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ANNEALED LYAPOUNOV EXPONENTS AND LARGE DEVIATIONS IN A POISSONIAN POTENTIAL II

By Alain-Sol SZNITMAN

Abstract. – We derive a large deviation principle for the position at large times t of d-dimensional annealed Brownian motion in a Poissonian potential in critical scale $t^{d/d+2}$, $d \geq 2$. The rate function is one of the Lyapounov norms constructed in the previous paper. Our large deviation results have a natural application to the study of Brownian motion with a constant drift in a Poissonian potential. They enable to study the transition which occurs between the "small drift" and "large drift" regime.

0. Introduction

The present article is a continuation of [11]. Our principal motivation here is the derivation of a large deviation principle in the critical scale $t^{d/d+2}$ for the position at time t of an "annealed Brownian motion" moving in a Poissonian potential or among Poissonian traps, in dimension $d \geq 2$. Namely, if $Z.$ denotes a canonical Brownian motion, and P_0 the Wiener measure on $C(\mathbb{R}_+, \mathbb{R}^d)$, $\mathbb{P}$ the law of the Poisson cloud of constant intensity $\nu > 0$, on the space Ω of simple pure point measures on $\mathbb{R}^d$, the annealed weighted measure is

$$
\begin{aligned}
(0.1)\ Q_t(dw, d\omega) &= \frac{1}{S_t} \exp\Big\{ -\int_0^t V(Z_s(w), \omega) ds \Big\} P_0(dw)\, \mathbb{P}(d\omega)\ , \\
&\quad \text{(soft obstacle case)}\ , \\
&= \frac{1}{S_t}\, 1\{T > t\}\ P_0(dw)\, \mathbb{P}(d\omega), \qquad \text{(trap or hard obstacle case)}\ ,
\end{aligned}
$$

with S_t the normalizing constant, $V(x,\omega) = \sum_i W(x - x_i)$ the Poissonian potential. Here $W \geq 0$, the shape function, is bounded compactly supported not a.s. equal to zero, and $\omega = \sum_i \delta_{x_i}$ is the typical cloud configuration. In the hard obstacle case, T denotes the entrance time of $Z.$ in the obstacle set $\bigcup_i \overline{B}(x_i, a)$, $a > 0$. In fact, the hard obstacle case corresponds to the singular shape function $W_{\text{h.o.}}(x) = \infty\ 1\{|x| \leq a\}$.

The special role of the scale $t^{d/d+2}$ is linked to the asymptotic behavior of S_t (Donsker-Varadhan [2]):

$$S_t = \exp\{-c(d,\nu)\, t^{d/d+2}(1+o(1))\}\,, \tag{0.2}$$

where $c(d,\nu)$ is an explicit constant. It is also reflected, as shown in [11], Theorem 2.4, in the asymptotic behavior as $t \to \infty$:

$$\begin{aligned} \overline{r}(t,0,\varphi(t)v) &= \exp\{-c(d,\nu)\, t^{d/d+2}(1+o(1))\}, \quad \varphi(t) = o(t^{d/d+2}),\ v \in \mathbb{R}^d\,, \\ &= \exp\{-\beta_0(v)\, \varphi(t)(1+o(1))\}, \quad \text{if } t^{d/d+2} = o(\varphi(t)),\ \varphi(t) = o(t)\,, \end{aligned} \tag{0.3}$$

provided $\overline{r}(t,x,y) = \mathbb{E}[r(t,x,y,\omega)]$, with r the kernel of the Schrödinger semigroup $e^{t(\frac{1}{2}\Delta - V)}$ (soft obstacle case) or Dirichlet heat kernel on $\mathbb{R}^d \backslash \bigcup_i \overline{B}(x_i,a)$ of $e^{\frac{t}{2}\Delta}$ (hard obstacle case). Here, $\beta_0(x)$ is one of the Lyapounov exponents introduced in [11]. The $\beta_\lambda(x)$, $\lambda \geq 0$, $x \in \mathbb{R}^d$, are norms in the x variable which for instance in the soft obstacle case measure the exponential decay in the x-direction of the $\mathbb{P}$-average of the λ-Green function $\left(-\frac{1}{2}\Delta + \lambda + V(\cdot,\omega)\right)^{-1}(0,\cdot)$. Our main purpose in this article is

THEOREM. – $d \geq 2$.

(0.4) $Z_t/t^{d/d+2}$ satisfies under Q_t a large deviation principle at rate $t^{d/d+2}$ with rate function $\beta_0(\cdot)$.

In the hard obstacle case, (0.4) improves our previous results in [6], where upper and lower bounds with different rate functions were derived. As an application of (0.4) we find the asymptotics in (0.3), for the critical $\varphi(t) = t^{d/d+2}$:

$$\overline{r}(t,0,t^{d/d+2}\,v) = \exp\{-(c(d,\nu) + \beta_0(v))\, t^{d/d+2}(1+o(1))\}, \qquad t \to \infty\,. \tag{0.5}$$

A natural application of the large deviation results obtained in [11] and here, is the study of the long time behavior of "annealed Brownian motion with a constant drift h", in a Poissonian potential or among Poissonian traps, that is when Q_t is replaced by

$$Q_t^h(dw,d\omega) = \frac{S_t}{\widetilde{S}_t^h}\, e^{h\cdot Z_t}\, Q_t(dw,d\omega) \tag{0.6}$$

with $\widetilde{S}_t^h$ the normalizing constant. This question in the trap case appeared in the physical literature (Grassberger-Procaccia [5]) with special interest for the transition of regime which occurs between the small h and large h situation. This was later investigated in Eisele-Lang [3], where it was shown that below a certain threshold ($|h| \leq \kappa$), $\widetilde{S}_t^h$ has no exponential growth in t, whereas above this threshold $\widetilde{S}_t^h$ has indeed an exponential growth in t.

We showed in [6] (trap case) that when $|h|$ is small

$$\begin{aligned} \widetilde{S}_t^h &= \exp\{-c(d,\nu)\, t^{d/d+2}(1+o(1))\}, \quad d \geq 2 \\ &= \exp\{-c(d,\nu - |h|)\, t^{1/3}(1+o(1))\}, \quad d = 1,\ |h| < \nu = \text{ critical threshold}\,, \end{aligned} \tag{0.7}$$

(the one dimensional case is special for reasons which further below will become clear). Here we are able to give a finer description of the transition which occurs in both the trap case and soft obstacle case. Namely if $h = |h|\, e$, $h \neq 0$, the critical threshold in the direction e is

$$\kappa(e) = \inf\{\beta_0(x),\ x \cdot e \geq 1\} = 1/\sup\{x \cdot e,\ \beta_0(x) = 1\} > 0\,. \tag{0.8}$$

In fact in the trap case or when $W(\cdot)$ is rotationally invariant, this threshold is independent of e. Among various results, we describe the large deviation principle satisfied by Z_t/t under Q_t^h, $h \in \mathbb{R}^d$, $d \geq 1$, and characterize the rate of exponential growth in time of $\widetilde{S}_t^h$, which is strictly positive when $|h| > \kappa(e)$. When $d \geq 2$, and $|h| < \kappa(e)$, we show that the first line of (0.7) holds and

(0.9) $\quad Z_t/t^{d/d+2}$ satisfies a large deviation principle at rate $t^{d/d+2}$ with rate function $\beta_0(x) - h \cdot x$ under Q_t^h .

We shall now give some indications on the proof of (0.4). The lower estimate was already proven in [11]. The upper estimate is the most delicate part. We have to prove with a suitable uniformity in x estimates of the type:

$$\begin{aligned}\varlimsup_{t\to\infty} t^{-d/d+2} \log\Big(\mathbb{E} \times E_0\Big[H(x\,t^{d/d+2}) \leq t,\ \exp\Big\{-\int_0^t V(Z_s,\omega)ds\Big\}\Big]\Big)\\ \leq -c(d,\nu) - \beta_0(x)\,,\end{aligned} \tag{0.10}$$

where $H(y)$ stands for the entrance time of $Z_\cdot$ in $B(y) \overset{\text{def}}{=} \overline{B}(y,1)$.

The proof strongly uses the "method of enlargement of obstacles", *see* [12] for a review. We construct a certain coarse grained picture of the cloud configurations, which enables us to single out "big holes" or "clearings" of size $\sim t^{1/d+2}$ which occur within the cloud. We thus have a description of the space in terms of forest and clearings. The first reduction step follows [6] (and also [9] for the $\mathbb{P}$-a.s. or "quenched situation"). It shows that by adjusting parameters coming in the definition of the coarse grained picture, we need only consider situations where a fixed number of clearings occur within distance t of the origin. These clearings are used as "resting places" by the process which spends most of its time there ($\geq (1-\eta)\,t$) and does not perform too many excursions between clearings and forest ($\leq \eta\, t^{d/d+2}$). The cost attached to the time spent in clearings is responsible for the term $-c(d,\nu)$ in (0.10). Let us also mention that since $d \geq 2$, in contrast to the one dimensional situation, the scale of the clearings ($\sim t^{1/d+2}$) is negligible with respect to the scale of the "big excursion" ($\sim t^{d/d+2}$) which is produced at some point by the process.

The main novelty here is that in contrast to [6] or even [9], we have a fine control on the cost attached to the excursions. This cost is responsible for the term $\beta_0(x)$ in (0.10). Here we extract from the sequence of excursions out of the clearings into the forest a sequence of excursions which never visits twice the same clearing and eventually ends up near $x\,t^{d/d+2}$. We then "piece together" these excursions performing what might be called "reconstructive path surgery", and produce a Brownian path that goes from 0 to somewhere near $x\,t^{d/d+2}$. In this process we obtain an upperbound on the cost attached to excursion which is given, up to correction terms by:

$$\mathbb{E} \otimes E_0\Big[\exp\Big\{-\int_0^{H(x\,t^{d/d+2})} V(Z_s,\omega)ds\Big\}\Big] = \exp\{-\beta_0(x)\,t^{d/d+2}(1+o(1))\}\,,$$

where the last equality comes from Theorem 1.3 of [11].

