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ANNEALED LYAPOUNOV EXPONENTS AND LARGE DEVIATIONS IN A POISSONIAN POTENTIAL I

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ABSTRACT. – We study annealed Brownian motion moving in a Poissonian cloud of killing spheres of fixed radius (hard obstacles) or in a Poissonian potential (soft obstacles). "Annealed" refers to the fact that statistical weights of interest are averaged both with respect to the path and environment measures. We construct Lyapounov exponents which for instance in the soft obstacle case measure the directional exponential decay of the environment averaged Green's function. These exponents come naturally in the description of certain large deviation principles which govern the large time position of annealed Brownian motion in a Poissonian potential, as well as in certain large time asymptotics of the associated heat kernel.

0. Introduction

The goal of the present article is to develop a series of large deviation results governing the long time behaviour of an "annealed Brownian particle" moving in a Poissonian potential or among Poissonian traps. Here "annealed" refers to the fact that we "average out" the Poissonian environment. Our results also apply to various asymptotics for the "annealed kernel" of the naturally associated Schrödinger semigroup.

We investigate the behaviour of a d-dimensional, $d \geq 1$, canonical Brownian motion $Z.$, under the annealed weighted measures

$$\begin{aligned} (0.1) \quad Q_t(dw, d\omega) &= \frac{1}{S_t} \exp\Big\{ - \int_0^t V(Z_s(w), \omega) ds \Big\} P_0(dw)\, \mathbb{P}(d\omega) \\ &\quad \text{(in the soft obstacle case)} \\ &= \frac{1}{S_t}\, 1\{T > t\}\, P_0(dw)\, \mathbb{P}(d\omega) \quad \text{(in the hard obstacle case)}\,, \end{aligned}$$

where S_t is the normalizing constant. Moreover, $V(x, \omega) = \sum_i W(x - x_i)$ for $\omega = \sum_i \delta_{x_i}$, is the Poissonian potential. Here W is bounded, non negative compactly supported, non degenerate. P_0 is the Wiener measure, $\mathbb{P}$ the law of the Poisson cloud of constant intensity $\nu > 0$. In the hard obstacle case, $T(w, \omega)$ is the entrance time of $Z.$ in the trap configurations $\bigcup_i \overline{B}(x_i, a)$, $a > 0$. The hard obstacle case in fact corresponds to the singular potential $W_{\text{h.o.}}(x) = \infty \cdot 1(|x| \leq a)$.

The large deviation properties of Z_t under Q_t involve certain "annealed Lyapounov coefficients" $\beta_\lambda(x)$, $\lambda \geq 0$, $x \in \mathbb{R}^d$ in the following fashion:

THEOREM:

(0.2) *Z_t/t satisfies a large deviation principle at rate t under Q_t with rate function*

$$J(x) = \sup_{\lambda \geq 0} \, (\beta_\lambda(x) - \lambda) \, . \tag{0.3}$$

If, as $t \to \infty$, $\varphi(t) = o(t)$ and $t^{d/d+2} = o(\varphi(t))$,

(0.4) *$Z_t/\varphi(t)$ satisfies a large deviation principle at rate $\varphi(t)$ with rate function $\beta_0(\cdot)$.*

The coefficients $\beta_\lambda(x)$ just mentioned are continuous, concave increasing in λ, and define norms in the x variable. In the soft obstacle case, they describe the (nondegenerate) exponential decay in the direction x of the $\mathbb{P}$-average of the λ-Green's function $\mathbb{E}\left[\left(-\frac{1}{2}\,\Delta + \lambda + V(\cdot,\omega)\right)^{-1}(0,x)\right]$, and satisfy a "shape theorem" type statement:

THEOREM:

$$\text{for } M > 0, \ \lim_{x \to \infty} \ \sup_{0 \leq \lambda \leq M} \frac{1}{|x|} \Big| - \log f_\lambda(x) - \beta_\lambda(x) \Big| = 0, \ \textit{with} \tag{0.5}$$

$$\begin{aligned} f_\lambda(x) &= E_0\Big[\exp\Big\{ -\nu \int \Big(1 - e^{-\int_0^{H(x)} W(Z_s - y)ds}\Big)\, dy - \lambda H(x)\Big\}, \\ &\qquad H(x) < \infty\Big], \ (\textit{soft obstacles}) \\ &= E_0[\exp\{-\nu \mid S^a_{H(x)} \mid - \lambda H(x)\}, \ H(x) < \infty], \ (\textit{hard obstacles}); \end{aligned} \tag{0.6}$$

here $H(x)$ denotes the entrance time of $Z_.$ in the closed ball $\overline{B}(x,1)$ and

$$S^a_u = \bigcup_{0 \leq s \leq u} \overline{B}(Z_s, a), \ \textit{for } u \geq 0 \, , \tag{0.7}$$

is the Wiener sausage of radius a in time u around $Z_.$.

Here the theory runs parallel, but with different critical scales ($t^{d/d+2}$ instead of $t/(\log t)^{2/d}$), and distinct Lyapounov coefficients, to the "quenched situation", (*i.e.* $\mathbb{P}$-almost sure), for soft obstacles, *see* [10]. Indeed for the quenched problem, the corresponding large deviation results rely on Lyapounov coefficients $\alpha_\lambda(x)$, $\lambda \geq 0$, $x \in \mathbb{R}^d$, which describe the (non degenerate) exponential decay in the direction x of the λ-Green's function $\left(-\frac{1}{2}\,\Delta + \lambda + V(\cdot,\omega)\right)^{-1}(0,x)$, for $\mathbb{P}$-a.e. ω. The coefficients $\alpha_\lambda(x)$ also satisfy a type of "shape theorem".

Let us give some comments about the coefficients $\beta_\lambda(x)$. The non degeneracy of $\beta_0(\cdot)$ ($\leq \beta_\lambda(\cdot)$), stems here as in [10] from exponential estimates of first passage percolation, (*see* Kesten [5]). There is also an immediate comparison $\beta_\lambda(\cdot) \leq \alpha_\lambda(\cdot)$ in the case of soft obstacles. We show that when $W(\cdot)$ is "substantial" this inequality is strict. However, it is an interesting question, untouched here, related to the theory of directed polymers, *see* Bolthausen [1], to determine whether for "small Poissonian potentials" and high enough dimension, the coefficients $\alpha_\lambda(\cdot)$ and $\beta_\lambda(\cdot)$ may coincide.

The special role of the scale $t^{d/d+2}$ in (0.4) comes from the asymptotic behaviour (Donsker-Varadhan [4]) of the normalizing constant S_t:

$$S_t = \exp\{-c(d,\nu)\, t^{d/d+2}(1+o(1))\},\ t \to \infty, \quad \text{where} \tag{0.8}$$

$$c(d,\nu) = \inf_U\{\nu|U| + \lambda(U)\} = (\nu\omega_d)^{2/d+2}\left(\frac{d+2}{2}\right)\left(\frac{2\lambda_d}{d}\right)^{d/d+2}, \tag{0.9}$$

if U runs over bounded open sets with negligible boundary in $\mathbb{R}^d$, $\lambda(U)$ is the principal Dirichlet eigenvalue of $-\frac{1}{2}\Delta$ in U, $\omega_d = |B(0,1))|$, and $\lambda_d = \lambda(B(0,1))$.

In fact the scales we investigate here in (0.2), (0.4), are such that the large deviation results are directly related to large deviation asymptotics for the average $\overline{r}(t,x,y) = \mathbb{E}[r(t,x,y,\omega)]$ of the kernel $r(t,x,y,\omega)$ of the Schrödinger semigroup $e^{t(\frac{1}{2}\Delta - V)}$ (soft obstacles), or of the Dirichlet heat kernel $e^{t\,\frac{1}{2}\Delta_{\mathrm{Dir},\mathbb{R}^d\setminus\cup_i \overline{B}(x_i,a)}}$ (hard obstacles).

THEOREM. – *If* $v \in \mathbb{R}^d$, *as* $t \to \infty$,

$$\text{if } \varphi(t) = o(t^{d/d+2}),\ \lim_{t\to\infty} t^{-d/d+2}\log \overline{r}(t,0,\varphi(t)v) = -c(d,\nu), \tag{0.10}$$

$$\text{if } \varphi(t) = o(t),\ t^{d/d+2} = o(\varphi(t)),\ \lim_{t\to\infty} \varphi(t)^{-1}\log \overline{r}(t,0,\varphi(t)v) = -\beta_0(v), \tag{0.11}$$

$$\lim_{t\to\infty} r^{-1}\log \overline{r}(t,0,tv) = -J(v)\,. \tag{0.12}$$

The study of what happens in the critical scale $t^{d/d+2}$ is the main object of the follow up of the present paper. On this question, we content ourselves here with the lowerbound part of the large deviation principle (0.4), when $\varphi(t) = t^{d/d+2}$.

It should be mentioned that asymptotic properties, notably the semiclassical approximation, as well as bounds on the Schrödinger semigroups can be found in the literature, *see* for instance Davies [2], Li-Yau [6], Simon [7] to quote a few. However, to our knowledge the existing results have different goals and are not well adapted to the study of the phenomena we describe here.

Let us finally point out that in the sequel of this paper, our large deviation results find a natural application to the long time behaviour of annealed Brownian motion with a constant drift h among Poissonian traps or potentials. They enable to relate the transition of regime which occurs between the small and large $|h|$ situation, with the Lyapounov exponents $\beta_\lambda(\cdot)$ introduced here.

I. Annealed Lyapounov coefficients

The main goal of this section is to construct and give estimates on the "annealed Lyapounov coefficients". The construction is similar in spirit to [10] section I, where the "quenched Lyapounov coefficients" were introduced. When "similar" becomes "identical", we shall then refer to [10].

Let us first recall our notations. We denote by $\mathbb{P}$ the law on the space Ω of simple point Radon measures on $\mathbb{R}^d$, $d \geq 1$, for which the canonical point process is a Poisson cloud of constant intensity $\nu > 0$. To each point of the cloud we attach a "soft obstacle" by translating to the point a fixed bounded, non negative, not almost surely equal to zero measurable function $W(\cdot)$. We assume the function $W(\cdot)$ is supported in the closed ball $\overline{B}(0,a)$, $a > 0$, and $a = a(W)$ is the minimal possible choice. For $\omega = \sum_i \delta_{x_i} \in \Omega$, we define

$$V(x,\omega) = \sum_i W(x - x_i) = \int W(x-y)\,\omega(dy),\ x \in \mathbb{R}^d\,. \tag{1.1}$$

The case of hard obstacles or killing traps corresponds formally and in fact for most formulas exactly to the choice of the singular potential $W_{\text{h.o.}}(x) = \infty \cdot 1(\{|x| \leq a\}$. Our canonical Brownian motion is denoted by $Z_.(w)$ for $w \in C(\mathbb{R}_+, \mathbb{R}^d)$, P_x, $x \in \mathbb{R}^d$ is the Wiener measure starting from x, and $(\theta_t)_{t \geq 0}$ is the canonical shift on $C(\mathbb{R}_+, \mathbb{R}^d)$. For $C \subseteq \mathbb{R}^d$ a closed subset H_C is the entrance time of $Z_.$ in C: $H_C = \inf\{s \geq 0,\ Z_s \in C\}$, and for U an open subset, T_U is the exit time of $Z_.$ from U: $T_U = \inf\{s \geq 0,\ Z_s \notin U\}$.

