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THE BINARY ADDITIVE DIVISOR PROBLEM

BY YOICHI MOTOHASHI

ABSTRACT. – Here we try to develop a comprehensive account of the theory of the binary additive divisor problem. The results contain the hitherto best estimates of the error terms in the asymptotic formulas for both the ordinary and the dual versions of the problem.

1. Introduction and statement of results

Our aim in the present paper is to investigate the asymptotic behavior of the sums

$$D(N;f) = \sum_{n=1}^{N} d(n)d(n+f), \qquad (f \geq 1)$$
$$D(N) = \sum_{n=1}^{N-1} d(n)d(N-n)$$

as N tends to infinity, where $d(n)$ is the divisor function, and in the second sum N is an integer. These belong to the additive divisor problem, and have a rich history. We thus begin our discussion with a review of former results on $D(N;f)$; the history of $D(N)$ will be given later.

The first result on $D(N;f)$ is due to Ingham [11], who showed the asymptotic formula

$$D(N;f) = (1+o(1))\frac{6}{\pi^2}\sigma_{-1}(f)N(\log N)^2,$$

where $\sigma_a(n)$ denotes the sum of the ath powers of the divisors of n. Then Estermann [8] was able to improve this to an asymptotic expansion by exploiting his own finding of a relation between $D(N;f)$ and the Kloosterman sum

$$S(m,n;l) = \sum_{\substack{d=1 \\ (d,l)=1}}^{l} e(\frac{1}{l}(md+m\bar{d})),$$

where $e(x) = \exp(2\pi i x)$ and $d\bar{d} \equiv 1 \pmod{l}$ as usual. As is remarked in [1], p. 185 a minor modification of Estermann's argument could yield, uniformly for $1 \leq f \leq \sqrt{N}$,

$$E(N;f) \ll N^{\frac{11}{12}} f^{\frac{1}{6}} (\log N)^3, \tag{1.1}$$

where

$$E(N;f) = D(N;f) - N\sum_{i=0}^{2}(\log N)^i \sum_{j=0}^{2} c_{ij} \sum_{d|f} d^{-1}(\log d)^j \tag{1.2}$$

with certain absolute constants c_{ij}.

The importance of the uniformity with respect to the shift parameter f was observed for the first time by Atkinson in his paper quoted above, where he needed a result of the type of (1.1) to estimate certain 'non-diagonal' parts of his formula for a form of the fourth power mean of the Riemann zeta-function $\zeta(s)$ on the critical line. In retrospect [1] was the first instance of the infusion of the theory of Kloosterman sums into the theory of $\zeta(s)$, which has recently become one of the most important topics in analytic number theory. Stronger estimates of $S(m,n\,;l)$ yield better bounds of $E(N;f)$, which in turn give finer results for the fourth power mean of $\zeta(s)$. Thus Weil's estimate

$$|S(m,n\,;l)| \le l^{\frac{1}{2}} d(l) \min\{(m,l)^{\frac{1}{2}}, (n,l)^{\frac{1}{2}}\}, \tag{1.3}$$

which is the best possible for individual l, led Heath-Brown [10] (via Estermann's argument) to

$$E(N;f) \ll N^{\frac{5}{6}+\varepsilon} \qquad (1 \le f \le N^{\frac{5}{6}}) \tag{1.4}$$

and

$$\int_M^{2M} E(N;f)^2 dN \ll M^{\frac{5}{2}+\varepsilon} \qquad (1 \le f \le N^{\frac{3}{4}}) \tag{1.5}$$

uniformly in f; here and in the sequel ε denotes an arbitrary small positive constant whose value may differ at each occurrence. These enabled him to establish

$$E_2(T) \ll T^{\frac{7}{8}+\varepsilon} \tag{1.6}$$

for the error term $E_2(T)$ in the asymptotic formula for the fourth power mean of $\zeta(s)$, i.e.,

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^4 dt = T\,P_4(\log T) + E_2(T), \tag{1.7}$$

where P_4 is a polynomial of the 4th degree.

However the whole situation was changed very drastically by the appearance of Kuznetsov's trace formulas [19][21], which transform sums of Kloosterman sums into bilinear forms of the Fourier coefficients of cusp forms over the full modular group. This is due to the fact that the estimation of $E(N;f)$ depends on $S(m,n\,;l)$ with variable l's in a way that reminds us that the binary Goldbach problem depends on the distribution of primes in arithmetic progressions with variable modulus. The first application of Kuznetsov's trace formulas to $D(N;f)$ was undertaken by Deshouillers and Iwaniec [5], who obtained, for each fixed $f \ge 1$,

$$E(N;f) \ll N^{\frac{2}{3}+\varepsilon}, \tag{1.8}$$

which is a substantial improvement upon (1.4) save for the non-uniformity in f. They gave the details only in the case $f = 1$ for the sake of simplicity, but their argument in fact yields the above. It should be remarked that to show (1.8) they needed also a large sieve inequality for the Fourier coefficients of cusp-forms, which itself was a consequence of Kuznetsov's trace-formulas, and which had been obtained by Iwaniec in his work [16] on the fourth power mean of $\zeta(s)$.

Then in the important work [22] Kuznetsov himself applied his trace-formulas to the following generalization of $D(N;f)$:

$$A_f(\alpha,\beta;W) = \sum_{n=1}^{\infty} \sigma_\alpha(n)\sigma_\beta(n+f)W(\frac{n}{f}),$$

where α,β are complex numbers and the weight $W(x)$ is a smooth function which is to satisfy certain decay condition as x tends to $+0$ or $+\infty$. What he obtained is an explicit formula which expresses $A_f(\alpha,\beta;W)$ in terms of a bilinear-form of L-functions attached to holomorphic and non-holomorphic cusp-forms, providing α,β are in a domain determined by W. This enabled Kuznetsov to state the estimate

$$E(N;f) \ll f^{\frac{1}{6}}(N\log N)^{\frac{2}{3}}d(f) \tag{1.9}$$

uniformly for $1 \le f \le N^{\frac{4}{5}}(\log N)^{-1}$, which is to be compared with (1.8). It induced also a new result on the fourth power mean of $\zeta(s)$. Thus Zavorotnyi [31] could use Kuznetsov's explicit formula for $A_f(0,0;W)$ with suitably chosen W's to prove the following improvement upon (1.6):

$$E_2(T) \ll T^{\frac{2}{3}+\varepsilon}. \tag{1.10}$$

But Kuznetsov's argument in [22] is, unfortunately, highly sketchy, and does not seem to suit the value of the results claimed there. Especially it lacks the proper procedure of the analytic continuation of an intermediate spectral decomposition [formula (117)], which in our view is the most crucial step to complete the proof of his remarkable Theorem 3.5.

Under these circumstances it seems very desirable for us to have a rigorous proof of his formula for $A_f(\alpha,\beta;W)$, at least for the sake of giving it a sound base for its future applications. We shall undertake this task by pursuing an approach that is somewhat different from Kuznetsov's; in fact it is closely related to the argument we have developed in our recent works [24][25] on the fourth power mean of $\zeta(s)$. Then, as a sort of reward, we are able to use safely the explicit formula for $A_f(0,0;W)$ (see Theorem 3 below) to prove

THEOREM 1. – *We have, uniformly for* $1 \le f \le N^{\frac{10}{7}}$,

$$\begin{aligned} \sum_{n=1}^{N} d(n)d(n+f) =& \frac{6}{\pi^2}\int_0^{N/f} m(x;f)dx \\ &+ O\{(N(N+f))^{\frac{1}{3}+\varepsilon} + f^{\frac{9}{40}}(N(N+f))^{\frac{1}{4}+\varepsilon} + f^{\frac{7}{10}}N^{\varepsilon}\}. \end{aligned} \tag{1.11}$$

Here

$$m(x;f) = \sigma(f)\log x\log(x+1) + \{\sigma(f)(2\gamma - 2\frac{\zeta'}{\zeta}(2) - \log f) + 2\sigma'(f)\}\log(x(x+1)) + \sigma(f)\{(2\gamma - 2\frac{\zeta'}{\zeta}(2) - \log f)^2 - 4(\frac{\zeta'}{\zeta})'(2)\} + 4\sigma'(f)(2\gamma - 2\frac{\zeta'}{\zeta}(2) - \log f) + 4\sigma''(f) \tag{1.12}$$

with obvious abbreviations, where γ is the Euler constant, and

$$\sigma^{(\nu)}(f) = \sum_{d|f} d(\log d)^\nu.$$

The main term in (1.11) is different from that in (1.2), for our range of uniformity is much wider; if f is less than N then our main term reduces to that in (1.2) with an admissible error. In particular we have

COROLLARY 1:

$$E(N;f) \ll N^{\frac{2}{3}+\varepsilon} \qquad (1 \le f \le N^{\frac{20}{27}})$$

uniformly in f.

COROLLARY 2:

$$D(N;f) = (1+o(1))\frac{6}{\pi^2}\int_0^{N/f} m(x;f)dx \qquad (1 \le f \le N^{\frac{10}{7}-\varepsilon})$$

uniformly in f.

We now turn to the sum $D(N)$, which may be called a dual of $D(N;f)$. Though research on $D(N)$ was not so intensive as on $D(N;f)$, perhaps because of no apparent relation with the mean values of $\zeta(s)$, the history of $D(N)$ can be traced much farther than that of $D(N;f)$. For, the origin of the problem of $D(N)$ may be found in the explicit formulas like

$$\sum_{n=1}^{N-1} \sigma_3(n)\sigma_3(N-n) = \frac{1}{120}(\sigma_7(N) - \sigma_3(N)),$$

which is due to Jacobi. Ramanujan once tried to extend this to the sum

$$\sum_{n=1}^{N-1} \sigma_\alpha(n)\sigma_\beta(N-n),$$

and conjectured that it would be

$$(1+o(1))\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}\frac{\zeta(\alpha+1)\zeta(\beta+1)}{\zeta(\alpha+\beta+2)}\sigma_{\alpha+\beta+1}(N), \tag{1.13}$$

providing α, $\beta > 0$. Then Ingham [11] took up $D(N)$, and proved

$$D(N) = (1 + o(1))\frac{6}{\pi^2}\sigma_1(N)(\log N)^2, \tag{1.14}$$

which obviously corresponds to his result on $D(N; f)$ quoted above. But it was again Estermann who made the first essential work on $D(N)$. In [7] he could relate $D(N)$ with Kloosterman sums, and obtained

$$E(N) \ll N^{\frac{7}{8}}(\log N)^{\frac{23}{4}}\sigma_{-\frac{3}{4}}(N),$$

where

$$E(N) = D(N) - N\sum_{i=0}^{2}(\log N)^i\sum_{j=0}^{2}d_{ij}\sum_{d|N}d^{-1}(\log d)^j \tag{1.15}$$

with certain absolute constants d_{ij}. As Halberstam showed later in the work [9] on the above conjecture of Ramanujan, Estermann's argument could have yielded

$$E(N) \ll N^{\frac{3}{4}}(\log N)^3 \tag{1.16}$$

if it was combined with (1.3). After this there had been virtually no research on $D(N)$ until Kuznetsov [22] (Theorem 3.5 with $w_1 \equiv 0$) found an explicit formula for

$$B_N(\alpha, \beta; W_0) = \sum_{n=1}^{\infty}\sigma_\alpha(n)\sigma_\beta(N-n)W_0\left(\frac{n}{N}\right),$$

where W_0 is a smooth function with a support in the unit interval. As a matter of fact he did not consider $B_N(\alpha, \beta; W_0)$ separately from $A_f(\alpha, \beta; W)$, since we have $A_{-N}(\alpha, \beta; W) = B_N(\alpha, \beta; W_0)$ with a suitable choice of W. But we make this separation, because of a reason that will become apparent in the last two sections of the present paper. Kuznetsov's formula expresses $B_N(\alpha, \beta; W_0)$ in terms of cusp form L-functions in much the same way as in the case of $A_f(\alpha, \beta; W)$, though his brief argument again does not seem to suit the value of the result. We shall give a rigorous proof of his claim, and as its application prove an improvement upon the long standing result (1.16):

THEOREM 2. – *Let $E(N)$ be as above. Then we have*

$$E(N) \ll N^{0.7+\varepsilon}.$$

Theorems 1 and 2 are deep in the sense that they depend not only on the spectral resolution of the non-Eucledian Laplacian but also on the hitherto best estimates of the Fourier coefficients of holomorphic and non-holomorphic cusp-forms.

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2. Explicit formulas

In this section we shall first introduce some basic terminology from the theory of automorphic forms, with which the formulas for $D(N;f)$ and $D(N)$ are to be formulated. Then we shall give some further discussion on them.

Thus, let $\{\lambda_j = \kappa_j^2 + \frac{1}{4}; \kappa_j > 0\ (j = 1, 2, 3\ldots)\} \cup \{0\}$ be the discrete spectrum of the non-Eucledian Laplacian acting on the space of all non-holomorphic automorphic functions with respect to the full modular group. Let φ_j be the Mass wave form attached to the eigenvalue λ_j so that $\{\varphi_j\}$ forms an orthonormal base of the space spanned by all cusp forms, and each φ_j is an eigen-function of every Hecke operator $T(n)$ ($n = -1$ or $n \geq 1$). The latter means that we have, for $n \geq 1$,

$$\begin{aligned}(T(n)\varphi_j)(z) &= \frac{1}{\sqrt{n}} \sum_{ad=n} \sum_{b=1}^{d} \varphi_j\left(\frac{az+b}{d}\right)\\ &= t_j(n)\varphi_j(z)\end{aligned}$$

with a certain real number $t_j(n)$, and

$$\begin{aligned}(T(-1)\varphi_j)(z) &= \varphi_j(-\bar{z})\\ &= \varepsilon_j \varphi_j(z)\end{aligned}$$

with $\varepsilon_j = \pm 1$. The $t_j(n)$'s appear also in the Fourier expansion of φ_j:

$$\varphi_j(x+iy) = \rho_j(1)\sqrt{y} \sum_{n \neq 0} t_j(n) K_{i\kappa_j}(2\pi|n|y) e(nx),$$

where K_ν is the K-Bessel function of order ν. With the first Fourier coefficient $\rho_j(1)$ we set

$$\alpha_j = |\rho_j(1)|^2 (\cosh \pi\kappa_j)^{-1}.$$

Then Kuznetsov [21] has shown

$$\sum_{\kappa_j \leq K} \alpha_j t_j(n)^2 \ll K^2 + n^{\frac{1}{2}+\varepsilon}, \tag{2.1}$$

where the implied constant depends only on ε. Also we have Serre's bound

$$|t_j(n)| \leq d(n) n^{\frac{1}{5}}; \tag{2.2}$$

for the proof of this important fact see [23][28].

The Hecke L-function $H_j(s)$ attached to φ_j is defined by

$$H_j(s) = \sum_{n=1}^{\infty} t_j(n) n^{-s},$$

which converges absolutely for $\mathrm{Re}(s) > 1$. The multiplicative property of $t_j(n)$ can be expressed in terms of $H_j(s)$: We have, for any ν,

$$\sum_{n=1}^{\infty} \sigma_\nu(n) t_j(n) n^{-s} = \zeta(2s-\nu)^{-1} H_j(s) H_j(s-\nu), \tag{2.3}$$

providing the left side converges absolutely. This is an analogue of Ramanujan's identity

$$\sum_{n=1}^{\infty} \sigma_\nu(n)\sigma_\mu(n) n^{-s} = \zeta(s)\zeta(s-\nu)\zeta(s-\mu)\zeta(s-\nu-\mu)/\zeta(2s-\nu-\mu). \tag{2.4}$$

On the other hand $H_j(s)$ can be continued to an entire function, and it satisfies a functional equation which implies, in particular,

$$H_j(s) \ll \kappa_j^c \tag{2.5}$$

uniformly for bounded s. Here and in the sequel the letter c stands for a constant whose value may differ at each occurrence, and whose dependency on other parameters (e.g., on s in the above) may easily be inferred from the context. Though (2.5) serves for most purpose below, we need also the following statistical result:

$$\sum_{\kappa_j \le K} \alpha_j H_j(\tfrac{1}{2})^4 \ll K^2 (\log K)^{20},$$

whose proof is given in [26].

Next we turn to holomorphic cusp-forms. Thus, let $\{\varphi_{j,k};\ 1 \le j \le \vartheta(k)\}$ be the orthonormal base, consisting of eigen-functions of all Hecke operators $T_k(n)$, of the Petersson unitary space of holomorphic cusp-forms of the even weight $2k$ with respect to the full modular group. This means in particular that we have

$$\begin{aligned}(T_k(n)\varphi_{j,k})(z) &= \frac{1}{\sqrt{n}} \sum_{ad=n} \left(\frac{a}{d}\right)^k \sum_{b=1}^{d} \varphi_{j,k}\left(\frac{az+b}{d}\right) \\ &= t_{j,k}(n)\varphi_{j,k}(z)\end{aligned}$$

with a certain real number $t_{j,k}(n)$. These appear also in the Fourier expansion of $\varphi_{j,k}(z)$:

$$\varphi_{j,k}(z) = \rho_{j,k}(1) \sum_{n=1}^{\infty} t_{j,k}(n) e(nz).$$

We put

$$\alpha_{j,k} = 16\Gamma(2k)(4\pi)^{-2k-1} |\rho_{j,k}(1)|^2.$$

Then corresponding to (2.1) we have

$$\sum_{j=1}^{\vartheta(k)} \alpha_{j,k} \ll k. \tag{2.7}$$

As before we define the Hecke L-series $H_{j,k}(s)$ by

$$H_{j,k}(s)=\sum_{n=1}^{\infty}t_{j,k}(n)n^{-s}.$$

This converges absolutely for $\mathrm{Re}(s)>1$, for we have the famous bound

$$|t_{j,k}(n)|\le d(n) \tag{2.8}$$

due to Deligne [3]. We have also the analogue of (2.3):

$$\sum_{n=1}^{\infty}\sigma_\nu(n)t_{j,k}(n)n^{-s}=\zeta(2s-\nu)^{-1}H_{j,k}(s)H_{j,k}(s-\nu). \tag{2.9}$$

Again $H_{j,k}(s)$ is entire, and for bounded s

$$H_{j,k}(s)\ll k^c \tag{2.10}$$

uniformly in j. Finally, we have the analogue of (2.6):

$$\sum_{k\le K}\sum_{j=1}^{\vartheta(k)}\alpha_{j,k}H_{j,k}(\tfrac12)^4\ll K^2(\log K)^c. \tag{2.11}$$

This follows from a large sieve inequality due to Deshouillers and Iwaniec [6], (1.28) with a little refinement.