The piecing together of this extracted sequence of excursions is delicate. One needs to control "joining costs", where an excursion arriving to a clearing is linked to the excursion departing from the clearing. Let us briefly explain the nature of the difficulty. For the simplicity of the argument, let us consider the trap case, although the same problem exists for soft obstacles. The difficulty is that if one brutally performs the linkage between this extracted sequence of excursions, the involved cost can very well be infinite. Indeed it can very well occur that the arriving point and the departing point do not belong to the same component of the complement in the clearing of the obstacle set $(\bigcup_i \overline{B}(x_i, a))$. So any path linking the arriving point to the departing point is killed by the obstacles and corresponds in this case to an infinite cost.

I. Critical large deviation principle

We shall use in this section the notations recalled in the introduction, and in any case follow the notations of [11] (*see* section I). Let us simply mention for the reader's convenience that H_A, $A \subseteq \mathbb{R}^d$ closed set, (resp. T_U, $U \subseteq \mathbb{R}^d$ open set) stand for the entrance time of $Z.$ in A (resp. exit time of $Z.$ from U), and we shall also use the special notation recalled in (0.10). In most cases we shall use the soft obstacle notation, the trap case being recovered by using the singular shape function $W_{\text{h.o.}}(x) = \infty\, 1\{|x| \le a\}$. Finally, we recall that in the soft obstacle case $a = a(W) > 0$, is the minimal choice such that $W(\cdot) = 0$ outside $\overline{B}(0, a)$. Our goal is now the

Proof of (0.4). – The lower bound part of the large deviation principle is proven in Theorem 2.1 (*see* (2.9)) of [11]. We are therefore only concerned here with the upper bound part. Observe that $\beta_0(\cdot)$ has compact sublevel sets (it is a seminorm on $\mathbb{R}^d$), and by (2.10) of [11] we have exponential tightness. So we need only prove that for A compact in $\mathbb{R}^d$, $0 \notin A$

$$\begin{aligned}(1.1)\qquad &\overline{\lim_{t\to\infty}}\, t^{-d/d+2} \log\Big(\mathbb{E} \times E_0\Big[Z_t \in t^{d/d+2} A,\ \exp\Big\{-\int_0^t V(Z_s,\omega)ds\Big\}\Big]\Big)\\ &\quad \le -c(d,\nu) - \inf\nolimits_A \beta_0(\cdot)\,.\end{aligned}$$

Now covering $t^{d/d+2} A$ by a number at most polynomially growing with t of closed balls of radius 1, our claim (0.4) follows from

THEOREM 1.1. – $(d \ge 2)$ *For* $0 < r_1 < r_2$, *in the hard and soft obstacle case*

$$\begin{aligned}(1.2)\qquad \overline{\lim_{t\to\infty}} \sup_{r_1 \le |\overline{x}| \le r_2} \Big\{ t^{-d/d+2} \log\Big(\mathbb{E} \otimes E_0\Big[H(\overline{x}\, t^{d/d+2}) \le t,\\ \exp\Big\{-\int_0^t V(Z_s,\omega)ds\Big\}\Big]\Big) + c(d,\nu) + \beta_0(\overline{x})\Big\} \le 0\,.\end{aligned}$$

(the term "$\exp\{...\}$" *means* "$1\{T > t\}$" *in the hard obstacle case).*

Proof of Theorem 1.1. – As pointed out in the introduction, the proof hinges on a coarse grained description of the cloud which enables to define forest and clearing parts. We first introduce our notations for this coarse grained picture. We partition $\mathbb{R}^d$ in large boxes of size $t^{1/d+2}$: for $t > 1$, and $m \in \mathbb{Z}^d$, we define

$$C_m = \{z \in \mathbb{R}^d,\ m_i\, t^{1/d+2} \le z_i < (m_i + 1)\, t^{1/d+2},\ i = 1, ..., d\}\ , \tag{1.3}$$

(the t dependence is dropped in the notation).

To describe what we mean by clearings and forest, we shall need three parameters b, δ, r. We pick $b > a$, $0 < \delta < 1$. For $\omega = \sum_j \delta_{x_j} \in \Omega$, a cloud configuration, we say that $x_i \in C_m$ is good (at level t), if for all balls $C = \overline{B}(x_i, 10^{\ell+1}\, b)$, with $0 \le \ell$, $10^{\ell+1}\, b < \frac{1}{2}\, t^{1/d+2}$,

$$|C_m \cap C \cap (\bigcup_{x_j \in C_m} \overline{B}(x_j, b))\,| \ge \frac{\delta}{3^d}\, |\, C_m \cap C|\ , \tag{1.4}$$

$|\cdot|$ denotes the volume.

Observe that the actual soft or hard obstacle plays no role here. Then Good_m stands for the set of good points in C_m at level t, and $\mathrm{Good} = \bigcup_{m \in \mathbb{Z}^d} \mathrm{Good}_m$. The good obstacles are in the precise sense given above well surrounded by obstacles. We also chop identically each segment $[m_i\, t^{1/d+2},\ (m_i + 1)\, t^{1/d+2}]$ in at most $\left[\frac{\sqrt{d}}{b}\, t^{1/d+2}\right] + 1$ segments of length $\frac{b}{\sqrt{d}}$, except may be for the "last one".

We now introduce our third parameter $r \in \left(0, \frac{1}{2}\right)$. We can now define what clearing boxes and forest boxes are in the following fashion.

We first define the event Cl_m "there is a clearing of size $r\, t^{1/d+2}$ in the box C_m", via:

$$Cl_m = \{\omega \in \Omega,\ |\widetilde{U}_m(\omega)\,| \ge 2^{-d}\, |B(0, r\, t^{1/d+2})|\}\ , \tag{1.5}$$

where $\widetilde{U}_m(\omega)$ denotes the open subset of $\overset{\circ}{C}_m$ obtained by taking the complement in the interior of C_m of the closed subboxes which receive a good point of C_m. We define $U_m(\omega)$ in a analogous fashion, except that "good point" is replaced by "point". Then using a covering lemma (*see* (2.14) of [6]),

$$|\widetilde{U}_m| \le |U_m| + \delta\, t^{d/d+2}\ . \tag{1.6}$$

It is may be helpful to say here that the natural probabilistic estimates are derived for $|U_m|$, and (1.6) roughly says that "if $|\widetilde{U}_m|$ is big, then $|U_m|$ is big".

We now let $\mathcal{A}(\omega)$ (here again the t dependence is dropped in the notation) stand for the closed set which is the union of closed cubes $\overline{C}_m$, $m \in \mathbb{Z}^d$, where "there is a clearing of size $r\, t^{1/d+2}$", that is

$$1_{\mathcal{A}(\omega)}(z) = \sum_{m \in \mathbb{Z}^d} 1_{\overline{C}_m}(z)\, 1_{Cl_m}(\omega)\ . \tag{1.7}$$

We let $\mathcal{A}^1(\omega)$ stand for the open set of points in $\mathbb{R}^d$ at distance smaller than $t^{1/d+2}$ from $\mathcal{A}(\omega)$, ($\mathcal{A}^1$ is empty if A is empty). In the sequel we shall mainly be concerned with what happens within distance t of the origin. To this end, we introduce for $t > 1$:

$$(1.8) \qquad \mathcal{T} = (-\gamma t^{1/d+2}, \gamma t^{1/d+2})^d, \qquad \mathcal{T}_1 = (-2\gamma t^{1/d+2}, 2\gamma t^{1/d+2}), \ \gamma = [t^{\frac{d+1}{d+2}}] \,.$$

Let us mention now that the point of the above construction directly coming from the "method of enlargement of obstacles" is that one has a good lower estimate of the principal Dirichlet eigenvalue $\lambda_{-\frac{1}{2}\Delta+V}(\mathcal{T})$ of $-\dfrac{1}{2}\Delta + V(\cdot,\omega)$ in $\mathcal{T}$ (soft obstacles) or $\lambda_{-\frac{1}{2}\Delta}(\mathcal{T}\backslash \bigcup_i \overline{B}(x_i,a))$ (hard obstacles) in terms of the principal Dirichlet eigenvalue $\lambda_{-\frac{1}{2}\Delta}(\Theta_b)$ of $-\dfrac{1}{2}\Delta$ in the coarse grained open set

$$(1.9) \qquad \Theta_b = \mathcal{T} \cap \mathcal{A}^1 \backslash \bigcup_{x_i \in \text{Good}} \overline{B}(x_i, b) \,.$$

We shall use these estimates further below under the form of uniform exponential moments. We also introduce the open sets $\widetilde{\mathcal{U}}(\omega)$ (resp. $\mathcal{U}(\omega)$) as the complement in $\mathcal{T}\cap(\bigcup_{C_m \cap \mathcal{A}^1(\omega)\neq\phi} C_m)^0$, of the closed subboxes included in some $\overline{C}_m$ with $C_m \cap \mathcal{A}^1(\omega) \neq \phi$, which receive a good point (resp. a point) of C_m. $\widetilde{\mathcal{U}}(\omega)$ will be easier to handle than $\Theta_b(\omega)$ and satisfies:

$$(1.10) \qquad \Theta_b(\omega) \subseteq \widetilde{\mathcal{U}}(\omega) \,.$$

Our next purpose is to describe excursions of $Z.$ in and out of the clearings. We shall only be concerned with trajectories of $Z.$ which do not leave $\mathcal{T}$, and therefore the knowledge of the restriction of $\mathcal{A}(\omega)$ to $\mathcal{T}_1$ will be sufficient for our purpose. However the various possible shapes of the restriction of $\mathcal{A}$ or $\mathcal{A}^1$ to $\mathcal{T}_1$ can still be quite complicated, and to alleviate the task of describing the excursions of the process $Z.$ between forest and clearings, we shall do some regrouping and embed each component of $\mathcal{A}^1 \cap \mathcal{T}$ in a system of three concentric balls $\mathcal{B}^{\text{int}}$, $\mathcal{B}$, $\mathcal{B}^{\text{ext}}$, the various balls $\mathcal{B}^{\text{ext}}$ being pairwise disjoint, each component of $\mathcal{A}^1 \cap \mathcal{T}$ being contained in some $\mathcal{B}^{\text{int}}$. To this end we prove