The annealed Lyapounov coefficients $\beta_\lambda(x)$, $\lambda \geq 0$, $x \in R^d$, will first come as a measure of the rate of exponential decay in the direction x of the function:

$$y \to E_\otimes E_0\Big[\exp\Big\{-\int_0^{H(y)} (\lambda + V)(Z_s(w),\omega)ds\Big\},\ H(y) < \infty\Big] \text{ (soft obstacles)}$$
$$y \to E_\otimes E_0[\exp\{-\lambda H(y)\},\ H(y) < T] \text{ (hard obstacles)},$$

with the notations:

$$H(y) = H_{B(y)},\ and\ B(y) = \overline{B}(y,1), \tag{1.2}$$

and for $w \in C(\mathbb{R}_+, \mathbb{R}^d)$, $\omega = \sum_i \delta_{x_i} \in \Omega$,

$$T_{(w,\omega)} = H_{\bigcup_i \overline{B}(x_i,a)}\,. \tag{1.3}$$

In other words T is the entrance time of $Z_.$ in the hard obstacles. To simplify notations, for $x, y \in \mathbb{R}^d$, $\lambda \geq 0$, $\omega \in \Omega$, we define in the soft obstacle case

$$e_\lambda(x,y,\omega) = E_x\Big[\exp\Big\{-\int_0^{H(y)} (\lambda + V)(Z_s,\omega)ds\Big\},\ H(y) < \infty\Big] \in (0,1] \tag{1.4}$$

$$
\begin{aligned}
(1.5)\quad f_\lambda(x) &= \mathbb{E}[e_\lambda(0,x,\omega)] \\
&= E_0\Big[\exp\Big\{-\nu\int\Big(1-e^{-\int_0^{H(x)} W(Z_s-y)ds}\Big)dy - \lambda H(y)\Big\},\ H(y)<\infty\Big] \\
&= E_{-x}\Big[\exp\Big\{-\nu\int\Big(1-e^{-\int_0^{H(0)} W(Z_s-y)ds}\Big)\,dy - \lambda H(0)\Big\},\ H(0)<\infty\Big],
\end{aligned}
$$

using translation invariance in the last step.

In the hard obstacle case the quantities are now

$$
(1.6)\qquad e^*_\lambda(x,y,\omega) = E_x[e^{-\lambda H(y)},\ H(y)<T] \in [0,1]
$$

$$
\begin{aligned}
(1.7)\qquad f^*_\lambda(x) &= \mathbb{E}[e^*_\lambda(0,x,\omega)] = E_0[\exp\{-\nu\,|\,S^a_{H(x)}\,|\,-\lambda H(x)\},\ H(x)<\infty] \\
&= E_{-x}[\exp\{-\nu\,|\,S^a_{H(0)}\,|\,-\lambda H(0)\},\ H(0)<\infty]\,,
\end{aligned}
$$

with the notation of (0.7).

Remark 1.1. – 1) When $W(\cdot)$ is spherically symmetric, and in the hard obstacle case, it is clear from the last expression in (1.5), and (1.7) that $f_\lambda(\cdot)$ and $f^*_\lambda(\cdot)$ are spherically symmetric. Moreover, an application of the strong Markov property shows that $f_\lambda(\cdot)$ and $f^*_\lambda(\cdot)$ are decreasing functions of the radius.

2) The function $f_\lambda(\cdot)$ is continuous. This simply follows from the continuity of $x \to e_\lambda(x,y,\omega)$ (Lemma 1.1 of [10]). The same is true for $f^*_\lambda(\cdot)$. This last point in view of 1) and (1.7) can be argued on the ground that for $\epsilon>0$, $M>0$, $\lim_{y\to y_0} E_y[\exp\{-M\,H_{\partial B(0,|y_0|)}\}] = 1$. □

We have proven in [10], the existence of coefficients $\alpha_\lambda(x)$ which are seminorms in x, such that:

$$
(1.8)\qquad \mathbb{P}-a.s.,\ \lim_{|x|\to\infty} \frac{1}{|x|}\,|\,-\log e_\lambda(0,x,\omega) - \alpha_\lambda(x)\,| = 0\,.
$$

We now come to the first step in the construction of the "annealed coefficients". The $\mathbb{P}$ integration over the cloud configuration enables to treat the case of hard obstacles. For the quenched problem, the functions $e^*_\lambda(0,\cdot,\omega)$ can very well be equal to zero outside a bounded set, for $\mathbb{P}$-a.e. ω (in the absence of "infinite cluster" in the complement of obstacles). This prevents the existence of non singular analogues of (1.8) in this case. In what follows we shall drop the * in the notation, which distinguishes between the soft and hard obstacle case, whenever this causes no ambiguity. In most cases the hard obstacle case is safely obtained by replacing in formulas $W(\cdot)$ by the singular potential $W_{\text{h.o.}}(\cdot) = \infty\cdot 1\{|\cdot|\le a\}$.

For $\lambda\ge 0$, $x\in\mathbb{R}^d$, (soft or hard obstacles) we define

$$
(1.9)\qquad b_\lambda(x) = -\inf_{z\in B(0)} \log f_\lambda(x-z) \ge 0\,.
$$

In view of Remark 1.1, when W is symmetric or with hard obstacles

$$b_\lambda(x) = -\log f_\lambda(y) \text{ for any } y \text{ with } |y| = |x| + 1 \,. \tag{1.10}$$

PROPOSITION 1.2. – *There exists a function $\beta_\lambda(x)$: $[0,\infty] \times \mathbb{R}^d \to [0,\infty)$, increasing in λ, positively homogeneous of degree one, convex in x, such that:*

$$\beta_\lambda(x) = \inf_{n \geq 1} \; \frac{1}{n} \; b_\lambda(nx), \; \lambda \geq 0, \; x \in \mathbb{R}^d \,. \tag{1.11}$$

$$\lim_{x \to \infty} \; \tfrac{1}{|x|} \; | \; b_\lambda(x) - \beta_\lambda(x) \; | = 0, \text{ (soft and hard obstacles). Moreover,} \tag{1.12}$$

$$\beta_\lambda(x) \leq k(d,\nu,a,\lambda)|x|, \; where \tag{1.13}$$

$$k(d,\nu,a,\lambda) = \min_{r>0} \, (\nu\, \omega_{d-1}(a+r)^{d-1} + \sqrt{2(\lambda_{d-1} + \lambda r^2)}/r), \; d \geq 2,$$
$$= \nu + \sqrt{2\lambda}, \; d = 1,$$

with the notations of (0.9). *In fact when $d = 1$, $\beta^*(x) = (\nu + \sqrt{2\lambda})|x|$. In case of soft obstacles one also has*

$$\beta_\lambda(x) \leq \alpha_\lambda(x) \leq \sqrt{2(\lambda_d + \lambda + \|W\|_\infty \, \nu \omega_d (a+2)^d)} \; |x| \,. \tag{1.14}$$

Proof. – The first fact is that $b_\lambda(\cdot)$ satisfies a (deterministic) subadditive property

$$b_\lambda(x+y) \leq b_\lambda(x) + b_\lambda(y), \; x, y \in \mathbb{R}^d \,, \tag{1.15}$$

which comes from the following two observations:

– Under P_z, $(z \in B(0))$, one way to enter $B(x+y)$ is to enter $B(x)$ and then enter $B(x+y)$.

– We have a "positive correlation inequality" under $\mathbb{P}$ in (1.5) which stems from the elementary inequality:

$$1 - e^{-(a+b)} = 1 - e^{-a} + 1 - e^{-b} - (1-e^{-a})(1-e^{-b}) \leq 1 - e^{-a} + 1 - e^{-b}, \; a, b \geq 0 \,. \tag{1.16}$$

This inequality is then applied with

$$a = \int_0^{H(x)} W(Z_s - y) ds, \qquad b = \int_{H(x)}^{H(x)+H(x+y)\circ\theta_{H(x)}} W(Z_s - y) ds.$$

Our claim (1.15) simply follows then from an application of the strong Markov property. The hard obstacle case is handled in an analogous fashion, using $|S^a_{H(x+y)}| \leq | S^a_{H(x)} | + |S^a_{H(x+y)} \circ \theta_{H(x)} |$, when $H(x) + H(x+y) \circ \theta_{H(x)} < \infty$.

The second easy fact is

$$\sup_{|x| \leq 1} \; b_\lambda(x) < \infty \,. \tag{1.17}$$

Indeed, provided $|y| = 2$, using (1.10) and Remark 1.1, for $|x| \leq 1$:

$$\begin{aligned}\exp\{-b_\lambda(x)\} \geq \exp\{-b^*_\lambda(x)\} \geq f^*_\lambda(y) = E_y[e^{-(\nu|S^a_{H(0)}|+\lambda H(0))},\ H(0) < \infty] \\ \geq \exp\{-\nu|B(0,a+4)|\}\ E_y[H(0) < T_{B(0,4)},\ e^{-\lambda H(0)}] > 0\,.\end{aligned}$$

The claims (1.11), (1.12) now easily follow from (1.15), (1.17): one first defines $\beta_\lambda(x) = \lim_{u\to\infty} \frac{1}{u}\, b_\lambda(ux) = \inf_{u\geq 1} \frac{1}{u}\, b(ux)$, as a directional limit. The function $\beta_\lambda(\cdot)$ is then homogeneous of degree one and convex. One then patches the convergence in the various directions (in a much simpler way than for the shape theorem defining $\alpha_\lambda(\cdot)$, *see* [10] after (1.21)).

The inequality (1.14) follows from:

$$\begin{aligned}-b_\lambda(x) = \inf_{z\in B(0)} \log f_\lambda(x-z) \geq \inf_{z\in B(0)} \mathbb{E}[\log e_\lambda(z,x,\omega)] \\ \geq \mathbb{E}[\inf_{z\in B(0)} \log e_\lambda(z,x,\omega)]\,.\end{aligned}$$

Now $-\inf_{z\in B(0)} e_\lambda(z,nx,\omega) = a(0,nx,\omega)$ in the notations of [10], (1.8), converges in $L^1(\mathbb{P})$ to $\alpha_\lambda(x)$. This together with (1.11) yield (1.14).

There now remains to prove (1.13). It clearly suffices to consider the case of hard obstacles, and using rotation invariance and homogeneity, we can choose $x = e_1$ the first vector in the canonical basis of $\mathbb{R}^d$. For $u > 1$, $r > 0$, denote by $C_{u,r}$ the cylinder (interval when $d = 1$), $C_{u,r} = (-\sqrt{u},\ u + \sqrt{u}) \times B^{d-1}(r)$, where $B^{d-1}(r)$ stands for the $(d-1)$ dimensional ball in $\mathbb{R}^{d-1}$, when $d \geq 2$, and is omitted when $d = 1$. Observe that for $t > 0$, in view of (1.10),

$$\begin{aligned}&\exp\{-b^*_\lambda((u-1)e_1)\} = f^*_\lambda(ue_1)\\ &\geq E_0[\exp\{-\nu(|S^a_{H(ue_1)} \circ \theta_t| + |S^a_t|) - \lambda(H(ue_1)\circ\theta_t + t)\},\ T_{C_{u,r}} > t,\\ &Z^1_t \in [u, u+1],\ H(ue_1)\circ\theta_t < \infty]\end{aligned}$$

if $Z^1_\cdot$ denotes the first coordinate of Z.

$$\begin{aligned}&\geq \inf\{f^*_\lambda(-x),\ x\in[0,1]\times B^{d-1}(r)\} \times \exp\{-\nu|C^a_{u,r}|\}\cdot e^{-\lambda t}\cdot\\ &P^{d-1}_0[T_{B^{d-1}(r)} > t] \times P^1_0[T_{(-\sqrt{u},\,u+\sqrt{u})} > t,\ Z_t \in [u,u+1]]\,,\end{aligned}$$

if $|C^a_{u,r}|$ denotes the volume of the a-neighborhood of $C_{u,r}$, and P^{d-1}_0 and P^1_0 stand respectively for the $d-1$ and 1 dimensional Wiener measure, (when $d = 1$, the fourth term involving P^{d-1}_0 is absent). Picking $t = \rho u$, $\rho > 0$, we find

$$\begin{aligned}(1.18)\ \beta^*(e_1) = \lim_{u\to\infty} \frac{1}{(u-1)} - \log(f^*(ue_1)) \leq \overline{\lim_{u\to\infty}} \frac{1}{(u-1)}\ \nu|C^a_{u,r}| + \lambda\rho + \frac{\lambda_{d-1}}{r^2}\ \rho\\ + \overline{\lim_{u\to\infty}} \frac{1}{(u-1)} - \log P^1_0\Big[T_{(-\sqrt{u},\,u+\sqrt{u})} > t,\ Z_t \in [u,\,u+1]\Big]\,.\end{aligned}$$