We may now state our version of Kuznetsov's formulas for $A_f(0,0;W)$ and $B_N(0,0;W_0)$:

THEOREM 3. – *If W is a C^∞ function with a compact support on the positive real axis, then we have, for any $f\ge 1$,*

$$\begin{aligned}&\sum_{n=1}^{\infty}d(n)d(n+f)W\left(\frac{n}{f}\right)\\&\quad=\frac{6}{\pi^2}\int_0^\infty m(x;f)W(x)dx+\frac{f^{\frac12}}{\pi}\int_{-\infty}^{\infty}\frac{f^{-ir}\sigma_{2ir}(f)|\zeta(\frac12+ir)|^4}{|\zeta(1+2ir)|^2}\Theta(r;W)dr\\&\quad+f^{\frac12}\sum_{j=1}^{\infty}\alpha_jt_j(f)H_j(\tfrac12)^2\Theta(\kappa_j;W)\\&\quad+\frac14 f^{\frac12}\sum_{k=6}^{\infty}\sum_{j=1}^{\vartheta(k)}(-1)^k\alpha_{j,k}t_{j,k}(f)H_{j,k}(\tfrac12)^2\Xi_0(k-\tfrac12;W).\end{aligned} \tag{2.12}$$

Here $m(x;f)$ is defined by (1.12), *and*

$$\Theta(r;W)=\frac12\mathrm{Re}\left\{\left(1+\frac{i}{\sinh(\pi r)}\right)\Xi_0(ir;W)\right\},$$

where

$$\Xi_0(\xi;W) = \frac{\Gamma(\xi+\frac{1}{2})^2}{\Gamma(2\xi+1)} \int_0^\infty x^{-\frac{1}{2}-\xi} F\Big(\xi+\tfrac{1}{2}, \xi+\tfrac{1}{2}; 2\xi+1; -\frac{1}{x}\Big) W(x)dx \tag{2.13}$$

with the hypergeometric function F.

THEOREM 4. – *If W_0 is a C^∞ function with a compact support on the open unit interval, then we have, for any integer $N \geq 2$,*

$$\begin{aligned}
\sum_{n=1}^{N-1} d(n)d(N-n)W_0\Big(\frac{n}{N}\Big) &= \frac{6}{\pi^2}\int_0^1 n(x;N)W(x)dx \\
&+ \frac{N^{\frac{1}{2}}}{\pi}\int_{-\infty}^{\infty} \frac{N^{-ir}\sigma_{2ir}(N)|\zeta(\frac{1}{2}+ir)|^4}{|\zeta(1+2ir)|^2}\Delta(r;W_0)dr \\
&+ N^{\frac{1}{2}}\sum_{j=1}^{\infty}\alpha_j t_j(N)H_j(\tfrac{1}{2})^2\Delta(\kappa_j;W_0) \\
&+ \frac{1}{4}N^{\frac{1}{2}}\sum_{k=6}^{\infty}\sum_{j=1}^{\vartheta(k)}(-1)^k\alpha_{j,k}t_{j,k}(N)H_{j,k}(\tfrac{1}{2})^2\Xi_+(k-\tfrac{1}{2};W_0).
\end{aligned} \tag{2.14}$$

Here $n(x;N)$ is defined to be the result of replacing $\log(1+x)$ *by* $\log(1-x)$ *and f by N in the definition* (1.12) *of $m(x;f)$. Also*

$$\begin{aligned}
&\Delta(r;W_0) \\
&= -\frac{1}{2}\int_0^1 \Big[\Big\{\log x + 2\gamma + 2\mathrm{Re}\frac{\Gamma'}{\Gamma}(\tfrac{1}{2}+ir) - \frac{1}{\cosh(\pi r)}\Big\}F(\tfrac{1}{2}+ir,\tfrac{1}{2}-ir;1;x) \\
&\qquad + \Big\{\frac{\partial}{\partial\alpha}F(\tfrac{1}{2}+\alpha+ir,\tfrac{1}{2}+\alpha-ir;1+2\alpha;x)\Big\}_{\alpha=0}\Big]W_0(x)dx, \\
&\Xi_+(k-\tfrac{1}{2};W_0) \\
&= \int_0^1\Big\{\frac{\partial}{\partial\alpha}\frac{\Gamma(k+\alpha)}{\Gamma(1+2\alpha)\Gamma(k-\alpha)}F(k+\alpha,1-k+\alpha;1+2\alpha;x)x^\alpha\Big\}_{\alpha=0}W_0(x)dx,
\end{aligned} \tag{2.15}$$

where γ is the Euler constant.

The combination of Theorems 3 and 4 is essentially equivalent to the most interesting case of Theorem 3.5 of Kuznetsov [22] (i.e., $s=\nu=\frac{1}{2}$ there). It should be remarked that in applications the requirement of the compactness of the supports of the respective weight functions is by no means restrictive than Kuznetsov's corresponding assumptions, and in fact it can be replaced by a condition more general than his.

Also, the explicit formula (2.12) should be compared with our result on the fourth power mean of $\zeta(s)$ ([25, Theorem]): If $0<\Delta<T(\log T)^{-1}$ then there exist absolute constants

$c(a,b;k,l)$ such that, with an obvious abuse of notation,

$$
\begin{aligned}
(\Delta\sqrt{\pi})^{-1}\int_{-\infty}^{\infty}|\zeta(\tfrac12+i(T+t))|^4e^{-(t/\Delta)^2}dt &= (\Delta\sqrt{\pi})^{-1}\\
\times\int_{-\infty}^{\infty}\mathrm{Re}\Big\{\sum_{\substack{a,b,k,l\ge0\\ ak+bl\le4}}c(a,b;k,l)\Big(\frac{\Gamma^{(a)}}{\Gamma}\Big)^k\Big(\frac{\Gamma^{(b)}}{\Gamma}\Big)^l(\tfrac12+i(T+t))\Big\}e^{-(t/\Delta)^2}dt\\
+\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{|\zeta(\frac12+it)|^6}{|\zeta(1+2ir)|^2}\vartheta(t;T,\Delta)dt+\sum_{j=1}^{\infty}\alpha_jH_j(\tfrac12)^3\vartheta(\kappa_j;T,\Delta)\\
+\sum_{k=6}^{\infty}\sum_{j=1}^{\vartheta(k)}(-1)^k\alpha_{j,k}H_{j,k}(\tfrac12)^3\theta(k-\tfrac12;T,\Delta)+O(T^{-1}(\log T)^2).
\end{aligned}\tag{2.16}
$$

Here

$$
\begin{aligned}
\vartheta(r;T,\Delta)&=\mathrm{Re}\big\{\big(1+\frac{i}{\sinh(\pi r)}\big)\theta(ir;T,\Delta)\big\};\\
\theta(\xi;T,\Delta)&=\frac{\Gamma(\frac12+\xi)^2}{\Gamma(1+2\xi)}\int_0^{\infty}x^{-1-\xi}(1+x)^{-\frac12}\cos\big(T\log\big(1+\frac1x\big)\big)\\
&\quad\times\exp\big(-\big(\frac{\Delta}{2}\log\big(1+\frac1x\big)\big)^2\big)F\big(\tfrac12+\xi,\tfrac12+\xi;1+2\xi;-\frac1x\big)dx.
\end{aligned}
$$

The formula (2.16) has the appearance that it might be considered as a result of averaging (2.12) over the shift parameter f. In fact, in the context of the fourth power mean of $\zeta(s)$ the expression (2.12) corresponds to the contribution of the individual non-diagonal term, and this observation led Zavorotnyi to (1.10). However, according to our experience such an averaging of (2.12) cannot yield a result as explicit as our (2.16); instead we need an approach that is an extension of Atkinson's method [2] on the mean square of $\zeta(s)$.

From Theorems 3 and 4 we shall deduce the following consequences on $D(N;f)$ and $D(N)$, respectively:

THEOREM 5. – *If we have the bound*

$$
|t_j(n)|\le n^{\alpha}\qquad (n\ge1),\tag{2.17}
$$

then we have, uniformly for $1\le f\le N^{2/(1+2\alpha)}$,

$$
\begin{aligned}
D(N;f)&=\frac{6}{\pi^2}\int_0^{N/f}m(x;f)dx\\
&\quad+O\{(N(N+f))^{\frac13+\varepsilon}+f^{\frac18+\frac12\alpha}(N(N+f))^{\frac14+\varepsilon}+f^{\frac12+\alpha}N^{\varepsilon}\}.
\end{aligned}\tag{2.18}
$$

THEOREM 6. – *We have, on* (2.17),

$$
E(N)\ll N^{\frac12+\alpha+\varepsilon}.\tag{2.19}
$$

From these and (2.2) Theorems 1 and 2 follow immediately. To prove (2.18) we shall use (2.12) as it stands. However, to prove (2.19) we shall not use (2.14) but rather an intermediate result which implies (2.14). For, there is a certain difficulty in applying the saddle point method to $\Delta(r;W_0)$; in contrast $\Theta(r;W)$ does not cause such a difficulty.

Returning to the analogy between (2.12) and (2.16) we remark that (2.16) yields not only

$$E_2(T) \ll T^{\frac{2}{3}}(\log T)^c,$$

which is a small improvement upon (1.10), but also the following two important consequences:

$$E_2(T) = \Omega(\sqrt{T}) \tag{2.20}$$

and

$$\int_0^T E_2(t)^2 dt \ll T^2(\log T)^c. \tag{2.21}$$

Proofs of these can be found in our joint papers [13]-[15] with Ivić. Thus one may expect that similar results on $E(N;f)$ would hold. In fact Ivić conjectured an analogue of (2.20) for $E(N;f)$ when f is fixed. Confirming this we shall prove

THEOREM 7. – *For each fixed $f \geq 1$ we have*

$$E(N;f) = \Omega(\sqrt{N}).$$

Also we have obtained, jointly with Ivić, the analogues of (2.21) for $E(N;f)$ and $E(N)$ (thus improving (1.5)); for the details see [15].

We shall prove Theorems 3 and 4 by using Kuznetsov's trace formulas, and as has been remarked above our argument is close to that of our former work [25]. But, it should be stressed that there is an alternative approach (*the inner product argument*) to binary additive divisor problems in general. This was first indicated by Selberg [29], and later considered by Kuznetsov [20] with more details. Then Tahtadjan and Vinogradov [30] carried out full details. They applied Selberg's spectral theory directly to a modification of the Eisenstein series, and obtained a meromorphic continuation of the additive divisor zeta-series

$$\sum_{n=1}^{\infty} d(n)d(n+f)n^{-s},$$

which could yield a proof of (1.8). On this matter see also Deshouillers [4]. Recently Jutila [18] used the argument of [30] to extend his theory [17] of transforming trigonometrical sums involving the divisor and the allied functions. In this context, the argument of [30] seems to have more possibilities than the one developed in the present paper. But, for the original additive divisor problems $D(N;f)$ and $D(N)$ our argument that exploits the inner structure of the divisor function has so far been able to yield results deeper than those obtainable by the method of Tahtadjan and Vinogradov.

3. Spectral decomposition

In this section we shall show spectral decompositions of $A_f(\alpha, \beta; W)$ and $B_N(\alpha, \beta; W_0)$ in the domain

$$R(b) = \{(\alpha,\, \beta);\, 0 > \mathrm{Re}(\alpha) > b,\, 2b - 2 > \mathrm{Re}(\alpha + \beta)\}, \tag{3.1}$$

where b is an arbitrary fixed negative number. Since our interest is in the values of $A_f(\alpha,\, \beta; W)$ and $B_N(\alpha,\, \beta; W_0)$ at the origin $(0, 0)$ which is not in $R(b)$, we shall have to continue analytically these spectral decompositions to a neighbourhood of the origin. That will be carried out in the next section. Thus, in the present section we shall always assume that $(\alpha,\, \beta) \in R(b)$. We shall treat only $A_f(\alpha, \beta; W)$ in great detail, for $B_N(\alpha, \beta; W_0)$ is quite similar as far as the spectral decomposition is concerned.

To begin with we quote, as Kuznetsov did in [22], an identity of Ramanujan: For $\mathrm{Re}(\nu) < 0,\ n \geq 1$, we have

$$\sigma_\nu(n) = \zeta(1 - \nu) \sum_{l=1}^{\infty} l^{\nu-1} c_l(n), \tag{3.2}$$

where

$$c_l(n) = \sum_{\substack{h=1 \\ (h,l)=1}}^{l} e\Big(\frac{h}{l} n\Big). \tag{3.3}$$

We apply (3.2) to the factor $\sigma_\beta(n + f)$ in the sum $A_f(\alpha, \beta; W)$, getting

$$A_f(\alpha, \beta; W) = \zeta(1 - \beta) \sum_{l=1}^{\infty} l^{\beta-1} \sum_{\substack{h=1 \\ (h,l)=1}}^{l} e\Big(\frac{h}{l} f\Big) \sum_{n=1}^{\infty} \sigma_\alpha(n) e\Big(\frac{h}{l} n\Big) W\Big(\frac{n}{f}\Big). \tag{3.4}$$

We then introduce the Mellin transform of W:

$$w(s) = \int_0^\infty W(x) x^{s-1} dx. \tag{3.5}$$

Since $W \in C_c^\infty(0,\, \infty)$, $w(s)$ is entire and of rapid decay in any vertical strip. The latter means that for any fixed $B > 0$ we have

$$w(s) \ll (1 + |s|\,)^{-B}, \tag{3.6}$$

if $\mathrm{Re}(s)$ is bounded. This fact will be used constantly in the sequel. Then we have the inversion formula

$$W(x) = \frac{1}{2\pi i} \int_{(a)} w(s) x^{-s} ds$$

for any $x>0$, where the symbol (a) denotes the straight line $\mathrm{Re}(s)=a$. This transforms (3.4) into

$$(3.7)\quad A_f(\alpha,\beta;W)=\zeta(1-\beta)\sum_{l=1}^{\infty}l^{\beta-1}\sum_{\substack{h=1\\(h,l)=1}}^{l}e\Big(\frac{f}{l}h\Big)\frac{1}{2\pi i}\int_{(a)}f^s w(s)D\big(s,\alpha;e\big(\frac{h}{l}\big)\big)ds,$$

where $a>1$ is to be sufficiently large, and

$$D\big(s,\alpha;e\big(\frac{h}{l}\big)\big)=\sum_{n=1}^{\infty}\sigma_a(n)e\Big(\frac{h}{l}n\Big)n^{-s},\qquad (h,l)=1.$$

We are going to shift the contour in (3.7) to the left. For this sake we quote some facts about $D\big(s,\alpha;e\big(\frac{h}{l}\big)\big)$: For each fixed $\alpha\neq 0$ this is a meromorphic function of s, which has simple poles at $s=1$ and $1+\alpha$ with the residues $\zeta(1-\alpha)l^{\alpha-1}$ and $\zeta(1+\alpha)l^{-\alpha-1}$, respectively, and has no singularities elsewhere. We have also the functional equation

$$(3.8)\quad \begin{aligned}D\big(s,\alpha;e\big(\frac{h}{l}\big)\big)&=2(2\pi)^{2s-2-\alpha}l^{\alpha-2s+1}\Gamma(1-s)\Gamma(1+\alpha-s)\\&\times\Big\{\cos\Big(\frac{\pi\alpha}{2}\Big)D\big(1-s,-\alpha;e\big(\frac{\bar h}{l}\big)\big)-\cos\Big(\pi\Big(s-\frac{\alpha}{2}\Big)\Big)D\big(1-s,-\alpha;e\big(-\frac{\bar h}{l}\big)\big)\Big\},\end{aligned}$$

where $h\bar h\equiv 1 \pmod l$. These facts can be proved easily by expressing $D\big(s,\alpha;e\big(\frac{h}{l}\big)\big)$ in terms of Hurwitz zeta-functions.

We now shift the contour in (3.7) to $\mathrm{Re}(s)=b$, where b is as in (3.1); note that we have (3.6) and that (3.8) implies that $D\big(s,\alpha;e\big(\frac{h}{l}\big)\big)$ is of polynomial order with respect to s if $\mathrm{Re}(s)$ is bounded. We then have

$$(3.9)\qquad A_f(\alpha,\beta;W)=U_1(\alpha,\beta)+A_1(\alpha,\beta),$$

where $U_1(\alpha,\beta)$ is the contribution of residues and $A_1(\alpha,\beta)$ the rest. The above facts about $D\big(s,\alpha;e\big(\frac{h}{l}\big)\big)$ imply that

$$U_1(\alpha,\beta)=\zeta(1-\beta)\sum_{l=1}^{\infty}l^{\beta-1}\big\{\zeta(1-\alpha)l^{\alpha-1}c_l(f)fw(1)+\zeta(1+\alpha)l^{-\alpha-1}c_l(f)f^{1+\alpha}w(1+\alpha)\big\},$$

where $c_l(f)$ is defined by (3.3). Thus by (3.2) we have

$$(3.10)\quad \begin{aligned}U_1(\alpha,\beta)=&f^{\alpha+\beta}\sigma_{1-\alpha-\beta}(f)w(1)\frac{\zeta(1-\alpha)\zeta(1-\beta)}{\zeta(2-\alpha-\beta)}\\&+f^{\beta}\sigma_{1+\alpha-\beta}(f)w(1+\alpha)\frac{\zeta(1+\alpha)\zeta(1-\beta)}{\zeta(2+\alpha-\beta)}.\end{aligned}$$

As for $A_1(\alpha,\beta)$ we use (3.8), and replace $D\big(1-s,-\alpha;\big(\pm\frac{\bar h}{l}\big)\big)$ by their absolutely convergent Dirichlet series; we note that $(\alpha,\beta)\in R(b)$ gives the absolute convergence throughout. Hence, after some exchange of the order of sums and integrals, we obtain

$$(3.11)\qquad A_1(\alpha,\beta)=A_1^{+}(\alpha,\beta)+A_1^{-}(\alpha,\beta).$$

Here

$$(3.12)\quad A_1^{\pm}(\alpha,\beta)=2(2\pi)^{\beta-1}\zeta(1-\beta)f^{\frac{1}{2}(\alpha+\beta+1)}\sum_{n=1}^{\infty}\sigma_{-\alpha}(n)n^{\frac{1}{2}(\alpha+\beta-1)}K_{\pm}(f,n;\alpha,\beta),$$

where

$$(3.13)\qquad K_{\pm}(f,n;\alpha,\beta)=\sum_{l=1}^{\infty}\frac{1}{l}S(\pm f,n;l)\psi_{\pm}\Big(\frac{4\pi\sqrt{fn}}{l};\alpha,\beta\Big)$$

with

$$(3.14)\quad \psi_+(x;\alpha,\beta)=\frac{1}{2\pi i}\cos\Big(\frac{\pi\alpha}{2}\Big)\int_{(b)}\Gamma(1-s)\Gamma(1+\alpha-s)w(s)\Big(\frac{x}{2}\Big)^{2s-\alpha-\beta-1}ds,$$

$$(3.15)\quad \begin{aligned}&\psi_-(x;\alpha,\beta)\\&=-\frac{1}{2\pi i}\int_{(b)}\Gamma(1-s)\Gamma(1+\alpha-s)\cos\Big(\pi\Big(s-\frac{\alpha}{2}\Big)\Big)w(s)\Big(\frac{x}{2}\Big)^{2s-\alpha-\beta-1}ds,\end{aligned}$$

To $K_{\pm}(f,n;\alpha,\beta)$ we apply the following versions of Kuznetsov's trace formulas:

LEMMA 1. – *Let $\varphi\in C^3(0,\infty)$ satisfy the conditions, for $\nu=0,1,2,3,$*

$$(3.16)\qquad \begin{cases}\varphi^{(\nu)}(x)\ll x^{\frac{1}{2}+\eta-\nu} \text{ as } x\to+0,\\ \varphi^{(\nu)}(x)\ll x^{-1-\eta-\nu} \text{ as } x\to+\infty,\end{cases}$$

where η is an arbitrary small positive constant. Then we have, for any $m,n\ge1$,

$$\begin{aligned}\sum_{l=1}^{\infty}\frac{1}{l}S(m,n;l)\varphi\Big(\frac{4\pi\sqrt{mn}}{l}\Big)&=\sum_{j=1}^{\infty}\alpha_j t_j(m)t_j(n)\hat\varphi(\kappa_j)\\&+\sum_{k=6}^{\infty}\sum_{j=1}^{\vartheta(k)}\alpha_{j,k}t_{j,k}(m)t_{j,k}(n)\hat\varphi\big(i(\tfrac{1}{2}-k)\big)+\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sigma_{2ir}(m)\sigma_{2ir}(n)}{(mn)^{ir}|\zeta(1+2ir)|^2}\hat\varphi(r)dr,\end{aligned}$$

where

$$\hat\varphi(r)=\frac{\pi i}{2\sinh(\pi r)}\int_0^{\infty}(J_{2ir}(x)-J_{-2ir}(x))\frac{\varphi(x)}{x}dx$$

with $J_{2ir}(x)$ being the J-Bessel function of order $2ir$.

