$ is order-preserving.

**Proof.** Let $x_1, x_2 \in M(v, \mu)$ be two points such that $x_1 \leq x_2$. If $x_1 \in D\left(\begin{pmatrix} i_0 \\ j_0 \end{pmatrix}\right)(v, \mu)$ and $x_2 \in D\left(\begin{pmatrix} i_1 \\ j_1 \end{pmatrix}\right)$ with $\begin{pmatrix} i_0 \\ j_0 \end{pmatrix} \neq \begin{pmatrix} i_1 \\ j_1 \end{pmatrix}$, the result follows.

Assume $\Gamma(v, \mu)(x_1) = \Gamma_1$, and $\Gamma(v, \mu)(x_2) = \Gamma_2$ are such that

$$\left(\begin{pmatrix} i^1_0 \\ j^1_0 \end{pmatrix}, \ldots, \begin{pmatrix} i^1_k \\ j^1_k \end{pmatrix}\right) = \left(\begin{pmatrix} i^2_0 \\ j^2_0 \end{pmatrix}, \ldots, \begin{pmatrix} i^2_k \\ j^2_k \end{pmatrix}\right)$$

and $\begin{pmatrix} i^1_{k+1} \\ j^1_{k+1} \end{pmatrix} \neq \begin{pmatrix} i^2_{k+1} \\ j^2_{k+1} \end{pmatrix}$. 


If there is an even number of \( \frac{1}{i_j} \)'s among the \( \left( \frac{1}{i_0}, \ldots, \frac{1}{i_k} \right) \), then the restriction of the map \( G_k(v, \mu) \) to the interval that contains \([x_1, x_2]\) preserves orientation. This implies that \( G_k(v, \mu)(x_1) \leq G_k(v, \mu)(x_2) \) and therefore \( \left( \frac{i_{k+1}}{j_{k+1}} \right) < \left( \frac{i_{k+1}}{j_{k+1}} \right) \). By the definition of the order relation in \( \Sigma_0 \) this implies \( \Gamma_1 < \Gamma_2 \).

If there is an odd number of \( \frac{1}{i_j} \)'s among the \( \left( \frac{1}{i_0}, \ldots, \frac{1}{i_k} \right) \), then the restriction \( G_k(v, \mu)(\cdot) \) to the interval \( D\left( \left( \frac{1}{i_0'} \ldots, \frac{1}{i_k'} \right) \right) \), which contains \([x_1, x_2]\), reverses orientation. This implies that \( G_k(v, \mu)(x_1) > G_k(v, \mu)(x_2) \) and therefore \( \left( \frac{i_{k+1}}{j_{k+1}} \right) > \left( \frac{i_{k+1}}{j_{k+1}} \right) \). By the definition of the order relation in \( \Sigma_0 \) we obtain \( \Gamma_1 < \Gamma_2 \).

Let us now consider two bisequences \( \Gamma_1, \Gamma_2 \in \Sigma_0 \) such that \( \Gamma(v, \mu)(x_1) = \Gamma_1, \Gamma(v, \mu)(x_2) = \Gamma_2 \), some \( x_1, x_2 \in M(v, \mu) \).

**Lemma 9.** If \( \Gamma_1 < \Gamma_2 \), then \( x_1 < x_2 \).

**Proof.** The proof is easy and left to the reader. ■

Let \( \Gamma \in \Sigma_0 \) be any realizable sequence and denote by \( \mu_\Gamma = \inf \{ \mu; \Theta(v, \mu) = \Gamma \} \). Let \( \Gamma_2 \in \Sigma_0 \) be any admissible bisequence at the level \( (v, \mu_\Gamma(v)) \) such that \( \Gamma_2 > \Gamma \).

**Lemma 10.** \( \Gamma_2 \) is realizable.

**Proof.** Denote by \( x_1(v, \mu) \in M(v, \mu), x_2(v, \mu) \in M(v, \mu) \) two points which satisfy \( \Gamma(v, \mu)(x_1(v, \mu)) = \Gamma \) and \( \Gamma(v, \mu)(x_2(v, \mu)) = \Gamma_2 \). We have \( x_1(v, \mu) < x_2(v, \mu) \) and \( x_1(v, \mu_\Gamma(v)) = \xi^{\mu_\Gamma(v)}1 \Gamma_1 \Gamma(v, \mu) \). Since \( \mu \mapsto \xi^{\mu_\Gamma(v)}1 \Gamma(v, \mu) \) is an increasing map we can find a parameter value \( \mu_2 > \mu_\Gamma(v) \) such that \( x_2(v, \mu_2) = \xi^{\mu_\Gamma(v)}1 \mu_2 \). This implies \( x_2(v, \mu_2) = \Gamma(v, \mu)(G(v, \mu_2, 1 - \delta)) = \sigma_0 \circ (\Gamma(v, \mu_2)(1 - \delta)) = \Theta(v, \mu_2) \). That is \( \Gamma_2 \) is realizable. ■

**Remark 6.** 1) Let \( \Gamma \in \Sigma_0 \) be any realizable sequence and \( \mu_\Gamma(v) = \inf \{ \mu; \Theta(v, \mu) = \Gamma \} \). Let \( \Gamma_2 \in \Sigma_0; \Gamma_2 \leq \Gamma \) be any bisequence which is not realizable for \( \xi^{-(\mu_\Gamma(v))}(1 - \delta) \leq \mu \leq \mu_\Gamma(v) \) then \( \Gamma_2 \) is not realizable at all, that is there no exists \( \xi^{-(\mu_\Gamma(v))}(1 - \delta) \leq \mu \leq \xi^{-(\mu_\Gamma(v))} \) such that \( \Theta(v, \mu) = \Gamma_2 \).

2) Assume \((v, \mu_1), (v, \mu_2) \in B(\kappa) \) satisfy \( \xi^{\mu_1}1 \mu_1 \in M(v, \mu_1) \), \( \xi^{\mu_1}1 \mu_2 \in M(v, \mu_2) \). If \( \mu_1 < \mu_2 \) then we have \( \Theta(v, \mu_1) = \Gamma(v, \mu_1)(\xi^{\mu_1}1 \mu_1 \leq \Theta(v, \mu_2) = \Gamma(v, \mu_2)(\xi^{\mu_1}1 \mu_2) \).

3) Assume \((v, \mu_1), (v, \mu_2) \in B(\kappa) \) satisfy \( \xi^{\mu_1}1 \mu_1 \in M(v, \mu_1) \), \( \xi^{\mu_1}1 \mu_2 \in M(v, \mu_2) \) and \( \Theta(v, \mu_1) < \Theta(v, \mu_2) \) then we have \( \mu_1 < \mu_2 \).

2.10.

Let \( \Gamma \in \Sigma_2 \) be any periodic bisequence which is realizable.

