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THE BASED RING OF THE LOWEST TWO-SIDED CELL OF AN AFFINE WEYL GROUP, II

BY NANHUA XI ⁽¹⁾

ABSTRACT. — We show that the lowest based ring of an affine Weyl group W is very interesting to understand some simple representations of the corresponding Hecke algebra H_{q_0} ($q_0 \in \mathbb{C}^*$) even when q_0 is a root of 1.

Let H_{q_0} be the Hecke algebra (over $\mathbb{C}$) attached by Iwahori and Matsumoto [IM] to an affine Weyl group W and to a parameter $q_0^2 \in \mathbb{C}^*$.

When q_0 is not a root of 1 or $q_0^2 = 1$, the simple H_{q_0} -modules have been classified (*see* [KL2]). However we know little about the simple H_{q_0} -modules when q_0 is a root of 1. In this paper we give some discussion to the representations of H_{q_0} with q_0 a root of 1. Namely, let J be the asymptotic Hecke algebra defined in [L3, III]. There exists a natural injection $\phi_{q_0}: H_{q_0} \rightarrow J$. Let $K(J)$ [resp. $K(H_{q_0})$] be the Grothendieck group of J -modules (resp. H_{q_0} -modules) of finite dimension over $\mathbb{C}$, then ϕ_{q_0} induces a surjective homomorphism $(\phi_{q_0})_*: K(J) \rightarrow K(H_{q_0})$, when q_0 is not a root of 1 or $q_0^2 = 1$, $(\phi_{q_0})_*$ is an isomorphism (*loc. cit.*). For each two-sided cell c of W , we can define the direct summand $K(J_c)$ [resp. $K(H_{q_0,c})$] of $K(J)$ [resp. $K(H_{q_0})$]. Thus $(\phi_{q_0})_*$ induces a homomorphism $(\phi_{q_0})_{*,c}: K(J_c) \rightarrow K(H_{q_0,c})$. The map $(\phi_{q_0})_{*,c}$ remains surjective and is an isomorphism if q_0 is not a root of 1 or $q_0^2 = 1$. In this paper we mainly discuss the map $(\phi_{q_0})_{*,c_0}$, where c_0 is the lowest two-sided cell of W .

1. Introduction

1.1. Let G be a simply connected, almost simple complex algebraic group and T a maximal torus. Let $P \subseteq X = \text{Hom}(T, \mathbb{C}^*)$ be the root lattice. The Weyl group $W_0 = N_G(T)/T$ of G acts on X in a natural way and this action is stable on P . Thus we can form the affine Weyl group $W_a = W_0 \ltimes P$, which is a normal subgroup of the extended affine Weyl group $W = W_0 \ltimes X$. There exists a finite abelian subgroup Ω of W such that $W = \Omega \ltimes W_a$. Let $S = \{r_0, r_1, \dots, r_n\}$ be the set of simple reflections of W_a with $r_0 \notin W_0$. Then we have a standard length function l on W_a which can be extended

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to W by defining $l(\omega w) = l(w)$ for any $\omega \in \Omega$, $w \in W_a$. We keep the same notation for the extension of l .

1.2. For any $u = \omega_1 u_1$, $w = \omega_2 w_1$, $\omega_1, \omega_2 \in \Omega$, $u_1, w_1 \in W_a$, we define $P_{u, w}$ to be P_{u_1, w_1} , as in [KL 1] if $\omega_1 = \omega_2$ and $P_{u, w}$ to be zero if $\omega_1 \neq \omega_2$. We say that $u \leq_{LR} w$ or $u \leq_L w$ if $u_1 \leq_{LR} w_1$, or $u_1 \leq_L w_1$ in the sense of [KL 1], we say that $u \leq_R w$ if $u^{-1} \leq_L w^{-1}$. These relations generate equivalence relations $\sim_{LR}$, $\sim_L$, $\sim_R$ in W , respectively, and the corresponding equivalence classes are called two-sided cells, left cells, right cells of W , respectively. The relation $\leq_{LR}$ (resp. $\leq_L$, $\leq_R$) in W then induces a partial order $\leq_{LR}$ (resp. $\leq_L$, $\leq_R$) in the set of two-sided (resp. left, right) cells of W . We extend the Bruhat order $\leq$ in W_a to W by defining $u \leq w$ if and only if $\omega_1 = \omega_2$ and $u_1 \leq w_1$.

Let q be an indeterminate and let $A = \mathbb{C}[q, q^{-1}]$. Let H be the Hecke algebra of W over A , that is a free A -module with basis T_w ($w \in W$) and multiplication defined by

$$(T_r - q^2)(T_r + 1) = 0 \quad \text{if } r \in S \quad \text{and} \quad T_w T_{w'} = T_{ww'} \quad \text{if } l(ww') = l(w) + l(w').$$

For each $w \in W$, let

$$C_w = q^{-l(w)} \sum_{u \leq w} P_{u, w}(q^2) T_u \in H.$$

And we write

$$C_w C_u = \sum_z h_{w, u, z} C_z \in H, \quad h_{w, u, z} \in A.$$

For each $z \in W$, there is a well defined integer $a(z) \geq 0$ such that

$$\begin{aligned} q^{a(z)} h_{w, u, z} &\in \mathbb{C}[q] \quad \text{for all } w, u \in W \\ q^{a(z)-1} h_{w, u, z} &\notin \mathbb{C}[q] \quad \text{for some } w, u \in W \end{aligned}$$

(see [L 3, I, 7.3]). We have $a(z) \leq l(w_0)$, where w_0 is the longest element of W_0 . It is known that

$$c_0 = \{ w \in W \mid a(w) = l(w_0) \}$$

is a two-sided cell of W (see [S, I]) which is the lowest one for the partial order $\leq_{LR}$.