LEMMA 2. – *Let $\varphi\in C^3(0,\infty)$ satisfy the conditions, for $\nu=0,1,2,3,$*

$$(3.17)\qquad \begin{cases}\varphi^{(\nu)}(x)\ll x^{1+\eta-\nu} \text{ as } x\to+0,\\ \varphi^{(\nu)}(x)\ll x^{-1-\eta-\nu} \text{ as } x\to+\infty,\end{cases}$$

where η is as above. Then we have, for any m, $n \geq 1$,

$$\sum_{l=1}^{\infty} \frac{1}{l} S(-m, n\,; l) \varphi(\frac{4\pi\sqrt{mn}}{l})$$
$$= \sum_{j=1}^{\infty} \varepsilon_j \alpha_j t_j(m) t_j(n) \check{\varphi}(\kappa_j) + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sigma_{2ir}(m)\sigma_{2ir}(n)}{(mn)^{ir}|\zeta(1+2ir)|^2} \check{\varphi}(r) dr,$$

where

$$\check{\varphi}(r) = 2\cosh(\pi r) \int_0^{\infty} K_{2ir}(x) \varphi(x) \frac{dx}{x}$$

with $K_{2ir}(x)$ being the K-Bessel function of order $2ir$.

For a detailed proof of these see our manuscript [27].

We apply Lemmas 1 and 2 to $K_{\pm}(f, n\,; \alpha, \beta)$, respectively. We consider first $K_+(f, n\,; \alpha, \beta)$. Thus we have to see if (3.16) is satisfied by $\psi_+(x; \alpha, \beta)$ when $(\alpha, \beta) \in R(b)$. Because of (3.6) we have obviously $\psi_+(x; \alpha, \beta) \in C^3(0, \infty)$. On the other hand $(\alpha, \beta) \in R(b)$ implies $\mathrm{Re}(2s - \alpha - \beta - 1) > 1$ in the integrand of (3.14), which means that the first condition in (3.16) is also satisfied by $\psi_+(x; \alpha, \beta)$. Further, we may move the contour in (3.14) to the left as much as we want without encountering any singularities, and hence the second condition in (3.16) is satisfied by $\psi_+(x; \alpha, \beta)$. Thus we can apply Lemma 1 to $K_+(f, n; \alpha, \beta)$ when $(\alpha, \beta) \in R(b)$. We have

$$\begin{aligned} K_+(f, n\,; \alpha, \beta) &= \sum_{j=1}^{\infty} \alpha_j t_j(f) t_j(n) \hat{\psi}_+(\kappa_j; \alpha, \beta) \\ &\quad + \sum_{k=6}^{\infty} \sum_{j=1}^{\vartheta(k)} \alpha_{j,k} t_{j,k}(f) t_{j,k}(n) \hat{\psi}_+\big(i(\tfrac{1}{2} - k); \alpha, \beta\big) \\ &\quad + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sigma_{2ir}(f)\sigma_{2ir}(n)}{(fn)^{ir}|\zeta(1+2ir)|^2} \hat{\psi}_+(r; \alpha, \beta) dr, \end{aligned} \tag{3.18}$$

where

$$\hat{\psi}_+(r; \alpha, \beta) = \frac{\pi i}{2\sinh(\pi r)} \int_0^{\infty} (J_{2ir}(x) - J_{-2ir}(x)) \psi_+(x; \alpha, \beta) \frac{dx}{x}. \tag{3.19}$$

To transform this integral we consider, in view of (3.14),

$$\int_0^{\infty} J_{2ir}(x) \int_{(b)} \Gamma(1-s)\Gamma(1+\alpha-s) w(s) \big(\frac{x}{2}\big)^{2s-\alpha-\beta-2} ds dx. \tag{3.20}$$

We are going to exchange the order of integrals. For this sake let us assume temporarily that besides (3.1)

$$2 < 2b - \mathrm{Re}(\alpha + \beta) < \tfrac{5}{2}. \tag{3.21}$$

Then we move the contour of the last integral to $\mathrm{Re}(s) = b - \frac{1}{2}$. The resulting double integral is absolutely convergent, for $J_{2ir}(x) \ll 1$ as $x \to +0$ and $J_{2ir}(x) \ll x^{-\frac{1}{2}}$ as $x \to +\infty$ if r is real and fixed. Hence (3.20) is equal to

$$\int_{(b-\frac{1}{2})} \Gamma(1-s)\Gamma(1+\alpha-s)w(s) \int_0^\infty J_{2ir}(x)\left(\frac{x}{2}\right)^{2s-\alpha-\beta-2} dx ds$$
$$= \int_{(b-\frac{1}{2})} \frac{\Gamma\left(s-\frac{1}{2}(\alpha+\beta+1)+ir\right)}{\Gamma\left(\frac{1}{2}(3+\alpha+\beta)-s+ir\right)} \Gamma(1-s)\Gamma(1+\alpha-s)w(s)ds,$$

which is regular in $R(b)$. But the original double integral (3.20) represents also a regular function in $R(b)$. This can be seen by dividing the range of the variable x into two parts according to $0 \le x < 1$ and $x \ge 1$, say. Obviously the first part yields a regular function in $R(b)$. For those x in the second part we move the contour in the s-integral to the far left, getting an integral function of α and β. Thus we may drop (3.21), and see that (3.20) is equal to the last integral for any $(\alpha, \beta) \in R(b)$. Hence, after some rearrangement, we have

$$\begin{aligned}(3.22)\qquad &\hat{\psi}_+(r;\alpha,\beta)\\ &= \frac{1}{4\pi i}\cos\left(\frac{\pi\alpha}{2}\right)\int_{(b)} \sin\left(\pi\left(s-\tfrac{1}{2}(\alpha+\beta)\right)\right)\Gamma\left(s-\tfrac{1}{2}(\alpha+\beta+1)+ir\right)\\ &\quad \times \Gamma\left(s-\tfrac{1}{2}(\alpha+\beta+1)-ir\right)\Gamma(1-s)\Gamma(1+\alpha-s)w(s)ds\end{aligned}$$

for real r and $(\alpha, \beta) \in R(b)$.

As for $\hat{\psi}_+\left(i\left(\frac{1}{2}-k\right);\alpha,\beta\right)$ we note that $J_{1-2k}(x) = -J_{2k-1}(x)$ for any integer $k \ge 1$. Then we can show as above that for any integer $k \ge 1$ and $(\alpha, \beta) \in R(b)$

$$\begin{aligned}(3.23)\qquad \hat{\psi}_+\left(i\left(\tfrac{1}{2}-k\right);\alpha,\beta\right) &= \frac{(-1)^k}{4i}\cos\left(\frac{\pi\alpha}{2}\right)\\ &\times \int_{(b)} \frac{\Gamma\left(s+k-1-\frac{1}{2}(\alpha+\beta)\right)}{\Gamma\left(k+1-s+\frac{1}{2}(\alpha+\beta)\right)}\Gamma(1-s)\Gamma(1+\alpha-s)w(s)ds.\end{aligned}$$

Now we introduce, for the sake of a later purpose, a function of three complex variables:

$$\begin{aligned}(3.24)\qquad \Psi_+(\xi;u,v) = \frac{1}{4\pi i}&\cos\left(\frac{\pi u}{2}\right)\int_{-\infty i}^{\infty i} \sin\left(\pi\left(s-\tfrac{1}{2}(u+v)\right)\right)\\ &\times \Gamma\left(s-\tfrac{1}{2}(u+v+1)+\xi\right)\Gamma\left(s-\tfrac{1}{2}(u+v+1)-\xi\right)\Gamma(1-s)\\ &\times \Gamma(1+u-s)w(s)ds.\end{aligned}$$

Here the path is such that the poles of the first two gamma-factors in the integrand and those of the other two gamma-factors are separated to the left and the right, respectively, by the path, and ξ, u, v are assumed to be such that the path can be drawn. When $\mathrm{Re}(\xi) = 0$ and $(u, v) \in R(b)$ we can take $\mathrm{Re}(s) = b$ as the path in (3.24). Thus we have

$$(3.25)\qquad \hat{\psi}_+(r;\alpha,\beta) = \Psi_+(ir;\alpha,\beta)$$

instead of (3.22).

We define another function of three complex variables by

$$(3.26)\quad \Xi_0(\xi;u,v)=\frac{1}{2\pi i}\int_{-\infty i}^{\infty i}\frac{\Gamma\big(\xi-\frac{1}{2}(u+v+1)+s\big)}{\Gamma\big(\xi+\frac{1}{2}(u+v+3)-s\big)}\Gamma(1-s)\Gamma(1+u-s)w(s)ds,$$

where the path separates the poles of $\Gamma\big(\xi-\frac{1}{2}(u+v+1)+s\big)$ and those of $\Gamma(1-s)\Gamma(1+u-s)$ to the left and the right of the path, respectively. If ξ, u, v are such that $\Psi_+(\xi;u,v)$ is well-defined, then we have

$$(3.27)\qquad \Psi_+(\xi;u,v)=-\frac{\pi\cos\big(\frac{1}{2}\pi u\big)}{4\sin(\pi\xi)}\big\{\Xi_0(\xi;u,v)-\Xi_0(-\xi;u,v)\big\}.$$

In fact, for such ξ, u, v we may use the path of (3.24) in the defining integrals of $\Xi_0(\pm\xi;u,v)$ too, and the rest of the proof of (3.27) is a simple application of the relation $\Gamma(s)\Gamma(1-s)=\pi/\sin(\pi s)$. We should note also that (3.23) can be written as

$$(3.28)\qquad \hat{\psi}_+\big(i(\tfrac{1}{2}-k);\alpha,\beta\big)=(-1)^k\frac{\pi}{2}\cos\big(\frac{\pi\alpha}{2}\big)\Xi_0\big(k-\tfrac{1}{2};\alpha,\beta\big)$$

for any integer $k\geq 1$ and $(\alpha,\beta)\in R(b)$.

Next we consider $K_-(f,n;\alpha,\beta)$ briefly. As in the case of $\psi_+(x;\alpha,\beta)$ we see easily that $\psi_-(x;\alpha,\beta)$ satisfies (3.17). Hence Lemma 2 gives

$$(3.29)\qquad \begin{aligned} K_-(f,n;\alpha,\beta)=&\sum_{j=1}^{\infty}\varepsilon_j\alpha_j t_j(f)t_j(n)\check{\psi}_-(\kappa_j;\alpha,\beta)\\ &+\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sigma_{2ir}(f)\sigma_{2ir}(n)}{(fn)^{ir}|\zeta(1+2ir)|^2}\check{\psi}_-(r;\alpha,\beta)dr,\end{aligned}$$

where

$$\check{\psi}_-(r;\alpha,\beta)=2\cosh(\pi r)\int_0^{\infty}K_{2ir}(x)\psi_-(r;\alpha,\beta)\frac{dx}{x}$$

Inserting (3.15) into the last integral we get an absolutely convergent double integral, since we have (3.6) and $K_{2ir}(x)\ll|\log x|$ as $x\to+0$ and $K_{2ir}\ll e^{-x}$ as $x\to\infty$ providing r is real. Thus, exchanging the order of integrals, we have

$$\begin{aligned}\check{\psi}_-(r;\alpha,\beta)=&-\frac{1}{4\pi i}\cosh(\pi r)\int_{(b)}\cos\big(\pi\big(s-\tfrac{1}{2}\alpha\big)\big)\Gamma(1-s)\Gamma(1+\alpha-s)w(s)\\ &\times\int_0^{\infty}K_{2ir}(x)\big(\frac{x}{2}\big)^{2s-\alpha-\beta-2}dxds.\end{aligned}$$

The inner integral is equal to $\Gamma\big(s-\frac{1}{2}(\alpha+\beta+1)+ir\big)\Gamma\big(s-\frac{1}{2}(\alpha+\beta+1)-ir\big)$. Hence, as in (3.25) we obtain, for real r and $(\alpha,\beta)\in R(b)$,

$$(3.30)\qquad \check{\psi}_-(r;\alpha,\beta)=\Psi_-(ir;\alpha,\beta),$$

where

$$(3.31)\qquad \begin{aligned}\Psi_-(\xi;u,v)=&-\frac{1}{4\pi i}\cos(\pi\xi)\int_{-\infty i}^{\infty i}\cos\big(\pi\big(s-\tfrac{1}{2}u\big)\big)\Gamma\big(s-\tfrac{1}{2}(u+v+1)+\xi\big)\\ &\times\Gamma\big(s-\tfrac{1}{2}(u+v+1)-\xi\big)\Gamma(1-s)\Gamma(1+u-s)w(s)ds;\end{aligned}$$

the path is as in (3.24). We have also the following counterpart of (3.27):

$$\Psi_-(\xi;u,v) = \frac{\pi}{4\sin(\pi\xi)} \times \left\{ \sin\left(\pi\left(\xi - \tfrac{1}{2}\nu\right)\right)\Xi_0(\xi;u,v) + \sin\left(\pi\left(\xi + \tfrac{1}{2}\nu\right)\right)\Xi_0(-\xi;u,v)\right\}, \tag{3.32}$$

providing ξ, u, v are such that $\Psi_-(\xi;u,v)$ is well-defined.

Before inserting (3.18) and (3.29) into (3.12) we make an important observation that $\Xi_0(\xi;u,v)$ is of rapid decay when $(u,v) \in R(b)$ is bounded and ξ tends to infinity on the imaginary axis. To show this we note that in (3.26) we may use the path which is the result of connecting the points $b-\infty i$, $b-\frac{1}{2}|\xi|i$, $-c-\frac{1}{2}|\xi|i$, $-c+\frac{1}{2}|\xi|i$, $b+\frac{1}{2}|\xi|i$, $b+\infty i$ with straight lines, where $c>0$ is to be chosen sufficiently large. Then (3.6) and Stirling's formula give the desired result. Similarly we can show that $\Xi_0\left(k-\frac{1}{2};u,v\right)$ is of rapid decay when $(u,v) \in R(b)$ and the integer k tends to $+\infty$. Then the relations (3.25), (3.27), (3.28), (3.30), (3.31) and (3.32) imply that $\hat{\psi}_+(r;\alpha,\beta)$, $\check{\psi}_-(r;\alpha,\beta)$, $\hat{\psi}_+\left(i\left(\frac{1}{2}-k\right);\alpha,\beta\right)$ are of rapid decay as functions of real r and integral $k \geq 1$, providing $(\alpha,\beta) \in R(b)$ is bounded. Having this we insert (3.18) and (3.29) into (3.12), getting multiple sums which are absolutely and uniformly convergent for all bounded $(\alpha,\beta) \in R(b)$; note that we need here (2.1), (2.7) and (2.8) (naturally (2.8) can be replaced by some statistical result like (2.1)). Hence, after some exchanges of the order of summation we find that (3.9) with (3.11) can be replaced by

$$A_f(\alpha,\beta;W) = U_1(\alpha,\beta) + A_c(\alpha,\beta) + A_d(\alpha,\beta) + A_h(\alpha,\beta). \tag{3.33}$$

Here, with an obvious abuse of notation,

$$A_d(\alpha,\beta) = 2(2\pi)^{\beta-1} f^{\frac{1}{2}(\alpha+\beta+1)} \sum_{j=1}^{\infty} \alpha_j t_j(f) \times H_j\left(\tfrac{1}{2}(1-\alpha-\beta)\right) H_j\left(\tfrac{1}{2}(1+\alpha-\beta)\right)(\Psi_+ + \varepsilon_j \Psi_-)(i\kappa_j;\alpha,\beta), \tag*{(3.34)$_d$}$$

$$A_h(\alpha,\beta) = \frac{1}{4}(2\pi)^{\beta} f^{\frac{1}{2}(\alpha+\beta+1)} \cos\left(\frac{\pi\alpha}{2}\right) \sum_{k=6}^{\infty}\sum_{j=1}^{\vartheta(k)} (-1)^k \alpha_{j,k} t_{j,k}(f) \times H_{j,k}\left(\tfrac{1}{2}(1-\alpha-\beta)\right) H_{j,k}\left(\tfrac{1}{2}(1+\alpha-\beta)\right)\Xi_0\left(k-\tfrac{1}{2};\alpha,\beta\right), \tag*{(3.34)$_h$}$$

$$A_c(\alpha,\beta) = -4i(2\pi)^{\beta-2} f^{\frac{1}{2}(\alpha+\beta+1)} \times \int_{(0)} \frac{f^{-\xi}\sigma_{2\xi}(f) Z(\xi;\alpha,\beta)}{\zeta(1+2\xi)\zeta(1-2\xi)} (\Psi_+ + \Psi_-)(\xi;\alpha,\beta)\,d\xi, \tag*{(3.34)$_c$}$$

where

$$Z(\xi;\alpha,\beta) = \zeta\left(\tfrac{1}{2}(1-\alpha-\beta)+\xi\right)\zeta\left(\tfrac{1}{2}(1-\alpha-\beta)-\xi\right) \times \zeta\left(\tfrac{1}{2}(1+\alpha-\beta)+\xi\right)\zeta\left(\tfrac{1}{2}(1+\alpha-\beta)-\xi\right). \tag{3.35}$$

In the above we have used (2.3), (2.4), (2.9) as well as (3.25), (3.28), (3.30). This ends our discussion of $A_f(\alpha,\beta;W)$ when $(\alpha,\beta) \in R(b)$.