Assume \( \mu_\Gamma(v) = \inf \{ \mu; \Theta(v, \mu) = \Gamma \} \).
Let $F = F^r$, for $1 < k < \#(\Gamma_0) - 1$. Suppose $F > T$, for some $j$. By Lemma 21 we have that $T$ is realizable. In fact denote by $x_j(v,\mu) \in M(v,\mu)$ a point which satisfies $T(v,\mu)(x_j(v,\mu)) = T_j$. By (2.11) we know that $D(\Gamma_j)(v,\mu)$ is a hyperbolic, attracting, fixed point of the map $G_{\infty}(v,\mu)$, for $\mu > \mu_{2T_0}(v)$ or $\mu > \mu_{T_0}(v)$. Since the $C^1$-surface $C_T = \{(v,\mu; x_j(v,\mu))/\mu \geq \mu_{T_0}(v) \}$ is transversal to $S_0 = \{(v,\mu; x_j(v,\mu))/\mu \in B(k_0)\}$ we have that $S_0 \cap C_T$, define a $C^1$ surface contained in $U \times [1 - \delta,1]$ and parametrized by $\{(v,\Gamma_j(v),x_j(v,\Gamma_j(v)))/v \in V\}$.

Let us denote by $X_T(v)$ the vector field associated to $(v,\Gamma_j(v)) \in B(k_0)$.

Let $\sigma(x_j(v,\Gamma_j(v))) \subset U$ be the hyperbolic, periodic orbit associated to the point $x_j(v,\Gamma_j(v))$. We have

$$\gamma_0(\sigma_0(X_T(v))) \subset W^s(\sigma(x_j(v,\Gamma_j(v)))),$$

that is, the associated vector field $X_T(v)$ represents a contracting singular cycle.

(B) Let $X \in \Sigma_0, X > \Gamma$ be any admissible bisequence, at the level $(v,\mu_\Gamma(v))$, such that $\sigma_0^k(X) = \Gamma$, some $k \in \mathbb{N}$.

Let us denote by $x_X(v,\mu) \in M(v,\mu)$ a point which satisfies $\Gamma(v,\mu)(x_X(v,\mu)) = X$. We have: $\sigma_0^k \circ \Gamma(v,\mu)(x_X(v,\mu)) = \sigma_0^k(X) = \Gamma$. That is: $\Gamma(v,\mu)G_k(v,\mu)(x_X(v,\mu)) = \Gamma(v,\mu)(p_T(v,\mu))$ (here $p_T(v,\mu)$ denotes the fixed point of the map $G_{\infty}(v,\mu)$ which satisfies $p_T(v,\mu) \in D(\Gamma)(v,\mu)$. In particular, $G_k(v,\mu)(x_X(v,\mu)) \in D(\Gamma)(v,\mu)$. That is $x_X(v,\mu) = G^{-k}(v,\mu)(D(\Gamma)(v,\mu))$. From here we conclude that, for $\mu > \mu_{2T_0}(v)$ or $\mu > \mu_{T_0}(v)$, the point $x_X(v,\mu)$ is a pre-image of the hyperbolic, repelling, fixed point $p_T(v,\mu)$. So in particular

$$C_X = \{(v,\mu; x_X(v,\mu));(v,\mu) \in B(k_0), \mu > \mu_{2T_0}(v), \mu > \mu_{T_0}(v)\}$$

is a $C^1$-surface transversal to $S_0$. Therefore the intersection $S_0 \cap C_X$ defines a $C^1$-surface, $C_X^0$, contained in $U \times [1 - \delta,1]$ and parametrized by

$$\{(v,\sigma_0^0(x_X(v,\mu)),x_X(v,\sigma_0^0(x_X(v,\mu))));v \in V\}.$$

Denote by $X_X(v)$ the vector field associated to $(v,\sigma_0^0(x_X(v,\mu))) \in B(k_0)$.

Let $\sigma(p_T(v,\sigma_0^0(x_X(v,\mu)))) \subset U$ be the hyperbolic, periodic orbit associated to the point $p_T(v,\sigma_0^0(x_X(v,\mu))) \in M(v,\sigma_0^0(x_X(v,\mu)))$. We have

$$\gamma_0(\sigma_0(X_X(v))) \subset W^s(\sigma(p_T(v,\sigma_0^0(x_X(v,\mu))))),$$

that is, the vector field $X_X(v)$ has a contracting singular cycle.

2.11.

Let $\Gamma \in \Sigma_0$ be any realizable bisequence. Assume $\mu_\Gamma = \mu_\Gamma(v)$ is the parameter value which satisfies $\Theta(v,\mu_\Gamma(v)) = \Gamma$ and $x_\Gamma = x_\Gamma(v,\mu) \in M(v,\mu)$ be a point which satisfies

$$\Gamma(v,\mu)(x_\Gamma(v,\mu)) = \Gamma.$$  

(A) Assume $\Gamma \in \text{Per}(\sigma)$. In this case we have $\Gamma \in \Sigma_1$ or $\Gamma \in \Sigma_2$ or there is $k \in \mathbb{N}$ such that $\sigma_0^k(\Gamma) = \Sigma_2$. In all the cases, as we have seen in (2.6), (2.7) (2.8) and (2.10), we
known that associated to $\Gamma$ we can find a $C^1$-surface $C^0 = \{(v, C_\Gamma(v)); v \in V\} \subset B(k_0)$ such that: the vector field $X_\Gamma(v)$, which represents the point $(v, C_\Gamma(v)) \in C^0$, presents a contracting singular cycle or a homoclinic orbit for the singularity $\sigma_0(X_\Gamma(v))$ or a saddle-node or a flip bifurcation.

(B) Suppose that $\Gamma \not\in \text{Per}(\sigma)$ and that there is $k \in \mathbb{N}$ such that $\sigma_0^k(\Gamma) \in \text{Per}(\sigma)$. In this situation, as we have seen in (2.6) and (2.10), we know that associated to $\Gamma$, we can find a $C^1$-surface $C^0 = \{(v, C_\Gamma(v)); v \in V\} \subset B(k_0)$ such that: the vector field $X_\Gamma(v)$, which represents the point $(v, C_\Gamma(v)) \in C^0$, presents a contracting singular cycle.

(C) Suppose $\Gamma \not\in \text{Per}(\sigma)$ and $\sigma_0^k(\Gamma) \not\in \text{Per}(\sigma)$, for any $k \in \mathbb{N}$. In this case we can find a sequence of realizable sequences $\Gamma_k \in \text{Per}(\sigma_0)$, $\Gamma_k < \Gamma$, such that

(i) $\lim_{k \to \infty} \Gamma_k = \Gamma$

(ii) $\mu_{\Gamma}(v) \to \mu_{\Gamma}(v), \mu_{\Gamma}(v) < \mu_{\Gamma}(v)$ and

(iii) $(\mu_{\Gamma}(v))$ is a Cauchy sequence of maps in the $C^1$-uniform topology (this can be proved as in (2.9)).

In this case, associated to $\Gamma$, we find a $C^1$-surface $\{(v, C_\Gamma(v)); v \in V\}$ such that the vector field which represents the point $(v, C_\Gamma(v)) \in C^0$ satisfies that the trajectory $\gamma_0(X_\Gamma(v))$ has recurrent behavior in the neighborhood $U$.