Now, it is easy to see that the first term of the right member of (1.18) is equal to $\nu\omega_{d-1}(a+r)^{d-1}$, when $d \geq 2$, and ν, when $d = 1$. Now by the method of images (*see* (1.13) in [8]),

$$\begin{aligned} &P_0^1\Big[T_{(-\sqrt{u},\,u+\sqrt{u})} > t,\ Z_t \in [u, u+1]\Big] \\ &\geq P_0^1\Big[Z_t \in [u,u+1]\Big] - P^1_{-2\sqrt{u}}\Big[Z_t \in [u,u+1]\Big] \\ &\quad - P^1_{2\sqrt{u}+2u}\Big[Z_t \in [u,u+1]\Big] \\ &\geq (2\pi t)^{-1/2}\Big(\exp\Big\{-\frac{(u+1)^2}{2t}\Big\} - \exp\Big\{-\frac{(u+2\sqrt{u})^2}{2t}\Big\} \\ &\qquad - \exp\Big\{-\frac{(u+2\sqrt{u}-1)^2}{2t}\Big\}\Big), \end{aligned}$$

from this, we conclude (as in [8] (1.13)), that the last term of (1.18) is smaller than $1/2\rho$. We thus obtain

$$\begin{aligned} \beta^*(e_1) &\leq \nu\,\omega_{d-1}(a+r)^{d-1} + \lambda\,\rho + \frac{\lambda_{d-1}}{r^2}\,\rho + \frac{1}{2\rho},\ d \geq 2, \\ &\leq \nu + \lambda\rho + \frac{1}{2\rho},\ d = 1, \end{aligned}$$

optimizing, we find (1.13), ($k(d,\nu,a,0)$ is $k(d,\nu,a)$ in the notations of [8]). In the one dimensional case, we also clearly have:

$f^*(ue_1) \leq \exp\{-\nu(u-1)\}\ E_0[\exp\{-\lambda\,T_{B(ue_1)}] = \exp\{-(\nu+\sqrt{2\lambda})(u-1)\}$, from which $\beta^*(e_1) = \nu + \sqrt{2\lambda}$ follows. □

We shall now reinforce the convergence statement (1.12) and prove:

THEOREM 1.3. – *For* $M > 0$,

(1.19) $$\lim_{x\to\infty}\ \sup_{0\leq\lambda\leq M}\ \frac{1}{|x|}\ |-\log f_\lambda(x) - \beta_\lambda(x)| = 0,\ \textit{(soft and hard obstacles).}$$

$\beta_\lambda(x)$ is a jointly continuous function, concave increasing in the λ variable.

Proof. – We shall begin with the proof of (1.19), with λ fixed. Observe for $|z| \leq 1$, $|x| > 3$, and $\lambda \geq 0$, in the soft and hard obstacle case,

$$\begin{aligned} f_\lambda(x-z) = &\ \mathbb{E}[e_\lambda(z,x,\omega)] \\ &\geq \exp\{-\nu|B(0,a+2)|\}\ E_z[e^{-\lambda T_{B(0,2)}}\,f_\lambda(x - Z_{T_{B(0,2)}})],\ \text{using}(1.16), \\ &\geq \exp\{-\nu|B(0,a+2)|\}\ E_z[\exp\{-\lambda\,E_z[T_{B(0,2)}/Z_{T_{B(0,2)}}]\}\ f_\lambda(x - Z_{T_{B(0,2)}})], \\ &\text{using Jensen's inequality for conditional expectations.} \end{aligned}$$

On the other hand, we also have:

(1.21) $$f_\lambda(x) \leq E_0[f_\lambda(x - Z_{T_{B(0,2)}})] = \int_{\partial B(0,2)} f_\lambda(x-y)\,d\sigma(y),$$

with $d\sigma$ the normalized surface measure on $\partial B(0,2)$. Now we have for $|z| \leq 1$:

$$E_z[T_{B(0,2)}/Z_{T_{B(0,2)}}] = h(z, Z_{T_{B(0,2)}}),\ P_z\text{-a.s., where} \tag{1.22}$$

$$h(z,y) = \frac{1}{v(z,y)} \int_0^\infty \int_{B(0,2)} p_{B(0,2)}(s,z,z')\, v(z',y)\, dz' ds, \quad z \in B(0,2),\ y \in \partial B(0,2)\ . \tag{1.23}$$

Here $v(z,y)$ is the density with respect to the normalized surface measure of $P_z[Z_{T_{B(0,2)}} \in dy]$, that is $v(z,y) = 2^{d-2}\,\dfrac{4-|z|^2}{|y-z|^d}$, for $d \geq 2$, and $v(z,2) = \dfrac{z+2}{2}$, $v(z,-2) = \dfrac{2-z}{2}$, for $d=1$, and $p_U(s,z,z')$ stands for the Dirichlet heat kernel in U:

$$p_U(s,z,z') = (2\pi s)^{-d/2} \exp\Big\{-\frac{(z-z')^2}{2s}\Big\}\, E^s_{z,z'}[T_{B(0,2)} > s], \quad s > 0, z, z' \in \mathbb{R}^d\ , \tag{1.24}$$

provided $E^s_{z,z'}$ stands for the Brownian bridge measure in time s from z to z', ($p_U(s,z,z')$ is zero if z or z' is not in U). It is classical and follows from semigroup considerations that $\sup_{z,z'} p_{B(0,2)}(s,z,z')$ decays exponentially in s. Moreover, $v(\cdot,y)$ for $y \in \partial B(0,2)$ belongs to $L^1(B(0,2))$. So from (1.20) and (1.21)

$$f_\lambda(x)/f_\lambda(x-z) \leq \exp\{\nu|B(0,a+2)|\} \cdot \sup_{z\in\overline{B}(0,1),\, y\in\partial B(0,2)} (e^{-\lambda h(z,y)} v(z,y))^{-1} < \infty\ .$$

In view of (1.12), this proves (1.19) for fixed λ.

The rest of the argument now closely follows that of Theorem 1.4 of [10]. Thanks to proposition 1.2, $y \to \beta_\lambda(y)$ is a convex uniformly continuous function for bounded λ. In view of (1.19), for fixed λ, it is also a concave non decreasing function of λ. Its only point of discontinuity in λ, might be $\lambda = 0$. If we show it is upper semicontinuous in λ, it will then be continuous in λ, and thanks to the uniform continuity in y, jointly continuous.

Now by Remark 1.1, $f_\lambda(\cdot)$ (hard or soft obstacles) is continuous, and therefore $b_\lambda(\cdot)$ is continuous. The upper semicontinuity of $\beta_\lambda(\cdot)$, now follows from (1.11).

The uniformity in λ in (1.19), now follows from the same Dini type argument as in Theorem 1.4 of [10]. One uses that for $x \neq 0$, $\dfrac{1}{|x|}\,|-\log f_\lambda(x) - \beta_\lambda(x)| = \left|-\dfrac{1}{|x|}\log f_\lambda(x) - \beta_\lambda\Big(\dfrac{x}{|x|}\Big)\right|$, together with $\lambda \to -\dfrac{1}{|x|}\log f_\lambda(x)$ is increasing and $\beta_\lambda(z)$ is jointly continuous increasing in λ on $[0,\infty) \times \partial B(0,1)$. □

We shall now prove that $\beta_0(y) \neq 0$ for $|y| \neq 0$, we also give an example of a "large W" for which $\beta_\lambda(\cdot) < \alpha_\lambda(\cdot)$. Another quite interesting question which we do not discuss here is to know whether the equality $\beta_\lambda(\cdot) = \alpha_\lambda(\cdot)$, is possible in some cases (for instance for small potentials, large λ, rarefied clouds and high enough dimension).

THEOREM 1.4. – *There is a constant* $\gamma(d,\nu,W) > 0$, *such that*

$$\max(\sqrt{2d},\gamma)\ |x| \le \beta_\lambda(x),\ for\ \lambda \ge 0,\ x \in \mathbb{R}^d\ . \tag{1.25}$$

When $W = c1_{\overline{B}(0,a)}$, $\lambda \ge 0$, *if a is large, for sufficiently large c*

$$\beta_\lambda(x) < \alpha_\lambda(x),\ for\ x \ne 0\ . \tag{1.26}$$

Proof. – We start with the proof of (1.25). With no loss of generality we treat the soft obstacle case. The lowerbound $\sqrt{2\lambda}\,|x| \le \beta_\lambda(x)$ is standard. We shall prove that for a $\gamma(d,\nu,W) > 0$:

$$\gamma|x| \le \beta_0(x) \le \beta_\lambda(x)\ . \tag{1.27}$$

The strategy of the proof follows ideas of first passage percolation, as in Proposition 2.3 of [9]. We chop $\mathbb{R}^d$ in cubic boxes of side length $\ell > 0$, centered at the points $\ell \cdot q$, $q \in \mathbb{Z}^d$:

$$\text{Box }(\ell\cdot q) = \Big\{z \in \mathbb{R}^d, -\frac{\ell}{2} \le z_i - \ell q_i < \frac{\ell}{2}\,,\ for\ \ i = 1,\dots,d\Big\}\ .$$

For $x \in \mathbb{R}^d$, $\omega \in \Omega$ we define the occupation function through

$$\begin{aligned} Oc_\omega(x) = 1,\ &\text{ if the open cube of side length } \frac{\ell}{4} \text{ with same center as the unique} \\ &\text{ Box } (\ell\cdot q) \text{ containing } x \text{ receives a point from } \omega \\ = 0,\ &\text{ otherwise.} \end{aligned}$$

It is shown in (2.50) of [9], using a suitable discrete supermartingale based on the successive displacements of $Z.$ at $\|\ \|$-distance $\frac{3\ell}{4}$, ($\|x\| \overset{\text{def}}{=} \sup_i |x_i|$), that for $v > 0$

$$E_0\Big[\exp\Big\{-\int_0^{T_v} V(Z_s,\omega)ds\Big\}\Big] \le \chi^{\widetilde{N}(v)}\ , \tag{1.28}$$

where $T_v = T_{B(0,v)}$,

$$\chi = \sup_{\|z\|\le\frac{\ell}{2},\,\|x\|\le\frac{\ell}{8}} E_z\Big[\exp\Big\{-\int_0^U W(Z_s - x)ds\Big\}\Big],\ \text{with} \tag{1.29}$$

$$U = \inf\Big\{s \ge 0,\ \|Z_s - Z_0\| \ge \frac{3\ell}{4}\Big\},\ \text{and} \tag{1.30}$$

$$\widetilde{N}(v) = \inf_{\widetilde{D}(v)} N_{Oc}(\widetilde{X})\ , \tag{1.31}$$

is the minimum occupation number $N_{Oc}(\widetilde{X}) = \sum_{j=0}^{m-1} Oc_\omega(\widetilde{X}(j))$, for a discrete path $\widetilde{X}(j)_{0\le j\le m}$, which never visits twice the same Box $(\ell\cdot q)$, has successive steps in neighboring boxes (*i.e.* $\|q - q'\| \leq 1$), has final location $\widetilde{X}(m)$ satisfying $|\widetilde{X}(m)| \geq v - 2\sqrt{d}\,\ell$, and has $m \geq \left[\dfrac{v}{\sqrt{d}\,2\ell}\right] - 1$ steps.

Now by Proposition 2.2 of [9], there exist a constant $p_T > 0$ depending on d only, such that, there exist positive constants C, D, E depending only on d and $p = \exp\left\{-\nu\, \dfrac{\ell^d}{4^d}\right\}$ such that

$$p < p_T \;\Rightarrow\; \mathbb{P}[\inf_{\mathcal{L}_m}\{N_{O^c}(\widetilde{X}) \leq C \cdot m\}] \leq D\, \exp\{-E\, m\}\,, \tag{1.32}$$

provided $\mathcal{L}_m$ is the family of discrete paths of m steps with successive steps in neighboring boxes, never visiting twice the same box, and starting at the origin, (on a similar statement in a bond percolation context, *see* Proposition 2.8 of Kesten [5]).

It is may be helpful to say here that $p_T(1) = 1$, and that $p_T(d)$, for $d \geq 2$ is the critical independent site percolation constant, for the adjacency relation $q \sim q'$, if $\|q - q'\| \leq 1$, such that when the probability p of vacancy of a site is smaller than p_T, the expected size of the vacant cluster at the origin is finite.