We now turn to $B_N(\alpha, \beta; W_0)$, but we shall be brief. We first introduce the Mellin transform

$$w_0(s) = \int_0^1 W_0(x) x^{s-1} dx. \tag{3.36}$$

Since the integral is over the unit interval, we see that for any $B > 0$

$$w_0(s) \ll (1+|s|)^{-B} \tag{3.37}$$

whenever $\mathrm{Re}(s) > -B$; note that this is much stronger than (3.6). We then follow the argument leading to (3.9)-(3.11), and get, in $R(b)$,

$$B_N(\alpha, \beta; W_0) = V_1(\alpha, \beta) + B_1^+(\alpha, \beta) + B_1^-(\alpha, \beta). \tag{3.38}$$

Here

$$\begin{aligned} V_1(\alpha, \beta) =& N^{\alpha+\beta}\sigma_{1-\alpha-\beta}(N) w_0(1) \frac{\zeta(1-\alpha)\zeta(1-\beta)}{\zeta(2-\alpha-\beta)} \\ &+ N^{\beta}\sigma_{1+\alpha-\beta}(f) w_0(1+\alpha) \frac{\zeta(1+\alpha)\zeta(1-\beta)}{\zeta(2+\alpha-\beta)} \end{aligned} \tag{3.39}$$

and

$$B_1^{\pm}(\alpha, \beta) = 2(2\pi)^{\beta-1}\zeta(1-\beta) N^{\frac12(\alpha+\beta+1)} \sum_{n=1}^{\infty} \sigma_{-\alpha}(n) n^{\frac12(\alpha+\beta-1)} L_{\pm}(N, n; \alpha, \beta), \tag{3.40}$$

where

$$L_{\pm}(N, n; \alpha, \beta) = \sum_{l=1}^{\infty} \frac{1}{l} S(\pm N, n; l) \varphi_{\pm}\Big(\frac{4\pi\sqrt{Nn}}{l}; \alpha, \beta\Big); \tag{3.41}$$

$$\begin{aligned} &\varphi_+(x; \alpha, \beta) \\ &= -\frac{1}{2\pi i}\int_{(b)} \Gamma(1-s)\Gamma(1+\alpha-s)\cos\big(\pi(s-\tfrac12\alpha)\big) w_0(s) \big(\frac{x}{2}\big)^{2s-\alpha-\beta-1} ds, \end{aligned} \tag{3.42}$$

$$\varphi_-(x; \alpha, \beta) = \frac{1}{2\pi i}\cos\big(\frac{\pi\alpha}{2}\big)\int_{(b)} \Gamma(1-s)\Gamma(1+\alpha-s) w_0(s) \big(\frac{x}{2}\big)^{2s-\alpha-\beta-1} ds. \tag{3.43}$$

We apply Lemmas 1 and 2 to $L_{\pm}$, respectively. We have

$$\begin{aligned} L_+(N, n; \alpha, \beta) =& \sum_{j=1}^{\infty} \alpha_j t_j(N) t_j(n) \hat{\varphi}_+(\kappa_j; \alpha, \beta) \\ &+ \sum_{k=6}^{\infty}\sum_{j=1}^{\vartheta(k)} \alpha_{j,k} t_{j,k}(N) t_{j,k}(n) \hat{\varphi}_+\big(i(\tfrac12 - k); \alpha, \beta\big) \\ &+ \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sigma_{2ir}(N)\sigma_{2ir}(n)}{(Nn)^{ir}|\zeta(1+2ir)|^2} \hat{\varphi}_+(r; \alpha, \beta) dr. \end{aligned} \tag{3.44}$$

In this we have, for real r and $(\alpha, \beta) \in R(b)$,

$$\hat{\varphi}_+(r;\alpha,\beta) = \Phi_+(ir;\alpha,\beta), \tag{3.45}$$

where

$$\begin{aligned}\Phi_+(\xi;u,v) = &-\frac{1}{4\pi i}\int_{-\infty i}^{\infty i}\cos\big(\pi\big(s-\tfrac12 u\big)\big)\sin\big(\pi\big(s-\tfrac12(u+v)\big)\big)\\ &\times\Gamma\big(s-\tfrac12(u+v+1)+\xi\big)\Gamma\big(s-\tfrac12(u+v+1)-\xi\big)\\ &\times\Gamma(1-s)\Gamma(1+u-s)w_0(s)ds\end{aligned} \tag{3.46}$$

with the same path as in (3.24). The analogue of (3.27) for $\Phi_+(\xi;u,v)$ is

$$\Phi_+(\xi;u,v) = \frac{\pi}{4\sin(\pi\xi)}\big\{\Xi_+(\xi;u,v)-\Xi_+(-\xi;u,v)\big\}. \tag{3.47}$$

where

$$\begin{aligned}\Xi_+(\xi;u,v) = &\frac{1}{2\pi i}\int_{-\infty i}^{\infty i}\frac{\Gamma\big(\xi-\frac12(u+v+1)+s\big)}{\Gamma\big(\xi+\frac12(u+v+3)-s\big)}\\ &\times\Gamma(1-s)\Gamma(1+u-s)\cos\big(\pi\big(s-\tfrac12 u\big)\big)w_0(s)ds\end{aligned} \tag{3.48}$$

with the same contour as in (3.26). Also, corresponding to (3.28), we have

$$\hat{\varphi}_+\big(i\big(\tfrac12-k\big);\alpha,\beta\big) = (-1)^k\frac{\pi}{2}\Xi_+\big(k-\tfrac12;\alpha,\beta\big). \tag{3.49}$$

On the other hand we have

$$\begin{aligned}L_-(N,n;\alpha,\beta) = &\sum_{j=1}^{\infty}\varepsilon_j\alpha_j t_N(f)t_j(n)\check{\varphi}_-(\kappa_j;\alpha,\beta)\\ &+\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sigma_{2ir}(N)\sigma_{2ir}(n)}{(Nn)^{ir}|\zeta(1+2ir)|^2}\check{\varphi}_-(r;\alpha,\beta)dr,\end{aligned} \tag{3.50}$$

In this we have

$$\check{\varphi}_-(r;\alpha,\beta) = \Phi_-(ir;\alpha,\beta), \tag{3.51}$$

where

$$\begin{aligned}\Phi_-(\xi;u,v) = &\frac{1}{4\pi i}\cos(\pi\xi)\cos\big(\frac{\pi u}{2}\big)\int_{-\infty i}^{\infty i}\Gamma\big(s-\tfrac12(u+v+1)+\xi\big)\\ &\times\Gamma\big(s-\tfrac12(u+v+1)-\xi\big)\Gamma(1-s)\Gamma(1+u-s)w_0(s)ds\end{aligned} \tag{3.52}$$

with the same path as in (3.34). Then, on the condition

$$\sin\frac{\pi}{2}(u+v)\sin\frac{\pi}{2}(u-v) \neq 0, \tag{3.53}$$

we have the following analogue of (3.32):

$$\Phi_-(\xi;u,v) = -\frac{\pi\cot(\pi\xi)\cos\left(\frac{1}{2}\pi u\right)}{4\sin\frac{1}{2}\pi(u+v)\sin\frac{1}{2}\pi(u-v)}\left\{\Xi_-(\xi;u,v)-\Xi_-(-\xi;u,v)\right\}. \tag{3.54}$$

where

$$\begin{aligned}\Xi_-(\xi;u,v) =& \frac{1}{2\pi i}\int_{-\infty i}^{\infty i}\frac{\Gamma\left(\xi-\frac{1}{2}(u+v+1)+s\right)}{\Gamma\left(\xi+\frac{1}{2}(u+v+3)-s\right)}\\ &\times\Gamma(1-s)\Gamma(1+u-s)\sin\left(\pi\left(s+\tfrac{1}{2}(v-u)\right)\right)w_0(s)ds\end{aligned} \tag{3.55}$$

with the same contour as in (3.26).

The relation (3.54) needs a proof. For this sake we note that if ξ, u, v are such that $\Phi_-(\xi;u,v)$ is well-defined, then we have

$$\begin{aligned}\int_{-\infty i}^{\infty i}&\sin(\pi s)\sin(\pi(s-u))\Gamma\left(s-\tfrac{1}{2}(u+v+1)+\xi\right)\\ &\times\Gamma\left(s-\tfrac{1}{2}(u+v+1)-\xi\right)\Gamma(1-s)\Gamma(1+u-s)w_0(s)ds=0,\end{aligned} \tag{3.56}$$

where the path is as in (3.24). In fact this integrand is regular on the right of the contour. Thus by virtue of (3.37) we get (3.56) after shifting the path to $\mathrm{Re}(s)=+\infty$. But we have

$$\sin(\pi s)\sin(\pi(s-u))+\sin\frac{\pi}{2}(u+v)\sin\frac{\pi}{2}(u-v) = \sin\pi\left(s+\tfrac{1}{2}(v-u)\right)\sin\pi\left(s-\tfrac{1}{2}(u+v)\right).$$

Hence, if (3.53) holds, then the integral in (3.52) is equal to

$$\begin{aligned}&\frac{1}{\sin\frac{1}{2}\pi(u+v)\sin\frac{1}{2}\pi(u-v)}\int_{-\infty i}^{\infty i}\sin\pi\left(s+\tfrac{1}{2}(v-u)\right)\sin\pi\left(s-\tfrac{1}{2}(u+v)\right)\\ &\qquad\times\Gamma\left(s-\tfrac{1}{2}(u+v+1)+\xi\right)\Gamma\left(s-\tfrac{1}{2}(u+v+1)-\xi\right)\Gamma(1-s)\Gamma(1+u-s)w_0(s)ds\\ &\quad=\frac{\pi^2 i}{\sin(\pi\xi)\sin\frac{1}{2}\pi(u+v)\sin\frac{1}{2}\pi(u-v)}\left\{\Xi_-(-\xi;u,v)-\Xi_-(\xi;u,v)\right\},\end{aligned}$$

whence we have (3.54).

Next we observe that for any bounded $(\alpha,\beta)\in R(b)$ the functions $\hat{\varphi}_+(r;\alpha,\beta)$ and $\check{\varphi}_-(r;\alpha,\beta)$ are of rapid decay when r tends to $\pm\infty$ on the real axis. This can be proved by using in (3.45) and (3.51) (or equivalently in (3.46) and (3.52)) the path that we have used in proving the rapid decay of $\Xi_0(\xi;u,v)$. (Here it should be stressed that $\Xi_\pm(\xi;u,v)$ are not of rapid decay when ξ tends to infinity on the imaginary axis.) On the other hand, to prove the rapid decay of $\hat{\varphi}_+\left(i\left(\frac{1}{2}-k\right);\alpha,\beta\right)$ we use the relation (3.49); we need only to shift the path to $\mathrm{Re}(s)=-c$ with a large $c>0$.

Having these we insert (3.44) and (3.50) into (3.40). Then we get the following analogue of (3.33)-(3.34): For any $(\alpha,\beta)\in R(b)$

$$B_N(\alpha,\beta;W_0) = V_1(\alpha,\beta)+B_c^+(\alpha,\beta)+B_c^-(\alpha,\beta)+B_d(\alpha,\beta)+B_h(\alpha,\beta), \tag{3.57}$$

where V_1 is as in (3.39), and

$$(3.58)_d \quad \begin{aligned} B_d(\alpha, \beta) &= 2(2\pi)^{\beta-1} N^{\frac{1}{2}(\alpha+\beta+1)} \\ &\times \sum_{j=1}^{\infty} \alpha_j t_j(N) H_j\big(\tfrac{1}{2}(1-\alpha-\beta)\big) H_j\big(\tfrac{1}{2}(1+\alpha-\beta)\big)(\Phi_+ + \varepsilon_j \Phi_-)(i\kappa_j; \alpha, \beta), \end{aligned}$$

$$(3.58)_h \quad \begin{aligned} B_h(\alpha, \beta) = \frac{1}{4}(2\pi)^{\beta} N^{\frac{1}{2}(\alpha+\beta+1)} \cos\big(\frac{\pi\alpha}{2}\big) \sum_{k=6}^{\infty} \sum_{j=1}^{\vartheta(k)} (-1)^k \alpha_{j,k} t_{j,k}(N) \\ \times H_{j,k}\big(\tfrac{1}{2}(1-\alpha-\beta)\big) H_{j,k}\big(\tfrac{1}{2}(1+\alpha-\beta)\big) \Xi_+\big(k - \tfrac{1}{2}; \alpha, \beta\big), \end{aligned}$$

$$(3.58)_c \quad B_c^{\pm}(\alpha, \beta) = -4i(2\pi)^{\beta-2} N^{\frac{1}{2}(\alpha+\beta+1)} \int_{(0)} \frac{N^{-\xi}\sigma_{2\xi}(N) Z(\xi; \alpha, \beta)}{\zeta(1+2\xi)\zeta(1-2\xi)} \Phi_{\pm}(\xi; \alpha, \beta) d\xi,$$

where $Z(\xi; \alpha, \beta)$ is defined by (3.35). This ends our discussion on $B_N(\alpha, \beta; W_0)$ when $(\alpha, \beta) \in R(b)$.

Before closing this section we remark that the functions Ξ_0 and $\Xi_{\pm}$ are introduced in order to make clear the process of analytic continuation of $A_c(\alpha, \beta)$ and $B_c^{\pm}(\alpha, \beta)$, which is to be developed in the next section.

4. Analytic continuation

Now we have to continue analytically the spectral decompositions (3.33) and (3.57) to a neighbourhood of the origin, and finish our proof of Theorems 3 and 4.

We deal first with $A_f(\alpha, \beta; W)$. Obviously our problem is equivalent to studying the analytical properties of $\Psi_{\pm}(\xi; \alpha, \beta)$ as functions of three complex variables. To this end we introduce the sets

$$D_{\pm} = \left\{ (\xi, u, v) \in \mathbb{C}^3; \begin{array}{l} \text{neither } \pm\xi - \frac{1}{2}(u+v-1) \text{ nor } \pm\xi + \frac{1}{2}(u-v+1) \\ \text{are equal to non-positive integers} \end{array} \right\}$$

and put

$$D_0 = D_+ \cap D_- .$$

These are domains in $\mathbb{C}^3$; that is, they are open and arcwise connected. The latter can be shown by simply connecting two points of respective sets by a straight line with possible indents. By the definitions (3.24) and (3.31), $\Psi_{\pm}(\xi; \alpha, \beta)$ are well-defined and regular at each point of D_0. Then by a routine argument we can show that they are single valued regular functions over D_0. Namely, starting at a point of D_0 they can be continued analytically to any point of D_0 and the result is always given by their original integral representations with a suitable choice of the contour. As a matter of fact $\Psi_{\pm}(\xi; u, v)$ are meromorphic over the entire $\mathbb{C}^3$, but we shall not use the notion of meromorphy to avoid any ambiguities that may be caused by the complicated nature of the polar sets of these functions.

We then confine (u, v) in an arbitrary fixed bounded set of $\mathbb{C}^2$, and assume that ξ is in a fixed vertical strip of $\mathbb{C}$. Obviously we have $(\xi, u, v) \in D_0$ if $|\mathrm{Im}(\xi)|$ is sufficiently large. On this situation one can show that $\Psi_\pm(\xi; u, v)$ are of rapid decay with respect to ξ uniformly for such (u, v) when ξ tends to infinity. To show this we use, in (3.24) and (3.31), the path that is the result of connecting the points $c - \infty i$, $c - \frac{1}{2}|\xi|i$, $-c - \frac{1}{2}|\xi|i$, $-c + \frac{1}{2}|\xi|i$, $c + \frac{1}{2}|\xi|i$, $c + \infty i$ with straight lines, where $c > 0$ is sufficiently large. Then the result follows from (3.6) and Stirling's formula.

Next we define, for each $\xi \in \mathbb{C}$,

$$Q_0(\xi) = \{ (u, v) \in \mathbb{C}^2;\ (\xi, u, v) \in D_0 \}, \tag{4.1}$$

which is never empty. We put

$$Q_d = \bigcap_{j=1}^{\infty} Q_0(i\kappa_j).$$

This is obviously a domain in $\mathbb{C}^2$. One should observe that $Q_d \supset R(b)$ for any negative b, and $(0, 0) \in D_d$. Then we consider $A_d(\alpha, \beta)$. The summands in it are all regular over Q_d because of (2.1) (with $n = 1$), (2.5) and the rapid decay of $\Psi_\pm$ mentioned above. Hence $A_d(\alpha, \beta)$ exists as a regular function over Q_d.

We next consider $A_h(\alpha, \beta)$. This time we need to know the analytical properties of $\Xi_0(\xi; u, v)$. One may show without difficulty that $\Xi_0(\xi; u, v)$ exists as a regular function over D_+ and that it is of rapid decay with respect to ξ uniformly for all bounded (u, v) when ξ tends to $+\infty$ in any fixed horizontal strip. We then define $Q_+(\xi)$ analogously to (4.1), and

$$Q_h = \bigcap_{k=6}^{\infty} Q_+\big(k - \tfrac{1}{2}\big).$$

This is a domain in $\mathbb{C}^2$, which contains the origin and the set $R(b)$ for any negative b. The sum defining $A_h(\alpha, \beta)$ is uniformly convergent for all bounded $(\alpha, \beta) \in Q_h$ because of (2.7), (2.8) and (2.10) as well as the rapid decay of $\Xi_0\big(k - \frac{1}{2}; u, v\big)$. Hence $A_h(\alpha, \beta)$ exists as a regular function over Q_h.

Thus we have shown that $A_d(\alpha, \beta) + A_h(\alpha, \beta)$ exists as a regular function over the domain $Q_d \cap Q_h$. This implies that $A_c(\alpha, \beta) + U_1(\alpha, \beta)$ is a regular function over $Q_d \cap Q_h$, for all other members in (3.33) are regular there. We shall make this fact more explicit in terms of $A_c(\alpha, \beta)$.