1.3. Let $\gamma_{w, u, z}$ be the constant term of $q^{a(z)} h_{w, u, z} \in \mathbb{C}[q]$. We have $\gamma_{w, u, z} \in \mathbb{N}$. Moreover (see [L 3, II])

$$(a) \quad \gamma_{w, u, z} \neq 0 \Rightarrow \begin{matrix} w \sim_L u^{-1}, & u \sim_L z, & w \sim_R z. \end{matrix}$$

Let J be the $\mathbb{C}$ -vector space with basis $(t_w)_{w \in W}$. This is an associative $\mathbb{C}$ -algebra with multiplication

$$t_w t_u = \sum_z \gamma_{w, u, z} t_z.$$

It has a unit element $1 = \sum_{d \in \mathcal{D}} t_d$, where $\mathcal{D} = \{d \in W_a \mid a(d) = l(d) - 2 \deg P_{e,d}\}$ (e is the unit of W) (see [L 3, II]).

For each two-sided cell c of W , let J_c be the subspace of J spanned by t_w , $w \in c$, then $J = \bigoplus_c J_c$, where the sum is over the set of all two-sided cells of W . By (a) we see that J_c is a two-sided ideal of J and in fact is an associative $\mathbb{C}$ -algebra with unit $\sum_{d \in \mathcal{D} \cap c} t_d$.

1.4. For each $q_0 \in \mathbb{C}^*$, we denote $H_{q_0} = H \otimes_A \mathbb{C}$, where $\mathbb{C}$ is an A -algebra with q acting as scalar multiplication by q_0 . We shall denote $T_w \otimes 1$, $C_w \otimes 1$ in H_{q_0} again by T_w , C_w . We also use the notation $h_{w,u,z}$ for the specialization at $q_0 \in \mathbb{C}^*$ of $h_{w,u,z}$.

The A -linear map $\phi: H \rightarrow J \otimes_{\mathbb{C}} A$ defined by

$$\phi(C_w) = \sum_{\substack{d \in \mathcal{D} \\ z \in W \\ a(z) = a(d)}} h_{w,d,z} t_z$$

is a homomorphism of A -algebra with 1 (see [L 3, II]). Let $\phi_{q_0}: H_{q_0} \rightarrow J$ be the induced homomorphism for any $q_0 \in \mathbb{C}^*$.

Any (left) J -module E gives rise, via $\phi_{q_0}: H_{q_0} \rightarrow J$, to a (left) H_{q_0} -module E_{q_0} . We denote by $K(J)$ [resp. $K(H_{q_0})$] the Grothendieck group of (left) J -modules (resp. H_{q_0} -modules) of finite dimension over $\mathbb{C}$. The correspondence $E \rightarrow E_{q_0}$ defines a homomorphism $(\phi_{q_0})_*: K(J) \rightarrow K(H_{q_0})$.

We similarly define $K(J_c)$ for any two-sided cell c of W . Then we have $K(J) = \bigoplus_c K(J_c)$, where the sum is over the set of all two-sided cells of W . Now we define $K(H_{q_0})_c$. For any simple H_{q_0} -module M , we attach to M a two-sided cell c_M of W by the following two conditions:

$$\begin{aligned} C_w M &\neq 0 \text{ for some } w \in c_M \\ C_w M &= 0 \text{ for any } w \text{ in a two-sided cell } c \text{ with } c \underset{LR}{\leq} c_M, c \neq c_M. \end{aligned}$$

Then c_M is well defined since there are only a finite number of two-sided cells in W . Let $K(H_{q_0})_c$ be the subgroup of $K(H_{q_0})$ spanned by simple H_{q_0} -modules M with $c_M = c$. Obviously we have $K(H_{q_0}) = \bigoplus_c K(H_{q_0})_c$. Thus for a two-sided cell c of W ,

$(\phi_{q_0})_*$ induces a homomorphism

$$(\phi_{q_0})_{*,c}: K(J_c) \rightarrow K(H_{q_0})_c.$$

The following result is due to Lusztig (see [L 3, III, 1.9 and 3.4]).

PROPOSITION 1.5. — *The map $(\phi_{q_0})_{*,c}$ is surjective for any $q_0 \in \mathbb{C}^*$, moreover, $(\phi_{q_0})_{*,c}$ is an isomorphism when q_0 is not a root of 1 or $q_0^2 = 1$.*

Now we state a conjecture.

CONJECTURE 1.6. — The map $(\phi_{q_0})_{*,c}$ is injective if $(\phi_{q_0})_{*,c'}$ is injective for some two-sided cell c' of W with $c' \leq c$.
LR

By proposition 1.6 one knows that $(\phi_{q_0})_{*,c}$ is injective is equivalent to that $(\phi_{q_0})_{*,c}$ is bijective.

We mainly discuss $(\phi_{q_0})_{*,c_0}$, where c_0 is the lowest two-sided cell of W . We prove that if $\sum_{w \in W_0} q_0^{2l(w)} \neq 0$, then $(\phi_{q_0})_{*,c_0}$ is injective (see Theorem 3.4) and show that $(\phi_{q_0})_{*,c_0}$ is likely not injective if $\sum_{w \in W_0} q_0^{2l(w)} = 0$ (see Theorem 3.6).

1.7. Let H'_{q_0} be the subalgebra of H_{q_0} spanned by T_w , $w \in W_0$. And let J' be the subspace of J spanned by t_w , $w \in W_0$. J' is a $\mathbb{C}$ -algebra with unit $\sum_{d \in \mathcal{D} \cap W_0} t_d$. Let

$\phi'_{q_0}: H'_{q_0} \rightarrow J'$ be defined by

$$\phi'_{q_0}(C_w) = \sum_{\substack{d \in \mathcal{D} \cap W_0 \\ z \in W_0 \\ a(d) = a(z)}} h_{w,d,z}(q_0) t_z, \quad w \in W_0,$$

then ϕ'_{q_0} is a $\mathbb{C}$ -algebra homomorphism preserving 1.