LEMMA 1.2. – *Let $C(1), ..., C(N)$, be closures of $N \geq 1$ distinct boxes of type (1.3). There exist an integer $L \in [1,N]$, and a sequence $(\mathcal{B}_j^{\text{int}}, \mathcal{B}_j; \mathcal{B}_j^{\text{ext}})_{1\leq j\leq L}$, such that for each j, $\mathcal{B}_j^{\text{int}}$, $\mathcal{B}_j$, $\mathcal{B}_j^{\text{ext}}$ are concentric closed balls, obtained by successively doubling the radius of $\mathcal{B}_j^{\text{int}}$, and*

(1.11) *the $\mathcal{B}_i^{\text{ext}}$, $1 \leq i \leq L$ are pairwise disjoint,*

(1.12) *the set of points at distance smaller or equal to $(\sqrt{d}+2)\, t^{1/d+2}$ from some $C(i)$, $i \in [1,N]$ is contained in $\bigcup_{1\leq j\leq L} \mathcal{B}_j^{\text{int}}$,*

(1.13) *the $\mathcal{B}_j^{\text{ext}}$ have radii no larger than $4 \times 9^{N-1}\Big(\dfrac{3}{2}\sqrt{d}+2\Big)\, t^{1/d+2}$.*

Proof. – Without loss of generality, we assume $t = 1$. We then first consider the closed balls $\mathcal{B}_j^{\text{int}}$, $j \in J_1 = \{1, ..., N\}$, with same centers x_j as the $C(j)$ and radius $R = \frac{3\sqrt{d}}{2} + 2$. We then construct by successively doubling the radii the various $\mathcal{B}_j$, $\mathcal{B}_j^{\text{ext}}$, $j \in J_1$. If (1.11) holds, we are finished since (1.12)-(1.13) is automatic.

Otherwise we pick $j_1 \neq j_2$ such that $\mathcal{B}_{j_1}^{\text{ext}} \cap \mathcal{B}_{j_2}^{\text{ext}} \neq \phi$. Then the mutual distance of the respective centers is no larger than $8R$. We now discard the concentric balls corresponding to j_2, that is we define $J_2 = J_1 \backslash \{j_2\}$, and for j_1 we pick the triplet of closed balls with centers x_{j_1} and respective radii $9R$, $18R$, $36R$. Then (1.12) is fulfilled, and if (1.11) hold we are finished. Otherwise we proceed inductively, picking $j_1 \neq j_2$ in J_2 such that $\mathcal{B}_{j_1}^{\text{ext}} \cap \mathcal{B}_{j_2}^{\text{ext}} \neq \phi$, the radii of the triplets of balls of index j_1 being larger than those with index j_2. We delete j_2 from J_2, and multiply by 9 the radii of the triplets associated to j_1. Obviously, this procedure stops after at most $(N - 1)$ steps, yielding the desired sequence of triplets of concentric balls satisfying (1.11), (1.12), (1.13). □

From now on we assume $t > 1$ large enough so that $\mathcal{B}(0, t^{d/d+2}\, r_2) \subseteq \mathcal{T}$, r_2 the constant appearing in (1.2). For each $\omega \in \Omega$, we consider the collection of cubes $\overline{C}_m \subseteq \overline{\mathcal{T}}_1$, where $Cl_m(\omega)$ occurs ("there is a clearing of size $r\, t^{1/d+2}$ "). To this collection of cubes if it is not already there we add the closed cube containing $\overline{x}\, t^{d/d+2}$ (*see* (1.2) for the notation). We denote by $n(\omega) + 1$, $(n(\omega) \geq 0)$, the number of cubes in this collection. Thanks to the lemma just proved we construct a collection of triplets of concentric balls $(\mathcal{B}_j^{\text{int}}(\omega), \mathcal{B}_j(\omega), \mathcal{B}_j^{\text{ext}}(\omega))_{1 \leq j \leq L(\omega)}$ where $L(\omega) \leq n(\omega) + 1$, which satisfy (1.11)-(1.13), with respect to the above mentioned collection of cubes. We assume the labelling is picked in such a way that $\mathcal{B}_{L(\omega)}^{\text{int}}$ contains $\overline{x}\, t^{d/d+2}$, and therefore all points at distance less than $(\sqrt{d} + 2)\, t^{1/d+2}$ from the cube C_m containing $\overline{x}\, t^{d/d+2}$.

We are now ready to describe the excursions of the path $Z_.$. We let $\mathcal{F}_s^+ = \bigcap_{\epsilon > 0} \sigma(Z_u,\ 0 \leq u \leq s + \epsilon)$, $s \geq 0$, stand for the canonical right continuous filtration on $C(\mathbb{R}_+, \mathbb{R}^d)$, and recall that $(\theta_u)_{u \geq 0}$ denotes the canonical shift on $C(\mathbb{R}_+, \mathbb{R}^d)$. We are now ready to define two sequences of $(\mathcal{F}_s^+)$-stopping times, namely the "returns" to $\mathcal{B}_j^{\text{int}}(\omega)$ and the "departures" from $\mathcal{B}_j(\omega)$:

$$R_1 = \inf\{u \geq 0,\ Z_u \in \bigcup_{1 \leq j \leq L(\omega)} \mathcal{B}_j^{\text{int}}(\omega)\} \leq \infty\,, \tag{1.14}$$

$$D_1 = \inf\{u \geq R_1,\ Z_u \in \bigcup_{1 \leq j \leq L(\omega)} \partial\mathcal{B}_j(\omega)\} \leq \infty\,, \tag{1.15}$$

and by induction for $k \geq 1$,

$$R_{k+1} = R_1 \circ \theta_{D_k} + D_k,\ D_{k+1} = D_1 \circ \theta_{D_k} + D_k, \text{ so that}$$
$$0 \leq R_1 \leq D_1 \leq R_2 \leq D_2 \leq ... \leq R_k \leq D_k \leq ... \leq \infty\,,$$

and except for the first each inequality is strict if the term in the left member of the inequality is finite. We then introduce

$$L_t = \frac{1}{t} \cdot \sum_{i \geq 0} (R_{i+1} \wedge t - D_i \wedge t), \text{ with the notation } D_0 = 0\,, \tag{1.17}$$

$$N_t = \sum_{i \geq 1} 1\{R_i \leq t\}\,, \tag{1.18}$$

So N_t measures in a sense the number of excursions between $\bigcup_j \mathcal{B}_j^{\text{int}}$ and $(\bigcup_j \mathcal{B}_j)^c$ up to time t, and L_t the fraction of time spent in "returns" to $\bigcup_j \mathcal{B}_j^{\text{int}}$.

Let us start with the estimate of the quantity under the logarithm in (1.2). For $0 < \eta < 1$, and $n_0 \geq 1$, we have:

$$\begin{aligned}
(1.19)\quad \mathbb{E} \otimes E_0\Big[H(\overline{x}\, t^{d/d+2}) \leq t,\ \exp\Big\{-\int_0^t V(Z_u,\omega)du\Big\}\Big] &\leq P_0[T_{\mathcal{T}} \leq t] \\
+ \mathbb{E} \otimes \mathbb{E}_0\Big[T_{\mathcal{T}} > t,\ \exp\Big\{-\int_0^t V(Z_u,\omega)du\Big\},\ &|\mathcal{A} \cap \mathcal{T}_1| \geq n_0\, t^{d/d+2}, \\
\text{or } L_t \geq \eta, \text{or } N_t \geq [\eta\, t^{d/d+2}]\Big] & \\
+ \mathbb{E} \otimes \mathbb{E}_0\Big[T_{\mathcal{T}} > t,\ \exp\Big\{-\int_0^t V(Z_u,\omega)du\Big\},\ &H(\overline{x}\, t^{d/d+2}) \leq t,\ |\mathcal{A} \cap \mathcal{T}_1| \\
\leq n_0\, t^{d/d+2},\ L_t \leq \eta,\ N_t < [\eta\, t^{d/d+2}]\Big] &\stackrel{\text{def}}{=} A_1 + A_2 + A_3\,.
\end{aligned}$$

The first term A_1 is easy to control, indeed, by standard Brownian motion estimates:

$$\lim_{t \to \infty} t^{-d/d+2} \log P_0[T_{\mathcal{T}} \leq t] = -\infty\,. \tag{1.20}$$

So for our purpose of proving (1.2), A_1 is negligible. Now the term A_2 was essentially studied in Theorem 2.1 of [6] in the case of hard obstacles, and in appendix A of [9] for soft obstacles. The method employed yields here:

$$\overline{\lim_{r \to 0}}\ \overline{\lim_{n_0 \to \infty}}\ \overline{\lim_{b \to \infty, \delta \to 0}}\ \overline{\lim_{t \to \infty}}\ t^{-d/d+2} \log A_2 = -\infty\,. \tag{1.21}$$

Let us simply mention for the reader's convenience that the control on $|\mathcal{A} \cap \mathcal{T}_1|$ comes from the fact that there is an, at most polynomially growing in t, number of possible choices for n_0 boxes $C_m \subset \overline{\mathcal{T}}_1$. Moreover, for $m \neq m'$ Cl_m and $Cl_{m'}$ are independent, and in view of (1.6), by a counting argument

$$\begin{aligned}
\mathbb{P}[Cl_m] \leq \mathbb{P}[|U_m| \geq t^{d/d+2}(2^{-d}\,|B(0,r)| - \delta)] &\leq 2^{(\frac{\sqrt{d}}{b}\, t^{1/d+2}+1)^d} \\
&\exp\{-\nu\, t^{d/d+2}(2^{-d}\,|B(0,r)| - \delta)\}\,.
\end{aligned}$$

As for the controls involving N_t and L_t, the main ingredients in (1.21), are the fact that it is "costly" to perform excursions in the forest, that is $\mathcal{A}^c$, because, with an "adequate choice of parameters", for $z \in \mathcal{A}^c(\omega)$, uniformly in ω:

$$E_z\Big[\exp\Big\{-\int_0^{H_{4r\, t^{1/d+2}}} V(Z_u,\omega)du\Big\}\Big] \leq 1 - c_1(d)$$

(*see* (A.7) of [9]), with $H_c = \inf\{u \geq 0,\ |Z_u - Z_0| \geq c\}$, and by an exponential estimate (*see* (A.10) of [9]), it is "costly" to spend too much time without reaching the clearing set $\mathcal{A}(\omega)$.