To this end we introduce a large parameter $P > 0$ which is to satisfy the condition

$$\zeta(s) \neq 0 \qquad \text{for} \quad \mathrm{Im}(s) = \pm 2P.$$

Then we divide the range of integration in $(3.34)_c$ into two parts according to $|\xi| > P$ and $|\xi| \le P$, and denote the corresponding parts of $A_c(\alpha, \beta)$ by $A_{c,1}(\alpha, \beta)$ and $A_{c,2}(\alpha, \beta)$ so that

$$A_c(\alpha, \beta) = A_{c,1}(\alpha, \beta) + A_{c,2}(\alpha, \beta), \tag{4.2}$$

where we have $(\alpha,\ \beta) \in R(b)$ of course. We consider $A_{c,1}(\alpha,\ \beta)$ with $(\alpha,\ \beta) \in R(b) \cap T_P$ where

$$T_P = \big\{(u,v);\ |u|, |v| < P/3\big\}.$$

We observe that if $\mathrm{Re}(\xi) = 0$, $|\mathrm{Im}(\xi)| \geq P$, then $Z(\xi;\alpha,\ \beta)$ is regular and $O(|\xi|^c)$ in T_P; where c depends only on P. Hence $A_{c,1}(\alpha,\ \beta)$ is a regular function over T_P. As for $A_{c,2}(\alpha,\ \beta)$ we argue more carefully. We first transform it by using (3.27) and (3.32), and apply the functional equation of $\zeta(s)$ to the factor $\zeta(1-2\xi)^{-1}$. Then we have, for $(\alpha,\ \beta)\ \in\ R(b)$,

$$\begin{aligned}A_{c,2}(\alpha,\ \beta) &= 2i(2\pi)^{\beta-2} f^{\frac{1}{2}(\alpha+\beta+1)} \int_{-Pi}^{Pi} (2\pi)^{2\xi} f^{-\xi} \sigma_{2\xi}(f) \\ &\times \Big\{ \cos\big(\frac{\pi\alpha}{2}\big) - \sin\big(\pi\big(\xi - \frac{\beta}{2}\big)\big)\Big\} Z(\xi;\alpha,\ \beta)\big\{\zeta(1+2\xi)\zeta(2\xi)\big\}^{-1} \Xi_0(\xi;\alpha,\ \beta) d\xi.\end{aligned} \tag{4.3}$$

In this we move the path to L_P which is the result of connecting the points $-Pi$, $[P]+\frac{1}{4}-Pi$, $[P]+\frac{1}{4}+Pi$, Pi by straight lines. Since $(\xi,\alpha,\ \beta) \in D_+$ if $\mathrm{Re}(\xi) \geq 0$ and $(\alpha,\ \beta) \in R(b)$, the singularities of the integrand which we encounter in this procedure come only from $Z(\xi;\alpha,\ \beta)\Gamma(1-2\xi)\zeta(2\xi)^{-1}$, and they are all poles and located at

$$-\tfrac{1}{2}(\alpha+\beta+1), \quad -\tfrac{1}{2}(\beta-\alpha+1), \tag{4.4}$$

and

$$\tfrac{1}{2}\rho \ \ (|\mathrm{Im}\rho| < P), \quad \tfrac{1}{2}n \ \ (2 \leq n \leq [P]), \tag{4.5}$$

where ρ runs over complex zeros of $\zeta(s)$; note our choice of P. To avoid any coincidence of the points in (4.4) with those in (4.5) we assume further that $(\alpha,\ \beta) \in Q_e$ where

$$Q_e = \big\{(u,v);\ \text{neither } u+v \text{ nor } u-v \text{ are equal to integers}\big\}.$$

This is a technical condition, and will soon be eliminated. Then we see that the poles given in (4.4) are simple whenever $(\alpha,\ \beta) \in R(b) \cap T_P \cap Q_e$, and there we have

$$A_{c,2}(\alpha,\ \beta) = U_2(\alpha,\ \beta) + Y(\alpha,\ \beta) + A^*_{c,2}(\alpha,\ \beta). \tag{4.6}$$

Here $U_2(\alpha,\ \beta)$ and $Y(\alpha,\ \beta)$ are the contributions of the residues at the poles given in (4.4) and (4.5), respectively, and $A^*_{c,2}(\alpha,\ \beta)$ has the same expression as the right side of (4.3) but with the contour L_P. In (4.6) the term $A^*_{c,2}(\alpha,\ \beta)$ is regular over T_P. For, $Z(\xi;\alpha,\beta)$ and $\Xi_0(\xi;\alpha,\beta)$ are regular over $L_P \times T_P$; observe that $(\xi,\alpha,\beta) \in D_+$ there. To define a domain where $Y(\alpha,\ \beta)$ is regular we introduce the sets

$$Q_z = \bigcap_\rho \big[Q_+(\rho/2) \cap \big\{(u,v);\ Z(\rho/2;u,v) \neq \infty\big\}\big],$$

$$Q_t = \bigcap_n \big[Q_+(n/2) \cap \big\{(u,v);\ Z(n/2;u,v) \neq \infty\big\}\big],$$

where ρ runs over complex zeros of $\zeta(s)$. Then $Q_z \cap Q_t$ is a domain in $\mathbb{C}^2$, which contains the origin and $R(b) \cap Q_e$. But, the residues at the poles listed in (4.5) are obviously all regular over $Q_z \cap Q_t$, and so is $Y(\alpha, \beta)$. Gathering the above discussion we have the decomposition

$$
\begin{aligned}
A_f(\alpha,\beta;W) =& U_1(\alpha, \beta) + U_2(\alpha, \beta) + A^*_{c,2}(\alpha, \beta) \\
& + A_{c,1}(\alpha, \beta) + Y(\alpha, \beta) + A_d(\alpha, \beta) + A_h(\alpha, \beta)
\end{aligned}
\tag{4.7}
$$

for any $(\alpha, \beta) \in S_P$ where

$$S_P = Q_d \cap Q_h \cap Q_z \cap Q_t \cap T_P.$$

It should be stressed that we have dropped the condition $(\alpha, \beta) \in Q_e$. This is legitimate. For, all members, except for $U_1(\alpha, \beta)$ and $U_2(\alpha, \beta)$, in (4.7) are regular in S_P. Obviously S_P is a domain in $\mathbb{C}^2$ such that $S_P \cap R(b) \neq \emptyset$ and $(0, 0) \in S_P$. Hence (4.7) gives an analytic continuation of (3.33) to a neighbourhood of the origin. Having this we now suppose in the expression (4.7) that α, β are small. We then move the contour L_P in $A^*_{c,2}(\alpha, \beta)$ back to the original segment $[-Pi, Pi]$. This time we encounter only the poles listed in (4.4), and get

$$A^*_{c,2}(\alpha, \beta) = -Y(\alpha, \beta) + A^{**}_{c,2}(\alpha, \beta), \tag{4.8}$$

where $A^{**}_{c,2}(\alpha, \beta)$ has the same expression as the right side of (4.3) but with different (α, β). Obviously $A^{**}_{c,2}(\alpha, \beta)$ is regular when α, β are small. Namely we have, in a neighbourhood of the origin,

$$A_f(\alpha,\beta;W) = U(\alpha, \beta) + \tilde{A}_c(\alpha, \beta) + A_d(\alpha, \beta) + A_h(\alpha, \beta), \tag{4.9}$$

where $U(\alpha, \beta) = U_1(\alpha, \beta) + U_2(\alpha, \beta)$ and $\tilde{A}_c(\alpha, \beta) = A_{c,1}(\alpha, \beta) + A^{**}_{c,2}(\alpha, \beta)$. We note that $\tilde{A}_c(\alpha, \beta)$ is regular at the origin, and has the same representation as $(3.34)_c$ but with different (α, β). The relation (4.9) implies in particular that $U(\alpha, \beta)$ is regular at the origin. This ends the analytic continuation of (3.33) to a neighbourhood of the origin.

Now, setting $(\alpha, \beta) = (0, 0)$ in (4.9) we obtain

$$\sum_{n=1}^{\infty} d(n)d(n+f)W\big(\frac{n}{f}\big) = U(0, 0) + \tilde{A}_c(0, 0) + A_d(0, 0) + A_h(0, 0). \tag{4.10}$$

Let us express $U(0,0)$ explicitly in terms of the weight W. By the definition of $U_2(\alpha, \beta)$ we have, after some rearrangement,

$$
\begin{aligned}
U_2(\alpha, \beta) =& \sigma_{\alpha+\beta+1}(f)\frac{\zeta(1+\alpha)\zeta(1+\beta)}{\Gamma(-\beta)\zeta(2+\alpha+\beta)}\Xi_0\big(-\tfrac{1}{2}(\alpha+\beta+1); \alpha, \beta\big) \\
& + f^{\alpha}\sigma_{-\alpha+\beta+1}(f)\frac{\zeta(1-\alpha)\zeta(1+\beta)}{\Gamma(-\beta)\zeta(2-\alpha+\beta)}\Xi_0\big(-\tfrac{1}{2}(-\alpha+\beta+1); \alpha, \beta\big)
\end{aligned}
\tag{4.11}
$$

at least if $(\alpha, \beta) \in R(b) \cap Q_e$; this we suppose for a while. But we have, for such (α, β),

$$\Xi_0\big(-\tfrac{1}{2}(\alpha+\beta+1); \alpha, \beta\big) = \frac{1}{2\pi i}\int_{-\infty i}^{\infty i} \Gamma(s-\alpha-\beta-1)\Gamma(1+\alpha-s)w(s)ds,$$

$$\Xi_0\big(-\tfrac{1}{2}(-\alpha+\beta+1); \alpha, \beta\big) = \frac{1}{2\pi i}\int_{-\infty i}^{\infty i} \Gamma(s-\beta-1)\Gamma(1-s)w(s)ds,$$

where the path in either integral is, by definition, such that it separates the poles of the first gamma-factor from those of the second of the respective integrand. To these we insert (3.5), and exchange the order of integral. We then get

$$\tag{4.12} \begin{cases} \Xi_0\big(-\tfrac{1}{2}(\alpha+\beta+1);\alpha,\beta\big)=\Gamma(-\beta)\displaystyle\int_0^\infty W(x)x^\alpha(1+x)^\beta dx,\\ \Xi_0\big(-\tfrac{1}{2}(-\alpha+\beta+1);\alpha,\beta\big)=\Gamma(-\beta)\displaystyle\int_0^\infty W(x)(1+x)^\beta dx.\end{cases}$$

This shows that we may now drop the condition $(\alpha,\beta)\in R(b)\cap Q_e$. Thus, from (3.10),(4.11) and (4.12), we find that

$$U(\alpha,\beta)=\int_0^\infty W(x)\mu_f(x;\alpha,\beta)dx,$$

where

$$\begin{aligned}&\mu_f(x;\alpha,\beta)\\ &=\sigma_{1+\alpha+\beta}(f)\frac{\zeta(1+\alpha)\zeta(1+\beta)}{\zeta(2+\alpha+\beta)}x^\alpha(1+x)^\beta+f^\alpha\sigma_{1-\alpha+\beta}(f)\frac{\zeta(1-\alpha)\zeta(1+\beta)}{\zeta(2-\alpha+\beta)}(1+x)^\beta\\ &\quad+f^\beta\sigma_{1+\alpha-\beta}(f)\frac{\zeta(1+\alpha)\zeta(1-\beta)}{\zeta(2+\alpha-\beta)}x^\alpha+f^{\alpha+\beta}\sigma_{1-\alpha-\beta}(f)\frac{\zeta(1-\alpha)\zeta(1-\beta)}{\zeta(2-\alpha-\beta)}.\end{aligned}$$

Then by a standard argument we get

$$\mu_f(x;0,0)=\lim_{(\alpha,\beta)\to(0,0)}\mu_f(x;\alpha,\beta)=\frac{6}{\pi^2}m(x;f),$$

where $m(x;f)$ is defined by (1.12). Hence we have

$$\tag{4.13} U(0,0)=\frac{6}{\pi^2}\int_0^\infty W(x)m(x;f)dx,$$

which is the first term on the right side of (2.12).

On the other hand we have

$$\tag{4.14} \begin{cases}\tilde{A}_c(0,0)=\pi^{-2}f^{\frac12}\displaystyle\int_{-\infty}^\infty f^{-ir}\sigma_{2ir}(f)\frac{|\zeta(\frac12+ir)|^4}{|\zeta(1+2ir)|^2}(\Psi_++\Psi_-)(ir;0,0)dr,\\ A_d(0,0)=\pi^{-1}f^{\frac12}\displaystyle\sum_{j=1}^\infty\alpha_jt_j(f)H_j(\tfrac12)^2(\Psi_++\Psi_-)(i\kappa_j;0,0),\\ A_h(0,0)=\dfrac14 f^{\frac12}\displaystyle\sum_{k=6}^\infty\sum_{j=1}^{\vartheta(k)}(-1)^k\alpha_{j,k}t_{j,k}(f)H_{j,k}(\tfrac12)^2\Xi_0(k-\tfrac12;0,0)\,;\end{cases}$$

on the right side of $A_d(0,0)$ we have used the fact that $H_j(\frac12)=0$ if $\varepsilon_j=-1$ which is a consequence of the functional equation of $H_j(s)$. But, we have, by (3.27) and (3.32),

$$\tag{4.15} (\Psi_++\Psi_-)(ir;0,0)=\frac{\pi}{2}\mathrm{Re}\big\{\big(1+\frac{i}{\sinh(\pi r)}\big)\Xi_0(ir;0,0)\big\},$$

providing r is real. Further by (3.26) we have

$$\Xi_0(\xi;0,0)=\frac{1}{2\pi i}\int_{(a)}\frac{\Gamma(\xi-\frac{1}{2}+s)}{\Gamma(\xi+\frac{3}{2}-s)}\Gamma(1-s)^2w(s)ds$$

if $\mathrm{Re}(\xi)>-\frac{1}{2}$, where $\frac{1}{2}-\mathrm{Re}(\xi)<a<1$. Then by (3.5) we get

$$\Xi_0(\xi;0,0)=\frac{\Gamma(\xi+\frac{1}{2})^2}{\Gamma(2\xi+1)}\int_0^\infty W(x)x^{-\frac{1}{2}-\xi}F\big(\xi+\tfrac{1}{2},\xi+\tfrac{1}{2};2\xi+1;-\frac{1}{x}\big)dx, \tag{4.16}$$

where F is the hypergeometric function. Finally, collecting (4.10) and (4.13)-(4.16) we end the proof of Theorem 3.

We now turn to the proof of Theorem 4, which is somewhat different from that of Theorem 3. The main difference occurs when we try to find an analytic continuation of $B_c^{\pm}(\alpha,\beta)$ to a neighbourhood of the origin; otherwise there is not much difference. Thus, following the above argument on $A_f(\alpha,\beta;W)$ up to the point where we started the discussion of $A_c(\alpha,\beta)$, we can conclude without difficulty that $B_d(\alpha,\beta)$ and $B_h(\alpha,\beta)$ are regular in $Q_d\cap Q_h$, and hence $B_c^+(\alpha,\beta)+B_c^-(\alpha,\beta)+V_1(\alpha,\beta)$ is regular there. As before we have to realize the analytic continuation of $B_c^{\pm}(\alpha,\beta)$ in their own term. But this time we have to treat them separately, since we do not have the analogue of (4.3) for $B_c^+(\alpha,\beta)+B_c^-(\alpha,\beta)$.

We consider $B_c^+(\alpha,\beta)$ first. As in (4.2) we divide it into two parts:

$$B_c^+(\alpha,\beta)=B_{c,1}^+(\alpha,\beta)+B_{c,2}^+(\alpha,\beta). \tag{4.17}$$

The mode of division is just as before, and thus $B_{c,1}^+(\alpha,\beta)$ is regular for $(\alpha,\beta)\in T_P$. As for $B_{c,2}^+(\alpha,\beta)$ we use (3.47) and the functional equation of $\zeta(s)$, getting

$$\begin{aligned}B_{c,2}^+(\alpha,\beta)=&-2i(2\pi)^{\beta-2}N^{\frac{1}{2}(\alpha+\beta+1)}\int_{-Pi}^{Pi}(2\pi)^{2\xi}N^{-\xi}\sigma_{2\xi}(N)Z(\xi;\alpha,\beta)\\&\times\Gamma(1-2\xi)\big\{\zeta(1+2\xi)\zeta(1-2\xi)\big\}^{-1}\Xi_+(\xi;\alpha,\beta)d\xi,\end{aligned} \tag{4.18}$$

providing $(\alpha,\beta)\in R(b)$. Keeping (α,β) in $R(b)\cap T_P$ we move the contour to L_P introduced above. We encounter poles at the points listed in (4.4) and (4.5). Then, on the condition that $(\alpha,\beta)\in R(b)\cap Q_e\cap T_P$, we have

$$B_{c,2}^+(\alpha,\beta)=V_2^+(\alpha,\beta)+Y^+(\alpha,\beta)+B_{c,2}^{+*}(\alpha,\beta), \tag{4.19}$$

where $V_2^+(\alpha,\beta)$ and $Y^+(\alpha,\beta)$ are the contributions of the residues at the poles (4.4) and (4.5), respectively, and $B_{c,2}^{+*}(\alpha,\beta)$ has the same expression as (4.18) but with the contour L_P. The terms $Y^+(\alpha,\beta)$ and $B_{c,2}^{+*}(\alpha,\beta)$ are regular in $Q_z\cap Q_t\cap T_P$. On the other hand we have, in (4.19),

$$\begin{aligned}&V_2^+(\alpha,\beta)\\&=-\frac{\sigma_{\alpha+\beta+1}(N)}{2\cos\frac{1}{2}\pi\beta\cos\frac{1}{2}\pi(\alpha+\beta)}\frac{\zeta(1+\alpha)\zeta(1+\beta)}{\Gamma(-\beta)\zeta(2+\alpha+\beta)}\Xi_+(-\tfrac{1}{2}(\alpha+\beta+1);\alpha,\beta)\\&\quad-\frac{N^{\alpha}\sigma_{\beta-\alpha+1}(N)}{2\cos\frac{1}{2}\pi\beta\cos\frac{1}{2}\pi(\alpha-\beta)}\frac{\zeta(1-\alpha)\zeta(1+\beta)}{\Gamma(-\beta)\zeta(2-\alpha+\beta)}\Xi_+(-\tfrac{1}{2}(\beta-\alpha+1);\alpha,\beta).\end{aligned} \tag{4.20}$$

But we have

$$\Xi_+\big(-\tfrac{1}{2}(\alpha+\beta+1);\alpha,\beta\big) = -\cos\big(\frac{\pi\alpha}{2}\big)\Gamma(-\beta)\int_0^1 W_0(x)x^\alpha(1-x)^\beta dx,$$

$$\Xi_+\big(-\tfrac{1}{2}(\beta-\alpha+1);\alpha,\beta\big) = -\cos\big(\frac{\pi\alpha}{2}\big)\Gamma(-\beta)\int_0^1 W_0(x)(1-x)^\beta dx,$$

which are consequences of (3.36) and (3.48). Thus (4.20) is replaced by

$$\begin{aligned} V_2^+(\alpha,\beta) =& \frac{\cos\big(\frac{1}{2}\pi\alpha\big)\sigma_{\alpha+\beta+1}(N)}{2\cos\frac{1}{2}\pi\beta\cos\frac{1}{2}\pi(\alpha+\beta)}\frac{\zeta(1+\alpha)\zeta(1+\beta)}{\zeta(2+\alpha+\beta)}\int_0^1 W_0(x)x^\alpha(1-x)^\beta dx \\ &+\frac{\cos\big(\frac{1}{2}\pi\alpha\big)N^\alpha\sigma_{\beta-\alpha+1}(N)}{2\cos\frac{1}{2}\pi\beta\cos\frac{1}{2}\pi(\alpha-\beta)}\frac{\zeta(1-\alpha)\zeta(1+\beta)}{\zeta(2-\alpha+\beta)}\int_0^1 W_0(x)(1-x)^\beta dx, \end{aligned}$$

which is obviously meromorphic in each variable α and β. This means that $B_{c,2}^+(\alpha,\beta)$ exists as a regular function over the domain

$$Q_z\cap Q_t\cap T_P\cap\big\{(u,v);V_2^+(u,v)\neq\infty\big\},$$

and there we have

$$B_c^+(\alpha,\beta)=V_2^+(\alpha,\beta)+Y^+(\alpha,\beta)+B_{c,1}^+(\alpha,\beta)+B_{c,2}^{+*}(\alpha,\beta). \tag{4.21}$$