Observe now, that under P_0, $T_{|y|-1} \leq H(y)$, when $|y| > 1$. So from (1.28) follows:

$$f_0(y) \leq \mathbb{E}_{\otimes} E_0\Big[\exp\Big\{-\int_0^{T_{|y|-1}} V(Z_s, \omega)ds\Big\}\Big] \leq \mathbb{E}[\chi^{\widetilde{N}(|y|-1)}]\,.$$

Now if we make sure that $\ell > 8a$, from the inequality $a + \dfrac{\ell}{2} + \dfrac{\ell}{8} < \dfrac{3\ell}{4}$, and the fact that W is not a.s. equal to zero, one concludes that $0 < \chi < 1$. If we also make sure that $\exp\left\{-\nu\, \dfrac{\ell^d}{4^d}\right\} < p_T$, then (1.32) holds, and it is plain that for any x with $|x| = 1$.

$$-\beta_0(x) = \lim_{n\to\infty} \frac{1}{n}\, \log f_0(nx) \leq \overline{\lim_{n\to\infty}}\, \frac{1}{n}\, \log \mathbb{E}[\chi^{\widetilde{N}(n-1)}] \stackrel{\text{def}}{=} -\gamma < 0\,, \tag{1.33}$$

thanks to the definition of $\widetilde{N}(v)$. This proves (1.25).

Let us now prove (1.26). We now have $W = c 1_{\overline{B}(0,a)}$. We choose $a > \dfrac{11}{8}\,\sqrt{d}\,\ell$, where ℓ is such that $\exp\left\{-\nu\, \dfrac{\ell^d}{4^d}\right\} < p_T$. Then,

$$\chi(c) = E_0[\exp\{-cU\}] < 1\,. \tag{1.34}$$

By a Borel Cantelli argument using (1.32),

$$\mathbb{P}\text{-a.s.} \;\underline{\lim}_{m\to\infty}\; \inf_{\mathcal{L}_m}\{N_{O^c}(\widetilde{X})\}/m \geq C\,.$$

It now follows from (1.28) and (1.8), that for x, with $|x| = 1$,

$$-\alpha_0(x) \leq \overline{\lim}\, \frac{1}{n}\, \log E_0\Big[\exp\Big\{-\int_0^{T_{n-1}} V(Z_s, \omega)ds\Big\}\Big] \leq -\frac{C}{2\sqrt{d}\,\ell}\, \log(1/\chi)\,. \tag{1.35}$$

Now for $\lambda \geq 0$, by (1.13), and (1.35), for any x with $|x| = 1$, $\beta_\lambda(x) \leq k(d, \nu, a, \lambda) < \frac{C}{2\sqrt{d}\,\ell} \log 1/\chi(c) \leq \alpha_0(x)$, provided c is chosen large enough, since $\lim_{c\to\infty} \chi(c) = 0$. □

We shall now relate, in the case of soft obstacles the coefficients $\beta_\lambda(x)$ to the exponential decay of the $\mathbb{P}$-averaged λ-Green's function of $-\frac{1}{2}\Delta + V$, in the direction x. We define for $t > 0$, $x, y \in \mathbb{R}^d$, $\omega \in \Omega$,

$$(1.36)\quad r(t,x,y,\omega) = (2\pi t)^{-d/2} \exp\Big\{ -\frac{(x-y)^2}{2t}\Big\}\, E^t_{x,y}\Big[\exp\Big\{-\int_0^t V(Z_s,\omega)ds\Big\}\Big]\,,$$

Then r is the kernel of a C^0-self adjoint semigroup on $L^2(\mathbb{R}^d, dx)$ which is for instance known to coincide with e^{-tH}, if H is the Friedrichs extension of $-\frac{1}{2}\Delta + V$ on $C_0^\infty(\mathbb{R}^d)$, *see* for instance [7]. We denote by $g_\lambda(x,y,\omega)$ the λ-Green's function:

$$(1.37)\qquad g_\lambda(x,y,\omega) = \int_0^\infty e^{-\lambda s}\, r(s,x,y,\omega)ds,\ \lambda \geq 0,\ x,y \in \mathbb{R}^d,\ \omega \in \Omega\,.$$

THEOREM 1.5. – *For $M > 0$,*

$$(1.38)\qquad \lim_{x\to\infty} \sup_{0\leq\lambda\leq M} \frac{1}{|x|}\, |-\log(\mathbb{E}[g_\lambda(0,x,\omega)]) - \beta_\lambda(x)| = 0\,.$$

Proof. – By the same Dini type argument as in Theorem 1.3, it is enough to prove (1.38) for fixed λ. By (1.37) of [10], there is a set $\widetilde{\Omega}$ of full $\mathbb{P}$-measure, on which the $e_\lambda(\cdot,y,\omega)$ can be represented in terms of the $g_\lambda(\cdot,y,\omega)$ via:

(1.39) for $\omega \in \widetilde{\Omega}$, $\lambda \geq 0$, $y \in \mathbb{R}^d$, there are bounded positive measure $e_y^{\lambda,\omega}(dz)$ on $B(y)$, such that for $x \in \mathbb{R}^d$, $e_\lambda(x,y,\omega) = \int_{B(y)} g_\lambda(x,z,\omega)\, e_y^{\lambda,\omega}(dz)\,,$

and the $g_\lambda(z,z',\omega)$ are finite continuous outside the diagonal ($d \geq 2$), and everywhere ($d = 1$).

The $e_y^{\lambda,\omega}(dz)$ are in fact the $(\lambda + V)(\cdot,\omega)$ equilibrium measures of $B(y)$. From (1.39) we immediately deduce that for $|y| > 1$:

$$(1.40)\qquad \begin{aligned} &e_\lambda(0,y,\omega)/A \leq g_\lambda(0,y,\omega),\ \text{provided}\\ &A = e_y^{\lambda,\omega}(B(y))\, \sup_{B(y)} g_\lambda(0,\cdot,\omega)/\inf_{B(y)} g_\lambda(0,\cdot,\omega)\end{aligned}$$

We can then use the estimate (1.43) and the end of the proof of Theorem 1.6 of [10], to find:

$$(1.41)\quad A \leq K(d)\, \exp\{(\lambda + \sup_{B(y,2)} V(\cdot,\omega))(1+H)\} \sup_{\overline{B}(0,1)\times\partial B(0,2)} v(\cdot,\cdot)/\inf_{\overline{B}(0,1)\times\partial B(0,2)} v(\cdot,\cdot)\,,$$

with $H = \sup_{\overline{B}(0,1)\times\partial B(0,2)} h(\cdot,\cdot)$, with the notations of (1.23).

Using now Hölder's inequality, we find for $p, q \in (1, \infty)$, $\frac{1}{p} + \frac{1}{q} = 1$

$$\mathbb{E}[e_\lambda(0,y,\omega)^{1/p}] \le \mathbb{E}[e_\lambda(0,y,\omega)/A]^{1/p}\ \mathbb{E}[A^{q/p}]^{1/q}, \text{ and from (1.40)}$$
$$\log(\mathbb{E}[g_\lambda(0,y,\omega)]) \ge p\ \log(\mathbb{E}[e_\lambda(0,y,\omega)]) - \frac{p}{q}\ \log \mathbb{E}[A^{q/p}]$$
$$(\text{we used here } e_\lambda^{1/p} \ge e_\lambda, \text{ since } e_\lambda \in (0,1]).$$

The last term in the last inequality is bounded by a fixed finite constant in view of (1.41), since $\sup_{B(y,2)} V(\cdot,\omega) \le \|W\|_\infty \cdot \omega(\overline{B}(y,a+2))$. It follows that

$$(1.42)\quad \overline{\lim_{y\to\infty}} \Big(-\frac{1}{|y|}\ \log \mathbb{E}[g_\lambda(0,y,\omega)] - \beta_\lambda\big(\frac{y}{|y|}\big)\Big) \le$$
$$\overline{\lim_{y\to\infty}} \frac{p}{|y|}\ | - \log f_\lambda(y) - \beta_\lambda(y)| + (p-1) \sup_{\partial B(0,1)} \beta_\lambda(\cdot) = (p-1) \sup_{\partial B(0,1)} \beta_\lambda(\cdot)\,,$$

thanks to (1.19). Letting now p tend to 1, we see the left member of (1.42) is non positive. Let us now prove the lim inf counter part of (1.42). We have

$$(1.43)\qquad \begin{aligned} &g_\lambda(0,y,\omega) \le e_\lambda(0,y,\omega) \times B, \text{ with} \\ &B = \frac{1}{e_y^{\lambda,\omega}(B(y))} \sup_{B(y)} g_\lambda(0,\cdot,\omega)/ \inf_{B(y)} g_\lambda(0,\cdot,\omega)\,. \end{aligned}$$

Let us first consider the case when $d \ge 3$, or $\lambda > 0$. Just as in [10], after (1.44), $e_y^{\lambda,\omega}(B(y)) \ge \text{cap}\ (B(y))$ or $\text{cap}_\lambda(B(y))$, the capacity or λ-capacity of the ball $B(y)$ relative to Brownian motion. Consequently,

$$(1.44)\quad B \le K'(d,\lambda) \cdot \exp\{(\lambda + \sup_{B(y,2)} V)\ H\} \cdot \sup_{\overline{B}(0,1)\times\partial B(0,2)} v(\cdot,\cdot)/ \inf_{\overline{B}(0,1)\times\partial B(0,2)} v(\cdot,\cdot)\,.$$

From Hölder's inequality we deduce that for $p, q \in (1,\infty)$, with $\frac{1}{p} + \frac{1}{q} = 1$,

$$(1.45)\qquad \begin{aligned} \log(\mathbb{E}(g_\lambda(0,y,\omega)]) &\le \frac{1}{p}\ \log \mathbb{E}[e_\lambda^p(0,y,\omega)] + \frac{1}{q}\ \log \mathbb{E}[B^q] \\ &\le \frac{1}{p}\ \log \mathbb{E}[e_\lambda(0,y,\omega)] + \frac{1}{q}\ \log \mathbb{E}[B^q]\,. \end{aligned}$$

Now an entirely similar argument as before shows that

$$\underline{\lim_{y\to\infty}} \Big(-\frac{1}{|y|} \cdot \log(\mathbb{E}[g_\lambda(0,y,\omega)]) - \beta_\lambda\Big(\frac{y}{|y|}\Big)\Big) \ge 0\,.$$

This finishes the proof of (1.38), when $d \ge 3$ or $\lambda \ge 0$. In the case $d \le 2$, and $\lambda = 0$, we replace in the right member of (1.44) the constant $K'(d,\lambda)$ (upperbound on $1/e_y^{\lambda,\omega}(B(y))$ by:

$$\inf_{\mu\in M_1(B(y))} \int_{B(y)\times B(y)} g(z,z',\omega)\ \mu(dz)\,\mu(dz')\,,$$

where $g(z,z',\omega) = g_0(z,z',\omega)$, which is finite for $\omega \in \widetilde{\Omega}$, $z \neq z'$, by Lemma 1.5 of [10].

Now the proof proceeds as before, provided we can show the quantity in (1.45) has finite moments of any order (which do not depend on y, by translation invariance). So it suffices to show that

$$\text{(1.46)} \qquad \mathbb{E}\Big[\Big(\int_{B(0)\times B(0)} g(z,z',\omega)\, dz dz'\Big)^q\Big] < \infty, \text{ for } q > 1, \text{ integer .}$$

Using translation invariance and Jensen's inequality:

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\frac{1}{|B(0)|} \int_{B(0)} \int_{B(0)} g(z,z',\omega)\, dz dz'\Big)^q\Big] \le \mathbb{E}\Big[\Big(\int_{\mathbb{R}^d} g(0,z',\omega)\, dz'\Big)^q\Big] \\
&= \int_0^\infty ds_1 \dots \int_0^\infty ds_q\, \mathbb{E}\Big[E_0\Big[\exp\Big\{-\int_0^{s_1} V(Z_{u_1},\omega) du_1\Big\}\Big] \dots \\
&\qquad\qquad E_0\Big[\exp\Big\{-\int_0^{s_q} V(Z_{u_q},\omega) du_q\Big\}\Big]\Big] \\
&\le \int_0^\infty ds_1 \dots \int_0^\infty ds_q\, \mathbb{E}_\otimes E_0\Big[\exp\Big\{-\int_0^{s_1} qV(Z_{u_1},\omega) du_1\Big\}\Big]^{1/q} \dots \\
&\quad \mathbb{E}_\otimes E_0\Big[\exp\Big\{-\int_0^{s_q} qV(Z_{u_q},\omega) du_q\Big\}\Big]^{1/q} \\
&\le \Big(\int_0^\infty ds_1\, \mathbb{E}_\otimes E_0\Big[\exp\Big\{-\int_0^{s} qV(Z_u,\omega) du\Big\}\Big]^{1/q}\Big)^q .
\end{aligned}$$

Now by Donsker-Varadhan's result (0,8), with qW instead of W, the last quantity is finite. It is clear that the argument we have just presented works in any dimension, but we only need it for $d = 1, 2$. □

COROLLARY 1.6:

$$\text{(1.47)} \qquad \textit{for } \lambda \ge 0,\ x \in \mathbb{R}^d,\ \beta_\lambda(x) = \beta_\lambda(-x) \textit{ (soft and hard obstacles)}$$

and $\beta_\lambda(\cdot)$ *are seminorms on* $\mathbb{R}^d$.