(D) Let now $s(v, \mu)$ be any pre image of the points $b(v, \mu)$ or $a(v, \mu)$ in the closure of the set $M(v, \mu)$, such that $s(v, \mu) \geq \xi^{-1}k_0$ for some $\xi^{-1}(k_0 - \delta) \leq \mu \leq \xi^{-1}(k_0 - 1)$. In this situation the $C^1$-surface $\{(v, \mu, s(v, \mu))\} = \mathcal{S}$ is transversal to $S_0$ and, therefore, the intersection $\mathcal{S} \cap S_0$ defines a $C^1$-surface $\mathcal{S}_0$ (resp. $\mathcal{S}$) parametrized by $\{(v, b(v), S(v, b(v))); v \in V\}$ (resp. $\{(v, \bar{a}(v), S(v, \bar{a}(v))); v \in V\}$). Let $X_\Gamma(v)$ (resp. $X_\bar{a}(v)$) denote the vector field associated to $(v, b(v)) \in B(k_0)$ (resp. $v, \bar{a}(v)) \in B(k_0)$). This vector field satisfies that

$$\gamma_0(\sigma_0(X_\Gamma(v))) \subset W^s(\sigma_1(X_\Gamma(v)))$$

(resp. $\gamma_0(\sigma_0(X_\bar{a}(v))) \subset W^s(\sigma_1(X_\bar{a}(v))))$. That is presents a contracting singular cycles.

This completes the proof of Theorem 1.

An easy consequence of the results in (2.7) through (2.11) is

**Corollary 4.** $\Gamma_0 \cup \Gamma_1$ is a dense subset of $B(k_0)$, any $k_0 \geq n_0$.

**3. Proof of Theorem 2**

Without loss of generality, we may assume that the family $\{X_\mu\}$ such that $X_{\mu=0} \in \mathcal{N}$ is given by $\{(v, \mu); -\varepsilon_0 < \mu < \varepsilon_0\}$ for some $\bar{v} \in V$ and $\varepsilon_0 > 0$ small.

We let $L(\mu; y)$ denote the map $L(\bar{v}, \mu; y)$ given by

$$L(\mu; y) = \begin{cases} \xi y, & 0 \leq y \leq \xi^{-1} \\ \mu - J(\mu; y)(y - (1 - \delta))^{\alpha}, & 1 - \delta \leq y \leq b(\mu) \\ \mu - K(\mu; y)(1 - y)^{\alpha}, & a(\mu) \leq y \leq 1, \end{cases}$$

where $a(\mu) = 1 - \delta^2(\mu), b(\mu) = 1 - \delta + \delta^1(\mu), \delta^i(\mu) = A^i(\mu)^{1/\alpha}, i = 1, 2; J$ and $K$ are $C^2$-map in the $\mu$-variable, $C^3$ in the $y$-variable for $y \neq 1 - \delta, 1$ and whose derivatives are small with $\mu$ small.

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Also $J(\mu, y) > 0$ and $K(\mu; y) > 0$ for any $(\mu, y), 0 \leq \mu \leq \mu_0 = \xi^{-n_0}; y \in I_1(\mu) \cup I_2(\mu)$. 

Given $0 \leq \mu \leq \mu_0$ we define $\Lambda(\mu) = \{y \in [0, 1]/L_n^\mu(y) \in \cup_{i=0}^n I_i(\mu), \text{for all } n \geq 0\}$.

Let $\Gamma_0 = \{\mu \in [0, \mu_0]/1 \not\in \Lambda(\mu)\}$ and $\Gamma_1 = \{\mu \in [0, \mu_0]/1 \in \Lambda(\mu) \text{ and there exists an}\}$ hyperbolic attracting periodic orbit for the map $L_\mu(\cdot)$}. Here $L_\mu(y) = L(\mu; y)$.

As we have seen in Chapter II, $\mu \in \Gamma_0 \cup \Gamma_1$ implies that the associated vector field $X(\tilde{v}, \mu)$ is structurally stable in $U$. Let $H = \Gamma_0 \cup \Gamma_1$ and $B = [0, \mu_0] \setminus H$.

Theorem 2 will follow from the following

**THEOREM 2'.** $- m(H \cap [0, \mu_0]) = 0$. (Here $m$ denotes the Lebesgue measure.)

Using the Lebesgue density theorem it is enough to prove that given $0 \leq \mu \leq \mu_0$ we have

\[
\lim_{\varepsilon \to 0} \frac{m(B \cap [\mu - \varepsilon, \mu + \varepsilon])}{2\varepsilon} < 1.
\]

3.1.
For $\mu \in [0, \mu_0]$, define $L_1(\mu) = L(\mu; 1)$ and $L_{n+1}(\mu) = L(\mu; L_n(\mu))$.

We have $L_{i+1}(\mu) = \xi L_i(\mu)$, for any $1 \leq i \leq n_0$ and $L_{n_0+1}(\mu) = \xi^{n_0} \mu$. Hence these maps satisfy:

a) $L_i'(\mu) > 0$ and $L_i''(\mu) = 0, \mu \in [0, \mu_0], 1 \leq i \leq n_0 + 1$,

b) $L_i'(\mu) \leq L_i'(0), 0 \leq \mu \leq \mu_0, 1 \leq i \leq n_0 + 1$.

For any $k \geq n_0 + 2$, let $I_k = I_k^1 \cup \cdots \cup I_k^{n_k}$ be the domain of definition of the map $L_k$.

Let $I_k = [\nu_0, \nu_1]$ be a component of the domain $I_k$ that satisfies $L_i'(\mu) \neq 0$, for $1 \leq i \leq k - 1$ and any $\mu \in I_k^1$.

**LEMMA 11.** - The map $L_k$ satisfies one and only one of the following possibilities:

(i) there exists a unique $\tilde{v} \in I_k^1$ such that $L_k'(\tilde{v}) = 0$ and $L_k''(\tilde{v}) < 0$ or

(ii) $L_k'(\mu) \neq 0$ and $L_k''(\mu) = 0$ for any $\mu \in I_k^1$ or

(iii) $L_k'(\mu) \neq 0$ and $L_k''(\mu) < 0$ for any $\mu \in I_k^1$.

**Proof.** - See the appendix.

**COROLLARY 5.** - Let $I = [\nu_0, \nu_1] \subset I_k^1$ be an interval and assume $L_i'(\mu) \neq 0$ for $\mu \in I, 1 \leq i \leq k$. Then for any $\alpha, \beta, \nu_0 \leq \alpha \leq \beta \leq \nu_1$ we have $L_k'(\alpha) \geq L_k'(\beta)$.

**Proof.** - Let $X(\mu) = \frac{L_k'(\mu)}{L_k''(\nu_0)}$, $\mu \in I$. We have $X(\nu_0) = 1$.

If $L_k'(\mu) < 0$, then $X'(\mu) = \frac{L_k''(\mu)}{L_k'(\nu_0)} > 0$ and $X$ is an increasing map. So $X(\alpha) \leq X(\beta)$ and hence $L_k'(\alpha) \geq L_k'(\beta)$.

If $L_k'(\mu) > 0$, then $X'(\mu) = \frac{L_k''(\mu)}{L_k'(\nu_0)} < 0$ and $X$ is a decreasing map. In particular, $X(\alpha) \leq X(\beta)$ and hence $L_k'(\alpha) \geq L_k'(\beta)$.

3.2.
We note that $[0, \mu_0] = \{0\} \cup \cup_{k=n_0}^{\infty} \xi^{-k}[\xi^{-1}, 1]$.

Let $k \geq n_0$ be a given number and $I_k = \xi^{-k}[\xi^{-1}, 1]$. For any given $\mu \in I_k$ we have $\xi^{-1} < \xi^k \mu \leq 1$. Clearly that it is enough to prove that $m(B \cap I_k) = 0$, for any $k \geq n_0$. 