Next we consider $B_c^-(\alpha,\beta)$. We have the decomposition

$$B_c^-(\alpha,\beta)=B_{c,1}^-(\alpha,\beta)+B_{c,2}^-(\alpha,\beta) \tag{4.22}$$

analogously to (4.17). In this $B_{c,1}^-(\alpha,\beta)$ is regular for $(\alpha,\beta)\in T_P$. To discuss the continuation of $B_{c,2}^-(\alpha,\beta)$ we assume that $(\alpha,\beta)\in R(b)\cap Q_e$. In particular $(u,v)=(\alpha,\beta)$ satisfies (3.53), and we can use (3.54). We have, for such (α,β),

$$\begin{aligned} B_{c,2}^-(\alpha,\beta) =& i(2\pi)^{\beta-1}N^{\frac{1}{2}(\alpha+\beta+1)}\frac{\cos\big(\frac{1}{2}\pi\alpha\big)}{\sin\frac{1}{2}\pi(\alpha+\beta)\sin\frac{1}{2}\pi(\alpha-\beta)} \\ &\times\int_{-Pi}^{Pi}\frac{N^{-\xi}\sigma_{2\xi}(N)Z(\xi;\alpha,\beta)}{\zeta(1+2\xi)\zeta(1-2\xi)}\cot(\pi\xi)\Xi_-(\xi;\alpha,\beta)d\xi. \end{aligned} \tag{4.23}$$

Keeping (α,β) in $R(b)\cap Q_e\cap T_P$ we move the contour to L_P again. We encounter poles at the points listed in (4.4) and (4.5), and get

$$B_{c,2}^-(\alpha,\beta)=V_2^-(\alpha,\beta)+Y^-(\alpha,\beta)+B_{c,2}^{-*}(\alpha,\beta), \tag{4.24}$$

where $V_2^-(\alpha,\beta)$ and $Y^-(\alpha,\beta)$ are the contributions of the residues at the poles (4.4) and (4.5), respectively, and $B_{c,2}^{-*}(\alpha,\beta)$ has the same expression as (4.23) but with the contour

L_P. The terms $Y^-(\alpha,\beta)$ and $B_{c,2}^{-*}(\alpha,\beta)$ are regular in $Q_z \cap Q_t \cap Q_e \cap T_P$. On the other hand we have, in (4.24),

$$(4.25)\quad \begin{aligned} V_2^-(\alpha,\beta) = &- \frac{\cos\left(\frac{1}{2}\pi\alpha\right)\sigma_{\alpha+\beta+1}(N)}{2\cos\frac{1}{2}\pi\beta\cos\frac{1}{2}\pi(\alpha+\beta)\sin\frac{1}{2}\pi(\alpha-\beta)}\frac{\zeta(1+\alpha)\zeta(1+\beta)}{\Gamma(-\beta)\zeta(2+\alpha+\beta)} \\ &\times \Xi_-(-\tfrac{1}{2}(\alpha+\beta+1);\alpha,\beta) \\ &- \frac{\cos(\frac{1}{2}\pi\alpha)N^\alpha\sigma_{\beta-\alpha+1}(N)}{2\cos\frac{1}{2}\pi\beta\cos\frac{1}{2}\pi(\alpha-\beta)\sin\frac{1}{2}\pi(\alpha+\beta)}\frac{\zeta(1-\alpha)\zeta(1+\beta)}{\Gamma(-\beta)\zeta(2-\alpha+\beta)} \\ &\times \Xi_-(-\tfrac{1}{2}(\beta-\alpha+1);\alpha,\beta). \end{aligned}$$

But, (3.36) and (3.55) give

$$\Xi_-(-\tfrac{1}{2}(\alpha+\beta+1);\alpha,\beta) = -\sin(\frac{\pi\alpha}{2})\Gamma(-\beta)\int_0^1 W_0(x)x^\alpha(1-x)^\beta dx\ ,$$

$$\Xi_-(-\tfrac{1}{2}(\beta-\alpha+1);\alpha,\beta) = \sin(\frac{\pi\alpha}{2})\Gamma(-\beta)\int_0^1 W_0(x)(1-x)^\beta dx.$$

Thus $V_2^-(\alpha,\beta)$ exists as a meromorphic function for each variable α and β. Hence $B_c^-(\alpha,\beta)$ exists as a regular function in the set

$$Q_z \cap Q_t \cap Q_e \cap T_P \cap \{(u,v); V_2^-(u,v) \neq \infty\},$$

and there we have

$$(4.26)\qquad B_c^-(\alpha,\beta) = V_2^-(\alpha,\beta) + Y^-(\alpha,\beta) + B_{c,1}^-(\alpha,\beta) + B_{c,2}^{-*}(\alpha,\beta).$$

Gathering the above considerations we get the decomposition

$$(4.27)\quad \begin{aligned} B_N(\alpha,\beta;W_0) = &V_1(\alpha,\beta) + V_2^+(\alpha,\beta) + V_2^-(\alpha,\beta) + B_{c,1}^+(\alpha,\beta) + B_{c,1}^-(\alpha,\beta) \\ &+ Y^+(\alpha,\beta) + Y^-(\alpha,\beta) + B_{c,2}^{+*}(\alpha,\beta) + B_{c,2}^{-*}(\alpha,\beta) + B_d(\alpha,\beta) + B_h(\alpha,\beta) \end{aligned}$$

in the domain $S_P \cap Q_e$; note that the sum of the first three terms on the right side is regular in $S_P \cap Q_e$, for all other terms of (4.27) are regular there. We remark that $S_P \cap Q_e$ contains points which are arbitrarily close to (0,0). This ends our analytic continuation of (3.57).

Now we suppose in (4.27) that α, β are small and $(\alpha,\beta) \in Q_e$. On this situation we move the contour L_P in $B_{c,2}^{\pm*}(\alpha,\beta)$ back to the original segment $[-Pi, Pi]$. This time we encounter poles given in (4.5) and at

$$(4.28)\qquad \frac{1}{2}(\alpha+\beta+1)\,,\quad \frac{1}{2}(\beta-\alpha+1),$$

which are close to $\frac{1}{2}$. We have

$$B_{c,2}^{\pm*}(\alpha,\beta) = V_3^\pm(\alpha,\beta) - Y^\pm(\alpha,\beta) + B_{c,2}^{\pm**}(\alpha,\beta),$$

where $V_3^{\pm}(\alpha,\beta)$ are the contributions of the poles (4.28), and $B_{c,2}^{\pm **}(\alpha,\beta)$ have the expression (4.18) and (4.23), respectively, but with different $(\alpha,\ \beta)$. Hence by (4.27) we have, on the present assumption,

$$B_N(\alpha,\beta;W_0) = V(\alpha,\beta) + \tilde{B}_c^+(\alpha,\beta) + \tilde{B}_c^-(\alpha,\beta) + B_d(\alpha,\beta) + B_h(\alpha,\beta), \tag{4.29}$$

where $V = V_1 + V_2^+ + V_2^- + V_3^+ + V_3^-$, and $\tilde{B}_c^{\pm}(\alpha,\beta)$ have the same expression as $(3.58)_c$, but with different $(\alpha,\ \beta)$. We note that in (4.29) all terms except for $V(\alpha,\ \beta)$ are obviously regular in a neighbourhood of the origin. That is, (4.29) holds for small $\alpha,\ \beta$ without the restriction $(\alpha,\ \beta) \in Q_e$, and in particular $V(\alpha,\ \beta)$ is regular at (0,0).

Namely, specializing (4.29) we get

$$\sum_{n=1}^{\infty} d(n)d(N-n)W_0(\frac{n}{N}) = V(0,0)+\tilde{B}_c^+(0,0)+\tilde{B}_c^-(0,0)+B_d(0,0)+B_h(0,0). \tag{4.30}$$

We now compute $V(0,0)$ explicitly in terms of $W_0(x)$. We note first that by definition

$$\begin{aligned} V_3^+(\alpha,\beta) =& \frac{\sigma_{\alpha+\beta+1}(N)}{2\cos\frac{1}{2}\pi\beta\cos\frac{1}{2}\pi(\alpha+\beta)} \frac{\zeta(1+\alpha)\zeta(1+\beta)}{\Gamma(-\beta)\zeta(2+\alpha+\beta)} \\ &\times \Xi_+\big(\tfrac{1}{2}(\alpha+\beta+1);\alpha,\beta\big) \\ &+ \frac{N^\alpha\sigma_{\beta-\alpha+1}(N)}{2\cos\frac{1}{2}\pi\beta\cos\frac{1}{2}\pi(\alpha-\beta)} \frac{\zeta(1-\alpha)\zeta(1+\beta)}{\Gamma(-\beta)\zeta(2-\alpha+\beta)} \\ &\times \Xi_+\big(\tfrac{1}{2}(\beta-\alpha+1);\alpha,\beta\big), \end{aligned} \tag{4.31}$$

$$\begin{aligned} V_3^-(\alpha,\beta) =& \frac{\cos(\frac{1}{2}\pi\alpha)\sigma_{\alpha+\beta+1}(N)}{2\cos\frac{1}{2}\pi\beta\cos\frac{1}{2}\pi(\alpha+\beta)\sin\frac{1}{2}\pi(\alpha-\beta)} \frac{\zeta(1+\alpha)\zeta(1+\beta)}{\Gamma(-\beta)\zeta(2+\alpha+\beta)} \\ &\times \Xi_-\big(\tfrac{1}{2}(\alpha+\beta+1);\alpha,\beta\big) \\ &+ \frac{\cos(\frac{1}{2}\pi\alpha)N^\alpha\sigma_{\beta-\alpha+1}(N)}{2\cos\frac{1}{2}\pi\beta\cos\frac{1}{2}\pi(\alpha-\beta)\sin\frac{1}{2}\pi(\alpha+\beta)} \frac{\zeta(1-\alpha)\zeta(1+\beta)}{\Gamma(-\beta)\zeta(2-\alpha+\beta)} \\ &\times \Xi_-\big(\tfrac{1}{2}(\beta-\alpha+1);\alpha,\beta\big), \end{aligned} \tag{4.32}$$

providing α,β are small and $(\alpha,\beta) \in Q_e$. We assume further that the points $\big(\frac{1}{2}(\alpha+\beta+1),\alpha,\beta\big)$ and $\big(\frac{1}{2}(\beta-\alpha+1),\alpha,\beta\big)$ are in D_0. Then we can use (3.47) and (3.54). We have, from (4.20) and (4.31),

$$\begin{aligned} V_2^+(\alpha,\beta) + V_3^+(\alpha,\beta) =& \frac{2}{\pi\cos(\frac{1}{2}\pi\beta)}\sigma_{\alpha+\beta+1}(N)\frac{\zeta(1+\alpha)\zeta(1+\beta)}{\Gamma(-\beta)\zeta(2+\alpha+\beta)} \\ &\times \Phi_+\big(\tfrac{1}{2}(\alpha+\beta+1);\alpha,\beta\big) \\ &+ \frac{2}{\pi\cos(\frac{1}{2}\pi\beta)}\sigma_{\beta-\alpha+1}(N)\frac{\zeta(1-\alpha)\zeta(1+\beta)}{\Gamma(-\beta)\zeta(2-\alpha+\beta)} \\ &\times \Phi_+\big(\tfrac{1}{2}(\beta-\alpha+1);\alpha,\beta\big). \end{aligned} \tag{4.33}$$

Similarly (4.25) and (4.32) give

$$
\begin{aligned}
V_2^-(\alpha,\beta)+V_3^-(\alpha,\beta) = & \frac{2}{\pi\cos(\frac{1}{2}\pi\beta)}\sigma_{\alpha+\beta+1}(N)\frac{\zeta(1+\alpha)\zeta(1+\beta)}{\Gamma(-\beta)\zeta(2+\alpha+\beta)} \\
& \times\Phi_-\big(\tfrac{1}{2}(\alpha+\beta+1);\alpha,\beta\big) \\
& +\frac{2}{\pi\cos(\frac{1}{2}\pi\beta)}\sigma_{\beta-\alpha+1}(N)\frac{\zeta(1-\alpha)\zeta(1+\beta)}{\Gamma(-\beta)\zeta(2-\alpha+\beta)} \\
& \times\Phi_-\big(\tfrac{1}{2}(\beta-\alpha+1);\alpha,\beta\big).
\end{aligned}
\tag{4.34}
$$

But, we have, by (3.46) and (3.52),

$$
\begin{aligned}
&(\Phi_++\Phi_-)\big(\tfrac{1}{2}(\alpha+\beta+1);\alpha,\beta\big) \\
&\quad=-\frac{1}{4\pi i}\int_{-\infty i}^{\infty i}\Big\{\sin\frac{\pi}{2}(\alpha+\beta)\cos\frac{\pi\alpha}{2}+\cos\pi(s-\frac{\alpha}{2})\sin\pi(s-\tfrac{1}{2}(\alpha+\beta))\Big\} \\
&\qquad\qquad\times\Gamma(s)\Gamma(s-\alpha-\beta-1)\Gamma(1-s)\Gamma(1+\alpha-s)w_0(s)ds \\
&\quad=-\frac{1}{4i}\int_{-\infty i}^{\infty i}\cos\pi(s-\alpha-\frac{\beta}{2})\Gamma(s-\alpha-\beta-1)\Gamma(1+\alpha-s)w_0(s)ds;
\end{aligned}
$$

in the first integral the path separates the poles of $\Gamma(s)\Gamma(s-\alpha-\beta-1)$ from those of $\Gamma(1-s)\Gamma(1+\alpha-s)$. We shift the contour in the last integral to $\mathrm{Re}(s)=+\infty$, while noticing (3.37). We then find that

$$
(\Phi_++\Phi_-)(\tfrac{1}{2}(\alpha+\beta+1);\alpha,\beta)=\frac{\pi}{2}\Gamma(-\beta)\cos\frac{\pi\beta}{2}\int_0^1 W_0(x)x^\alpha(1-x)^\beta dx,
\tag{4.35}
$$

providing the present condition on (α,β). In just the same way we get also

$$
(\Phi_++\Phi_-)(\tfrac{1}{2}(\beta-\alpha+1);\alpha,\beta)=\frac{\pi}{2}\Gamma(-\beta)\cos\frac{\pi\beta}{2}\int_0^1 W_0(x)(1-x)^\beta dx.
\tag{4.36}
$$

Collecting (3.39) and (4.33)-(4.36) we find that in a neighbourhood of the origin

$$
V(\alpha,\beta)=\int_0^1 W_0(x)\rho_N(x;\alpha,\beta)dx,
$$

where

$$
\begin{aligned}
\rho_N(x;\alpha,\beta)= & \sigma_{1+\alpha+\beta}(N)\frac{\zeta(1+\alpha)\zeta(1+\beta)}{\zeta(2+\alpha+\beta)}x^\alpha(1-x)^\beta \\
& +N^\alpha\sigma_{1-\alpha+\beta}(N)\frac{\zeta(1-\alpha)\zeta(1+\beta)}{\zeta(2-\alpha+\beta)}(1-x)^\beta \\
& +N^\beta\sigma_{1+\alpha-\beta}(N)\frac{\zeta(1+\alpha)\zeta(1-\beta)}{\zeta(2+\alpha-\beta)}x^\alpha \\
& +N^{\alpha+\beta}\sigma_{1-\alpha-\beta}(N)\frac{\zeta(1-\alpha)\zeta(1-\beta)}{\zeta(2-\alpha-\beta)}.
\end{aligned}
$$

Hence we obtain

$$V(0,0)=\frac{6}{\pi^2}\int_0^1 W_0(x)n(x;N)dx, \tag{4.37}$$

where $n(x;N)$ is defined in Theorem 4.

On the other hand we have in (4.30)

$$(4.38)_d \qquad B_d(0,0)=\frac{1}{\pi}N^{\frac12}\sum_{j=1}^{\infty}\alpha_j t_j(N)H_j(\tfrac12)^2(\Phi_++\Phi_-)(i\kappa_j;0,0),$$

$$(4.38)_h \qquad B_h(0,0)=\frac14 N^{\frac12}\sum_{k=6}^{\infty}\sum_{j=1}^{\vartheta(k)}(-1)^k\alpha_{j,k}t_{j,k}(N)H_{j,k}(\tfrac12)^2\Xi_+(k-\tfrac12;0,0),$$

$$(4.38)_c \qquad \begin{aligned}&(\tilde{B}_c^++\tilde{B}_c^-)(0,0)\\ &\quad=\frac{1}{\pi^2}N^{\frac12}\int_{-\infty}^{\infty}N^{-ir}\sigma_{2ir}(N)\frac{|\zeta(\frac12+ir)|^4}{|\zeta(1+2ir)|^2}(\Phi_++\Phi_-)(ir;0,0)dr.\end{aligned}$$

We have, by (3.46),(3.48) and (3.52), respectively,

$$\Phi_+(ir;0,0)=\frac{\pi i}{4}\int_{(\frac34)}\Gamma(s)^{-2}\Gamma(s-\tfrac12+ir)\Gamma(s-\tfrac12-ir)\cot(\pi s)w_0(s)ds, \tag{4.39}$$

$$\Xi_+(k-\tfrac12;0,0)=\frac{1}{2\pi i}\int_{(0)}\frac{\Gamma(k-1+s)}{\Gamma(k+1-s)}\Gamma(1-s)^2\cos(\pi s)w_0(s)ds, \tag{4.40}$$

$$\Phi_-(ir;0,0)=\frac{1}{4\pi i}\cosh(\pi r)\int_{(\frac34)}\Gamma(s-\tfrac12+ir)\Gamma(s-\tfrac12-ir)\Gamma(1-s)^2w_0(s)ds, \tag{4.41}$$

where r is real, and k is integral. We shift the path in (4.39)-(4.41) to $\mathrm{Re}(s)=+\infty$, while invoking (3.37). Inserting the results into $(4.38)_d$-$(4.38)_c$, and collecting (4.30), (4.37), we end the proof of Theorem 4. We do not give the details here. For, as we have remarked after the statement of Theorem 6 we shall not use Theorem 4 to prove Theorem 6; we shall apply the saddle point method to (4.39)-(4.41) without recoursing to (2.15), and prove Theorem 6 in the next section.

5. Specialization

In this section we shall prove Theorems 5, 6 and 7. We shall first deduce Theorem 5 from Theorem 3.

To this end let $0 < \delta < \frac{1}{4}$ and $p(x) = \exp(-\frac{1}{x}\exp(-\frac{1}{1-x}))$. We define $g(x)$ to be equal to 0 for $x \le \frac{1}{2}$ and $x \ge 1$, $p((x-\frac{1}{2})/\delta)$ for $\frac{1}{2} \le x \le \frac{1}{2}+\delta$, 1 for $\frac{1}{2}+\delta \le x \le 1-\delta$, $p((1-x)/\delta)$ for $1-\delta \le x \le 1$. We may set $W(x) = g(fx/M)$ in Theorem 3, where M is a large parameter. Then we have, uniformly for

$$M^{-1+\varepsilon} < \delta < \frac{1}{4}, \quad 1 \le f \le M^{1/\varepsilon}, \tag{5.1}$$

the relation

$$D(M;f) - D(M/2;f) = A_f(0,0;W) + O(\delta M^{1+\varepsilon}), \tag{5.2}$$

where $\varepsilon > 0$ is small and fixed. Thus Theorem 3 gives

$$E^*(M;f) - E^*(M/2;f) = e_1(M;f) + e_2(M;f) + e_3(M;f) + O(\delta M^{1+\varepsilon}), \tag{5.3}$$

where

$$E^*(M;f) = \sum_{n=1}^{M} d(n)d(n+f) - \frac{6}{\pi^2}\int_0^{M/f} m(x;f)dx, \tag{5.4}$$

and $e_\nu(M;f)$ ($\nu = 1,2,3$) corresponds to the $(\nu+1)$-th term on the right side of the expression (2.12).