As in 1.4 we define $K(H'_{q_0})$, $K(J')$, $K(H'_{q_0})_{c'}$, $K(J')_{c'}$, $(\phi'_{q_0})_{*}$, $(\phi'_{q_0})_{*,c'}$, etc., where c' is a two-sided cell of W_0 . We also have

PROPOSITION 1.8. — $(\phi'_{q_0})_{*,c'}$ is surjective for any $q_0 \in \mathbb{C}^*$. Moreover $(\phi_{q_0})_{*,c'}$ is an isomorphism when q_0 is not a root of 1 or $q_0^2 = 1$.

CONJECTURE 1.9. — $(\phi_{q_0})_{*,c'}$ is injective if $(\phi'_{q_0})_{*,c''}$ is injective for some two-sided cell c'' of W_0 with $c'' \leq c'$.
LR

When c' is the lowest two-sided cell of W_0 , it is easy to see that $(\phi'_{q_0})_{*,c'}$ is injective if and only if $\sum_{w \in W_0} q_0^{2l(w)} \neq 0$.

2. The two-side cell c_0 and the ring J_{c_0}

In this section we recall and prove some results on c_0 and J_{c_0} .

2.1. We denote by w_0 the longest element in W_0 . Let

$$\mathfrak{S} = \{w \in W \mid l(w w_0) = l(w) + l(w_0) \quad \text{and} \quad w w_0 r \notin c_0 \text{ for any } r \in S \cap W_0\}.$$

Then $\mathcal{D}_0 = \mathcal{D} \cap c_0 = \{w w_0 w^{-1} \mid w \in \mathfrak{S}\}$ and $|\mathfrak{S}| = |W_0|$ (see [S, II]).

Let $X^+ = \{w \in W \mid l(rx) > l(x) \text{ for any } r \in S'\}$, where $S' = S \cap W_0$. Let $x_i \in X^+$ ($i \in \{1, 2, \dots, n\} = I_0$) be the i -th basic dominant weight, then x_i has the properties: $l(x_i r_i) < l(x_i)$, $x_i r_j = r_j x_i$, $l(x_i r_j) = l(x_i) + 1$ if $i \neq j \in I_0$. We have

$$c_0 = \{w' w_0 x w^{-1} \mid w, w' \in \mathfrak{S}, x \in X^+\} \quad (\text{see [S, II]}).$$

Moreover $l(w' w_0 x w^{-1}) = l(w') + l(w_0) + l(x) + l(w^{-1})$.

LEMMA 2.2. — *Let $u \in c_0$, then $C_u = h C_{w_0} h'$ for some $h, h' \in H_{q_0}$, i.e., the two-sided ideal $\bigoplus_{u \in c_0} \mathbb{C} C_u$ of H_{q_0} is generated by the element C_{w_0} .*

Proof. — Write $u = w' w_0 w$ for some $w', w \in W$ such that $l(u) = l(w') + l(w_0) + l(w)$. We use induction on $l(u)$ to prove that C_u is in the two-sided ideal N of H_{q_0} generated by C_{w_0} .

When $l(u) = l(w_0)$, then $C_u = C_{w_0} C_{w_0} C_{w_0}$ for some $\omega, \omega' \in \Omega$. Now assume that $l(w') > 0$. Let $s \in S$ be such that $sw' \leq w'$, then

$$C_s \cdot C_{su} = C_u + \sum_{\substack{z \in c_0 \\ l(z) < l(u)}} a_z C_z, \quad a_z \in \mathbb{N} \quad (\text{see [KL 1]}).$$

By induction hypothesis we know that $C_u \in N$. Similarly we can prove that $C_u \in N$ if $l(w) > 0$. The lemma is proved.

COROLLARY 2.3. — *For a simple H_{q_0} -module M , we have $c_M = c_0$ if and only if $C_{w_0} M \neq 0$.*

For $w \in W$, set $L(w) = \{r \in S \mid rw \leq w\}$ and $R(w) = \{r \in S \mid wr \leq w\}$.

LEMMA 2.4. — (i) *Let w' be the longest element in the Weyl group generated by $L(w)$ (or $R(w)$), then $w = w' w''$ (or $w = w'' w'$) for some $w'' \in W$ and $l(w) = l(w') + l(w'')$.*

(ii) *Let w' be the longest element in the Weyl group W' generated by $S - L(w)$ [resp. $S - R(w)$], then $l(w' w) = l(w') + l(w)$ [resp. $l(w w') = l(w) + l(w')$].*

Proof. — (i) follows from $T_{w'} C_w = q^{l(w')} C_w$ or $C_w T_{w'} = q^{l(w')} C_w$.

(ii) follows from the fact that w is the shortest element in $W' w$ or $w W'$.

Let Γ_0 be the left cell in c_0 containing w_0 , then

$$\begin{aligned} \Gamma_0 &= \{w w_0 x \mid x \in X^+, w \in \mathfrak{S}\} \\ &= \{w \in W \mid R(w) = S'\} \end{aligned}$$

LEMMA 2.5. — *Any element $u \in \Gamma_0$ has the form $w x w_j$, where $w \in W_0$, $x = \prod_{i=1}^n x_i^{a_i} \in X^+$. w_j is the longest element in $W_j = \langle r_j \mid a_j = 0, j \in I_0 \rangle$, moreover $l(u) = l(w) + l(x) + l(w_j)$.*

Proof. — Choose $x = \prod_{i=1}^n x_i^{a_i} \in X^+$ such that $u \in \Gamma_0 \cap W_0 x W_0$.