Let us denote by $\mathcal{E}$ the event

$$\mathcal{E} = \{T_{\mathcal{T}} > t,\ N_t < [\eta\, t^{d/d+2}],\ L_t \leq \eta,\ H(\overline{x}\, t^{d/d+2}) \leq t\}\,. \tag{1.22}$$

As mentioned above, there is an, at most polynomial in t number of possible choices for at most n_0 distinct boxes $\overline{C}_m$ in $\overline{\mathcal{T}}_1$. To prove (1.2), it therefore suffices to check that:

$$\begin{aligned}(1.23)\quad &\overline{\lim_{\eta\to 0}}\ \overline{\lim_{r\to 0}}\ \overline{\lim_{n_0\to\infty}}\ \overline{\lim_{b\to\infty,\delta\to 0}}\ \overline{\lim_{t\to\infty}}\ \sup_{\#\mathcal{M}\leq n_0}\ \sup_{r_1\leq|\overline{x}|\leq r_2}\\ &t^{-d/d+2}\log\Big(\mathbb{E}\Big[\overline{\mathcal{A}\cap\mathcal{T}_1} = \mathcal{A}_{\mathcal{M}},\ E_0\Big[\mathcal{E},\ \exp\Big\{-\int_0^t V(Z_s,\omega)ds\Big\}\Big]\Big]\Big) + c(d,\nu)\\ &+\ \beta_0(\overline{x}) \leq 0\,,\end{aligned}$$

here $\mathcal{M}$ stands for a possibly empty subset of $\mathbb{Z}^d\cap\ [-2[t^{\frac{d+1}{d+2}}],\ 2[t^{\frac{d+1}{d+2}}] - 1]^d$, and $\mathcal{A}_{\mathcal{M}} = \bigcup_{m\in\mathcal{M}} \overline{C}_m$. Now the method of enlargement of obstacles (*see* Theorem 1.1) of [7] for hard obstacles, and Theorem A.1 of [8] for soft obstacles), yields the following exponential estimate: for $M > 0, \rho > 0$,

$$\overline{\lim_{r\to 0}}\ \sup_{b>a,0<\delta<1}\ \overline{\lim_{t\to\infty}}\ \sup_{z,\omega,\mathcal{T}} E_z[H] \leq K(d,M,\rho) < \infty,\ \text{with} \tag{1.24}$$

$$\begin{aligned}&H = \exp\{\lambda(T\wedge T_{\mathcal{T}})\},\ \text{for hard obstacles,}\\ (1.25)\quad &H = \int_0^{T_{\mathcal{T}}} V(Z_s,\omega)\ \exp\Big\{-\int_0^s V(Z_u,\omega)du + \lambda s\Big\}ds\\ &\quad +\ \exp\Big\{-\int_0^{T_{\mathcal{T}}} V(Z_s,\omega)ds + \lambda T_{\mathcal{T}}\Big\}\\ &\quad = 1 + \int_0^{T_{\mathcal{T}}} \lambda e^{\lambda s}\ \exp\Big\{-\int_0^s V(Z_u,\omega)du\Big\}ds,\ \text{for soft obstacles,}\end{aligned}$$

$$\text{with } \lambda = (\lambda_{-\frac{1}{2}\Delta}(\Theta_b)\wedge M\, t^{-2/d+2} - \rho\, t^{-2/d+2})_+\,. \tag{1.26}$$

In fact in what follows, we shall use (1.24), with λ replaced by $\widetilde{\lambda} \leq \lambda$, and the choice (from now on)

$$M = c(d,\nu) + 1 \tag{1.27}$$

$$\widetilde{\lambda} = (\lambda_{-\frac{1}{2}\Delta}(\widetilde{\mathcal{U}})\wedge(M\, t^{-2/d+2}) - \rho\, t^{-2/d+2})_+\,, \tag{1.28}$$

where $\widetilde{\mathcal{U}}$ has been introduced before (1.10), and $\widetilde{\lambda} \leq \lambda$, follows from (1.10). Let us now study for some ω such that $\overline{\mathcal{A}(\omega) \cap \mathcal{T}_1} = \mathcal{A}_{\mathcal{M}}$, $(\# \mathcal{M} \leq n_0)$, the expression under the E_0 expectation in (1.23). On the set $\mathcal{E}$, the condition $L_t \leq \eta$ implies that:

$$\sum_{i \geq 1} D_i \wedge t - R_i \wedge t \geq (1-\eta)\, t\,. \tag{1.29}$$

We define

$$J = \inf\{i \geq 1,\ Z_{R_i} \in \mathcal{B}_L^{\text{int}}\}\,, \tag{1.30}$$

where we recall that $\mathcal{B}_L^{\text{int}}$ is the ball $\mathcal{B}_j^{\text{int}}$ which contains $B(\overline{x}\, t^{d/d+2}, 2)$, and that the $(\mathcal{B}_j^{\text{int}}$, $\mathcal{B}_j$, $\mathcal{B}_j^{\text{ext}})_{1 \leq j \leq L}$ are deterministic functions of $\mathcal{A}_{\mathcal{M}}\Big(= \overline{\mathcal{A}(\omega) \cap \mathcal{T}_1}\Big)$ and $\overline{x}\, t^{d/d+2}$. Observe that for $t \geq \text{const}(n_0, r_1)$, as we shall now assume, $0 \notin \mathcal{B}_L^{\text{int}}$.

Our goal is now to split "excursion costs" and "resting costs" for the process. Observe that $\{H(\overline{x}\, t^{d/d+2}) \leq t\} \subseteq \{R_J \leq t\}$, so using (1.29),

$$\begin{aligned}
&1_{\mathcal{E}} \,\exp\Big\{ - \int_0^t V(Z_u,\omega)du\Big\} \leq \exp\Big\{ -\widetilde{\lambda}(1-\eta)\,t + \widetilde{\lambda} \sum_{i\geq 1} (D_i \wedge t - R_i \wedge t) \\
&\quad - \int_0^t V(Z_u,\omega)du\Big\}\ 1\{R_J \leq t < T_{\mathcal{T}}\}\ 1_{\mathcal{E}} \\
&\leq \exp\{-\widetilde{\lambda}(1-\eta)\,t\} \cdot \exp\Big\{\widetilde{\lambda} \cdot \sum_{i=1}^{J-1} (D \wedge T_{\mathcal{T}}) \circ \theta_{R_i} - \int_0^{R_J} V(Z_u,\omega)du\Big\} \\
&\quad \times 1\{1 \leq J \leq N_t < [\eta\, t^{d/d+2}]\} \cdot \exp\Big\{\widetilde{\lambda} \cdot \sum_{k=J}^{N_t} (D \wedge T_{\mathcal{T}}) \circ \theta_{R_k} \wedge (t-R_k)_+ \\
&\quad - \int_{R_J}^t V(Z_u,\omega)du\Big\} \cdot 1\{R_J \leq t < T_{\mathcal{T}}\},\ \text{with the notation } D = T_{\bigcup_{1\leq j \leq L} \mathcal{B}_j}\,.
\end{aligned} \tag{1.31}$$

Let us now define G, by replacing in the definition of H in (1.25), $T_{\mathcal{T}}$ by $D \wedge T_{\mathcal{T}}$ and λ by $\widetilde{\lambda}$ (*see* (1.28)), so that $1 \leq G \leq H$. From (1.31) follows:

$$\begin{aligned}
&1_{\mathcal{E}}\ \exp\Big\{ - \int_0^t V(Z_u,\omega)du\Big\} \\
&\leq \exp\{-\widetilde{\lambda}(1-\eta)\,t\}\ \exp\Big\{\widetilde{\lambda} \sum_{i=1}^{J-1} D \wedge T_{\mathcal{T}} \circ \theta_{R_i} - \int_0^{R_J} V(Z_u,\omega)du\Big\} \\
&\times 1\{1 \leq J \leq [\eta\, t^{d/d+2}],\ R_J < T_{\mathcal{T}}\} \cdot \prod_{k=1}^{[\eta\, t^{d/d+2}]} G \circ \theta_{R_k \wedge T_{\mathcal{T}}} \circ \theta_{R_J}\,.
\end{aligned} \tag{1.32}$$

Now thanks to the enlargement of obstacle estimate (1.24), and the fact that we wish to prove (1.23), with the successive lim sup operations indicated, with no loss of generality, we can restrict ourselves to values of the parameters ρ, r, b, δ, t, for which:

$$\text{(1.33)} \qquad \sup_{z,\omega} E_z[G] < K(d, M, \rho) \quad (\text{with } M = c(d,\nu) + 1, \text{ from } (1.27))\,.$$

If we now use the strong Markov property to the E_0 expectation of (1.32), as well as (1.33), we obtain:

$$E_0\Big[1_{\mathcal{E}}\, \exp\Big\{-\int_0^t V(Z_u,\omega)du\Big\}\Big] \le K(d,M,\rho)^{\eta\, t^{d/d+2}} \cdot \exp\{-\widetilde{\lambda}(1-\eta)\,t\}\,\cdot$$
$$E_0\Big[R_J < T_{\mathcal{T}},\ 1 \le J \le [\eta\, t^{d/d+2}],\ \exp\Big\{\widetilde{\lambda} \sum_{i=1}^{J-1} (D \wedge T_{\mathcal{T}}) \circ \theta_{R_i} - \int_0^{R_J} V(Z_u,\omega)du\Big\}\Big]\,.$$