Let $\Lambda(r, M/f)$ stand for $\Xi_0(ir;W)$ which is defined by (2.13) and is equal to $\Xi_0(ir;0,0)$ (cf. (4.16)) on the present specialization of W. Then we have, by Euler's representation of hypergeometric functions,

$$\Lambda(\xi, Z) = Z\int_{\frac{1}{2}}^{1} g(x)\int_0^1 (y(1-y))^{-\frac{1}{2}+i\xi}(Zx+y)^{-\frac{1}{2}-i\xi}dydx, \tag{5.5}$$

where $Z > 0$ and $\mathrm{Im}(\xi) < \frac{1}{2}$. By partial integration with respect to the variable x we have also, for $\nu \ge 0$,

$$\begin{aligned}\Lambda(\xi, Z) =& \frac{(-1)^{\nu-1}Z^{-\nu}}{(\frac{1}{2}-i\xi)(\frac{3}{2}-i\xi)\cdots(\nu+\frac{1}{2}-i\xi)} \\ &\times \int_{\frac{1}{2}}^{1} g^{(\nu+1)}(x)\int_0^1 (y(1-y))^{-\frac{1}{2}+i\xi}(Zx+y)^{\nu+\frac{1}{2}-i\xi}dydx.\end{aligned} \tag{5.6}$$

From (5.5) and (5.6) we can deduce the following estimates: If $Z \ge 1$, then we have, uniformly for δ,

$$\Lambda((\frac{1}{2}-k)i, Z) \ll Z^{1-k}2^{-k} \quad (k : \text{integer} \ge 2), \tag{5.7}$$

(5.8) $$\Lambda(\xi, Z) \ll Z^{\frac{1}{2}}(1+|\xi|)^{-\frac{3}{2}}(1+\delta|\xi|)^{-1} \quad (\xi : \text{real}).$$

If $0 < Z \le 1$, then we have, uniformly for δ,

(5.9) $$\Lambda((\tfrac{1}{2}-k)i, Z) \ll \min(k^{-2},\, k^{-1}\exp(-\frac{1}{4}kZ^{\frac{1}{2}})) \quad (k : \text{integer} \ge 2),$$

(5.10) $$\Lambda(\xi, Z) \ll (1+|\xi|)^{-2}(\log\frac{Z}{2})^{5} \quad (\xi : \text{real},\ |\xi| < Z^{-\frac{1}{2}}(\log\frac{Z}{2})^{2}),$$

(5.11) $$\Lambda(\xi, Z) \ll Z^{\frac{1}{4}}|\xi|^{-\frac{3}{2}}(1+\delta|\xi|Z^{\frac{1}{2}})^{-1} \quad (\xi : \text{real},\ |\xi| \ge Z^{-\frac{1}{2}}(\log\frac{Z}{2})^{2}).$$

Among these (5.7) follows from (5.5) immediately. To show (5.9) we use (5.6) with $\nu = 0$. We have

$$\Lambda((\tfrac{1}{2}-k)i, Z) = \frac{1}{k-1}\int_{\frac{1}{2}}^{1} g'(x)\int_{0}^{1}(\frac{y(1-y)}{Zx+y})^{k-1}dydx.$$

The maximum of the inner integrand is $O(\exp(-\frac{1}{4}k\sqrt{Z}))$, since $0 < Z \le 1$; and the integrand itself is $O((1-y)^{k-1})$, whence we have (5.9). To show (5.10) we use (5.6) with $\nu = 0$ again. We divide the inner integral into two parts according to $y \le Z^{\frac{1}{2}}(\log(Z/2))^{2}$ and $Z^{\frac{1}{2}}(\log(Z/2))^{2} < y \le 1$. In the first part the integrand is obviously $O(y^{-\frac{1}{2}}(Z+y)^{\frac{1}{2}})$, and in the second part it is $(1-y)^{-\frac{1}{2}+i\xi}(1+O(y^{-1}Z(1+|\xi|)))$, for we have $|\xi| < Z^{-\frac{1}{2}}(\log(Z/2))^{2}$. From these (5.10) follows. As for (5.8) and (5.11) we prove them by applying the saddle point method to the inner integral of (5.6) with $\nu = 0$ or 1. But we give the details only for (5.11), since (5.8) is easier than (5.11).

Thus we consider the integral

(5.12) $$\int_{0}^{1}(y(1-y))^{-\frac{1}{2}}(U+y)^{\nu+\frac{1}{2}}\exp(i\xi q(y,U))dy,$$

where $U = Zx \le 1$, $\xi \ge U^{-\frac{1}{2}}(\log(U/2))^{2}$, $q(y,U) = \log\{(y(1-y))/(U+y)\}$, and $\nu = 0$ or 1. The saddle point is at $y_0 = U^{\frac{1}{2}}(U^{\frac{1}{2}}+(U+1)^{\frac{1}{2}})^{-1}$, which is close to $U^{\frac{1}{2}}$ when U is small. We then move the path in (5.12) to $L = L_1 + L_2 + L_3$ where L_ν $(\nu = 1,2,3)$ are the segments $[0\,, y_0(1-r_0\omega)]$, $[y_0(1-r_0\omega),\, y_0(1+r_0\omega)]$, $[y_0(1+r_0\omega),\, 1]$, respectively. Here r_0 is a small positive constant, and $\omega = \exp(-\frac{1}{4}\pi i)$. We denote a point on L by $y(1+r\omega)$ where $0 \le y \le 1$. We have $r = r_0$ on L_1, $-r_0 \le r \le r_0$ on L_2, and $r = y_0(1-y)(y(1-y_0))^{-1}r_0$ on L_3. Then we note that we have

$$q(y(1+r\omega), U) = \log\frac{y(1-y)}{U+y} + \log s(r; y, U),$$

where

$$s(r; y, U) = 1 + \frac{U-2yU-y^{2}}{(U+y)(1-y)}r\omega + \frac{yU(U+1)}{(U+y)(U+y+yr\omega)(1-y)}r^{2}i.$$

This gives immediately that on $L_1 + L_3$ we have $\mathrm{Im}\{q(y(1 + r\omega), U)\} > cU^{\frac{1}{2}} r_0^2$ with a positive constant c. Thus the integral over $L_1 + L_3$ is $O(\exp(-cr_0^2 U^{\frac{1}{2}}\xi))$; note the lower bound of ξ which we have introduced above. On the other hand on L_2 we have $y = y_0$ and $\mathrm{Im}\{q(y_0(1 + r\omega), U)\} > cU^{\frac{1}{2}} r^2$. Hence the integral over L_2 is $O(U^{\frac{1}{2}(\nu+\frac{1}{2})}\xi^{-\frac{1}{2}})$, which is also a bound of the original integral (5.12). Inserting this into (5.6) with $\nu = 0$ or 1 we obtain (5.11).

We may now return to (5.3), and assume (2.17). We consider first the case $f \le M$. Then (5.8) gives readily

$$e_1(M; f) \ll M^{\frac{1}{2}} d(f). \tag{5.13}$$

Also (5.7) gives

$$\begin{aligned} e_3(M; f) &\ll f^{\frac{11}{2}} M^{-5} \sum_{k=6}^{\infty} \sum_{j=1}^{\vartheta(k)} \alpha_{j,k} |t_{j,k}(f)| H_{j,k}(\tfrac{1}{2})^2 2^{-k} \\ &\ll f^{\frac{11}{2}} M^{-5} d(f), \end{aligned} \tag{5.14}$$

for we have (2.7), (2.8) and (2.10). As for $e_2(M; f)$ we divide it into two parts $e_2^{(\nu)}(M; f)$, $\nu = 1, 2$, according to $\kappa_j \le f^{\frac{1}{4}-\alpha}$ and $f^{\frac{1}{4}-\alpha} < \kappa_j$, respectively, where α is as in (2.17). We have, by (2.17) and (5.8),

$$\begin{aligned} e_2^{(1)}(M; f) &\ll f^{\alpha} M^{\frac{1}{2}} \sum_{\kappa_j \le f^{1/4-\alpha}} \alpha_j H_j(\tfrac{1}{2})^2 \kappa_j^{-\frac{3}{2}} \\ &\ll f^{\frac{1}{8}+\frac{1}{2}\alpha} M^{\frac{1}{2}} (\log M)^c, \end{aligned} \tag{5.15}$$

where we have used (2.1) with $n = 1$ and (2.6). To $e_2^{(2)}(M; f)$ we apply (2.1) with $n = f$ and (2.6) as well as (5.8), getting

$$e_2^{(2)}(M; f) \ll (f^{\frac{1}{8}+\frac{1}{2}\alpha} M^{\frac{1}{2}} + M^{\frac{1}{2}} \delta^{-\frac{1}{2}} + \delta^{\frac{1}{2}} f^{\frac{1}{4}} M^{\frac{1}{2}})(\log M)^c. \tag{5.16}$$

Collecting (5.3), (5.13)-(5.16) and setting $\delta = M^{-\frac{1}{3}}$ which satisfies (5.1), we obtain

$$E^*(M; f) - E^*(M/2; f) \ll M^{\frac{2}{3}+\varepsilon} + f^{\frac{1}{8}+\frac{1}{2}\alpha} M^{\frac{1}{2}+\varepsilon} \quad (1 \le f \le M) \tag{5.17}$$

uniformly in f.

We consider next the case $M \le f < M^{2-\varepsilon}$. This time we use (5.9)-(5.11). Thus (5.10) and (5.11) give

$$e_1(M; f) \ll f^{\frac{1}{2}+\varepsilon}. \tag{5.18}$$

To $e_3(M; f)$ we apply (2.7), (2.8), (2.11) and (5.9), getting

$$e_3(M; f) \ll f^{\frac{1}{2}+\varepsilon}. \tag{5.19}$$

On the other hand (2.17), (5.10) and (5.11) give

$$
\begin{aligned}
e_2(M;f) &\ll f^{\frac{1}{2}+\alpha+\varepsilon} \sum_{\kappa_j\le (f/M)^{1/2}} \alpha_j H_j(\tfrac{1}{2})^2\kappa_j^{-2} \\
&\quad + (Mf)^{\frac{1}{4}} \sum_{\kappa_j> (f/M)^{1/2}} \alpha_j |t_j(f)| H_j(\tfrac{1}{2})^2 \kappa_j^{-\frac{3}{2}}(1+\delta\kappa_j(M/f)^{\frac{1}{2}})^{-1}.
\end{aligned}
\tag{5.20}
$$

In this the first sum is $O(M^\varepsilon)$ by (2.1) with $n=1$ and (2.6). To bound the second sum we consider two cases separately. First we suppose that $M^{2/(1+4\alpha)} \le f \le M^{2-\varepsilon}$. Then by (2.1) with $n=f$ and (2.6) the sum is $O(M^{\frac{1}{4}+\varepsilon}+\delta^{-\frac{1}{2}}f^{\frac{1}{4}}M^{-\frac{1}{4}+\varepsilon})$. On the other hand, when $M<f\le M^{2/(1+4\alpha)}$, we divide the sum into two parts according to $(f/M)^{\frac{1}{2}}\le\kappa_j\le f^{\frac{1}{4}-\alpha}$ or $f^{\frac{1}{4}-\alpha}<\kappa_j$. The first part is estimated by (2.1) with $n=1$, (2.6) and (2.17); we find that it is $O(f^{\frac{1}{8}+\frac{1}{2}\alpha+\varepsilon})$. The second part is, by (2.1) with $n=f$ and (2.6), $O(f^{\frac{1}{8}+\frac{1}{2}\alpha+\varepsilon}+\delta^{-\frac{1}{2}}f^{\frac{1}{4}}M^{-\frac{1}{4}+\varepsilon}+\delta^{\frac{1}{2}}M^{\frac{1}{4}+\varepsilon})$. Inserting these into (5.20) we see that

$$
\begin{aligned}
&e_2(M;f)\\
&\ll \begin{cases} f^{\frac{1}{2}+\alpha+\varepsilon}+f^{\frac{1}{2}+\varepsilon}\delta^{-\frac{1}{2}} & (M^{2/(1+4\alpha)}\le f\le M^{2-\varepsilon}) \\ (Mf)^{\frac{1}{4}}f^{\frac{1}{8}+\frac{1}{2}\alpha+\varepsilon}+f^{\frac{1}{2}+\varepsilon}\delta^{-\frac{1}{2}}+M^{\frac{1}{2}}f^{\frac{1}{4}+\varepsilon}\delta^{\frac{1}{2}} & (M\le f\le M^{2/(1+4\alpha)}). \end{cases}
\end{aligned}
\tag{5.21}
$$

Collecting (5.3), (5.18), (5.19) and (5.21), and setting $\delta=M^{-\frac{2}{3}}f^{\frac{1}{3}}$ which satisfies (5.1), we get

$$
\begin{aligned}
&E^*(M;f)-E^*(M/2;f)\\
&\ll \begin{cases} f^{\frac{1}{2}+\alpha+\varepsilon} & (M^{2/(1+4\alpha)}\le f\le M^{2/(1+2\alpha)}) \\ (Mf)^{\frac{1}{3}+\varepsilon}+(Mf)^{\frac{1}{4}+\varepsilon}f^{\frac{1}{8}+\frac{1}{2}\alpha} & (M\le f\le M^{2/(1+4\alpha)}). \end{cases}
\end{aligned}
\tag{5.22}
$$

From (5.4), (5.17) and (5.22) we immediately obtain the assertion of Theorem 5.

Now we turn to the proof of Theorem 7. Thus we fix f, and so we have, by (5.3) and (5.14),

$$E^*(M;f)-E^*(M/2;f)=e_1(M;f)+e_2(M;f)+O(\delta M^{1+\varepsilon}),$$

where δ is to be fixed later. We remark that in order to prove Theorem 7 it is enough to show that the above expression is $\Omega(M^{\frac{1}{2}})$, providing $\delta<M^{-\frac{1}{2}-2\varepsilon}$. To this end we introduce

$$F(N;f)=\int_{N/2}^{N}(E^*(M;f)-E^*(M/2;f))\frac{dM}{M},$$

and we shall show that $F(N;f)=\Omega(N^{\frac{1}{2}})$ with a $\delta<N^{-\frac{2}{3}}$. Obviously this will end the proof of Theorem 7.

We have

$$F(N;f)=F_1(N;f)+F_2(N;f)+O(\delta N^{1+\varepsilon}),\tag{5.23}$$

where

$$F_\nu(N;f)=\int_{N/2}^{N} e_\nu(M;f)\frac{dM}{M} \qquad (\nu=1,2).$$

We put, for $\xi>0$,

$$\eta(\xi;Z)=\frac{1}{2}\mathrm{Re}\{\frac{1}{\frac{1}{2}-i\xi}(1+\frac{i}{\sinh\pi\xi})(\lambda(\xi;Z)-\lambda(\xi;Z/2))\},$$

where

$$\lambda(\xi;Z)=\int_{\frac{1}{2}}^{1}\frac{g(x)}{x}\int_0^1 (y(1-y))^{-\frac{1}{2}+i\xi}(Zx+y)^{\frac{1}{2}-i\xi}dydx$$

with g being as above. Then we have

$$F_1(N;f)=\frac{1}{\pi}f^{\frac{1}{2}}\int_{-\infty}^{\infty} f^{-i\xi}\sigma_{2i\xi}(f)\frac{|\zeta(\frac{1}{2}+i\xi)|^4}{|\zeta(1+2i\xi)|^2}\eta(\xi;N/f)d\xi \tag{5.24}$$

and

$$F_2(N;f)=f^{\frac{1}{2}}\sum_{j=1}^{\infty}\alpha_j t_j(f)H_j(\tfrac{1}{2})^2\eta(\kappa_j;N/f). \tag{5.25}$$

We have, by partial integration as before,

$$\lambda(\xi;Z)=\frac{Z^{\frac{1}{2}-i\xi}}{\frac{3}{2}-i\xi}\int_{\frac{1}{2}}^{1}(\frac{g(x)}{x})'x^{\frac{3}{2}-i\xi}\int_0^1(y(1-y))^{-\frac{1}{2}+i\xi}(1+\frac{y}{Zx})^{\frac{3}{2}-i\xi}dydx. \tag{5.26}$$

Applying the saddle point method to the inner integral we get, for $Z\ge 1$,

$$\lambda(\xi;Z)\ll Z^{\frac{1}{2}}(1+|\xi|)^{-\frac{3}{2}}\ ,$$

or

$$\eta(\xi;Z)\ll Z^{\frac{1}{2}}(1+|\xi|)^{-\frac{5}{2}}$$

uniformly in ξ. This allows us to truncate (5.25): Thus let $K_0\ge 1$. Then we have

$$F_2(N;f)=f^{\frac{1}{2}}\sum_{\kappa_j\le K_0}\alpha_j t_j(f)H_j(\tfrac{1}{2})^2\eta(\kappa_j;N/f)+O(N^{\frac{1}{2}}K_0^{-\frac{1}{2}}(\log N)^c), \tag{5.27}$$

where we have used (2.1) and (2.6). On the other hand, when $Z\gg 1+|\xi|$ we may replace the last factor in the integrand of (5.26) by $1+O((1+|\xi|)/Z)$. This gives, for $Z\gg 1+|\xi|$,

$$\lambda(\xi;Z)=Z^{\frac{1}{2}-i\xi}\frac{\Gamma(\frac{1}{2}+i\xi)}{(\frac{1}{2}-i\xi)\Gamma(1+2i\xi)}(1-2^{-\frac{1}{2}+i\xi})+O(\frac{Z^{\frac{1}{2}}\delta}{(1+|\xi|)^{\frac{1}{2}}}+Z^{-\frac{1}{2}}),$$

where the implied constant is absolute. Hence, on the condition $K_0\le N$, (5.27) is transformed into

$$F_2(N;f)=N^{\frac{1}{2}}F_2^*(N;f)+O((N^{-\frac{1}{2}}K_0+N^{\frac{1}{2}}K_0^{-\frac{1}{2}}+N^{\frac{1}{2}}K_0^{\frac{1}{2}}\delta)(\log N)^c),$$

where

$$F_2^*(N;f)=\sum_{j=1}^{\infty}\alpha_j t_j(f)H_j(\tfrac{1}{2})^2\eta^*(\kappa_j;N/f);$$

$$\eta^*(\xi;Z)=\frac{1}{2}\mathrm{Re}\big\{\big(1+\frac{i}{\sinh\pi\xi}\big)\frac{\Gamma(\frac{1}{2}+i\xi)^2}{(\frac{1}{2}-i\xi)^2\Gamma(1+i\xi)}(1-2^{-\frac{1}{2}+i\xi})^2Z^{-i\xi}\big\}.$$

We then set $K_0=N^{\frac{2}{3}}$, and get

$$F_2(N;f)=N^{\frac{1}{2}}F_2^*(N;f)+O(N^{\frac{1}{6}}(\log N)^c), \tag{5.28}$$

providing

$$N^{-1+\varepsilon}\le\delta\le N^{-\frac{2}{3}}. \tag{5.29}$$

Similarly we have, on (5.29),

$$F_1(N;f)=\frac{1}{\pi}N^{\frac{1}{2}}\int_{-\infty}^{\infty}f^{-i\xi}\sigma_{2i\xi}(f)\frac{|\zeta(\frac{1}{2}+i\xi)|^4}{|\zeta(1+2i\xi)|^2}\eta^*(\xi;N/f)d\xi+O(N^{-\frac{1}{6}}).$$

But, by partial integration we see that this integral is $O((\log N)^{-1})$. Hence

$$F_1(N;f)\ll N^{\frac{1}{2}}/\log N, \tag{5.30}$$

provided (5.29).