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Given \( \mu \in I_k \) let \( D(v,\mu) \) denote the interval \( D(v,\mu) \) and the map \( G(v,\mu) \) as defined in (2.11).

Let \( J_0 = \xi^{-k}[1-\delta,1] \) and \( g_0 : J_0 \to [1-\delta,1] \) be the map \( g_0(\mu) = \xi^k \mu \).

Let us define, inductively,

\[
J_r = \left\{ \mu \in J_{r-1} / \mu \in \bigcup_{j=0}^{\infty} \bigcup_{i=1,2} D\left( v_i \right) \right\}
\]

and \( g_r : J_r \to [1-\delta,1] \) by \( g_r(\mu) = G_\mu(g_{r-1}(\mu)) \), \( r \geq 1 \).

Let \( J_r = [\nu_0, \nu_1] \) be a component of the domain \( J_r \) such that \( g_0(\mu) \neq 0 \), for \( 0 \leq i \leq r-1 \) and any \( \mu \in J_r \).

**Corollary 6.** For the map \( g_r \mid J_r \) we have one and only one of the following possibilities:

(i) there exists a unique \( \check{v} \in J_r \) such that \( g_0(\mu) = 0 \) and \( g_0(\mu) < 0 \), for any \( \mu \in J_r \) or

(ii) \( g_r(\mu) \neq 0 \) and \( g_r(\mu) < 0 \) for any \( \mu \in J_r \).

**Proof.** The proof follows from Lemma 11.

**Corollary 7.** Let \( J = [\nu_0, \nu_1] \subset J_r \subset J_r \) be an interval such that \( g_0(\mu) \neq 0 \), for \( 0 \leq i \leq r \) and \( \mu \in J_r \). Let \( \alpha, \beta \) be the parameter values such that \( \nu_0 \leq \alpha \leq \beta \leq \nu_1 \) we have \( g_r(\alpha) \geq g_r(\beta) \).

**Proof.** Similar to Corollary 5.

### 3.3.

Let us now consider a parameter value \( \mu \in J_r \) that satisfies: there is an interval \( [\alpha, \beta] \subset J_r \) such that \( \mu \in [\alpha, \beta] \) and \( g_0(\nu) \neq 0 \), \( 0 \leq i \leq r \), \( \nu \in [\alpha, \beta] \).

(A1) Let us assume \( g_r(\nu) > 0 \), \( \nu \in [\alpha, \beta] \);

\[
[b(\beta), a(\beta)] \subset [g_r(\alpha), g_r(\beta)] \text{ and } g_r(\mu) \in I_1(\mu)
\]

**Proposition 3.** There exists \( \bar{\mu} \in [\alpha, \beta] \) such that

\[
\frac{m(B \cap [\mu, \bar{\mu}])}{\bar{\mu} - \mu} \leq 1/3, \text{ for } k \text{ big enough.}
\]

**Proof.** Denote by \( \mu \leq \mu_1 \leq \mu_2 \leq \beta \) the parameter values which satisfy \( g_r(\mu_1) = b(\beta) \), and \( g_r(\mu_2) = a(\beta) \). We have \( g_r(\mu_2) - g_r(\mu) = \int_{\mu}^{\mu_2} g_r(\nu) d\nu \geq g_r(\mu_1)(\mu_2 - \mu_1) \) and \( g_r(\mu_1) - g_r(\mu) = \int_{\mu}^{\mu_1} g_r(\nu) d\nu \geq g_r(\mu_1)(\mu_1 - \mu) \).

Since \( g_r(\mu_1) - g_r(\mu) \leq b(\mu_1) - (1 - \delta) \) we have

\[
\frac{m(B \cap [\mu, \mu_2])}{\mu_2 - \mu} \leq \frac{\mu_1 - \mu}{\mu_2 - \mu_1} \leq \frac{b(\mu_1) - (1 - \delta)}{a(\beta) - b(\beta)},
\]

which can be taken smaller or equal to \( 1/3 \) for \( k \) big.

(A2) Assume \( g_r(\nu) < 0 \), \( \nu \in [\alpha, \beta] \); \( [b(\beta), a(\beta)] \subset [g_r(\alpha), g_r(\beta)] \) and \( g_r(\mu) \in I_1(\mu) \).

**Proposition 4.** There exists \( \bar{\mu} \in [\alpha, \mu] \) such that

\[
\frac{m(B \cap [\bar{\mu}, \mu])}{\mu - \bar{\mu}} \leq 1/3, \text{ for } k \text{ big enough.}
\]
Proof. - The proof is similar to that of Proposition 3. ■

(A3) Assume there is \( \left( \frac{i}{j} \right), j \neq 0 \), such that \( D\left( \frac{i}{j} \right)(\nu) \subset \left[ g_r(\alpha), g_r(\beta) \right] \).

Given \( \nu \in [\alpha, \beta] \) denote by \( I_1\left( \frac{i}{j} \right)(\nu) \) the interval contained in \( D\left( \frac{i}{j} \right)(\nu) \) such that \( G\left( \nu, I_1\left( \frac{i}{j} \right)(\nu) \right) = I_t(\nu) \), for \( t = 1, 2 \).

(A31) Assume that \( g_r(\mu) \in I_1\left( \frac{i}{j} \right)(\mu); i = 2 \) and \( g_r'(\nu) > 0 \), for \( \nu \in [\alpha, \beta] \). Denote by \( \mu < \mu_1 < \mu_2 < \beta \) the parameter values which satisfy \( G(\mu_1, g_r(\mu_1)) = b(\beta) \) and \( G(\mu_2, g_r(\mu_2)) = a(\beta) \), respectively. We have

**Proposition 5.** \(- \frac{m(B \cap [\mu, \mu_2])}{\mu_2 - \mu} \leq \frac{1}{3}, \text{ for } k \text{ big enough.} \)

Proof. - The proof is similar to that of Proposition 3. ■

(A32) Assume that \( g_r(\mu) \in I_1\left( \frac{i}{j} \right)(\mu); i = 2 \) and that \( g_r'(\nu) < 0 \), for \( \nu \in [\alpha, \beta] \). Let denote by \( \alpha < \mu_2 < \mu_3 < \mu \) the parameter values which satisfy \( G(\mu_1, g_r(\mu_1)) = b(\beta) \), \( G(\mu_2, g_r(\mu_2)) = a(\beta) \), respectively. We have:

**Proposition 6.** \(- \frac{m(B \cap [\mu_2, \mu_3])}{\mu_3 - \mu_2} \leq 1/3, \text{ for } k \text{ big enough.} \)

Proof. - The proof is similar to that of Proposition 3. ■

(A33) Assume that \( g_r(\mu) \in I_1\left( \frac{i}{j} \right)(\mu), i = 1 \) and that \( g_r'(\nu) > 0 \), for \( \nu \in [\alpha, \beta] \). Denote by \( \mu < \mu_1 < \mu_2 < \beta \) the parameter values which satisfy \( G(\mu_1, g_r(\mu_1)) = a(\beta) \) and \( G(\mu_2, g_r(\mu_2)) = b(\beta) \), respectively. We have:

**Proposition 7.** \(- \frac{m(B \cap [\mu_2, \mu_3])}{\mu_3 - \mu} \leq 1/3, \text{ for } k \text{ big enough.} \)

Proof. - The proof is similar to that of Proposition 3. ■

(A34) Assume that \( g_r(\mu) \in I_2\left( \frac{i}{j} \right)(\mu), i = 1 \) and \( g_r'(\nu) < 0 \) for \( \nu \in [\alpha, \beta] \). Let denote by \( \alpha < \mu_2 < \mu_1 < \mu \) the parameter values which satisfy \( G(\mu_2, g_r(\mu_2)) = b(\beta) \) and \( G(\mu_1, g_r(\mu_1)) = a(\beta) \), respectively.