This is already a partial split of costs as promised. It now follows that our claim (1.23) will follow if we prove:

$$\text{(1.34)} \qquad \begin{aligned} &\widetilde{\lim}\ \sup_{\#\mathcal{M}\le n_0}\ \sup_{r_1 \le |\overline{x}| \le r_2}\ t^{-d/d+2} \log\Big(\mathbb{E}\Big[\overline{\mathcal{A}\cap\mathcal{T}_1} = \mathcal{A}_{\mathcal{M}},\ \exp\{-\widetilde{\lambda}(1-\eta)\,t\} \\ &E_0\Big[R_J < T_{\mathcal{T}},\ J \le [\eta\, t^{d/d+2}], \\ &\exp\Big\{\widetilde{\lambda}\cdot \sum_{i=1}^{J-1} (D\wedge T_{\mathcal{T}})\circ\theta_{R_i} - \int_0^{R_J} V(Z_u,\omega)du\Big\}\Big]\Big]\Big) + c(d,\nu) + \beta_0(\overline{x}) \le 0\,, \\ &\text{provided } \widetilde{\lim} \text{ stands for “ } \overline{\lim_{\rho\to 0}}\ \overline{\lim_{\eta\to 0}}\ \overline{\lim_{r\to 0}}\ \overline{\lim_{n_0\to\infty}}\ \overline{\lim_{b\to\infty,\delta\to 0}}\ \overline{\lim_{t\to\infty}}\,.\text{ ”} \end{aligned}$$

We are now going to extract from all the excursions in the forest, a sequence of excursions which never visits twice the same $\mathcal{B}_j^{\text{int}}$, and eventually leads to $\mathcal{B}_L^{\text{int}}$ containing $\overline{x}\, t^{d/d+2}$. Namely we pick $\omega \in \Omega$, such that $\overline{\mathcal{A}\cap\mathcal{T}_1} = \mathcal{A}_{\mathcal{M}}$. Recall that this and $\overline{x}\, t^{d/d+2}$ completely determine the sequence $(\mathcal{B}_j^{\text{int}}, \mathcal{B}_j, \mathcal{B}_j^{\text{ext}})_{1\le j\le L}$.

Consider now the expression under E_0 expectation in (1.34). On the set $1 \le J \le [\eta\, t^{d/d+2}]$ and $R_J < T_{\mathcal{T}}$, we have the sequence $0 \le R_1 < R_2 < \ldots < R_{J-1} < R_J$, and the equality $R_1 = 0$ occurs only if $0 \in \bigcup_{1\le j\le L}, \mathcal{B}_j^{\text{int}}$ (in fact for $t \ge$ const (n_0, r_1), $0 \notin \mathcal{B}_L^{\text{ext}}$). For each trajectory of $Z_.$, we can define $j_1(w)$ the unique ball $\mathcal{B}_j^{\text{int}}$ to which Z_{R_1} belongs, $i_1(w)$ the last index i in $[1, J(w)]$ for which $Z_{R_i} \in \mathcal{B}_{j_1(w)}^{\text{int}}$, $j_2(w)$ the index of the next ball visited namely the unique j for which $Z_{R_{i_1(w)+1}} \in \mathcal{B}_j^{\text{int}}$, $i_2(w)$ the last index i in $[1, J(w)]$ for which $Z_{R_i} \in \mathcal{B}_{j_2(w)}^{\text{int}}$ and so on until $j_{\ell(w)}(w)$, $i_{\ell(w)}(w) = J(w) - 1$, $j_{\ell(w)+1}(w) = L$, $i_{\ell(w)+1}(w) = J$, with $0 \le \ell(w) \le L(\mathcal{M}, \overline{x}\, t^{d/d+2}) - 1$.

So there are no more than $L \le n_0 + 1$ possibilities for j, moreover $0 \le \ell \le L$ and $1 \le i_1 < i_2 < \ldots < i_\ell < i_{\ell+1} = J \le [\eta\, t^{d/d+2}]$; consequently the number of possibilities for the sequences $(i_1, j_1)\ldots(i_\ell, j_\ell)$ grows at most polynomially with t. Our claim (1.34) then follows if we prove:

$$(1.35)\quad \widetilde{\lim} \sup_{\#\mathcal{M}\le n_0} \sup_{r_1\le|\overline{x}|\le r_2} \sup_{0\le \ell\le L(\mathcal{M},\overline{x}\,t^{d/d+2})-1} \sup_{1\le i_1<\ldots<i_\ell\le[\eta\, t^{d/d+2}]} \sup_{j_1\ne\ldots\ne j_\ell\in[1,L-1]}$$
$$t^{-d/d+2}\log\Big(\mathbb{E}\Big[\overline{\mathcal{A}\cap\mathcal{T}_1}=\mathcal{A}_{\mathcal{M}},\ \exp\{-\widetilde{\lambda}(1-\eta)\,t\}$$
$$\times E_0\Big[\exp\Big\{-\int_0^{R_{i_\ell+1}} V(Z_u,\omega)du\Big\}$$
$$\times\exp\Big\{\widetilde{\lambda}\sum_{i=1}^{i_\ell}(D\wedge T_{\mathcal{T}})\circ\theta_{R_i}\Big\},\ \mathcal{G}\Big]\Big]\Big)+c(d,\nu)+\beta_0(\overline{x})\le 0,\ \text{with}$$

$$(1.36)\quad \mathcal{G}=\{Z_{R_1},\ Z_{R_{i_1}}\in\mathcal{B}_{j_1}^{\rm int},\ Z_{R_{i_1+1}},\ Z_{R_{i_2}}\in\mathcal{B}_{j_2}^{\rm int},\ldots,Z_{R_{i_\ell+1}}\in\mathcal{B}_L^{\rm int},\ R_{i_\ell+1}<T_{\mathcal{T}}\}$$

(in the case $\ell=0$, $\mathcal{G}=\{Z_{R_1}\in\mathcal{B}_L^{\rm int},\ R_1<T_{\mathcal{T}}\}$).

With the notation $i_0=0$, the term under E_0 expectation in (1.35) equals

$$(1.37)\quad \Big(\prod_{k=1}^{\ell}\exp\Big\{\widetilde{\lambda}\cdot\sum_{i=i_{k-1}+1}^{i_k}(D\wedge T_{\mathcal{T}})\circ\theta_{R_i}-\int_{R_{(i_{k-1}+1)}}^{D_{i_k}}V(Z_u,\omega)du\Big\}$$
$$\exp\Big\{-\int_{D_{i_k}}^{R_{i_k+1}}V(Z_u,\omega)du\Big\}\Big)\cdot\exp\Big\{-\int_0^{R_1}V(Z_u,\omega)du\Big\}\cdot 1_{\mathcal{G}}\,.$$

We can now define for $i\ge 1$, $\widetilde{D}_i$ the first exit time from $\bigcup_{1\le j\le L}\overset{\circ}{\mathcal{B}}{}_j^{\rm ext}$ after D_i, (recall the $\mathcal{B}_j^{\rm ext}$ are pairwise disjoint):

$$\widetilde{D}_i=T_{\bigcup_j \overset{\circ}{\mathcal{B}}{}_j^{\rm ext}}\circ\theta_{D_i}+D_i\le\infty,\qquad i\ge 1\,.$$

Obviously $D_{i_k}<\widetilde{D}_{i_k}<R_{i_k+1}=R_1\circ\theta_{\widetilde{D}_{i_k}}+\widetilde{D}_{i_k}$, for $1\le k\le\ell$, on $\mathcal{G}$. So the E_0 expectation of the expression in (1.37) is smaller than

$$(1.38)\quad E_0\Big[\Big\{-\int_0^{R_1}V(Z_u,\omega)du\Big\}\prod_{k=1}^{\ell}\exp\Big\{\widetilde{\lambda}\cdot\sum_{i=i_{k-1}+1}^{i_k}(D\wedge T_{\mathcal{T}})\circ\theta_{R_i}$$
$$-\int_{R_{i_{k-1}+1}}^{D_{i_k}}V(Z_u,\omega)du\Big\}$$
$$\exp\Big\{-\int_{\widetilde{D}_{i_k}}^{R_{i_k+1}}V(Z_u,\omega)du\Big\},\ \mathcal{G}\Big]\,.$$

Denote by $q_j(x,y)$ for $1\le j\le L-1$, $x\in\overset{\circ}{\mathcal{B}}{}_j^{\rm ext}$, $y\in\partial\overset{\circ}{\mathcal{B}}{}_j^{\rm ext}$, the Poisson kernel, that is the density of $Z_{T_{\overset{\circ}{\mathcal{B}}{}_j^{\rm ext}}}$ under P_x, with respect to the normalized surface measure $ds_j(y)$ on $\partial\overset{\circ}{\mathcal{B}}{}_j^{\rm ext}$, that is

$$q_j(x,y)=(\rho_j)^{d-2}\,\frac{|y-x_j|^2-|x-x_j|^2}{|y-x|^d},\qquad\text{if}\quad \mathcal{B}_j^{\rm ext}=\overline{B}(x_j,\rho_j)\,,$$

Then we can find $0 < \kappa_1(d) < 1 < \kappa_2(d) < \infty$, such that for any j:

$$\kappa_1(d) \le \inf_{\mathcal{B}_j^{\text{int}} \times \mathcal{B}_j^{\text{ext}}} q_j(x,y) \le \sup_{\mathcal{B}_j \times \partial\mathcal{B}_j^{\text{ext}}} q_j(x,y) \le \kappa_2(d)\,. \tag{1.39}$$

Now if we use in (1.38) the strong Markov property, the bound (1.33) for the terms involving $\widetilde{\lambda}$, and (1.39) between time D_{i_k} and $\widetilde{D}_{i_k}$, $1 \le k \le \ell$, we see that the expression in (1.40) is smaller than:

$$\begin{aligned} &K(d,M,\rho)^{\eta\, t^{d/d+2}}\, \kappa_2(d)^{L-1} \times E_0\Big[\exp\Big\{-\int_0^{R_1} V(Z_u,\omega)du\Big\},\ Z_{R_1} \in \mathcal{B}_{j_1}^{\text{int}}\Big] \\ &\prod_{k=1}^{\ell} \int_{\partial\mathcal{B}_{j_k}^{\text{ext}}} E_y\Big[\exp\Big\{-\int_0^{R_1} V(Z_u,\omega)du\Big\},\ Z_{R_1} \in \mathcal{B}_{j_{k+1}}^{\text{int}}\Big] ds_{j_k}(y)\,, \end{aligned} \tag{1.40}$$

with $j_{\ell+1} = L$, and $R_1 < \infty$ implicit in each expression.