Proof. – The symmetry of $\beta_\lambda(\cdot)$ is the only point to prove. In the soft obstacle case, it follows from (1.38), the symmetry of $g_\lambda(\cdot,\cdot)$, and translation invariance. In the case of hard obstacles it follows from (1.19), and the rotational symmetry of $f_\lambda(\cdot)$ (Remark 1.1). □

Remark 1.7. – From Theorem 1.3 and 1.7, and the nondegeneracy of $\beta_\lambda(\cdot)$, it is easy to deduce the following "shape theorem": for $0 < \epsilon < 1$, $0 \le \lambda$, for large enough t:

$$\text{(1.48)} \qquad \begin{aligned}
&\{\beta_\lambda(\cdot) \le (1-\epsilon)\,t\} \subseteq \{-\log f_\lambda(\cdot) \le t\} \subseteq \{\beta_\lambda(\cdot) \le (1+\epsilon)\,t\} \\
&\{\beta_\lambda(\cdot) \le (1-\epsilon)\,t\} \subseteq \{-\log(\mathbb{E}[g_\lambda(0,\cdot,\omega)]) \le t\} \subseteq \{\beta_\lambda(\cdot) \le (1+\epsilon)\,t\} . \quad \square
\end{aligned}$$

The coefficients $\beta_\lambda(x)$ in Theorem 1.3, measure a type of cost in going from 0 to a multiple of x. In the terminology of first passage percolation, (*see* for instance [5]), they are point to point constants. It is also possible to give a point to line or rather point to hyperplane version of the result:

COROLLARY 1.8. – *Let e be a unit vector of $\mathbb{R}^d$. For $u \geq 0$, define* $S_u = \inf \{s \geq 0,\ Z_s \cdot e \geq u\}$, *then for* $\lambda \geq 0$:

$$\begin{aligned}(1.49)\qquad &\lim_{u \to \infty} \frac{1}{u} \log \mathbb{E}_{\otimes} E_0 \Big[\exp \Big\{ - \int_0^{S_u} (\lambda + V)(Z_s, \omega) ds \Big\} \Big] \\ &= -\inf\{\beta_\lambda(x),\ x \cdot e \geq 1\} \\ &= -1/\sup\{x \cdot e,\ \beta_\lambda(x) = 1\}\end{aligned}$$

Proof. – The proof is a simple repetition of the proof of corollary 1.9 in [10]. □

We shall now close this section with a variational formula for $\beta_\lambda(\cdot)$, in the one dimensional case with soft obstacles. This formula sheds some light on the nature of the difference between the annealed coefficients $\beta_\lambda(\cdot)$ and the quenched coefficients $\alpha_\lambda(\cdot)$ introduced in [10]. Let us first introduce some notations. For $\ell < 0$, we define

(1.50)
$$F_{\lambda,\ell}(\omega) = -\log E_0 \Big[\exp \Big\{ - \int_0^{H_{\{1\}}} (\lambda + V)(Z_s, \omega) ds \Big\},\ H_{\{1\}} < H_{\{\ell\}} \Big],\ \lambda \geq 0,\ \omega \in \Omega\,.$$

We also define $F_\lambda(\omega)$, by an analogous formula to (1.50), except that the term, $H_{\{1\}} < H_{\{\ell\}}$, is now omitted. Then it was shown in (1.30) of [10], that for soft obstacles, when $d = 1$, $\lambda \geq 0$,

$$(1.51)\qquad \alpha_\lambda(1) = \mathbb{E}[F_\lambda]\,.$$

Let us now recall some notations concerning entropy. For I a non empty bounded interval of $\mathbb{R}$, we denote by π_I the canonical map from Ω onto Ω_I the set of pure point finite measures on I, endowed with its natural σ-algebra. We also define for $\mathbb{Q}$ a probability on Ω the entropy $H_I(\mathbb{Q}|\mathbb{P})$ of $\mathbb{Q}^I = \pi_I \circ \mathbb{Q}$ with respect to $\mathbb{P}^I = \pi_I \circ \mathbb{P}$, that is:

$$\begin{aligned}(1.52)\qquad H_I(\mathbb{Q}|\mathbb{P}) &= \int_{\Omega_I} \log \frac{d\,\mathbb{Q}^I}{d\mathbb{P}^I}\, d\,\mathbb{Q}^I,\ if\ d\,\mathbb{Q}^I << d\mathbb{P}^I, +\infty\ ,\ \text{otherwise} \\ &= \sup_{F_I} \Big\{ \int_\Omega F_I \circ \Pi_I\, d\,\mathbb{Q} - \log \Big(\int_\Omega \exp\{F_I \circ \Pi_I\}\, d\,\mathbb{P} \Big) \Big\}\,,\end{aligned}$$

where F_I runs over bounded measurable functions on Ω_I (*see* [3], p. 68).

Now for $\mathbb{Q}$ a probability on Ω which is invariant under the translations T_t, $t \in \mathbb{R}$, $T_t(\omega)(\cdot) = \omega(\cdot - t)$, one has a natural translation invariance property of H_I, as well as a superadditivity property due to the product character of $\mathbb{P}$. One then defines for a translation invariant $\mathbb{Q}$ on Ω

$$H(\mathbb{Q}|\mathbb{P}) = \lim_{|I| \to \infty} \frac{1}{|I|} H_I(\mathbb{Q}|\mathbb{P}) = \sup_{|I|} \frac{1}{|I|} H_I(\mathbb{Q}|\mathbb{P}) \in [0, \infty]\,.$$

From the last formula of (1.52) it is easy to deduce with a standard truncation argument that when $H(\mathbb{Q}|\mathbb{P}) < \infty$, $E^{\mathbb{Q}}[\omega(I)] < \infty$, I bounded interval of $\mathbb{R}$. Therefore, when $H(\mathbb{Q}|\mathbb{P}) < \infty$, using Jensen's inequality:

$$\begin{aligned}(1.53)\quad E^{\mathbb{Q}}[F_\lambda] &\le E^{\mathbb{Q}}[F_{\lambda,-1}] \\ &\le E^{\mathbb{Q}}[\{\lambda + \|W\|_\infty\,\omega([-1-a,\,1+a])\}\; E_0[H_{\{1\}}/H_{\{1\}} < H_{\{-1\}}]] \\ &\quad + \log(1/P_0[H_{\{1\}} < H_{\{-1\}}]) < +\infty\,,\end{aligned}$$

(of course $P_0[H_{\{1\}} < H_{\{-1\}}] = 1/2$). The coming formula should be contrasted with (1.51):

THEOREM 1.9. – $(d = 1)$. *Suppose that in addition W is continuous, then for $\lambda \ge 0$,*

$$(1.54)\qquad \beta_\lambda(1) = \inf_{\mathbb{Q}}\{E^{\mathbb{Q}}[F_\lambda] + H(\mathbb{Q}|\mathbb{P})\}\,,$$

where $\mathbb{Q}$ runs over the set of translation invariant probabilities on Ω.

Proof. – We shall first prove that $\beta_\lambda(1)$ is smaller than the right member of (1.54). Using the strong Markov property, for $\lambda \ge 0$, $\ell < 0$,

$$\begin{aligned}(1.55)\quad f_\lambda(n+1) &= \mathbb{E}\otimes E_0\Big[\exp\Big\{-\int_0^{H_{\{n\}}}(\lambda+V)(Z_s,\omega)ds\Big\}\Big] \\ &= \mathbb{E}\Big[\prod_{i=0}^{n-1}\exp\{-F_\lambda\circ T_{-i}\}\Big] \ge \mathbb{E}\Big[\exp\Big\{-\sum_{i=0}^{n-1}F_{\lambda,\ell}\circ T_{-i}\Big\}\Big]\end{aligned}$$

If now $\mathbb{Q}$ is translation invariant and $H(\mathbb{Q}|\mathbb{P}) < \infty$, the last term is bigger than:

$$E^{\mathbb{Q}}\Big[\exp\Big\{-\sum_{i=0}^{n-1}F_{\lambda,\ell}\circ T_{-i}\Big\}\Big/\frac{d\,\mathbb{Q}^{[-\ell-a,n+a]}}{d\,\mathbb{P}^{[-\ell-a,n+a]}}\Big] \ge \text{using Jensen's inequality}$$

$$\exp\Big\{-E^{\mathbb{Q}}\Big[\sum_{i=0}^{n-1}F_{\lambda,\ell}\circ T_{-i}\Big] - H_{[-\ell-a,n+a]}(\mathbb{Q}|\mathbb{P})\Big\}\,.$$

Taking the logarithm of the $(n+1)^{\text{th}}$ root of $f_\lambda(n+1)$, and letting n tend to infinity, we see that

$$\beta_\lambda(1) \le E^{\mathbb{Q}}[F_{\lambda,\ell}] + H(\mathbb{Q}|\mathbb{P})$$

letting ℓ tend to $-\infty$, we deduce that $\beta_\lambda(1)$ is smaller than the right member of (1.54).

Let us now prove the reverse inequality. To this end consider $\widetilde{\Omega} = \Omega_{[0,1]}^{\mathbb{Z}}$, endowed with the product topology, if $\Omega_{[0,1]}$ is endowed with the usual Polish topology generated by the maps $\omega \in \Omega_{[0,1]} \longrightarrow \int_0^1 f(x)d\omega(x)$, $f \in C_b([0,1])$. We have a natural map $\Psi : \Omega \to \widetilde{\Omega}$ defined by $\Psi(\omega) = (\omega_i)_{i\in\mathbb{Z}}$, with $\omega_i \in \Omega_{[0,1]}$ equal to the restriction to [0,1] of $T_{-i}(\omega)$.