Then the shortest element in $W_0 x W_0$ is $xw_J w_0$ and the shortest element in $\Gamma_0 \cap W_0 x W_0$ is xw_J by lemma 2.4 (i), where w_J is the longest element in $W_J = \langle r_j | a_j = 0, j \in I_0 \rangle$. The lemma is proved.

LEMMA 2.6. — (i) Let $J \subseteq K \subseteq I_0$, then in H_{q_0} we have $C_{w_J} C_{w_K} = C_{w_K} C_{w_J} = \eta_J C_{w_K}$, where w_J, w_K are the longest element in $W_J = \langle r_j | j \in J \rangle$, $W_K = \langle r_k | k \in K \rangle$, respectively, $\eta_J = q_0^{-l(w_J)} \sum_{w \in W_J} q_0^{2l(w)}$.

(ii) $C_{ww_J} = h C_{w_J}$, $C_{w_J w'} = C_{w_J} h'$ for some $h, h' \in H_{q_0}$ if $l(ww_J) = l(w) + l(w_J)$, $l(w_J w') = l(w_J) + l(w')$.

Proof. — First we prove (ii). We use induction on $l(w)$. Assume that $l(w) > 0$. Choose $r \in S$ such that $rw \leq w$, then

$$C_r C_{rw_J} = C_{ww_J} + \sum_{\substack{z \in W \\ l(z) < l(ww_J)}} a_z C_z, \quad a_z \in \mathbb{N} \quad (\text{see [KL 1]}).$$

Moreover $a_z \neq 0$ implies that $z \leq_L rw_J$. So $R(z) \supseteq \{r_j | j \in J\}$ (see [KL 1]).

By Lemma 2.4 we see that $z = z' w_J$ for some $z' \in W$ and $l(z) = l(z') + l(w_J)$. By induction hypothesis we know that $C_{ww_J} = h C_{w_J}$ for some $h \in H_{q_0}$. Similarly we have $C_{w_J w'} = C_{w_J} h'$ for some $h' \in H_{q_0}$.

(i) follows from $C_J C_J = \eta_J C_J$ and (ii).

COROLLARY 2.7. — Let x, w_J be as in 2.5, then in H_{q_0} we have

$$C_{w_0} C_{xw_J} = \eta_J \sum_{\substack{y \in X^+ \\ w_0 y \leq w_0 x}} a_{x,y} C_{w_0 y} \in \mathbb{C}, \quad a_{x,y} \in \mathbb{C} \quad \text{and} \quad a_{x,x} = 1.$$

Proof. — By 2.1 and 2.6(ii) we see that $C_{xw_J} = C_{w_J x} = C_{w_J} h$, where

$$h = \sum_{\substack{w \in W \\ l(w_J w) = l(w_J) + l(w) \\ w_J w \leq w_J x}} a_w T_w, \quad a_w \in \mathbb{C}, \quad a_x = q_0^{-l(x)}.$$

By (2.6(i)) we know that

$$(a) \quad C_{w_0} \cdot C_{xw_J} = C_{w_0} \cdot C_{w_J} h = \eta_J C_{w_0} h.$$

Note that $h_{w_0, xw_J, z} \neq 0$ implies that $z \sim_L xw_J$, $z \sim_R w_0$ (see [L 3, I]), we have $z \in \Gamma_0 \cap \Gamma_0^{-1} = \{w_0 y | y \in X^+\}$. So by (a) we get

$$C_{w_0} C_{xw_J} = \eta_J \sum_{y \in X^+} a_{x,y} C_{w_0 y}, \quad a_{x,y} \in \mathbb{C}.$$

Since $a_x = q_0^{-l(x)}$ and $l(w) < l(x)$ if $a_w \neq 0$, $w \neq x$. We have $a_{x,x} = 1$ and $a_{x,y} = 0$ if $l(y) > l(x)$ or $l(y) = l(x)$ but $x \neq y$. Let $w \in W$ be such that $a_w \neq 0$. Consider the expression

$$C_{w_0} \cdot T_w = \sum_{z^{-1} \in \Gamma_0} b_z C_z, \quad b_z \in \mathbb{C}.$$

Since $w_j w \leq w_j x$, we have $b_z \neq 0$ implies that $z \leq w_0 x$. Thus by (a) we know that $a_{x,y} \neq 0$ implies that $w_0 y \leq w_0 x$. The Corollary is proved.

2.8. For any $x \in X$, we choose $x', x'' \in X^+$ such that $x = x' x''^{-1}$ and then define $\tilde{T}_x = q_0^{-l(x')} T_{x'} (q_0^{-l(x'')} T_{x''})^{-1}$. $\tilde{T}_x$ is independent of the choices x' and x'' . We denote the conjugacy class of $x \in X$ in W by O_x and let $z_x = \sum_{x' \in O_x} \tilde{T}_{x'}$. z_x is in the center of

H_{q_0} . For $x \in X^+$, denote $d(x', x)$ the dimension of the x' -weight space $V(x)_{x'}$ of $V(x)$, where $V(x)$ is the irreducible representation of G with highest weight x . We set $S_x = \sum_{x' \in X^+} d(x', x) z_{x'}$, $x \in X^+$.

LEMMA 2.9. (see [X]). — In H_{q_0} we have $C_{w' w_0 w^{-1}} S_x = S_x C_{w' w_0 w^{-1}} = C_{w' w_0 x w^{-1}}$ for any $w', w \in \mathfrak{S}$, $x \in X^+$.

LEMMA 2.10. — Let $u \in \Gamma_0$, then

$$C_u = \sum_{\substack{y \in X^+ \\ I \subseteq I_0}} h_{I,y} C_{x_I w_I} S_y,$$

where $h_{I,y} \in H'_{q_0} = \bigoplus_{w \in W_0} \mathbb{C} T_w = \bigoplus_{w \in W_0} \mathbb{C} C_w$, $x_I = \prod_{i \in I} x_i$, $I' = I_0 - I$.