Denote by $\widetilde{\mathcal{U}}_{\mathcal{M}}(\omega)$ the open set complement in $\mathcal{T} \cap (\bigcup_{C_m \cap \mathcal{A}_{\mathcal{M}}^1 \ne \phi} C_m)^0$ of the closed subboxes included in some $\overline{C}_m$ with $C_m \cap \mathcal{A}_{\mathcal{M}}^1 \ne \phi$, which receive some good point of C_m. Then on the set $\overline{\mathcal{A} \cap \mathcal{T}^1} = \mathcal{A}_{\mathcal{M}}$, $\mathcal{U}(\omega) = \mathcal{U}_{\mathcal{M}}(\omega)$, and $\widetilde{\lambda}(\omega) = \widetilde{\lambda}_{\mathcal{M}}(\omega)$, if $\widetilde{\lambda}_{\mathcal{M}}$ is defined as $\widetilde{\lambda}$ in (1.28), with $\widetilde{\mathcal{U}}$ replaced by $\widetilde{\mathcal{U}}_{\mathcal{M}}$. So from (1.40), we see that the claim (1.35) follows if we prove:

$$\begin{aligned} \widetilde{\lim}\ \widetilde{\sup}\, t^{-d/d+2} \log\Big(\mathbb{E}\Big[&\exp\{-\widetilde{\lambda}_{\mathcal{M}}(\omega)(1-\eta)\,t\} \\ &\times E_0\Big[\exp\Big\{-\int_0^{R_1} V(Z_u,\omega)du\Big\}, Z_{R_1} \in \mathcal{B}_{j_1}^{\text{int}}\Big] \\ &\times \prod_{k=1}^{\ell} \int_{\partial\mathcal{B}_{j_k}^{\text{ext}}} E_y\Big[\exp\Big\{-\int_0^{R_1} V(Z_u,\omega)du\Big\}, \\ &Z_{R_1} \in \mathcal{B}_{j_{k+1}}^{\text{int}}\Big] ds_{j_k}(y)\Big]\Big) + c(d,\nu) + \beta_0(\overline{x}) \le 0\,, \end{aligned} \tag{1.41}$$

where $\widetilde{\sup}$ denotes the various suprema which appear in (1.35). Observe that $\widetilde{\lambda}_{\mathcal{M}}(\omega)$ is measurable with respect to the σ-field generated by the restriction of the cloud to the $\sqrt{d}\,t^{1/d+2}$ neighborhood of $\mathcal{A}_{\mathcal{M}}$. On the other hand, the other terms under the $\mathbb{E}$ expectation in (1.41) are measurable with respect to the restriction of the cloud to a closed a-neighborhood of $(\bigcup_{1 \le j \le L} \mathcal{B}_j^{\text{int}})^c$. Now as soon as $t^{1/d+2} > a$, these two σ-fields are independent under $\mathbb{P}$. Therefore our claim (1.41) follows from:

$$\widetilde{\lim} \sup_{\#\mathcal{M} \le n_0} t^{-d/d+2} \log \mathbb{E}[\exp\{-\widetilde{\lambda}_{\mathcal{M}}(\omega)(1-\eta)\,t\}] \le -c(d,\nu), \text{ and} \tag{1.42}$$

$$\begin{aligned} &\widetilde{\lim}\ \widetilde{\sup}\, t^{-d/d+2} \log\Big(\mathbb{E}\Big[E_0\Big[\exp\Big\{-\int_0^{R_1} V(Z_u,\omega)du\Big\},\ Z_{R_1} \in \mathcal{B}_{j_1}^{\text{int}}\Big] \cdot \\ &\prod_{k=1}^{\ell} \int_{\partial\mathcal{B}_{j_k}^{\text{ext}}} E_y\Big[\exp\Big\{-\int_0^{R_1} V(Z_u,\omega)du\Big\},\ Z_{R_1} \in \mathcal{B}_{j_{k+1}}^{\text{int}}\Big] ds_{j_k}(y)\Big]\Big) + \beta_0(\overline{x}) \le 0\,. \end{aligned} \tag{1.43}$$

We have now really split apart the "resting cost" and the "excursion cost". We begin with the proof of (1.43). The proof of (1.42) is in a sense classical, and essentially follows the argument to prove (2.16) in [7].

Define a sequence $\overline{R}_i$, $\overline{D}_i$, $i \geq 1$, exactly as in (1.14) - (1.16), except that the role of D_1 is replaced by that of $\widetilde{D}_1$, that is:

$$\begin{aligned} &\overline{R}_1 = R_1,\ \overline{D}_1 = T_{\bigcup_j \mathcal{B}_j^{\text{ext}}} \circ \theta_{R_1} + R_1 (= \widetilde{D}_1) \\ &\overline{R}_{i+1} = R_1 \circ \theta_{\overline{D}_i} + \overline{D}_i,\ 1 \leq i \\ &\overline{D}_{i+1} = \overline{D}_1 \circ \theta_{\overline{D}_i} + \overline{D}_i,\ 1 \leq i, \text{ so that} \\ &0 \leq \overline{R}_1 \leq \overline{D}_1 \leq \overline{R}_2 \leq \overline{D}_2 \leq \ldots \leq \infty\,. \end{aligned} \tag{1.44}$$

We are considering returns to $\mathcal{B}_j^{\text{int}}(\omega)$ and departures from $\mathcal{B}_j^{\text{ext}}(\omega)$ instead of departures from $\mathcal{B}_j(\omega)$. We shall now piece together the excursions and reconstruct a Brownian path, namely the expression under the $\mathbb{E}$ expectation in (1.43), thanks to (1.39) is smaller than:

$$\begin{aligned} &\kappa_1(d)^{-\ell}\, E_0\Big[\exp\Big\{-\int_0^{\overline{R}_1} V(Z_u,\omega)du\Big\},\ Z_{\overline{R}_1} \in \mathcal{B}_{j_1}^{\text{int}}\,, \\ &\prod_{k=1}^{\ell} \exp\Big\{-\int_{\overline{D}_k}^{\overline{R}_{k+1}} V(Z_u,\omega)du\Big\},\ Z_{\overline{R}_{k+1}} \in \mathcal{B}_{j_{k+1}}^{\text{int}}\Big] \leq \\ &\kappa_1(d)^{-\ell}\, E_0\Big[\exp\Big\{-\int_0^{\overline{R}_1} V(Z_u,\omega)du\Big\}\cdot 1\{Z_{\overline{R}_1} \in \mathcal{B}_{j_1}^{\text{int}}\}\,\cdot \\ &\prod_{k=1}^{\ell} \exp\Big\{-\nu\int dy\,\Big(1-e^{-\int_{\overline{R}_k}^{\overline{D}_k} W(Z_u-y)du}\Big) - \int_{\overline{D}_k}^{\overline{R}_{k+1}} V(Z_u,\omega)du\Big\} \\ &1(Z_{\overline{R}_{k+1}} \in \mathcal{B}_{j_{k+1}}^{\text{int}},\ \overline{R}_{k+1} < \infty)\Big]\ \cdot \prod_1^{\ell} (1/\min_{x\in\mathcal{B}_{j_k}^{\text{int}},\, y\in\partial\mathcal{B}_{j_k}^{\text{ext}}} E_{x,y}^{j_k}\Big[\exp\Big\{-\nu\int dy \\ &\Big(1-e^{-\int_0^{T_{j_k}} W(Z_u-y)du}\Big)\Big\}\Big] \stackrel{\text{def}}{=} A\cdot B\,, \end{aligned} \tag{1.45}$$

provided $E_{x,y}^{j_k}$, for $x \in \mathcal{B}_{j_k}^{\text{int}}$, $y \in \partial\mathcal{B}_{j_k}^{\text{ext}}$ denotes Brownian motion starting at x, conditioned to exit $\mathcal{B}_{j_k}^{\text{ext}}$ at y, and T_{j_k} is the exit time from $\mathcal{B}_{j_k}^{\text{ext}}$.

Observe now that the last term B is smaller than:

$$\begin{aligned} &\prod_1^{\ell} \Big(1/\min_{x\in\mathcal{B}_{j_k}^{\text{int}},\, y\in\partial\mathcal{B}_{j_k}^{\text{ext}}} E_{x,y}^{j_k}\,[\exp\{-\nu\,|\,S_{T_k}^a\,|\}]\Big) \\ &\leq \Big(\sup_{t^{1/d+2}\leq R\leq c(d,n_0)\,t^{1/d+2}}\ \max_{x\in\overline{B}(0,\frac{R}{4}),\, y\in\partial B(0,R)} 1/E_{x,y}^{B(0,R)}[\exp\{-\nu\,|\,S_{T_{B(0,R)}}^a\,|\}]\Big)^{n_0} \stackrel{\text{def}}{=} \widetilde{B}\,, \end{aligned} \tag{1.46}$$

where we recall that with the notations of (0.1) of [11], $S_u^a = \bigcup_{0\leq s\leq u} \overline{B}(Z_s,a)$ is the Wiener sausage of radius a in time u, and $c(d,n_0) = 4 \times 9^{n_0} \times \Big(\dfrac{3}{2}\sqrt{d}+2\Big)$, comes from (1.13) in lemma 1.2.