We have:

**Proposition 8.** \(- \frac{m(B \cap [\mu_2, \mu_3])}{\mu_3 - \mu} \leq 1/3, \text{ for } k \text{ big enough.} \)

Proof. - The proof is similar to that of Proposition 3. ■

(A35) Assume that \( g_r(\mu) \in I_2\left( \frac{i}{j} \right)(\mu), i = 2 \) and that \( g_r'(\nu) > 0 \), for \( \nu \in [\alpha, \beta] \) and, additionally, \( \left[ g\left( \frac{2}{j} \right)(\beta), z\left( \frac{2}{j-1} \right)(\beta) \right] \subset [g_r(\alpha), g_r(\beta)] \).
Denote by \( \mu < \mu_1 < \mu_2 < \beta \) the parameter values which satisfy \( g_r(\mu_1) = y \left( \binom{2}{j} \right)(\beta) \),
\[ g_r(\mu_1) = z \left( \binom{2}{j-1} \right)(\beta) , \]
respectively.

We have

**Proposition 9.** \(- \frac{m(B \cap [\mu, \mu_2])}{\mu_2 - \mu} \leq 1/3 \), for \( k \) big enough.

**Proof.** The proof is similar to that of Proposition 3. \( \blacksquare \)

\( A_36 \) Assume that \( i = 2; g_r(\mu) \in I_2 \left( \binom{i}{j} \right)(\mu) \) and that \( g'_r(\nu) < 0 \), for \( \nu \in [\alpha, \beta] \) and
\[ \left[ y \left( \binom{2}{j} \right)(\beta), z \left( \binom{2}{j-1} \right)(\beta) \right] \subset \left[ \alpha, \beta \right] . \]

Denote by \( \alpha < \mu_2 < \mu_1 < \mu \) the parameter values which satisfy \( g_r(\mu_2) = y \left( \binom{1}{j} \right)(\beta) \),
\[ g_r(\mu_1) = y \left( \binom{1}{j+1} \right)(\beta) , \]
respectively.

We have

**Proposition 10.** \(- \frac{m(B \cap [\mu_2, \mu])}{\mu_2 - \mu} \leq 1/3 \), for \( k \) big enough.

**Proof.** The proof is similar to that of Proposition 3. \( \blacksquare \)

\( A_37 \) Assume that \( i = 1; g_r(\mu) \in I_1 \left( \binom{i}{j} \right)(\mu) \); \( g'_r(\nu) > 0 \), for \( \nu \in [\alpha, \beta] \) and
\[ \left[ z \left( \binom{1}{j} \right)(\beta), y \left( \binom{1}{j+1} \right)(\beta) \right] \subset \left[ \alpha, \beta \right] . \]

Denote by \( \mu < \mu_1 < \mu_2 < \beta \) the parameter values which satisfy \( g_r(\mu_1) = z \left( \binom{1}{j} \right)(\beta) \)
and \( g_r(\mu_2) = y \left( \binom{1}{j+1} \right)(\beta) , \)
respectively.

We have

**Proposition 11.** \(- \frac{m(B \cap [\mu_2, \mu])}{\mu_2 - \mu} \leq 1/3 \), for \( k \) big enough. \( \blacksquare \)

\( A_38 \) Assume that \( i = 1; g_r(\mu) \in I_1 \left( \binom{i}{j} \right)(\mu) \); \( g'_r(\nu) < 0 \) for \( \nu \in [\alpha, \beta] \) and
\[ \left[ z \left( \binom{1}{j} \right)(\beta), y \left( \binom{1}{j+1} \right)(\beta) \right] \subset \left[ \alpha, \beta \right] . \]

Let denote by \( \alpha < \mu_2 < \mu_1 < \mu \) the parameter values which satisfy \( g_r(\mu_2) = y \left( \binom{1}{j+1} \right)(\beta) \) and \( g_r(\mu_1) = z \left( \binom{1}{j} \right)(\beta) , \)
respectively.

We have

**Proposition 12.** \(- \frac{m(B \cap [\mu_2, \mu])}{\mu_2 - \mu} \leq 1/3 \) for \( k \) big enough.

\( A_4 \) Assume that \( \left[ z \left( \binom{1}{0} \right)(\beta), y \left( \binom{1}{1} \right)(\beta) \right] \subset \left[ \alpha, \beta \right] \) and \( g_r(\mu) \in D \left( \binom{1}{0} \right)(\mu) . \)
(A41) Assume that \( g'(\nu) > 0 \) for \( \nu \in [\alpha, \beta] \).

Let denote by \( \mu < \mu_1 < \mu_2 < \beta \) the parameter values which satisfy \( g_r(\mu_1) = z \begin{pmatrix} 1 \\ 0 \end{pmatrix}(\beta) \) and \( g_r(\mu_2) = y \begin{pmatrix} 1 \\ 1 \end{pmatrix}(\beta) \), respectively.

We have

**Proposition 13.** \( -\frac{m(B \cap [\mu, \mu_2])}{\mu_2 - \mu} \leq 1/3 \), for \( k \) big enough.

**Proof.** The proof is similar to that of Proposition 3. \( \blacksquare \)

(A42) Assume that \( g'(\nu) < 0 \), for \( \nu \in [\alpha, \beta] \).

Denote by \( \alpha < \mu_2 < \mu_1 < \mu \) the parameter values that satisfy \( g_r(\mu_2) = y \begin{pmatrix} 1 \\ 1 \end{pmatrix}(\beta) \) and \( g_r(\mu_2) = z \begin{pmatrix} 1 \\ 0 \end{pmatrix}(\beta) \), respectively.

We have

**Proposition 14.** \( -\frac{m(B \cap [\mu_2, \mu])}{\mu - \mu_2} \leq 1/3 \), for \( k \) big enough.

**Proof.** The proof is similar to that of Proposition 3. \( \blacksquare \)

(3.4). Consider a parameter value \( \mu \in J_0 \) which satisfies: there exists \( r_0 \in \mathbb{N} \) that

\[ G^r_\mu(\xi^k \mu) \in \left( [1 - \delta, 1] \setminus \bigcup_{j=0}^{\infty} \bigcup_{i=1,2} D \begin{pmatrix} i \\ j \end{pmatrix}(\mu) \right) \]

In this case we have \( \mu \in \Gamma_0 \) or \( G^r_\mu(\xi^k \mu) = b(\mu) \) or \( G^r_\mu(\xi^k \mu) = a(\mu) \). It is clear that assertion \( (*) \) is true in any of the cases above. Let

\[ T = \left\{ \mu \in J_0 / g_r(\mu) \in \bigcup_{j=0}^{\infty} \bigcup_{i=1,2} D \begin{pmatrix} i \\ j \end{pmatrix}(\mu), \text{ for any } r \geq 0 \right\} \]

For a given \( \mu \in T \) we have three possibilities for the itinerary \( \Gamma_\mu \):

1. \( \Gamma_\mu \) is a periodic itinerary;
2. \( \Gamma_\mu \) is an itinerary which is eventually periodic and
3. \( \Gamma_\mu \) do not satisfies (1) and (2) above.