It is plain that $\Psi(\mathbb{P}) = \widetilde{\mathbb{P}}$, if $\widetilde{\mathbb{P}}$ denotes the infinite product of the Poisson cloud of constant intensity $\nu > 0$ on $[0,1]$. Moreover, if we define:

$$(1.56)\qquad \begin{aligned} &\widetilde{F}_\lambda(\widetilde{\omega}) = -\log E_0\Big[\exp\Big\{-\int_0^{H_{\{1\}}} (\lambda + V)(Z_s,\widetilde{\omega})ds\Big\}\Big], \text{ with} \\ &V(x,\widetilde{\omega}) = \sum_{i\in\mathbb{Z}} \int W(x-y)\, T_i(\omega_i)(dy),\ for\ \widetilde{\omega} = (\omega_i)_{i\in\mathbb{Z}}, \text{ then} \end{aligned}$$

$$(1.57)\qquad f_\lambda(n+1) = E^{\widetilde{\mathbb{P}}}\Big[\exp\Big\{-\sum_{i=0}^{n-1} \widetilde{F}_\lambda \circ \widetilde{T}_i\Big\}\Big],\ (\widetilde{T}_i \text{ discrete shift on } \widetilde{\Omega})\ .$$

Now $\widetilde{F}_\lambda$ is easily seen to be a continuous function on $\widetilde{\Omega}$, and therefore the map $\widetilde{R} \in M_1(\widetilde{\Omega}) \longrightarrow E^{\widetilde{R}}[\widetilde{F}]$ is a lower semicontinuous function on $M_1(\widetilde{\Omega})$, for the weak convergence topology. Applying the process level large deviation principle for the product measure $\widetilde{\mathbb{P}}$, *see* Deuschel-Stroock [3], p. 167 and p. 41, we obtain

$$(1.58)\qquad \beta_\lambda(1) \ge \inf_{\widetilde{\mathbb{Q}}} \{E^{\widetilde{\mathbb{Q}}}[\widetilde{F}] + \widetilde{H}(\widetilde{\mathbb{Q}}|\widetilde{\mathbb{P}})\}\ , \text{where}$$

$\widetilde{\mathbb{Q}}$ runs over probabilities on $\widetilde{\Omega}$ invariant under the (discrete) translations $\widetilde{T}_i$, and for such a probability (*see* also [3] p. 182).

$$(1.59)\qquad \begin{aligned} \widetilde{H}(\widetilde{\mathbb{Q}}|\widetilde{\mathbb{P}}) &= \lim_{n\to\infty} \frac{1}{n}\ H(\widetilde{\mathbb{Q}}^{[1,n]} \mid \widetilde{\mathbb{P}}^{[1,n]}), \text{ with obvious notations} \\ &= \sup_{n\ge 1} \frac{1}{n}\ H(\widetilde{\mathbb{Q}}^{[1,n]} \mid \widetilde{\mathbb{P}}^{[1,n]}). \end{aligned}$$

Consider now a translation invariant $\widetilde{\mathbb{Q}}$, with $\widetilde{H}(\widetilde{\mathbb{Q}}|\widetilde{\mathbb{P}}) < \infty$. Then $E^{\mathbb{Q}}[\omega_i(\{\underset{\sim}{u}\})] = 0$, for $i \in \mathbb{Z}$, $u \in [0,1]$, and we can find a unique $\mathbb{Q} \in M_1(\Omega)$ such that $\Psi \circ \mathbb{Q} = \widetilde{\mathbb{Q}}$. This $\mathbb{Q}$ is such that $T_i \circ \mathbb{Q} = \mathbb{Q}$ for $i \in \mathbb{Z}$, $E^{\mathbb{Q}}[\omega(\{s\})] = 0$, $s \in \mathbb{R}$ and

$$(1.60)\qquad E^{\widetilde{\mathbb{Q}}}[\widetilde{F}_\lambda] = E^{\mathbb{Q}}[F_\lambda],\ H_{[1,n+1]}(\mathbb{Q}|\mathbb{P}) = H(\widetilde{\mathbb{Q}}^{[1,n]} \mid \widetilde{\mathbb{P}}^{[1,n]})\ .$$

From the identity, for $u \in (0,1)$, $\lambda \ge 0$,

$$\begin{aligned} &E_{-u}\Big[\exp\Big\{-\int_0^{H_{\{1-u\}}} (\lambda+V)(Z_s,\omega)ds\Big\}\Big] = E_{-u}\Big[\exp\Big\{-\int_0^{H_{\{0\}}} (\lambda+V)(Z_s,\omega)ds\Big\}\Big]\cdot \\ &E_0\Big[\exp\Big\{-\int_0^{H_{\{1-u\}}} (\lambda+V)(Z_s,\omega)ds\Big\}\Big]\ , \end{aligned}$$

we deduce that for $u \in (0,1)$, $E^{\mathbb{Q}}[F_\lambda \circ T_u] = E^{\mathbb{Q}}[F_\lambda]$. However, if $\widetilde{\mathbb{Q}}_u = \Psi(T_u \circ \mathbb{Q})$, $u \in (0,1)$, then

$$(1.61)\qquad E^{\widetilde{\mathbb{Q}}_u}[\widetilde{F}_\lambda] = E^{\mathbb{Q}}[F_\lambda \circ T_u] = E^{\mathbb{Q}}[F_\lambda], \text{ and}$$

$$(1.62)\qquad \widetilde{H}(\widetilde{\mathbb{Q}}_u|\widetilde{\mathbb{P}}) = \lim_{n\to\infty} \frac{1}{n}\, H(\widetilde{\mathbb{Q}}_u^{[1,n]} \mid \mathbb{P}^{[1,n]}) = \lim_{n\to\infty} \frac{1}{n}\, H_{[1,n+1]}(T_u \circ \mathbb{Q}|\mathbb{P})$$
$$= \lim_{n\to\infty} \frac{1}{n}\, H_{[1-u,n+1-u]}(\mathbb{Q}|\mathbb{P}) = \widetilde{H}(\widetilde{\mathbb{Q}}|\mathbb{P})\,.$$

If we define $\overline{\mathbb{Q}} = \int_0^1 T_u \circ \mathbb{Q}\; du \in M_1(\Omega)$, then $\overline{\mathbb{Q}}$ is translation invariant, by (1.60), (1.61) $E^{\overline{\mathbb{Q}}}[F_\lambda] = E^{\widetilde{\mathbb{Q}}}[\widetilde{F}_\lambda]$, and

$$H(\overline{\mathbb{Q}}|\mathbb{P}) = \sup_{n\geq 1} \frac{1}{n}\, H_{[1,n+1]}(\overline{\mathbb{Q}}|\mathbb{P}) \leq \sup_{n\geq 1} \frac{1}{n} \int_0^1 H_{[1,n+1]}(T_u \circ \mathbb{Q}|\mathbb{P})du$$

(by an easy convexity argument)

$$\leq \int_0^1 \widetilde{H}(\widetilde{\mathbb{Q}}_u|\widetilde{\mathbb{P}})du = \widetilde{H}(\widetilde{\mathbb{Q}}|\widetilde{\mathbb{P}})\,.$$

So from (1.58) and the above construction, we see that $\beta_\lambda(1)$ is bigger than the right member of (1.54). This concludes our proof. □

II. Large deviation estimates

The object of the present section is the derivation of certain large deviation estimates on Z_t under the "annealed weighted measure" Q_t on $C(\mathbb{R}_+, \mathbb{R}^d) \times \Omega$, defined in (0.1). We shall only be concerned with displacements of order $\varphi(t)$ where $\varphi(t)$ is a scale between $t^{d/d+2}$ and t. As far as the scale $t^{d/d+2}$ is concerned, we shall only prove here the lower part of the large deviation principle, the upper estimate being the main object of the sequel of the present paper. As an application of the large deviation estimates we shall derive some long time asymptotics of the $\mathbb{P}$-averaged Schrödinger heat kernel (*see* (1.36)). As we are now going to see, deviations of order t for Z_t under Q_t are governed by

$$(2.1)\qquad J(x) = \sup_{\lambda\geq 0}\, (\beta_\lambda(x) - \lambda),\ x \in \mathbb{R}^d\,.$$

From the joint continuity of $\beta_\lambda(x)$, the upperbound (1.13), it is easy to argue that $J(\cdot)$ is continuous convex. Moreover, from the bounds (1.13), (1.25) follows:

$$(2.2)\quad \text{when } d\geq 2,\ \gamma|x| \vee \frac{x^2}{2} \leq J(x) \leq k(d,\nu,a,0)\,|x|,\ \mathit{for}\ |x| \leq \sqrt{2\lambda_{d-1}}/r_0$$
$$\leq \min_{r>0}\,(\nu\omega_{d-1}(a+r)^{d-1}\,|x| + \frac{\lambda_{d-1}}{r^2}) + \frac{x^2}{2},$$
$$\text{for } |x| > \sqrt{2\lambda_{d-1}}/r_0\,,$$

provided $r_0 > 0$ denotes the unique value of r for which $\nu\omega_{d-1}(a+r)^{d-1} + \sqrt{2\lambda_{d-1}}/r = k(d,\nu,a,0)$.

$$(2.3)\qquad \text{when } d=1,\ \gamma\,|x| \vee \frac{x^2}{2} \leq J(x) \leq J_{\text{hard obst}}(x) = \nu x + \frac{x^2}{2}\,.$$

Moreover, in the case of soft obstacles, we also have thanks to (1.14)

$$(2.4) \qquad J(x) \leq I(x) \stackrel{\text{def}}{=} \sup_{\lambda}(\alpha_\lambda(x) - \lambda) \leq \alpha\,|x| \ for\ |x| \leq \alpha$$
$$\leq \frac{1}{2}\,(x^2 + \alpha^2)\ for\ |x| > \alpha\,,$$

with $\alpha(d, \nu, W(\cdot)) = \sqrt{2(\lambda_d + \|W\|_\infty\, \nu(a+2)^d)}$.

$I(\cdot)$ is the rate function governing deviations of order t for the "quenched problem" (*see* Theorem 2.1 of [10]). Our main object here is the proof of

THEOREM 2.1. – *Under Q_t, Z_t/t obeys a large deviation principle at rate t with rate function $J(\cdot)$, as t tends to infinity, that is:*

$$(2.5) \qquad \overline{\lim_{t\to\infty}}\ t^{-1} \log Q_t(Z_t \in t\,A) \leq -\inf_{x\in A} J(x),\ for\ A \subseteq \mathbb{R}^d,\ closed\,.$$

$$(2.6) \qquad \underline{\lim}_{t\to\infty}\, t^{-1} \log Q_t(Z_t \in t\,O) \geq -\inf_{x\in 0} J(x),\ for\ O \subseteq \mathbb{R}^d,\ open\,.$$

Moreover, if $\varphi(t) : R_+ \to \mathbb{R}_+$ is such that when t tends to infinity

$$(2.7) \qquad t^{d/d+2} = o(\varphi(t))\ and\ \varphi(t) = o(t),\ then$$

(2.8) *$Z_t/\varphi(t)$ obeys a large deviation principle at rate $\varphi(t)$ with rate function $\beta_0(\cdot)$.*

(2.9) *If $\varphi(t) = t^{d/d+2}$, $Z_t/\varphi(t) = Z_t/t^{d/d+2}$ obeys the lower estimate part of the large deviation principle with rate $t^{d/d+2}$ and rate function $\beta_0(\cdot)$.*

Remark 2.2. – The most difficult point to prove here is (2.6). This partly comes from the lack of information on the coefficients $\beta_\lambda(x)$, for instance on their differentiability properties in the λ variable.

Proof. – The proof of the upperbound part of the large deviation principles (2.5), (2.8) is a simple repetition of the arguments used in Theorem 2.1 of [10]. The only point to be mentioned here is that from (1.28), (1.33), we deduce

$$(2.10) \quad \overline{\lim_{v\to\infty}}\ \frac{1}{v}\ \log \mathbb{E} \times E_0\Big[\exp\Big\{-\int_0^{T_v} V(Z_s, \omega)ds\Big\}\Big] \leq -\gamma < 0 \text{ (soft obstacles)},$$

and an analogous estimate for $v^{-1} \log(\mathbb{E} \times \mathbb{E}_0[T_v < T])$, for hard obstacles. This plays the role of (2.9) in [10], and yields exponential tightness.

We now come to the proof of the lower estimates. We shall write the formulas for soft obstacles. The case of hard obstacles should be understood, using the singular potential $W_{\text{h.o.}}(\cdot) = \infty \cdot 1\{|\cdot| \leq a\})$ and interpreting the corresponding formulas as in (1.5) - (1.7). In cases where ambiguity may arise, we shall explicitly write the hard obstacle formula. These remarks being made, our first step is to introduce a quantity which enables to estimate certain probabilities of crossings in the direction $v \in \mathbb{R}^d \backslash \{0\}$, at a certain velocity. More precisely, for $0 < \gamma_1 < \gamma_2 < \infty$, $v \in \mathbb{R}^d \backslash \{0\}$, $0 \leq n$, we consider the stopping time

$$S_{n,v,\gamma_1} = H(nv) \circ \theta_{n\,\gamma_1} + n\,\gamma_1, \text{ and the event} \tag{2.11}$$

$$A_{n,v,\gamma_1,\gamma_2} = \{S_{n,v,\gamma_1} \leq n\gamma_2\}\,. \tag{2.12}$$

In words $A_{n,v,\gamma_1,\gamma_2}$ means that $Z.$ has entered $B(nv)$ during the time interval $[n\gamma_1, n\gamma_2]$. Now for $n \geq 0$, $v \in \mathbb{R}^d \backslash \{0\}$, $0 < \gamma_1 < \gamma_2$, $\lambda \geq 0$, we define the nonnegative constants

$$\begin{aligned} (2.13)\quad & c(n, v, \gamma_1, \gamma_2, \lambda) = \\ & -\inf_{z \in B(0)} \log \mathbb{E}_{\otimes} E_z \Big[A_{n,v,\gamma_1,\gamma_2},\ \exp\Big\{-\int_0^{S_{n,v,\gamma_1}} (\lambda + V)(Z_s, \omega)ds\Big\}\Big] = \\ & -\inf_{z \in B(0)} \log E_z \Big[A_{n,v,\gamma_1,\gamma_2},\ \exp\Big\{-\lambda S_{n,v,\gamma_1} - \\ & \nu \int \Big(1 - e^{-\int_0^{S_{n,v,\gamma_1}} W(Z_s - y)ds}\Big) dy\Big\}\Big]\,, \end{aligned}$$

using the above mentioned convention for hard obstacles. The main interest for us of these quantities stems from:

LEMMA 2.3. – *For* $0 < \gamma_1 < \gamma_2 < \infty$, $\lambda \geq 0$, $v \in \mathbb{R}^d \backslash \{0\}$,

$$\frac{1}{n}\, c(n, v, \gamma_1, \gamma_2, \lambda) \underset{n \to \infty}{\longrightarrow} \delta(v, \gamma_1, \gamma_2, \lambda) \in [0, \infty)\,. \tag{2.14}$$

Moreover, if $(\gamma_1, \gamma_2) \cap [\beta'_\lambda(v)_+,\ \beta'_\lambda(v)_-] \neq \phi$, $\lambda > 0$, *then*

$$\delta(v, \gamma_1, \gamma_2, \lambda) \leq \beta_\lambda(v)\,, \tag{2.15}$$

provided $\beta'_\lambda(v)_+$ *(resp.* $\beta'_\lambda(v)_-$*) denote respectively the right and left derivative of the increasing concave function* $\beta_{.}(v)$ at $\lambda > 0$.