Observe that $\widetilde{B}$ does not depend on ω. Moreover, from the "positive correlation inequality" (1.16) of [11], and its corresponding version for hard obstacles

$$
\begin{aligned}
(1.47)\quad \mathbb{E}[A] \le \kappa_1(d)^{-\ell}\, E_0\Big[&\exp\Big\{-\nu \int dy\Big(1 - e^{-\int_0^{\overline{R}_{\ell+1}} W(Z_u - y)du}\Big)\Big\},\\
&Z_{\overline{R}_1} \in \mathcal{B}^{\rm int}_{j_1}, \ldots, Z_{\overline{R}_\ell} \in \mathcal{B}^{\rm int}_{j_\ell},\ Z_{\overline{R}_{\ell+1}} \in \mathcal{B}^{\rm int}_L,\ \overline{R}_{\ell+1} < \infty\Big]\\
\le \kappa_1(d)^{-\ell}\, E_0\Big[&\exp\Big\{-\nu \int dy\Big(1 - \exp\Big\{-\int_0^{H_{\mathcal{B}^{\rm int}_L}} W(Z_u - y)du\Big\}\Big)\Big\},\\
&H_{\mathcal{B}^{\rm int}_L} < \infty\Big]\,.
\end{aligned}
$$

Observe that $\mathcal{B}^{\rm int}_L \subseteq \overline{B}(\overline{x}\, t^{d/d+2},\ 2c(d,n_0)\, t^{1/d+2})$, where $c(d,n_0)$ is as in (1.46), and that $\mathcal{B}^{\rm int}_L$ can be covered by a collection of $c'(d,n_0)\, t^{d/d+2}$ balls of radius 1 with centers in $B(\overline{x}\, t^{d/d+2},\ 2c(d,n_0)\, t^{1/d+2} + 1) \overset{\rm def}{=} \widetilde{B}_t(\overline{x})$. Recall from (0.6) of [11] that with $H(x) = H_{\overline{B}(x,1)}$,

$$
\begin{aligned}
f_0(x) &= E_0\Big[\exp\Big\{-\nu \int dy\Big(1 - e^{-\int_0^{H(x)} W(Z_u - y)du}\Big)\Big\}\Big] \quad \text{(soft obstacles)}\\
&= E_0[\exp\{-\nu\, |\, S^a_{H(x)}\, |\}] \quad \text{(hard obstacles)}\,.
\end{aligned}
$$

It follows that the right most member of (1.47) is smaller than $c'(d,n_0)\, t^{d/d+2}\cdot \kappa_1(d)^{-n_0}$ $\sup\{f_0(x),\, x\in \widetilde{B}_t(\overline{x})\}$. Using Theorem 1.3 of [11], which relates $\beta_0(\cdot)$ to the exponential decay of $f_0(\cdot)$, we have

$$
\begin{aligned}
(1.48)\quad &\overline{\lim_{t\to\infty}}\ \widetilde{\sup}\ (t^{-d/d+2}\ \log \mathbb{E}[A] + \beta_0(\overline{x}))\\
&\le \overline{\lim_{t\to\infty}}\ \sup_{r_1 \le |\overline{x}| \le r_2}\ t^{-d/d+2}\ \sup\{\log f_0(x),\ x\in \widetilde{B}_t(\overline{x})\} + \beta_0(\overline{x}) \le 0\,.
\end{aligned}
$$

So our claim (1.43) will follow from the

LEMMA 1.3:

$$
\begin{aligned}
(1.49)\quad &\overline{\lim_{t\to\infty}}\ t^{-d/d+2} \sup_{t^{1/d+2}\le R\le c(d,n_0)\, t^{1/d+2}}\\
&\sup_{x\in \overline{B}(0,\frac{R}{4}),\, y\in\partial B(0,R)} \log(1/E^{B(0,R)}_{x,y}\ [\exp\{-\nu\, |\, S^a_{T_{B(0,R)}}\, |\}]) = 0\,.
\end{aligned}
$$

Proof. – Using Jensen's inequality, the quantity we study is smaller than

$$
\begin{aligned}
(1.50)\quad &\nu\, E^{B(0,R)}_{x,y}\Big[\,|S^a_{T_{B(0,R)}}|\,\Big] \le \nu\, |B(0,3a)| + \frac{\nu}{q_R(x,y)} \int_{B(0,R)\setminus B(y,3a)} dz\\
&E_x[H_{\overline{B}(z,a)} < T_{B(0,R)},\ q_R(Z_{H_{\overline{B}(z,a)}}, y)]\,,
\end{aligned}
$$

if $q_R(\cdot,\cdot)$ stands for the Poisson kernel relative to $B(0,R)$. Observe that when $z \in B(0,R)\backslash B(y,3a)$:

$$\max_{\overline{B}(z,a)} q_R(\cdot,y) \le R^{d-2}\,\frac{|y+z|+a}{(|y-z|-a)^{d-1}} \le 2^{d-1}\,R^{d-2}\,\frac{2R+a}{|y-z|^{d-1}}\,.$$

Consequently,

$$\overline{\lim_{R\to\infty}}\; R^{-d}\int_{B(0,R)\backslash B(y,3a)}\Big(\max_{\overline{B}(z,a)} q(\cdot,y)\Big)dz \le \text{const}(d) < \infty, \text{ and} \tag{1.51}$$

$$\overline{\lim_{R\to\infty}}\; R^{-d}\sup_{x\in\overline{B}(0,\frac{R}{4}),\,y\in\partial B(0,R)}\int_{B\left(x,\,\frac{R}{\log R}\right)}\max_{\overline{B}(z,a)}\cdot\, q(\cdot,y)dz = 0\,. \tag{1.52}$$

Now exactly as in (1.39), for $R>0$, $x\in\overline{B}\Big(0,\dfrac{R}{4}\Big)$, $y\in\partial B(0,R)$, $q(x,y)\ge\kappa_1(d)$. So in view of (1.51), (1.52) our claim (1.49) will follow from:

$$\lim_{R\to\infty}\;\sup_{x\in\overline{B}(0,\frac{R}{4}),\,z\in B(0,R)\backslash B(x,\frac{R}{\log R})} P_x[H_{\overline{B}(z,a)} < T_{B(0,R)}] = 0\,. \tag{1.53}$$

But this last point follows from:

$$\begin{aligned}P_x[H_{\overline{B}(z,a)} < T_{B(0,R)}] &\le P_x[H_{\overline{B}(z,a)} < T_{B(z,2R)}]\\ &= \frac{g(|x-z|)-g(2R)}{g(a)-g(2R)} \le \frac{g(R/\log R)-g(2R)}{g(a)-g(2R)}\,,\end{aligned}$$

with $R>e$, and $g(u)=u^{2-d}$, when $d\ge 3$, $g(u)=\log 1/u$, when $d=2$, for $u>0$. $\square$

The last point to prove is now (1.42), which essentially follow as in the proof of (2.16) of [7]. Namely, we first observe that there are at most $3^d\,n_0$ cubes C_m which intersect $\mathcal{A}^1_{\mathcal{M}}$. Moreover, if $\mathcal{U}_{\mathcal{M}}$ is defined analogously to $\widetilde{\mathcal{U}}_{\mathcal{M}}$ (before (1.41)), except that "good point" is replaced by "point", out of (1.6), we find

$$|\widetilde{\mathcal{U}}_{\mathcal{M}}| \le |\mathcal{U}_{\mathcal{M}}| + 3^d\,n_0\,\delta t^{d/d+2}\,. \tag{1.54}$$

It then follows by estimating the number of possibilities of $\widetilde{\mathcal{U}}_{\mathcal{M}}$ and $\mathcal{U}_{\mathcal{M}}\subseteq\widetilde{\mathcal{U}}_{\mathcal{M}}$ for a fixed $\mathcal{M}$ that

$$\begin{aligned}&\mathbb{E}[\exp\{-\widetilde{\lambda}_{\mathcal{M}}(\omega)(1-\eta)\,t\}]\\ &\le \sum_{\mathcal{U}_{\mathcal{M}}\subset\widetilde{\mathcal{U}}_{\mathcal{M}},|\widetilde{\mathcal{U}}_{\mathcal{M}}|\le|\mathcal{U}_{\mathcal{M}}|+3^d\,n_0\,\delta t^{1/d+2}} \exp\Big\{-\Big(\lambda_{-\frac{1}{2}\Delta}(\widetilde{\mathcal{U}}_{\mathcal{M}})\wedge(M\,t^{-2/d+2})-\rho\,t^{-2/d+2}\Big)_+\\ &\times(1-\eta)t\Big\}\cdot\mathbb{P}[\omega(\mathcal{U}_{\mathcal{M}})=0] \le \exp\{2\cdot 3^d\,n_0(\log 2)\Big(\frac{\sqrt{d}}{b}\,t^{1/d+2}+1\Big)^d\\ &-\inf_{\widetilde{\mathcal{U}}_{\mathcal{M}}}\Big\{\nu|\widetilde{\mathcal{U}}_{\mathcal{M}}| - 3^d\,n_0\,\delta t^{d/d+2} + \Big(\lambda_{-\frac{1}{2}\Delta}(\widetilde{\mathcal{U}}_{\mathcal{M}})\wedge(M\,t^{-2/d+2})-\rho\,t^{-2/d+2}\Big)_+(1-\eta)\,t\Big\}\,.\end{aligned} \tag{1.55}$$

From this it follows that

$$\overline{\lim_{t\to\infty}} \sup_{\#\mathcal{M}\le n_0} t^{-d/d+2} \log \mathbb{E}[\exp\{-\widetilde{\lambda}_{\mathcal{M}}(\omega)(1-\eta)\, t\}]$$
$$\le 2\cdot 3^d (\log 2)\, n_0 \Big(\frac{\sqrt{d}}{b}\Big)^d + 3^d\, n_0\, \delta + \rho \; - \; \underline{\lim}_{t\to\infty} \; \inf\nolimits_{\#\mathcal{M}\le n_0} \inf\nolimits_{\widetilde{\mathcal{U}}_{\mathcal{M}}} \{M(1-\eta)\}$$
$$\wedge \{\nu|t^{-1/d+2}\widetilde{\mathcal{U}}_{\mathcal{M}}| + (1-\eta)\, \lambda_{-\frac{1}{2}\Delta}(t^{-1/d+2}\,\widetilde{\mathcal{U}}_{\mathcal{M}})\}\,.$$