Proof. — By 2.5 we see that $u = w x w_j$, where $w \in W_0$, $x = \prod_{i=1}^n x_i^{a_i}$, $J = \{j \in I_0 \mid a_j = 0\}$.

We use induction on $l(u)$, when $w = e$, by 2.9 we see that $C_u = C_{x_J w_J} S_y$, where $J' = I_0 - J$, $y = \prod_{j \in J'} x_j^{a_j - 1}$, i.e. the lemma is true. Now assume that $l(w) > 0$ and choose $r \in S'$ such that $rw \leq w$, then

$$C_r \cdot C_{r w x w_j} = C_{w x w_j} + \sum_{\substack{z \in \Gamma_0 \\ l(z) < l(w x w_j)}} a_z C_z, \quad a_z \in \mathbb{N}.$$

By induction hypothesis we know that there exists $h_{I,y} \in H'_{q_0}$ such that $C_u = \sum_{\substack{y \in X^+ \\ I \subseteq I_0}} h_{I,y} C_{x_I w_I} S_y$. The lemma is proved.

2.11. Let R_G be the ring of the rational representations ring of G tensor with $\mathbb{C}$. Then R_G is a $\mathbb{C}$ -algebra with a $\mathbb{C}$ -basis $V(x)$, $x \in X^+$. Let $M_{\mathfrak{S} \times \mathfrak{S}}(R_G)$ be the $\mathfrak{S} \times \mathfrak{S}$ matrix

ring over R_G . Then we have

THEOREM 2.12 (see [X]). — *There is a natural isomorphism $J_{c_0} \xrightarrow{\sim} M_{\mathfrak{S} \times \mathfrak{S}}(R_G)$ such that $t_{w'w_0xw}^{-1} \rightarrow (m_{w_1, w_2}) \in M_{\mathfrak{S} \times \mathfrak{S}}(R_G)$, $w', w^{-1}, w_1, w_2 \in \mathfrak{S}$,*

$$m_{w_1, w_2} = \begin{cases} V(x) & \text{if } w_1 = w', \quad w_2 = w \\ 0 & \text{otherwise.} \end{cases}$$

Hereafter we identify J_{c_0} with $M_{\mathfrak{S} \times \mathfrak{S}}(R_G)$.

3. The homomorphism $(\phi_{q_0})_{*, c_0}$

3.1. For any semisimple conjugacy class s in G , we have a simple representation ψ_s of $J_{c_0} \simeq M_{\mathfrak{S} \times \mathfrak{S}}(R_G)$:

$$\begin{aligned} \psi_s: M_{\mathfrak{S} \times \mathfrak{S}}(R_G) &\rightarrow M_{\mathfrak{S} \times \mathfrak{S}}(\mathbb{C}) \\ (m_{w, w'}) &\rightarrow (tr(s, m_{w, w'})), \quad w, w' \in \mathfrak{S}. \end{aligned}$$

Any simple representation of J_{c_0} is isomorphic to some ψ_s (see [X]). Let E_s be the simple J_{c_0} -module providing the representation ψ_s . E_s gives rise, via

$$\phi_{q_0, c_0}: H_{q_0} \rightarrow J \rightarrow J_{c_0},$$

to an H_{q_0} -module E_{s, q_0} . Note that $\phi_{q_0, c_0}(S_x) = \sum_{w \in \mathfrak{S}} t_{ww_0xw}^{-1}$ for any $x \in X^+$, we see that S_x acts on E_{s, q_0} by scalar $tr(s, V(x))$.

PROPOSITION 3.2. — *The set $\Lambda = \{(\phi_{q_0})_{*, c_0}(E_s) \mid s \text{ semisimple conjugacy class of } G\} - \{0\}$ is a base of $K(H_{q_0})_{c_0}$.*

Proof. — It is easy to see that $(\phi_{q_0})_{*, c_0}(E_s) = \sum_M a_M M$, where the sum is over the set of composition factors M of E_{s, q_0} with $c_M = c_0$ and a_M is the multiplicity of M in E_{s, q_0} .

Now let $F_i = (\phi_{q_0})_{*, c_0}(E_{s_i}) \in \Lambda$, $1 \leq i \leq k$, and suppose that $\sum_{i=1}^k m_i F_i = 0$, $m_i \in \mathbb{Z}$. Let $F_i = \sum_{M_{ij}} a_{M_{ij}} M_{ij}$, M_{ij} simple H_{q_0} -module with $c_{M_{ij}} = c_0$. Since S_x acts on E_{s_i, q_0} by scalar $tr(s_i, V(x))$. S_x acts on M_{ij} by scalar $tr(s_i, V(x))$ if $a_{M_{ij}} \neq 0$. $F_i \in \Lambda$ implies that $a_{M_{ij}} \neq 0$ for some M_{ij} . Therefore $m_i = 0$. By 1.6 we know that $(\phi_{q_0})_{*, c_0}$ is surjective, hence Λ is a base of $K(H_{q_0})_{c_0}$. The proposition is proved.