Assume \( \Gamma_\mu \) is periodic. In this case we know (see (2.11)) that there is an interval \( [\alpha, \beta] \subset T \) such that \( \Gamma_\nu = \Gamma_\mu \), for any \( \nu \in [\alpha, \beta] \); \( \mu \in [\alpha, \beta] \) and \( B \cap [\alpha, \beta] \) is a finite number of points. So for these parameter values assertion \( (*) \) is true.

Assume \( \Gamma_\mu \) is eventually periodic. Under these circumstances it is easy to see that we can find an interval \( [\alpha, \beta] \subset J_0 \) and an index \( r \in \mathbb{N} \) such that:

1. \( \mu \in [\alpha, \beta] \);
2. \( \mu \neq 0, 0 \leq i \leq r \) for any \( \nu \in [\alpha, \beta] \) and
3. \( \mu / [\alpha, \beta] \) satisfies the conditions of one of the Propositions specified in (3.3) above.

It is clear that we can find a sequence of intervals \([\alpha_n, \beta_n] \subset [\alpha_{n-1}, \beta_{n-1}] \) and a sequence of indexes \( r_n > r_{n-1} \) such that (i), (ii) and (iii) hold for any of the given \( n \in \mathbb{N} \).
Therefore we can conclude the following

**Lemma 12.** There exists a sequence $\mu_n \rightarrow \mu$ such that

$$
\frac{m(B \cap [\mu, \mu_n])}{\mu_n - \mu} \leq \frac{1}{3} \quad \text{or} \quad \frac{m(B \cap [\mu_n, \mu])}{\mu - \mu_n} \leq \frac{1}{3},
$$

for $k$ big enough.

In particular, for any of these parameter values assertion (*) is true.

Assume $\Gamma$ satisfies (3) above. In this case we can find a sequence $\mu_n \rightarrow \mu$ such that $\Gamma_{\mu_n}$ satisfies (1) or (2) above. For these parameter values assertion (*) holds, therefore we conclude that it (*) is true for $\mu$.

This completes the proof of Theorem 2. \hfill \blacksquare

(3.5) Comments on the general case

Let us now consider the general case for contracting singular cycles. In his paper San Martin [8] introduces a nice idea with which to work in this case. Let us consider the periodic orbits $\sigma_1(X), \ldots, \sigma_r(X)$ that belong to the singular cycle $\Gamma$. Let $q_i(X) \in \sigma_i(X)$ be a point and $Q_i \subset M$ be a transversal section associated to $q_i(X), i = 1, \ldots, n$. Assume this cross section is parametrized by $\{(x_i, y_i); |x_i|, |y_i| \leq 1\}$ satisfying $W^s_{\sigma_i} \supseteq \{(x_i,0); |x_i| \leq 1\}$ and $W^u_{\sigma_i} \supseteq \{(0,y_i); |y_i| \leq 1\}$.

Let $p_i^j = p^j_i(X)$ be the first intersection between $\gamma^j_i(X)$ and $Q_{i+1}, i = 1, 2, \ldots, n-1; j = 1, 2$. We have $p_i^j = (x_{i+1}^j(X), 0)$ and assume $x_{i+1}^j > 0$. Denote by $q_i^j = q^j_i(X) = (0,y_i^j(X))$ the first intersection of the backward orbit of $p_i^j$ with $Q_i$.

We will assume $y_i^j(X) > 0, i = 1, \ldots, n-1; j = 1, 2$.

Since $p_i^j$ and $q_i^j$ are in the same orbit we can find horizontal strips $R_i^j(X) \ni q_i^j$ and neighborhoods $U_i^j \ni p_i^j$, so that the positive orbits of points at $R_i^j$ intersect $U_i^j$. This procedure define Poincaré maps $P_i^j : R_i^j \rightarrow U_i^j; i = 1, 2, \ldots, n-1; j = 1, 2$.

On the other hand, the positive orbit of points at a horizontal strip $R_i(X)$, containing $W^s(\sigma_i(X)) \cap Q_i$, turns around the closed orbit $\sigma_i(X)$ and then returns to $Q_i$. This define a return map $P_i : R_i \rightarrow Q_i, i = 1, \ldots, n$.

Denote by $q_n^j = q_n^j(X)$ the last intersection of the orbit $\gamma_n^j(X)$ with $Q_n, j = 1, 2$. Since $w(q_n^j) = \sigma_0(X)$ and $\alpha(q_n^j) = \sigma_n(X)$, there are horizontal strips $R_i^j(X) \ni q_i^j$ such that the positive orbit of points at $R_i^j$ pass first near $\sigma_0(X)$ and afterwards intersect $Q_1$. This define maps $P_n^j : R_n^j \rightarrow Q_1, j = 1, 2$.

Therefore the first return map $F_X$ is defined on $\bigcup_{i=1}^{n} (R_i \cup R_i^1 \cup R_i^2)$ with values on $\bigcup_{i=1}^{n} Q_i$ and its restriction to $R_i$ coincides with the Poincaré map associated to $\sigma_i(X)$.

The same construction applies to vector field $Y$, near enough to $X$ in the $C^r$-topology, $r \geq 3$.

From now and on the proof follows as in chapters II and III (3.1)-(3.4), that is: Give an explicit formula to the map $F_Y$; show that there is an invariant stable foliation for $F_Y$; change coordinates in the neighborhood $U$ and prove the result for the one-dimensional map associated to $F_Y$. 

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In this paragraph we prove Lemma 13. Let \( L(\mu; y) \) denote the map given by

\[
L(\mu; y) = \begin{cases} 
\xi y, & 0 \leq y \leq \xi^{-1} \\
\mu - J(\mu; y)(y - (1 - \delta))^\alpha, & 1 - \delta \leq y \leq b(\mu) \\
\mu - K(\mu; y)(1 - y)^\alpha, & a(\mu) \leq y \leq 1,
\end{cases}
\]

where \( a(\mu) = 1 - \delta^2(\mu), b(\mu) = 1 - \delta + \delta^1(\mu) ; \delta^i(\mu) = A^i\mu^{1/\alpha}, A^i > 0 \), for \( i = 1, 2 \); \( J \) and \( K \) are \( C^2 \)-maps in the \( \mu \)-variable, \( C^3 \) in the \( y \)-variable \( y \neq 1 - \delta, 1 \) and whose derivatives \( \frac{\partial J}{\partial \mu}, \frac{\partial^2 J}{\partial \mu^2}, \frac{\partial J}{\partial y}, \frac{\partial^2 J}{\partial y^2}, \frac{\partial K}{\partial \mu}, \frac{\partial^2 K}{\partial \mu^2}, \frac{\partial K}{\partial y}, \frac{\partial^2 K}{\partial y^2}, \frac{\partial K}{\partial \mu^2} \) are small numbers, with \( \mu \) small. Moreover \( J(\mu; y) > 0 \) and \( K(\mu; y) > 0 \), any \( (\mu; y), 0 \leq \mu \leq \mu_0 = \xi^{-n_0} \).