From this follows that the quantity in the left member of (1.42) is smaller

$$-\underline{\lim}_{\eta\to 0} \{M(1-\eta)\} \wedge \inf\nolimits_U \{\nu|U| + (1-\eta)\, \lambda_{-\frac{1}{2}\Delta}(U)\}\,, \tag{1.56}$$

where U runs over all bounded open sets of $\mathbb{R}^d$ with negligible boundary. Using the Faber-Krahn inequality, as in Donsker-Varadhan [2], balls are optimal in the infimum, which appears in (1.56). Since $M = c(d,\nu) + 1 = \inf\limits_U \{\nu|U| + \lambda_{-\frac{1}{2}\Delta}(U)\} + 1$, it is routine to see that the quantity in (1.56) is $-c(d,\nu)$. This finishes the proof of Theorem 1.1. □

Remark 1.4. – *Let us mention that the method we used to prove Theorem* 1.1 *can also be used to study how an excursion at distance $t^{d/d+2}$ is produced. For instance, it is an easy variation on the proof given here, to show using the lower estimate* (2.31) *of* [11], *that for $\epsilon > 0$, $\overline{x} \neq 0$,*

$$\begin{aligned}&\overline{\lim_{r\to 0}}\; \overline{\lim_{b\to\infty,\delta\to 0}}\; \overline{\lim_{t\to\infty}}\, t^{-d/d+2}\\ &\log(Q_t[H(\overline{x}\, t^{d/d+2}) \le t,\; \nu|\widetilde{\mathcal{U}}| + t\, \lambda_{-\frac{1}{2}\Delta}(\widetilde{\mathcal{U}})\\ &\quad \ge (c(d,\nu) + \epsilon)\, t^{d/d+2} / H(\overline{x}\, t^{d/d+2}) \le t]) < 0\end{aligned} \tag{1.57}$$

and a similar statement for hard obstacles.

So by picking r, b, δ adequately, given an excursion leading to $B(\overline{x}\, t^{d/d+2})$ occurs before time t, the Q_t conditional probability that

$$\lambda_{-\frac{1}{2}\Delta}(t^{-1/d+2}\,\widetilde{\mathcal{U}}) + \nu|t^{-1/d+2}\widetilde{\mathcal{U}}| \le c(d,\nu) + \epsilon \tag{1.58}$$

tends to 1 as t goes to infinity.

This last condition (1.58) for instance prevents the existence of more than one "big component" of $t^{-1/d+2}\,\widetilde{\mathcal{U}}$ (*i.e.* of volume $\ge c > 0$), if we pick ϵ small. □

We can now derive the asymptotic behavior of $\overline{r}(t, 0, t^{d/d+2}\, v)$ described in (0.5) (soft and hard obstacle case).

Theorem 1.5. – $(d \ge 2)$

$$\textit{for } v \in \mathbb{R}^d, \quad \lim_{t\to\infty} t^{-d/d+2} \log \overline{r}(t, 0, t^{d/d+2}\, v) = -c(d,\nu) - \beta_0(v)\,. \tag{1.59}$$

Proof. – We recall that (2.33) of [11], with the soft obstacle notation, says that for $0 < u < t$, $\rho > 0$, $x \in \mathbb{R}^d$:

$$E_0\Big[\exp\Big\{-\nu\int\Big(1-\exp\Big\{-\int_0^{t-u} W(Z_s-y)ds\Big\}\Big)dy\Big\}\, p_{B(x,2\rho)}(u,Z_{t-u},x),$$
$$Z_{t-u}\in B(x,\rho)\Big]\, e^{-\nu|B(0,2\rho+a)|}$$
$$\leq \overline{r}(t,0,x) \leq E_0\Big[\exp\Big\{-\nu\int\Big(1-\exp\Big\{-\int_0^{t-u} W(Z_s-y)ds\Big)dy\Big\}\, p(u,Z_{t-u},x)\Big],$$

with $p(\cdot,\cdot,\cdot)$ (resp. $p_U(\cdot,\cdot,\cdot)$) the usual heat kernel (resp. Dirichlet heat kernel relative to U) of $-\frac{1}{2}\,\Delta$. It suffices to consider $v \neq 0$, (else *see* Theorem 2.4 of [11]). For the lowerbound part of (1.59), we pick $u = 1$, $\rho = 2 + 2|v|$, and use (2.31) of [11]. As for the upperbound, exactly as in (2.34) of [11], we conclude that for $\epsilon > 0$, with $u = 1$ above

$$\begin{aligned}\overline{\lim_{t\to\infty}}\, t^{-d/d+2}\, \log \overline{r}(t,0,t^{d/d+2}\,v) &\leq \overline{\lim_{t\to\infty}}\, t^{-d/d+2}\, \log Q_t[Z_{t-1}\in \overline{B}(t^{d/d+2}\,v,\ \epsilon\, t^{d/d+2})] \\ &\quad + \log S_t) \\ &\leq -\inf\nolimits_{\overline{B}(v,2\epsilon)}\, \beta_0(\cdot) - c(d,\nu)\ ,\end{aligned}$$

which yields (1.59), letting ϵ tend to zero. □

II. Annealed Brownian motion with a drift in a Poissonian potential

The large deviation results we derived here and in [11] have a natural application to the study of annealed Brownian motion with a constant drift in a Poissonian potential, as already mentioned in the introduction. In particular we shall derive here a characterization of the (direction dependent) threshold for h, which separates the "small h" regime from the large h regime. Let us simply recall that $J(\cdot)$ with the notations of [11] is the rate function governing the large deviation property at rate t of Z_t/t under Q_t:

$$J(x) = \sup_{\lambda\geq 0}(\beta_\lambda(x) - \lambda)\ . \tag{2.1}$$

So with the notations of the introduction (*see* (0.6), (0.8)), we have:

THEOREM 2.1. – ($d \geq 1$, hard and soft obstacles)
Under Q_t^h, Z_t/t satisfies a large deviation principle at rate t, with rate function

$$J^h(x) = J(x) - hx + \sup\nolimits_y(h\cdot y - J(y)),\ \textit{moreover} \tag{2.2}$$

$$\lim_{t\to\infty} \frac{1}{t} \log \widetilde{S}_t^h = \sup_y (h \cdot y - J(y)) = 0, \qquad for \quad |h| \le \kappa(e) \tag{2.3}$$
$$> 0, \qquad for \quad |h| > \kappa(e)\,.$$

Proof. – The proof is entirely analogous to the proof of Theorem 3.1 of [10] in the quenched (i.e. $\mathbb{P}$ almost sure) case for soft obstacles. The transition of regime in (2.3), using the variational formula is related to the behavior of $J(\cdot)$ near the origin, namely

$$\text{for} \quad y \in \mathbb{R}^d, \quad \lim_{u\to 0} \frac{J(uy)}{u} = \beta_0(y), \quad see\ (3.10)\ of\ [10]. \quad \square \tag{2.4}$$

The large deviation results derived in section I enable to precise the description of what happens when $|h| < \kappa(e)$. The next theorem not only covers both the hard and soft obstacles situations, but it also goes up to the critical point $\kappa(e)$ and extends our results for small $|h|$ in [6].

THEOREM 2.2. – ($d \ge 2$, hard and soft obstacles)
For $|h| < \kappa(e)$

$$\lim_{t\to\infty} t^{-d/d+2} \log \widetilde{S}_t^h = -c(d,\nu), \ and \tag{2.5}$$

$$Z_t/t^{d/d+2} \ satisfies\ under Q_t^h\ a\ large\ deviation\ principle \tag{2.6}$$
$$at\ rate\ t^{d/d+2}\ with\ rate\ function\ \beta_0(x) - h\cdot x\,.$$

Proof. – The only point to prove is the exponential tightness result

$$\lim_{L\to\infty} \overline{\lim_{t\to\infty}} \ t^{-d/d+2} \log \mathbb{E} \otimes E_0 \Big[\exp\Big\{ h\cdot Z_t - \int_0^t V(Z_s,\omega)ds \Big\}, \tag{2.7}$$
$$h\cdot Z_t \ge L\, t^{d/d+2} \Big] = -\infty\,.$$

Indeed, (*see* for instance Ellis [4], p. 51 or Deuschel-Stroock [1], p. 43 and p. 51), it follows from (0.4) and (2.7) that

$$\lim_{t\to\infty} t^{-d/d+2} \log E^{Q_t}[\exp\{h\cdot Z_t\}] = \sup_{y\in\mathbb{R}^d} (h\cdot y - \beta_0(y)) = 0\,,$$

since $|h| < \kappa(e)$, (*see* (0.8)), as well as (2.6).
As for the proof of (2.7), the term under the logarithm in (2.7) equals

$$\exp\{L\, t^{d/d+2}\} \,\mathbb{E} \otimes E_0 \Big[\exp\Big\{ -\int_0^t V(Z_s,\omega)ds \Big\},\ e\cdot Z_t \ge \frac{L}{|h|}\, t^{d/d+2} \Big] \tag{2.8}$$
$$+ |h| \int_{\frac{L}{|h|}\, t^{d/d+2}}^{\infty} \exp\{|h|u\}\, \mathbb{E} \otimes E_0 \Big[\exp\Big\{ -\int_0^t V(Z_s,\omega)ds \Big\},\ e\cdot Z_t \ge u \Big] du\,.$$

Now from Corollary 1.8 of [11], we can pick $\gamma \in (|h|, \kappa(e))$ such that for large n

$$\mathbb{E} \otimes E_0 \left[\exp\left\{ - \int_0^t V(Z_s, \omega) ds \right\},\ Z_t \cdot e \geq u \right] \leq \exp\{-\gamma u\} .$$

From this for large enough t, the expression in (2.8) is smaller than $\dfrac{\gamma}{\gamma - |h|}$ $\exp\left\{ - \left(\dfrac{\gamma}{|h|} - 1 \right) L\, t^{d/d+2} \right\}$, and our claim (2.7) follows. □

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