Proof. – We shall keep $v, \gamma_1, \gamma_2, \lambda$, fixed for the moment, and shall only keep track of the n dependence in the notation. The claim (2.14) follows from a subadditive property of the constants $c(n)$, which we now explain.

Observe first, that for $n, m \geq 0$, using the notation $S_{n,m} = H_{(n+m)v} \circ \theta_{\gamma_1 m} + \gamma_1 m$ and $A_{n,m} = \{S_{n,m} \leq m\,\gamma_2\}$, for $z \in B(0)$:

$$\begin{aligned} & E_z \Big[A_{n+m},\ \exp\Big\{-\lambda S_{n+m} - \nu \int \Big(1 - e^{-\int_0^{S_{n+m}} W(Z_s - y)ds}\Big) dy\Big\}\Big] \geq \\ & E_z \Big[A_n \cap \theta_{S_n}^{-1}(A_{n,m}),\ \exp\Big\{-\lambda S_n - \lambda S_{n,m} \circ \theta_{S_n} - \nu \int \Big(1 - e^{-\int_0^{S_n} W(Z_s - y)ds} \cdot \\ & e^{-\int_{S_n}^{S_n + S_{n,m} \circ \theta S_n} W(Z_s - y)ds}\Big) dy\Big\}\Big]\,. \end{aligned}$$

If we now use (1.16) and the strong Markov property, we find: $c(n + m) \leq c(n) + c(m)$, for $n, m \geq 0$.

It is also very easy to check (by arguments easier but in the spirit of the proof of (1.13)), that $c(1) < \infty$. It readily follows that

$$\frac{c(n)}{n} \underset{n\to\infty}{\longrightarrow} \delta(v,\gamma_1,\gamma_2,\lambda) = \inf_{n\geq 1} \frac{c(n)}{n}\,. \tag{2.16}$$

This prove (2.14). Let us now prove (2.15). We first pick $(\rho,\eta) \in (0,1)$ such that:

$$\rho\,\beta'_\lambda(v)_+ + (1-\rho)\,\beta'_\lambda(v)_- + [-\eta,\eta] \subseteq (\gamma_1,\gamma_2)\,. \tag{2.17}$$

Then, for $x \in B(0)$, $\lambda > 0$, and similar methods as above

$$e^{-c(n)} = \inf_{x\in B(0)} E_x\Big[A_n,\ \exp\Big\{-\lambda S_n - \nu\int\big(1-e^{-\int_0^{S_n} W(Z_s-y)ds}\big)dy\Big\}\Big] \geq$$
$$\inf_{x\in B(0)} E_x\Big[H(\rho n v) \in \rho n[\beta'_\lambda(v)_+ - \eta,\ \beta'_\lambda(v)_+ + \eta],\ \exp\Big\{-\lambda H(\rho n v)$$
$$-\nu\int\Big(1-e^{-\int_0^{H(\rho n v)} W(Z_s-y)ds}\Big)dy\Big\}\Big]\cdot$$
$$\inf_{z\in B(0)} E_z\Big[H((1-\rho)nv) \in (1-\rho)\,n\,[\beta'_\lambda(v)_- - \eta,\ \beta'_\lambda(v)_- + \eta],\ \exp\Big\{-\lambda H((1-\rho)nv)$$
$$-\nu\int\Big(1-e^{-\int_0^{H((1-\rho)nv)} W(Z_s-y)ds}\Big)dy\Big\}\Big]\,.$$

If we now pick $0 < \lambda_2 < \lambda < \lambda_1$, then for $n \geq 1$:

$$\begin{aligned}
e^{-c(n)} \geq{}& \inf_{x\in B(0)} f_{\lambda_1}(\rho n v - x)\cdot \inf_{z\in B(0)} f_{\lambda_2}((1-\rho)(nv-z)\cdot \\
&\exp\{-(\lambda-\lambda_2)\,n\,(\beta'_\lambda(v)_- + \eta)\} \\
&\cdot \inf_{x\in B(0)} \Big\{E_x\Big[\frac{H(\rho n v)}{\rho n} \in [\beta'_\lambda(v)_+ - \eta,\ \beta'_\lambda(v)_+ + \eta],\ \exp\Big\{-\lambda_1 H(\rho n v) \\
&-\nu\int\Big(1-e^{-\int_0^{H(\rho n v)} W(Z_s-y)ds}\Big)dy\Big\}\Big]\ /\ f_{\lambda_1}(\rho n v - x)\Big\} \\
&\cdot \inf_{z\in B(0)} \Big\{E_x\Big[\frac{H((1-\rho)nv)}{(1-\rho)n} \in [\beta'_\lambda(v)_- - \eta,\ \beta'_\lambda(v)_- + \eta], \\
&\exp\Big\{-\lambda_2 H((1-\rho)nv) - \nu\int\Big(1-e^{-\int_0^{H((1-\rho)nv)} W(Z_s-y)ds}\Big)dy\Big\}\Big] \\
&/f_{\lambda_2}((1-\rho)nv - z)\Big\}\,.
\end{aligned} \tag{2.18}$$

Observe that (1.19) immediately implies that for $M > 0$,

$$\lim_{y\to\infty}\ \sup_{0\leq\widetilde{\lambda}\leq M}\ \sup_{x\in B(0)} \frac{1}{|y|}\,|-\log f_{\widetilde{\lambda}}(y-x) - \beta_{\widetilde{\lambda}}(y)\,| = 0\,. \tag{2.19}$$

If we pick $\lambda_1 > \lambda$ and $0 < \lambda_2 < \lambda$ such that $\beta'_{\lambda_1}(v)$ and $\beta'_{\lambda_2}(v)$ exist and respectively belong to $(\beta'_\lambda(v)_+ - \eta,\ \beta'_\lambda(v)_+]$ and $[\beta'_\lambda(v)_-,\ \beta'_\lambda(v) + \eta)$, it is a standard argument of large deviation theory, to infer from (2.19) that the fourth and fifth terms in the right member of (2.18) converge to 1, as n goes to infinity (*see* also (2.21) of [10]). It now follows that for such a choice $\delta(n, \gamma_1, \gamma_2, \lambda) \leq \rho\, \beta_{\lambda_1}(v) + (1-\rho)\, \beta_{\lambda_2}(v) + (\lambda - \lambda_2)(\beta'_\lambda(v)_- + \eta)$. If we now let λ_1 and λ_2 converge to λ, we precisely find (2.15). □

Let us now give a proof of the lowerbounds of the large deviation principles in Theorem 2.1. Let us first explain the strategy, for instance in the case of (2.6). The idea is that one possibility for Z_t to be in a $o(t)$ neighborhood of a point tv, is to first wait a time t_1, and be back in $B(0)$ at time t_1, then to travel in the v direction and be in $B(tv)$ at a time between $t_1 + \gamma_1 t$ and $t_1 + \gamma_2 t$, where γ_1 and γ_2 are picked close enough and $t_1 + \gamma_2 t$ is close to t, and let then $Z_.$ wait near tv until time t. We shall now see how the rate function $J(v)$ naturally appears when one optimizes the probability of a succession of such events.

We first explain how parameters are picked. Observe that for $v \neq 0$, $\beta'_\lambda(v) > 0$, and decreases to zero as λ tends to infinity, thanks to the bounds (1.13), (1.25), and the concavity of $\lambda \to \beta_\lambda(v)$. So for each fixed $v \neq 0$, and $n \geq 2$, we pick sequences $\gamma_1(n)$, $\gamma_2(n)$, $\lambda(n) > 0$, such that:

$$\begin{aligned}
&\text{when } \beta'_0(v) < 1,\ \beta'_{\lambda(n)}(v) \text{ exists},\ \lambda(n) \underset{n\to\infty}{\longrightarrow} 0,\\
&\gamma_1(n) = \beta'_{\lambda(n)}(v)\Big(1 - \frac{1}{n}\Big)\\
&\gamma_2(n) = \beta'_{\lambda(n)}(v)\Big(1 + \frac{1-\beta'_0(v)}{n}\Big) < 1\\
&\text{when } \beta'_0(v) \geq 1,\ \Big(1 - \frac{2}{n}\Big) \in [\beta'_{\lambda(n)}(v)_+,\ \beta'_{\lambda(n)}(v)_-]\\
&\gamma_1(n) = 1 - \frac{3}{n},\ \gamma_2(n) = 1 - \frac{1}{n}\,.
\end{aligned} \tag{2.20}$$

In the case of the lower estimate part of (2.8) and (2.9) we shall instead choose:

$$\begin{aligned}
&\overline{\lambda}(n) = \frac{1}{n},\ 0 < \overline{\gamma}_1(n) < \overline{\gamma}_2(n) \text{ are such that}\\
&\overline{\gamma}_1(n) < \beta_{\lambda_n}(v)_- < \overline{\gamma}_2(n),\ |\overline{\gamma}_2(n) - \overline{\gamma}_1(n)| < \frac{1}{n}\,.
\end{aligned} \tag{2.21}$$

It is easy to check that such choices are possible. Moreover, they satisfy the condition $(\gamma_1(n),\ \gamma_2(n)) \cap [\beta'_{\lambda(n)}(v)_+,\ \beta'_{\lambda(n)}(v)_-] \neq \phi$, (resp. $(\overline{\gamma}_1(n),\ \overline{\gamma}_2(n)) \cap [\beta'_{\lambda(n)}(v)_+,\ \beta'_{\lambda(n)}(v)_-] \neq \phi$), which appears in Lemma 2.3. We shall now begin with proof of (2.6). It clearly suffices to prove that for $v \neq 0$,

$$\underline{\lim}_{t\to\infty} t^{-1} \log Q_t(Z_t \in B(Z_t \in B([t]v, 2)) \geq -J(v)\,. \tag{2.22}$$

Define $t_1 = t - \gamma_2(n)[t] > 0$, so that $t_1 + \gamma_1(n)[t] + (\gamma_2(n) - \gamma_1(n))\,t \geq t$. Then for sufficiently large t we write with the notations of (2.11), (2.12):

$$\begin{aligned}
&E_0\Big[\exp\Big\{-\nu \int \Big(1 - e^{-\int_0^t W(Z_s - y)ds}\Big)dy\Big\},\ Z_t \in B([t]v, 2)\Big] \geq\\
&E_0\Big[\exp\Big\{-\nu \int \Big(1 - e^{-\int_0^t W(Z_s - y)ds}\Big)dy\Big\} \cdot 1\{Z_{t_1} \in B(0)\} \cdot 1_{A_{[t],v,\gamma_1(n),\gamma_2(n)}} \circ \theta_{t_1} \cdot\\
&1\{T_{B([t]v,2)} > (\gamma_2(n) - \gamma_1(n))t\} \circ \theta_{S_{[t],v,\gamma_1(n)}} \circ \theta_{t_1}\Big]\,.
\end{aligned}$$