Collecting (5.23), (5.28), (5.30) and choosing δ appropriately, we obtain

$$F(N;f)=N^{\frac{1}{2}}F_2^*(N;f)+O(N^{\frac{1}{2}}(\log N)^{-1}).$$

It only remains for us to appeal to [13, Lemma 3]. Then what is to be checked is to see whether the relevant non-vanishing condition holds or not in $F_2^*(N;f)$. Namely we need to have a κ such that

$$\sum_{\kappa_j=\kappa}\alpha_j t_j(f)H_j(\tfrac{1}{2})^2\neq 0. \tag{5.31}$$

But, by [26, Lemma] we have, as $K\to\infty$,

$$\sum_{j=1}^{\infty}\alpha_j t_j(f)H_j(\tfrac{1}{2})^2(\kappa_j^2+\tfrac{1}{4})\exp(-(\kappa_j/K)^2)=(1+o(1))2\pi^{-\frac{3}{2}}K^4\log K$$

if f is bounded. Obviously this implies (5.31), and we end our proof of Theorem 7.

Now we move to the proof of Theorem 6. Let p be the function introduced at the beginning of this section. We set, in (4.30), $W_0(x)$ to be the function which is equal to 0 for $0 \le x \le \delta$, $p((x-\delta)/\delta)$ for $\delta \le x \le 2\delta$, 1 for $2\delta \le x \le 1-2\delta$, $p((1-\delta-x)/\delta)$ for $1-2\delta \le x \le 1-\delta$, 0 for $1-\delta \le x \le 1$, where

$$N^{-1+\varepsilon} \le \delta < \tfrac{1}{8}. \tag{5.32}$$

We have

$$D(N) = B_N(0,0;W_0) + O(\delta N^{1+\varepsilon}). \tag{5.33}$$

Thus we have

$$E(N) = e_1(N) + e_2(N) + e_3(N) + O(\delta N^{1+\varepsilon}); \tag{5.34}$$

here $e_j(N)$ $(j=1,2,3)$ corresponds to $\tilde{B}_c(0,0), B_d(0,0), B_h(0,0)$, respectively, in the formula (4.30), where we have put $\tilde{B}_c = \tilde{B}_c^+ + \tilde{B}_c^-$. We denote by $\Phi_\pm(ir)$ and $\Xi_+(k-\frac{1}{2})$ the present specializations of $\Phi_\pm(ir;0,0)$ and $\Xi_+(k-\frac{1}{2};0,0)$, respectively. Then we see that our problem is reduced to the estimation of these quantities for real r and integral $k \ge 6$. We shall show the following estimates:

$$\Phi_\pm(ir) \ll r^{-2}(1+r\delta)^{-1}\log(r/\delta) \qquad (r:\ \text{real} \ge 1), \tag{5.35$_\pm$}$$

$$\Xi_+(k-\tfrac{1}{2}) \ll k^{-2}(1+\delta^{\frac{1}{4}}k^{\frac{1}{2}})^{-1}(\log k/\delta)^{\frac{3}{2}} \qquad (k:\text{integer} \ge 6). \tag{5.36}$$

The estimates $(5.35)_\pm$ give $e_1(N) \ll N^{\frac{1}{2}+\varepsilon}$. Also, (2.1) with $n=1$, (2.6), (2.17) and $(5.35)_\pm$ give $e_2(N) \ll N^{\frac{1}{2}+\alpha+\varepsilon}\delta^{-\varepsilon}$. Further, (2.8), (2.11), and (5.36) give $e_3(N) \ll N^{\frac{1}{2}+\varepsilon}\delta^{-\varepsilon}$. Hence, choosing δ appropriately, we obtain the assertion of Theorem 6.

Thus we have to prove $(5.35)_\pm$ and (5.36). We first treat $\Phi_+(ir)$, $r \ge 1$. For this sake let L be a large integer. Then we have, by (4.39),

$$\Phi_+(ir) = \frac{\pi i}{4}\int_{(L+\frac{1}{2})} \Gamma(s-\tfrac{1}{2}+ir)\Gamma(s-\tfrac{1}{2}-ir)\Gamma(s)^{-2}\cot(\pi s)w_0(s)ds + O(e^{-\frac{1}{2}\pi r})$$

with the present specialization of W_0, where the implied constant depends only on L. We divide this integral into three parts according to $\mathrm{Im}(s) \ge r+\sqrt{r}$, $|\mathrm{Im}(s)| < r+\sqrt{r}$, and $\mathrm{Im}(s) \le -r-\sqrt{r}$. By Stirling's formula we see readily that the second part is $O(r^{-\frac{1}{2}L})$ uniformly in δ. In the expression for the first part we may replace $\cot(\pi s)$ by $-i$ with a negligible error, and we denote the result by $\Phi_+^{(1)}(ir)$. Then we have

$$\Phi_+(ir) = 2\mathrm{Re}\{\Phi_+^{(1)}(ir)\} + O(r^{-\frac{1}{2}L}). \tag{5.37}$$

We note that we have, for any $\nu \ge 0$,

$$\Phi_+^{(1)}(ir) = (-1)^{\nu+1}\frac{\pi i}{4}\int_\delta^{1-\delta} W_0^{(\nu+1)}(x)G_\nu(x,r)x^{\nu+L}dx, \tag{5.38}$$

where

$$G_\nu(x,r)=\int_{r+\sqrt{r}}^{\infty}\frac{\Gamma(L+i(t+r))\Gamma(L+i(t-r))}{\Gamma(L+\frac12+it)\Gamma(L+\frac32+\nu+it)}x^{it}dt\,.$$

By Stirling's formula we have, after some simplification,

$$G_\nu(x,r)=c(L,\nu)\int_{r+\sqrt{r}}^{\infty}t^{-\nu-2}(1-(r/t)^2)^{L-\frac12}\exp(ig(x,r,t))dt+O(r^{-\nu-2}),$$

where

$$g(x,r,t)=(t+r)\log(t+r)+(t-r)\log(t-r)-2t\log t+t\log x\,;$$

$c(L,\nu)$ and the constant in the error term depend only on L and ν. By partial integration we have

$$G_\nu(x,r)\ll\int_{r+\sqrt{r}}^{\infty}\Big|\frac{\partial}{\partial t}\frac{t^{-\nu-2}(1-(r/t)^2)^{L-\frac12}}{\log(1-(r/t)^2)+\log x}\Big|dt+r^{-\nu-2}.$$

Here we should observe that we have $\log x<-\delta$. We compute explicitly this partial derivative, and also divide the range of integration at $t=r\delta^{-\frac12}$. Then we find, without difficulty, that the last integral is $O(r^{-2}\log(1/\delta))$ if $\nu=0$, and $O(r^{-\nu-2})$ if $\nu\ge1$, uniformly in x. Inserting these into (5.38), we get $(5.35)_+$ via (5.37).

We consider next $\Phi_-(ir)$, which is in fact easier than $\Phi_+(ir)$. We shift the contour of (4.41) with the present specialization of W_0 to $\mathrm{Re}(s)=(\log r)^{-1}$. We encounter simple poles at $s=\frac12\pm ir$, which contribute negligibly. On noting $w_0(s)\ll\log r$ on the new contour we get $\Phi_-(ir)\ll r^{-2}\log r$ simply by applying Stirling's formula to the absolute value of the integrand. Further, shifting the contour to $\mathrm{Re}(s)=-\frac12+(\log r/\delta)^{-1}$ and noting $w_0(s)\ll\delta^{-\frac12}$ there, we get also $\Phi_-(ir)\ll\delta^{-\frac12}r^{-3}$. Combining these estimates of $\Phi_-(ir)$ we get $(5.35)_-$.

We now turn to $\Xi_+(k-\frac12)$. In (4.40) we shift the path to $\mathrm{Re}(s)=-\nu$, where $\nu=1$ or 0. We divide the range of integration into three parts according to $\mathrm{Im}(s)\ge1$, $|\mathrm{Im}(s)|<1$ and $\mathrm{Im}(s)\le-1$. The second part is estimated by applying Stirling's formula to the absolute value of the integrand. Then we see that it is $O(k^{-2-2\nu}\delta^{-\nu}\log(1/\delta))$. For, we have $w_0(s)\ll\delta^{-\nu}\log(1/\delta)$ on the line $\mathrm{Re}(s)=-\nu$, which is a consequence of the representation

$$w_0(s)=\frac{(-1)^\nu}{s(s+1)...(s+\nu-1)}\int_\delta^{1-\delta}W_0^{(\nu)}(x)x^{s+\nu-1}dx. \tag{5.39}$$

Denoting the first part by $\Xi_+^{(1)}(k-\frac12)$ we have

$$\Xi_+(k-\tfrac12)=2\mathrm{Re}\{\Xi_+^{(1)}(k-\tfrac12)\}+O(k^{-2-2\nu}\delta^{-\nu}\log(1/\delta)). \tag{5.40}$$

Here we note that we have

$$\Xi_+^{(1)}(k-\tfrac12)=(-1)^{\nu+1}\frac{\pi}{4}\int_\delta^{1-\delta}W_0^{(\nu+1)}(x)I_\nu(x\,,k)dx, \tag{5.41}$$

where

$$I_\nu(x,k)=\int_1^\infty \frac{\Gamma(k-1-\nu+it)}{\Gamma(k+1+\nu-it)}\Gamma(\nu+1-it)\Gamma(-it)\cosh(\pi t)x^{it}dt\ ;$$

we have used (5.39) with ν replaced by $\nu+1$. By Stirling's formula we have, after some simplification,

$$I_\nu(x,k)=\int_1^\infty \frac{t^\nu}{(t^2+k^2)^{\nu+1}}\exp(ih(t\,;x\,,k))l_\nu(t;k)dt\ , \tag{5.42}$$

where

$$h(t\,;x,k)=t\log(1+(k/t)^2)+t\log x-(2k-1)\arctan(k/t)\ ;$$

$l_\nu(t\,;k)$ is regular and absolutely bounded in the region $|\arg(t-1)|\le\frac{1}{2}\pi-\varepsilon$ for any small $\varepsilon>0$, and moreover there we have, uniformly in k,

$$\frac{\partial}{\partial t}l_\nu(t;k)\ll|t|^{-2}\ . \tag{5.43}$$

With these we enter into the proof of (5.36). We see that our problem is reduced to the estimation of (5.42). We thus consider two cases separately according to $\delta\le x\le 2\delta$ and $1-2\delta\le x\le 1-\delta$, because of our present specialization of W_0. We shall treat the second case in detail; the first case will be briefly treated later. Thus we assume, for a while, that $1-2\delta\le x\le 1-\delta$. We first treat the case where $\delta^{\frac{1}{2}}k\le A\log(k/\delta)$ with a fixed constant A. To this end we divide $I_0(x,k)$ into three parts so that

$$I_0(x,k)=I_0^{(1)}(x,k)+I_0^{(2)}(x,k)+I_0^{(3)}(x,k), \tag{5.44}$$

where $I_0^{(1)}$ corresponds to $t\le\frac{1}{2}\delta^{-\frac{1}{2}}k$, $I_0^{(2)}$ to $\frac{1}{2}\delta^{-\frac{1}{2}}k\le t\le 2\delta^{-\frac{1}{2}}k$, $I_0^{(3)}$ to $2\delta^{-\frac{1}{2}}k\le t$ in the expression (5.42) with $\nu=0$. We have trivially

$$I_0^{(2)}(x,k)\ll\delta^{\frac{1}{2}}/k. \tag{5.45}$$

On the other hand we have, by partial integration,

$$I_0^{(1)}(x,k)\ll\int_1^{\frac{1}{2}\delta^{-\frac{1}{2}}k}\left|\frac{\partial}{\partial t}\frac{l_0(t;k)}{(t^2+k^2)\frac{\partial}{\partial t}h(t\,;x,k)}\right|dt+k^{-2}, \tag{5.46}$$

providing $\delta^{-\frac{1}{2}}k\ge 2$. Here we have used the fact that

$$\frac{\partial}{\partial t}h(t;x,k)=\log(1+(k/t)^2)+\log x-\frac{k}{t^2+k^2} \tag{5.47}$$

is a decreasing function of t, and its value at $t=\frac{1}{2}\delta^{-\frac{1}{2}}k$ is larger than δ. We then perform the partial differentiations in (5.46), and get without difficulty

$$I_0^{(1)}(x,k)\ll k^{-2}\log(1/\delta). \tag{5.48}$$

Similarly we can show

$$I_0^{(3)}(x,k) \ll k^{-2} \tag{5.49}$$

uniformly in δ. Collecting (5.44)-(5.49) we get

$$I_0(x,k) \ll k^{-2}\log(k/\delta) \quad (k \le A\delta^{-\frac{1}{2}}\log(k/\delta)) \tag{5.50}$$

uniformly for $1-2\delta \le x \le 1-\delta$, where A is an arbitrary fixed constant.

We next consider the case where $\delta^{\frac{1}{2}}k$ is large. For this sake we apply the saddle point method to $I_1(x,k)$. The saddle point $t = t_0$ is such that

$$\left\{\frac{\partial}{\partial t}h(t;x,k)\right\}_{t=t_0} = 0\,. \tag{5.51}$$

As we have observed already the right side of (5.47) is positive at $t = \frac{1}{2}\delta^{-\frac{1}{2}}k$. On the other hand it is negative at $t = 2\delta^{-\frac{1}{2}}k$. Thus t_0 is uniquely determined by (5.51), and it satisfies

$$\frac{1}{2}\delta^{-\frac{1}{2}}k < t_0 < 2\delta^{-\frac{1}{2}}k. \tag{5.52}$$

We move the path in (5.42) to $S = S_1 + S_2 + S_3$, where $S_1,\ S_2$ are the segments $[1,\ t_0(1-r_0\omega)]$, $[t_0(1-r_0\omega),\ t_0(1+r_0\omega)]$, respectively; S_3 is the half line which starts at $t_0(1+r_0\omega)$ and goes to $+\infty e^{i\theta}$ with $\theta = \arg(t_0(1+r_0\omega))$. Here r_0 is a small positive constant, and $\omega = \exp(-\frac{1}{4}\pi i)$. We denote the points on S by $y(1+r\omega)$ with $1 \le y < \infty$ and real r. Thus $r = -\big(t_0(t-1)/(t(t_0-1))\big)r_0$ on S_1, $-r_0 \le r \le r_0$ on S_2, and $r = r_0$ on S_3. We then note that

$$\begin{aligned} h(y(1+r\omega);k,x) &= y(1+r\omega)\big\{\log(1+(k/y)^2(1+r\omega)^{-2}) + \log x\big\} \\ &\quad - (2k-1)ky\int_0^{(1+r\omega)^{-1}} \frac{d\xi}{y^2+k^2\xi^2}. \end{aligned}$$

Expanding the right side into a power series in r, we get

$$\begin{aligned} h(y(1+r\omega);k,x) &= h(y;k,x) + y\frac{\partial}{\partial y}h(y;k,x)r\omega \\ &\quad + \frac{(k-1)ky^2+k^4}{(y^2+k^2)^2}yr^2 i + O\big(\frac{k^2y}{y^2+k^2}r^3\big), \end{aligned} \tag{5.53}$$

where the implied constant is absolute, providing $|r| \le 1$, say. Having this we may estimate the contributions of $S_j\,(j=1,2,3)$. If $1 \le y \le A$ with a large A then we take simply the absolute value of the integrand, and see that the contribution of the corresponding part of S_1 is $O(k^{-4})$. If $A \le y \le \sqrt{k}$ then (5.47) and (5.53) give $\mathrm{Im}(h(y(1+r\omega);k,x)) > cr_0\log k$ with a $c > 0$. The contribution of this part of S_1 is $O(k^{-cr_0A})$, which is obviously negligible, providing A is sufficiently large. If $\sqrt{k} \le y \le t_0$ then (5.53) implies $\mathrm{Im}(h(y(1+r\omega);k,x)) > ck^2t^{-1}r_0^2$. The contribution of this part of

S_1 is $O\big(k\exp(-c\delta^{\frac{1}{2}}kr_0^2)\big)$. This ends the treatment of S_1. The part S_3 can be treated in a similar way, and we can show that its contribution is bounded likewise. As for S_2 we see by (5.51) and (5.53) that its contribution is $O(k^{-\frac{5}{2}}\delta^{\frac{3}{4}})$. Gathering these estimates we find that

$$I_1(x,k) \ll k^{-\frac{5}{2}}\delta^{\frac{3}{4}} \tag{5.54}$$

uniformly for $1-2\delta \le x \le 1-\delta$, if $\delta^{\frac{1}{2}}k \ge A\log(k/\delta)$ with a sufficiently large A.

We still have to consider the case where $\delta \le x \le 2\delta$. If $\delta^{\frac{1}{2}}k \le A\log(k/\delta)$ then we have the analogue of (5.44) with $\delta^{-\frac{1}{2}}k$ being replaced by $\delta^{\frac{1}{2}}k$, which yields (5.50) for the interval $\delta \le x \le 2\delta$. If $\delta^{\frac{1}{2}}k > A\log(k/\delta)$ then we use again the saddle point method. But, the only difference is that this time the solution of (5.51) is in the interval $[\frac{1}{2}\delta^{\frac{1}{2}}k,\ 2\delta^{\frac{1}{2}}k]$. Otherwise the situation is quite similar to the above, and we get (5.54) for the interval $\delta \le x \le 2\delta$ too.

Hence we may use (5.50) and (5.54) throughout the interval $[\delta,\ 1-\delta]$, whence via (5.40) and (5.41) we end the proof of (5.36). This completes our proof of Theorem 6.

Addendum

D. Bump *et al.* (*Duke Math. J.*, 66, 75-81(1992)) have recently succeeded in reducing Serre's exponent $\frac{1}{5}$ to $\frac{5}{28}$. This entails obvious improvements upon our results stated in the first section.

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