COROLLARY 3.3. — *E_{s, q_0} has at most one composition factor to which the attached two-sided cell is c_0 . Moreover the multiplicity a_M is 1 if E_{s, q_0} has such a composition factor M .*

THEOREM 3.4. — *If $\sum_{w \in W_0} q_0^{2l(w)} = q_0^{l(w)} \eta_{I_0} \neq 0$, then $(\phi_{q_0})_{*, c_0}$ is injective, so $(\phi_{q_0})_{*, c_0}$ is an isomorphism by 1.6.*

Proof. — We have

$$\phi_{q_0, c_0}(C_{w_0}) = \sum_{\substack{w \in \mathfrak{S} \\ x \in X^+}} h_{w_0, ww_0w^{-1}, w_0xw^{-1}} t_{w_0xw^{-1}} \in J_{c_0}.$$

We identify J_{c_0} with $M_{\mathfrak{S} \times \mathfrak{S}}(R_G)$, then $\phi_{q_0, c_0}(C_{w_0}) = (m_{w', w}) \in M_{\mathfrak{S} \times \mathfrak{S}}(R_G)$ and

$$m_{w', w} = \begin{cases} \sum_{x \in X^+} h_{w_0, ww_0w^{-1}, w_0xw^{-1}} V(x), & \text{if } w' = e \\ 0 & \text{if } w' \neq e \end{cases}.$$

Note that $C_{w_0} C_{w_0} = \eta_{I_0} C_{w_0}$, we see that $m_{e, e} = \eta_{I_0} \neq 0$, where e is the unit in W . So $C_{w_0} E_{s, q_0} \neq 0$ for any semisimple conjugacy class s of G since $\psi_s \phi_{q_0, c_0}(C_{w_0}) \neq 0$. Now let $0 = F_0 \subseteq F_1 \subseteq \dots \subseteq F_k = E_{s, q_0}$ be a composition series of E_{s, q_0} and let i be the integer such that $C_{w_0} F_i \neq 0$ and $C_{w_0} F_{i-1} = 0$. Then $C_{w_0} M \neq 0$ where $M = F_i / F_{i-1}$, otherwise, $C_{w_0} F_i \subseteq F_{i-1}$, choose $v \in F_i$ such that $C_{w_0} v \neq 0$, we have $C_{w_0}^2 v = \eta_{I_0} C_{w_0} v \neq 0$. A contradiction, so $C_{w_0} M \neq 0$, i.e., $c_M = c_0$. That is to say $(\phi_{q_0})_{*, c_0}(E_s) \neq 0$. The theorem follows from proposition 3.2.

3.5. In the subsequent part of this section we assume that $\eta_{I_0} = 0$, i.e., $\sum_{w \in W_0} q_0^{2l(w)} = 0$.

Let $\Delta_{q_0} = \{I \subseteq I_0 \mid \eta_{I'} \neq 0 \text{ but } \eta_{I' \cup \{i\}} = 0 \text{ for any } i \in I\}$. Here we use the convention that I' always denotes the complement of I in I_0 i.e., $I' = I_0 - I$.

THEOREM 3.6. — *Let s be a semisimple conjugacy class of G , then $(\phi_{q_0})_{*, c_0}(E_s) = 0$ if and only if $\alpha_I = 0$ for all $I \in \Delta_{q_0}$, where*

$$\alpha_I = \sum_{x \in X^+} h_{w_0, xI w_I', w_0x} \text{tr}(s, V(x)) \text{ for any } I \subseteq I_0.$$

We need two lemmas.

LEMMA 3.7. — *The following conditions are equivalent.*

- (i) $C_{w_0} E_{s, q_0} = 0$.
- (ii) $\psi_s \phi_{q_0, c_0}(C_{w_0}) = 0$.
- (iii) $\sum_{x \in X^+} h_{w_0, ww_0w^{-1}, w_0xw^{-1}} \text{tr}(s, V(x)) = 0$ for all $w \in \mathfrak{S}$.
- (iv) $\sum_{x \in X^+} h_{w_0, ww_0, w_0x} \text{tr}(s, V(x)) = 0$ for all $w \in \mathfrak{S}$.
- (v) $\alpha_I = \sum_{x \in X^+} h_{w_0, xI w_I', w_0x} \text{tr}(s, V(x)) = 0$ for all $I \subseteq I_0$.
- (vi) $\alpha_I = \sum_{x \in X^+} h_{w_0, xI w_I', w_0x} \text{tr}(s, V(x)) = 0$ for all $I \in \Delta_{q_0}$.

Proof. — (i) and (ii) are obviously equivalent.

Note that $h_{w_0, ww_0w^{-1}, z} \neq 0$ implies that $z = w_0 x w^{-1}$ for some $x \in X^+$ and that $\phi_{q_0, c_0}(C_{w_0}) = (m_{w', w})$,

$$m_{w', w} = \begin{cases} \sum_{x \in X^+} h_{w_0, ww_0w^{-1}, w_0xw^{-1}} V(x), & \text{if } w' = e \\ 0, & \text{otherwise} \end{cases}$$

we see that (ii) $\Leftrightarrow$ (iii).

By theorem 2.9 in [X] we have $h_{w_0, ww_0, w_0x} = h_{w_0, ww_0w^{-1}, w_0xw^{-1}}$. So we have (iii) $\Leftrightarrow$ (iv).

By Lemma 2.4 (i) we see that $x_1 w_{I'} = w w_0$ for some $w \in W$. Using the method in [S] one knows that $w \in \mathfrak{S}$. Thus we have (iv) $\Rightarrow$ (v). Now we show that (v) $\Rightarrow$ (iv). Let $w \in \mathfrak{S}$, then $ww_0 \in \Gamma_0$, hence by 2.10

$$C_{ww_0} = \sum_{\substack{y \in X^+ \\ I \subseteq I_0}} h_{I, y} C_{x_1 w_{I'}} S_y, \quad h_{I, y} \in H'_{q_0}.$$

Since $C_{w_0} h_{I, y} = a_{I, y} C_{w_0}$ for some $a_{I, y} \in \mathbb{C}$, we have

$$\sum_{x \in X^+} h_{w_0, ww_0, w_0x} \text{tr}(s, V(x)) = \sum_{\substack{y \in X^+ \\ I \subseteq I_0}} a_{I, y} \alpha_I \text{tr}(s, V(y)) = 0.$$

Finally we prove that (v) and (vi) are equivalent.