Define \( L_1(\mu) = L(\mu; 1) = \mu \) and \( L_{n+1}(\mu) = L(\mu; L_n(\mu)), n \geq 1. \)

We have \( L_{i+1}(\mu) = \xi L_i(\mu), 1 \leq i \leq n_0 \) and \( L_{n+1}(\mu) = \xi^{n_0} \mu. \) Hence these maps satisfy:

(a) \( L_i'(\mu) > 0 \) and \( L_i''(\mu) = 0, \mu \in [0, \mu_0], 0 \leq i \leq n_0 + 1 \)

(b) \( L_i'(\mu) \leq L_i'(0), 0 \leq \mu \leq \mu_0. \)

For any \( k \geq n_0 + 2 \), let \( I_k = I_k^1 \cup I_k^2 \cup \cdots \cup I_k^n \) be the domain of definition of the map \( L_k. \)

Let \( I_k^i = [\nu_0, \nu_1] \) be a component of the domain \( I_k \) that satisfies \( L_i'(\mu) \neq 0, \) for \( 0 \leq i \leq k - 1 \) and \( \mu \in I_k^i. \)

**Lemma 13.** For the map \( L_k \) we have one and only one of the following:

(i) there exists only one \( \bar{\nu} \in I_k^i \) such that \( L_k'(\bar{\nu}) = 0 \) and \( L_k''(\bar{\nu}) < 0 \) or

(ii) \( L_k'(\mu) \neq 0 \) and \( L_k''(\mu) = 0 \) for \( \mu \in I_k^i, \) or

(iii) \( L_k'(\mu) \neq 0 \) and \( L_k''(\mu) < 0 \) for \( \mu \in I_k^i. \)

**Proof.** For \( L_{k-1}(\mu) \leq \xi^{-1}, \mu \in I_k^i, \) we have \( L_k(\mu) = \xi L_{k-1}(\mu) \) and the result follows by the induction hypothesis. Otherwise let us consider \( A = \bigcup_{\mu \in [0, \mu_0]} [\mu \times I_1(\mu)] \) and \( B = \bigcup_{\mu \in [0, \mu_0]} (\{\mu\} \times I_2(\mu)). \)

We must have \( A \cap (\text{Graph}(L_k/I_k^i)) \neq \emptyset \) or \( B \cap (\text{Graph}(L_k/I_k^i)) \neq \emptyset \) (only one of these intersections in non-empty).

**I)** Assume \( L_{k-1}'(\mu) < 0 \) for \( \mu \in I_k^i. \)

(i) We have \( L_{k-1}(\nu_0) = 1 \) and \( L_{k-1}(\nu_1) = a(\nu_1). \)

Under these conditions \( L_k(\mu) = L(\mu; L_{k-1}(\mu)) = \mu - K(\mu; L_{k-1}(\mu))(1 - L_{k-1}(\mu))^{\alpha}. \)

So

\[
L_k'(\mu) = -\frac{\partial K}{\partial \mu}(\mu; L_{k-1}(\mu))(1 - L_{k-1}(\mu))^{\alpha} \\
+ \left[-\frac{\partial K}{\partial y}(\mu; L_{k-1}(\mu))(1 - L_{k-1}(\mu))^{\alpha} \\
+ \alpha K(\mu; L_{k-1}(\mu))(1 - L_{k-1}(\mu))^{\alpha - 1}\right]L_{k-1}'(\mu)
\]
and
\[ L''(\mu) = (1 - L_{k-1}(\mu))^{\alpha-2}[-\alpha(\alpha - 1)K(\cdot, \cdot)(L_{k-1}(\mu))^2 + K_{\mu}(\cdot, \cdot)(1 - L_{k-1}(\cdot))^2 + 2\alpha K_{\mu}(\cdot, \cdot) L_{k-1}(\cdot)(1 - L_{k-1}(\cdot)) + K_y(\cdot, \cdot)(1 - L_{k-1}(\cdot))(L_{k-1}(\cdot))^2 + \alpha K(\cdot, \cdot)(1 - L_{k-1}(\cdot)) \cdot L''_{k-1}(\cdot) + 2K_{\mu y}(\cdot, \cdot) (1 - L_{k-1}(\cdot))^2 \cdot L'_{k-1}(\cdot) + 2\alpha K_{y}(\cdot, \cdot) (1 - L_{k-1}(\cdot))(L_{k-1}(\cdot))^2 + L_y(\cdot, \cdot)(1 - L_{k-1}(\cdot))^2 \cdot L''_{k-1}(\cdot)]].
\]

Since
\[ L_{k-1}(\mu) = \xi L_{k-2}(\mu) = \cdots = \xi^{j-1} L_{k-j}(\mu) = \xi^{j-1} [\mu - K(\mu; L_{k-j-1}(\cdot))(1 - L_{k-j-1}(\mu))^\alpha]\]
if \( \alpha(\mu) \leq L_{k-j-1}(\mu) \leq 1 \)
\[ = \xi^{j-1} [\mu - J(\mu; L_{k-j-1}(\cdot))(L_{k-j-1}(\mu) - 1 - \delta)^\alpha]\]
if \( 1 - \delta \leq L_{k-j-1}(\mu) \leq b(\mu). \)

Therefore we have
\[ L_{k-1}'(\mu) = \xi^{j-1} [1 - J(\cdot, \cdot)(L_{k-j-1}(\mu) - (1 - \delta))^\alpha - J_y(\cdot, \cdot)(L_{k-j-1}(\mu) - (1 - \delta))^\alpha \cdot L_{k-j-1}(\cdot) - \alpha J(\cdot, \cdot)(L_{k-j-1}(\mu) - (1 - \delta))^{\alpha-1} L_{k-j-1}'(\cdot)]\]

or
\[ L_{k-1}'(\mu) = \xi^{j-1} [1 - K(\cdot, \cdot)(1 - L_{k-j-1}(\cdot))^\alpha - K_y(\cdot, \cdot)(1 - L_{k-j-1}(\cdot))^\alpha L_{k-j-1}(\cdot) + \alpha K(\cdot, \cdot)(1 - L_{k-j-1}(\cdot))^\alpha - 1 L_{k-j-1}'(\cdot)],\]

depending on \( 1 - \delta \leq L_{k-j-1}(\mu) \leq b(\mu) \) or \( \alpha(\mu) \leq L_{k-j-1}(\mu) \leq 1 \), respectively. Since \( L_{k-1}'(\mu) < 0 \) we have
\[ L_{k-1}'(\mu) > \frac{1 - J_y(\cdot, \cdot)(L_{k-j-1}(\mu) - (1 - \delta))^\alpha [L_{k-j-1}(\mu) - (1 - \delta))]^{\alpha-1} \alpha J(\cdot, \cdot) + J_y(\cdot, \cdot)(L_{k-j-1}(\mu) - (1 - \delta))]}{[L_{k-j-1}(\mu) - (1 - \delta))]^{\alpha-1} [\alpha K(\cdot, \cdot) - K_y(\cdot, \cdot)(1 - L_{k-j-1}(\cdot))]}
\]
or
\[ -L_{k-j-1}'(\mu) > \frac{1 - K(\cdot, \cdot)(1 - L_{k-j-1}(\cdot))^\alpha}{(1 - L_{k-j-1}(\cdot))^\alpha - \alpha K(\cdot, \cdot) - K_y(\cdot, \cdot)(1 - L_{k-j-1}(\cdot))}\]

depending on \( 1 - \delta \leq L_{k-j-1}(\mu) \leq b(\mu) \) or \( \alpha(\mu) \leq L_{k-j-1}(\mu) \leq 1 \), respectively. In any case we get \( |L_{k-j-1}'(\mu)| \geq 20, \) for \( \mu \in I_k \).