Using (1.16) and the strong Markov property, this is bigger than:

$$(2.23)\quad E_0[\exp\{-\nu\,|S^a_{t_1}|\},\ Z_{t_1}\in B(0)]\cdot$$
$$\inf_{z\in B(0)} E_z\Big[A_{[t],v,\gamma_1(n),\gamma_2(n)},\ \exp\Big\{-\nu\int\Big(1-e^{-\int_0^{S_{[t],v,\gamma_1(n)}} W(Z_s-y)ds}\Big)dy\Big\}\Big]\cdot$$
$$\inf_{z\in B(0)} E_z[T_{B(0,2)}>(\gamma_2(n)-\gamma_1(n))\,t,\ \exp\{-\nu\,|\,S^a_{(\gamma_2(n)-\gamma_1(n))t}|\}]$$
$$\overset{\text{def}}{=} A_1\cdot A_2\cdot A_3\,.$$

It is a fairly standard calculation that

$$(2.24)\qquad \lim_{u\to\infty} u^{-d/d+2}\log E_0[\exp\{-\nu\,|\,S^a_u\,|\},\ Z_u\in B(0)]=-c(d,\nu)\,.$$

Indeed in view of (0.8), only a lower bound is required. One simply writes:

$$E_0[\exp\{-\nu\,|S^a_u|\},\ Z_u\in B(0)]\ge \exp\{-\nu\,|\,B(0,R_0\,u^{1/d+2}+a)|\}$$
$$\cdot\ P_0[T_{B(0,R_0\,u^{1/d+2})}>u,\ Z_u\in B(0)]\,,$$

where $R_0(d,\nu)$ is picked so that

$$\nu\,|B(0,R_0)|+\lambda(B(0,R_0))=c(d,\nu)\,.$$

By scaling the last term of (2.26) equals

$$P_0[T_{B(0,R_0)}>u^{d/d+2},\ Z_{u^{d/d+2}}\in B(0,u^{-1/d+2})]=\exp\Big\{-\frac{\lambda_d}{R_0^2}\,u^{d/d+2}(1+o(1))\Big\}\,,$$

using an eigenfunction expansion, and (2.24) follows. On the other hand,

$$A_3\ge\exp\{-\nu\,|B(0,a+2)|\}\ P_0[T_{B(0,2)}>(\gamma_2(n)-\gamma_1(n))t]$$
$$=\exp\Big\{-\frac{\lambda_d}{4}\,(\gamma_2(n))-\gamma_1(n)\ t(1+o(1))\Big\},\ \text{and}$$

$$(2.26)\qquad A_2\ge e^{\lambda(n)\gamma_1(n)[t]}\exp\{-c([t],v,\gamma_1(n),\gamma_2(n),\lambda(n))\}\,.$$

Collecting (2.24) - (2.26), and using (2.15) we find

$$(2.27)\ \varliminf_{t\to\infty} t^{-1}\log E_0\Big[\exp\Big\{-\nu\int\Big(1-e^{-\int_0^t W(Z_s-y)ds}\Big)dy\Big\},\ Z_t\in B([t]v,2)\Big]\ge$$
$$-\beta_{\lambda(n)}(v)+\gamma_1(n)\,\lambda(n)-\frac{\lambda_d}{4}(\gamma_2(n)-\gamma_1(n))\,.$$

Letting n tend to infinity, we see the left member of (2.29) is bigger than

$$-\varliminf_{n\to\infty}(\beta_{\lambda(n)}(v)-\gamma_1(n)\,\lambda(n))\,.$$

Now in view of (2.20), when $\beta_0'(v) < 1$, this last quantity is $\beta_0(v) = J(v)$. In the case $\beta_0'(v) \geq 1$, the sequence $\lambda(n)$ is decreasing and converges to a value $\lambda(\infty)$ such that

$$\beta'_{\lambda(\infty)}(v)_+ \leq 1 \leq \beta'_{\lambda(\infty)}(v)_- \tag{2.28}$$

(this last inequality being omitted when $\lambda(\infty) = 0$).

Then we have $\underline{\lim}\ (\beta_{\lambda(n)}(v) - \gamma_1(n)\,\lambda(n)) = \beta_{\lambda(\infty)}(v) - \lambda(\infty) = J(v)$, since by (2.28) $\lambda \to \beta_\lambda(v) - \lambda$ increases on $[0, \lambda(\infty)]$ and decreases on $[\lambda(\infty), \infty)$. So the left member of (2.27) is bigger than $-J(v)$. In view of (0.8), this yields our claim (2.22).

In case $\varphi(t)$ is given as in (2.7) or (2.9), it is enough to show that for $v \neq 0$

$$\underline{\lim}_{t\to\infty} \varphi(t)^{-1} \log Q_t(Z_t \in B([\varphi(t)]\,v, 2)) \geq -\beta_0(v)\,. \tag{2.29}$$

We now define $t_1 = t - \overline{\gamma}_2(n)[\varphi(t)]$, for sufficiently large t, so that $t_1 + \overline{\gamma}_1(n)[\varphi(t)] + (\gamma_2(n) - \gamma_1(n))\ \varphi(t) \geq t$, and similarly to (2.23)

$$E_0\Big[\exp\Big\{-\nu \int \Big(1-e^{-\int_0^t W(Z_s-y)ds}\Big)dy\Big)\Big\},\ Z_t \in B([\varphi(t)]\,v, 2)\Big] \geq \overline{A}_1{\cdot}\overline{A}_2{\cdot}\overline{A}_3\,, \tag{2.30}$$

with $\overline{A}_1$ as in (2.23),

$$\overline{A}_2 = \inf_{z\in B(0)} E_z\Big[A_{[\varphi(t)],v,\overline{\gamma}_1(n),\overline{\gamma}_2(n)},\ \exp\Big\{-\nu \int \Big(1 - e^{-\int_0^{S_{[\varphi(t)],v,\overline{\gamma}_1(n)}} W(Z_s-y)ds}\Big)dy\Big\}\Big]$$
$$\overline{A}_3 = \inf_{z\in B(0)} E_z[T_{B(0,2)} > (\overline{\gamma}_2(n) - \overline{\gamma}_1(n))\,\varphi(t),\ \exp\{-\nu\,|\,S^a_{(\overline{\gamma}_2(n)-\overline{\gamma}_1(n))\varphi(t)}|\}]\,.$$

From this, as before we deduce that

$$\begin{aligned}
&\underline{\lim}_{t\to\infty} \varphi(t)^{-1} \log\Big(\mathbb{E}_0\Big[\exp\Big\{-\nu \int \Big(1 - e^{-\int_0^t W(Z_s-y)ds}\Big)dy\Big\},\\
&Z_t \in B([\varphi(t)]\,v, 2)\Big)\Big] \geq -\beta_{\overline{\lambda}(n)}(v) + \overline{\lambda}(n)\,\overline{\gamma}_1(n) - \frac{\lambda_d}{4}\,(\overline{\gamma}_1(n) - \overline{\gamma}_1(n)),\\
&\text{if } t^{d/d+2} = o(\varphi(t)),\\
&-c(d,\nu) - \beta_{\overline{\lambda}(n)}(v) + \overline{\lambda}(n)\,\overline{\gamma}_1(n) - \frac{\lambda_d}{4}\,(\overline{\gamma}_2(n) - \overline{\gamma}_1(n)),\ if\ t^{d/d+2} = \varphi(t)\,.
\end{aligned} \tag{2.31}$$

Letting n tend to infinity we see that the left member of (2.30) is bigger than $-\beta_0(v)$, when $t^{d/d+2} = o(\varphi(t))$, and $-\beta_0(v) - c(d,\nu)$, when $\varphi(t) = t^{d/d+2}$. In view of (0.8), this finishes the proof of (2.29), and concludes the proof of Theorem 2.1. □

We now apply our results to study certain large t asymptotics of

$$\begin{aligned}
\overline{r}(t,x,y) &= \mathbb{E}(r(t,x,y,\omega))]\\
&= (2\pi t)^{-d/2} \exp\Big\{-\frac{(y-x)}{2t}{}^2\Big\}\\
&\quad E^t_{x,y}\Big[\exp\Big\{-\nu \int \Big(1 - e^{-\int_0^t W(Z_s-y)ds}\Big)dy\Big\}\Big] \text{ (soft obstacles)}
\end{aligned} \tag{2.32}$$

with the obvious interpretation of the last formula for hard obstacles.

THEOREM 2.4. – i) *if* $\varphi(t) = o(t^{d/d+2})$, *for* $v \in \mathbb{R}^d$,

$$\lim_{t\to 0} t^{-d/d+2} \log \overline{r}(t,0,\varphi(t)v) = -c(d,\nu).$$

ii) *if* $t^{d/d+2} = o(\varphi(t))$, $\varphi(t) = o(t)$, *for* $v \in \mathbb{R}^d$,

$$\lim_{t\to\infty} \varphi(t)^{-1} \log \overline{r}(t,0,\varphi(t)v) = -\beta_0(v)$$

iii) *for* $\qquad v \in \mathbb{R}^d,\ \lim_{t\to\infty} \log \overline{r}(t,0,tv) = -J(v).$

Proof. – With the notations of (1.24), using (1.16) we find that for $0 < u < t$, $x \in \mathbb{R}^d$, $\rho > 0$:

$$\begin{aligned}
&E_0\Big[\exp\Big\{-\nu\int\Big(1-e^{-\int_0^{t-u} W(Z_s-y)ds}\Big)dy\Big\}\cdot p_{B(x,2\rho)}(u,\, Z_{t-u},\, x),\\
(2.33)\qquad &Z_{t-u}\in B(x,\rho)\Big]\, e^{-\nu|B(0,2\rho+a)|} \le \overline{r}(t,0,x)\\
&E_0\Big[\exp\Big\{-\nu\int\Big(1-e^{-\int_0^{t-u} W(Z_s-y)ds}\Big)dy\Big\}\, p(u,\, Z_{t-u}, x)\Big]
\end{aligned}$$

with $p(\cdot,\cdot,\cdot) = p_{\mathbb{R}^d}(\cdot,\cdot,\cdot)$. We now apply (2.33) to $x = \varphi(t)v$, with various choices of u and ρ.

For i), we pick $u = 1$, $\rho = 2 + |v|$, the upperbound follows from (0.8), and the lowerbound from (2.24), when $v = 0$, and from (2.30) (applied to $\varphi'(\cdot) = \varphi(\cdot + 1)$ with $\varphi(t) = o(t^{d/d+2})$), when $v \neq 0$.

In the proof of ii) and iii) it suffices to consider $v \neq 0$. For the lowerbound, we pick $v = 1$ and $\rho = 2 + 2|v|$, and use (2.22) in the case of iii) and (2.29) with $\varphi'(\cdot) = \varphi(\cdot + 1)$ in the case of ii), since the normalizing constant S_t plays no role in view of (0.8). As for the upperbound, we pick $u = 1$. Then for $\epsilon > 0$, from the explicit form of $p(1,\cdot,\cdot)$

$$(2.34)\qquad \overline{\lim_{t\to\infty}}\, \varphi(t)^{-1} \log(E_0[p(1, Z_{t-1}, \varphi(t)v),\ Z_{t-1} \notin \overline{B}(\varphi(t)v,\ \epsilon\varphi(t))] = -\infty\ ,$$

with $\varphi(t) = t$ in case iii). It follows using again (0.8) that:

$$\begin{aligned}
\overline{\lim_{t\to\infty}}\, \varphi(t)^{-1} \log \overline{r}(t,0,\varphi(t)v) &\le \overline{\lim_{t\to\infty}}\, \varphi(t)^{-1} \log Q_{t-1}(Z_{t-1} \in \overline{B}(\varphi(t)v,\ \epsilon\varphi(t)))\\
&\le -\inf_{\overline{B}(v,\epsilon)} \beta_0(\cdot) \text{ by (2.8) with } \varphi'(\cdot) = \varphi(\cdot + 1) \text{ in case ii)}\ ,\\
&\le -\inf_{\overline{B}(v,2\epsilon)} J(\cdot) \text{ in case iii)}\ .
\end{aligned}$$

Letting ϵ tend to zero, we now find our claim. □

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