One direction is obvious. Now assume that (vi) holds. Let $J \subseteq I_0$. We use induction on $l(x_J)$ to prove that $\alpha_J = 0$. When $\eta_{J'} = 0$ or $J \in \Delta_{q_0}$ we have $\alpha_J = 0$ by 2.7 or by (vi). Suppose $\eta_{J'} \neq 0$ and $J \notin \Delta_{q_0}$. Choose $j \in J$ such that $\eta_{J' \cup \{j\}} \neq 0$. Let $K = J - \{j\}$, then $K' = J' \cup \{j\}$. We have

$$\begin{aligned} C_{w_0} C_{x_J w_{J'}} &= \frac{1}{\eta_{K'}} C_{w_0} C_{w_{K'}} C_{x_J w_{J'}} \quad (\text{by 2.6}) \\ &= \frac{\eta_{J'}}{\eta_{K'}} C_{w_0} (C_{w_{K'} x_K x_j} + \sum_{\substack{I \subseteq I_0 \\ y \in X^+}} h_{I, y} C_{x_1 w_{I'}} S_y), \quad h_{I, y} \in H'_{q_0} \quad (\text{by 2.6, 2.10}). \end{aligned}$$

Let $C_{w_0} h_{I, y} = a_{I, y} C_{w_0}$, $a_{I, y} \in \mathbb{C}$. By 2.7 we see that $a_{I, y} \eta_{I'} \neq 0$ implies that $l(x_1 y) < l(x_J)$. Obviously $l(x_K) < l(x_J)$. Using induction hypothesis we get

$$\alpha_J = \frac{\eta_{J'}}{\eta_{K'}} (\alpha_K \text{tr}(s, V(x_j)) + \sum_{\substack{I \subseteq I_0 \\ y \in X^+}} a_{I, y} \alpha_I \text{tr}(s, V(y))) = 0.$$

The lemma is proved.

LEMMA 3.8. — $(\phi_{q_0})_{*, c_0}(E_s) = 0$ if and only if $C_{w_0} E_{s, q_0} = 0$.

Proof. — The “if” part is obvious. The “only if” part need to do a little more.

Assume that $C_{w_0}E_{s, q_0} \neq 0$. By 3.7 we see that $\alpha_i \neq 0$ for some $I \subseteq I_0$. As in [LX] we define an automorphism $\alpha: W \rightarrow W$ by

$$\alpha(wx) = w_0 wx^{-1} w_0, \quad w \in W_0, \quad x \in X.$$

One verifies that α leaves stable W_0, X, S, S' . In particular, α induces a bijection $\alpha: I_0 \rightarrow I_0$ and an automorphism $\sigma: H_{q_0} \rightarrow H_{q_0}$ by defining $C_u \rightarrow C_{\alpha(u)}$, $u \in W$. Let $J = \alpha(I)$, we have $\alpha(x_J) = x_J$, $\alpha(w_{I'}) = w_{J'}$. Consider

$$\psi_s \phi_{q_0, c_0}(C_{x_J^{-1} w_{J'}}) = (n_{w', w}) \in M_{\mathfrak{S} \times \mathfrak{S}}(\mathbb{C}).$$

By 2.4 and 2.12, we know that $n_{w', w} = 0$ if $w' \neq e$ and

$$n_{e, w} = \sum_{x \in X^+} h_{x_J^{-1} w_{J'}, w w_0 w^{-1}, w_0 x w^{-1}} tr(s, V(x)).$$

In particular,

$$n_{e, e} = \sum_{x \in X^+} h_{x_J^{-1} w_{J'}, w_0, w_0 x} tr(s, V(x)).$$

We claim that $n_{e, e} = \alpha_i$. In fact, let ι be the antiautomorphism of H_{q_0} defined by $C_u \rightarrow C_{u^{-1}}$, $u \in W$. Apply ι to the equality

$$C_{w_0} C_{x_I w_{I'}} = \sum_{x \in X^+} h_{w_0, x_I w_{I'}, w_0 x} C_{w_0 x}.$$

We get

$$C_{x_I^{-1} w_{I'}} C_{w_0} = \sum_{x \in X^+} h_{w_0, x_I w_{I'}, w_0 x} C_{x^{-1} w_0}.$$

Apply σ to the above identity we obtain

$$C_{x_J^{-1} w_{J'}} C_{w_0} = \sum_{x \in X^+} h_{w_0, x_I w_{I'}, w_0 x} C_{w_0 x}.$$

Therefore $h_{x_J^{-1} w_{J'}, w_0, w_0 x} = h_{w_0, w_I w_{I'}, w_0 x}$ and $n_{e, e} = \alpha_i \neq 0$. By this and $n_{w', w} = 0$ if $w' \neq e$ we see that α_i is an eigenvalue of $\psi_s \phi_{q_0, c_0}(C_{x_J^{-1} w_{J'}})$. Let $0 \neq v \in E_{s, q_0}$ be such that $C_{x_J^{-1} w_{J'}} v = \alpha_i v$. Let F be the H_{q_0} -submodule of E_{s, q_0} generated by v . Then F has a maximal H_{q_0} -submodule F_0 which doesn't contain v . F/F_0 is an irreducible H_{q_0} -module. Moreover $C_{x_J^{-1} w_{J'}}(F/F_0) \neq 0$ since $v \notin F_0$. We have proved that $(\phi_{q_0})_{*, c_0}(E_s) \neq 0$.

Theorem 3.6 follows from 3.7 and 3.8.