Now consider the map \( \rho(\mu) \) given by
\[ \rho(\mu) = J_y(\mu; L_{k-j-1}(\mu))(L_{k-j-1}(\mu) - (1 - \delta))^\alpha + J_y(\cdot, \cdot)(L_{k-j-1}(\mu) - (1 - \delta))^\alpha + \alpha J(\cdot, \cdot)(L_{k-j-1}(\mu) - (1 - \delta))^{\alpha-1}] \times L_{k-j-1}'(\mu)\]
or

\[ \rho(\mu) = K_\mu(\mu; L_{k-1}(\mu))(1 - L_{k-1}(\mu))\alpha \]

\[ + [K_y(\mu; L_{k-1}(\mu))(1 - L_{k-1}(\mu))^{\alpha} \]

\[ - \alpha K(\mu; L_{k-1}(\mu))(1 - L_{k-1}(\mu))^{\alpha-1}] \times L_{k-1}(\mu), \]

depending on whether \( 1 - \delta \leq L_{k-1}(\mu) \leq b(\mu) \) or \( a(\mu) \leq L_{k-1}(\mu) \leq 1 \), respectively.

In the first case an easy computation, using the facts that \( L_{k-1}(\mu) \geq 20; L''_{k-1}(\mu) < 0 \)
and \( L_{k-1}(\mu) - (1 - \delta) > 0 \) gives \( \rho'(\mu) > 0 \), for \( \nu_0 \leq \mu \leq \nu_1 \).

Similarly in the second case we get \( \rho'(\mu) > 0 \).

Since \( L_{k-1}(\mu) = \xi^{\alpha-1}[1 - \rho(\mu)] \), we have:

\[ L'_{k}(\mu) = [1 - L_{k-1}(\mu)]^{\alpha-2}[-\alpha(\alpha - 1)(K(\mu; L_{k-1}(\mu))\xi^{\alpha-1}(1 - \rho(\mu))]^2 \]

\[ - K_\mu(\mu; L_{k-1}(\mu))(1 - L_{k-1}(\mu))^{\alpha} \]

\[ + 2\alpha K_y(\mu; L_{k-1}(\mu))\xi^{\alpha-1}(1 - \rho(\mu))(1 - L_{k-1}(\mu)) \]

\[ - K_\mu(\mu; L_{k-1}(\mu))(1 - L_{k-1}(\mu))^{\alpha} - (\xi^{\alpha-1}(1 - \rho(\mu)))^2 \]

\[ - \alpha K(\mu; L_{k-1}(\mu))\xi^{\alpha-1}\rho'(\mu) \]

\[ - 2K_\mu(\cdot; \cdot)(1 - L_{k-1}(\mu))^{\alpha} \xi^{\alpha-1}(1 - \rho(\mu)) \]

\[ + 2\alpha K_y(\cdot; \cdot)(1 - L_{k-1}(\mu))(\xi^{\alpha-1}(1 - \rho(\mu)))^2 \]

\[ + K_\mu(\cdot; \cdot)(1 - L_{k-1}(\mu))^2 \xi^{\alpha-1}\rho'(\mu)]; \]

which is clearly a negative number.

We note that \( L_k(\nu_0) = 1 \). Let us compute \( L'_k(\nu_1) \).

We have

\[ L'_k(\nu_1) = 1 + \nu_1^{1-\alpha}\left[ \alpha K^{1/\alpha} L_{k-1}(\nu_1) - \frac{K_y}{K} L_{k-1}(\nu_1)\nu_1^{1/\alpha} - \frac{K_\mu}{K} \nu_1^{1/\alpha} \right]. \]

Since \( L_{k-1}(\nu_1) < 0 \) and \( L_{k-1}(\nu_1) = a(\nu_1) \), we get \( L'_k(\nu_1) < 0 \).

Since \( L'_k(\mu) < 0 \), we find only one \( \bar{\nu} \in [\nu_0, \nu_1] \) such that \( L'_k(\bar{\nu}) = 0 \).

(ii) Assume \( L_{k-1}(\nu_0) < 1 \) and \( L_{k-1}(\nu_1) = a(\nu_1) \).

Similarly, as in (i) of above, we obtain \( L'_k(\mu) < 0 \) for \( \mu \in \tilde{I}_k^1 \). If \( L'_k(\nu_1) > 0 \) then there exists only one \( \bar{\nu} \in \tilde{I}_k^1 \) such that \( L'_k(\bar{\nu}) = 0 \).

As before we get \( L''_k(\mu) < 0 \) for \( \mu \in \tilde{I}_k^1 \). If \( L'_k(\nu_1) > 0 \) then \( L'_k(\mu) > 0 \) for \( \mu \in \tilde{I}_k^1 \). If \( L'_k(\nu_1) < 0 \) then there is only one \( \bar{\nu} \in \tilde{I}_k^1 \) such that \( L'_k(\bar{\nu}) = 0 \).

(iv) Assume \( L_{k-1}(\nu_0) < 1 \) and \( L_{k-1}(\nu_1) = a(\nu_1) \).

As before we prove that \( L'_k(\mu) \) is a decreasing map and we get the result.

(v) Assume \( L_{k-1}(\nu_0) = b(\nu_0) \) and \( L_{k-1}(\nu_1) = 1 - \delta \).

We proceed as in (i) to prove \( L''_k(\mu) < 0 \) and hence we obtain \( L'_k(\mu) \geq L_k(\nu_1) = 1 \), any \( \mu \in \tilde{I}_k^1 \).

(vi) Assume \( L_{k-1}(\nu_0) < b(\nu_0) \) and \( L_{k-1}(\nu_1) = 1 - \delta \).

In a similar way as in (i) we get \( L''_k(\mu) < 0 \) and then \( L'_k(\mu) \geq L'_k(\nu_1) = 1 \), any \( \mu \in \tilde{I}_k^1 \).

As before we get \( L'_k(\mu) < 0 \) and \( L'_k(\mu) \geq 1 \), any \( \mu \in \tilde{I}_k^1 \).
(viii) Assume $L_{k-1}(\nu_0) = b(\nu_0)$ and $L_{k-1}(\nu_1) > 1 - \delta$.
As before we get the result.

II) Assume $L'_{k-1}(\mu) > 0$ (non-constant) for $\mu \in I_k^i$.
As in Case (I) we have eight possibilities. We proceed as in (I)(i) to get the result in all of the cases.

III) The case $L'_{k-1}(\mu) = \text{constant}$, i.e., $\nu_0 = 0 \in I_k^i$ satisfies $L'_{k-1}(\mu) > 0$ and $L''_{k}(\mu) = 0$,
for $\mu \in I_k^i$.

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