3.9. There are two special cases. One is that $\eta_{I_0} = 0$ but $\eta_I \neq 0$ for any proper subset I of I_0 . In this case we have $\Delta_{q_0} = \{\{i\} \mid i \in I_0\}$. Let $i' = I - \{i\}$. By 2.7 we have $h_{w_0, x_I w_{I'}, w_0 x} = \eta_{i'} a_{i, x}$ for some $a_{i, x} \in \mathbb{C}$. Moreover, $a_{i, x} \neq 0$ implies that $w_0 x \leq w_0 x_i$ and $a_{i, x_i} = 1$. By this we see that the equation system

$$\alpha_{\{i\}} = \eta_{i'}, \quad \sum_{\substack{x \in X^+ \\ w_0 x \leq w_0 x_i}} a_{i, x} tr(s, V(x)) = 0, \quad i \in I_0$$

uniquely determines $\text{tr}(s, V(x_i))$, $i \in I_0$. In other words, there exists a unique semisimple conjugacy class s of G such that $\alpha_{\{i\}} = 0$ for all $i \in I_0$. By 3.6 we have got the following.

PROPOSITION. — *There exists a unique semisimple conjugacy class s of G such that $(\phi_{q_0})_{*, c_0}(E_s) = 0$ when $\eta_{I_0} = 0$ but $\eta_I \neq 0$ for any proper subset I of I_0 .*

When W is of type $\tilde{A}_n$. We can determine the semisimple conjugacy class s in the proposition explicitly. We have $a_{i,x} = 0$ if $x \neq x_i$ since x_i is a minimal dominant weight for any $i \in I_0$. So $\alpha_{\{i\}} = \eta_i \cdot \text{tr}(s, V(x_i))$. Let T be the diagonal subgroup of $G = \text{SL}_{n+1}(\mathbb{C})$. We may require that $x_i \in \text{Hom}(T, \mathbb{C}^*)$ is defined by $x_i(t) = t_1 t_2 \dots t_i$ where $t = \text{diag}(t_1, t_2, \dots, t_{n+1}) \in T$. Thus, we have

$$\text{tr}(s, V(x_i)) = \sum_{\substack{j_a \in I_0 \cup \{n+1\} \\ j_a \neq j_b \text{ if } a \neq b}} t_{j_1} t_{j_2} \dots t_{j_i}.$$

where $t = \text{diag}(t_1, t_2, \dots, t_{n+1}) \in s \cap T$, s a semisimple conjugacy class of G . Hence, $\text{tr}(s, V(x_i)) = 0$, $1 \leq i \leq n$ is equivalent to that t_i ($1 \leq i \leq n+1$) is the solution of the equation $\lambda^{n+1} + (-1)^{n+1} = 0$. So if $\eta_{I_0} = 0$ but $\eta_I \neq 0$ for any proper subset I of I_0 , $(\phi_{q_0})_{*, c_0}(E_s) = 0$ if and only if the eigenpolynomial of s is $\lambda^{n+1} + (-1)^{n+1}$.

Another special case is that $q_0 + q_0^{-1} = 0$. In this case $\Delta_{q_0} = \{I_0\}$. So $(\phi_{q_0})_{*, c_0}(E_s) = 0$ if and only if $\alpha_{I_0} = 0$. If we identify the set $\{\text{semisimple conjugacy classes of } G\}$ with $\mathbb{C}^n$ through the bijection

$$s \rightarrow (\text{tr}(s, V(x_1)), \text{tr}(s, V(x_2)), \dots, \text{tr}(s, V(x_n))),$$

then $\alpha_{I_0} = 0$ defines a hypersurface in $\mathbb{C}^n$. That is to say, the set $\{\text{semisimple conjugacy class } s \text{ of } G \mid (\phi_{q_0})_{*, c_0}(E_s) = 0\}$ is a variety of dimension $n-1$.

When W_0 is of rank 2, if $\eta_{I_0} = 0$, then either $\eta_I \neq 0$ for any proper subset $I \subseteq I_0$ or $q_0 + q_0^{-1} = 0$. The above discussion shows that $(\phi_{q_0})_{*, c_0}$ is an isomorphism if and only if $\eta_{I_0} \neq 0$.

3.10. In general it is difficult to compute $C_{w_0} C_{x_1 w_1}$ in H . Now we compute it for the simplest case: x_1 is the highest short root.

When $x_1 \in X^+$ is the highest short root, $x_1 w_1 = r_0 w_0$, and $w_0 x \leq w_0 x_1$, $x \in X^+$ implies that $x = e$ or x_1 . So by 2.7, in H we have

$$C_{w_0} C_{r_0 w_0} = C_{w_0} C_{x_1 w_1} = \sigma_{I'} (C_{w_0 x_1} + a C_{w_0}),$$

where $\sigma_{I'} \in A = \mathbb{C}[q, q^{-1}]$ is determined by $C_{w_1'} C_{w_1'} = \sigma_{I'} C_{w_1'}$, $a \in A$. We need to determine the coefficient a . Comparing the coefficient of T_e in both sides we get

$$q^{-l(w_0)-1} \sigma_{I_0} = q^{-l(w_0 w_1)} \sigma_{I'} P_{e, w_0 x_1}(q^2) + a q^{-l(w_0)} \sigma_{I'}.$$

i. e.

$$(a) \quad \sigma_{I_0} = q^{1-l(x_1)} \sigma_{I'} P_{w_0, w_0 w_1}(q^2) + a q \sigma_{I'}.$$

Using the formula 8.10 in [L2] we get the following

PROPOSITION 3.11. — *If x_1 is the highest short weight, then*

$$P_{w_0, w_0 x_1} = \begin{cases} \sum_{i=1}^n q^{e_i-1} & \text{for type } \tilde{A}_n, \tilde{D}_n, \tilde{E}_n. \\ 1 & \text{for type } \tilde{C}_n, \tilde{G}_2. \\ \frac{q^{2(n-1)} - 1}{q^2 - 1} & \text{for type } \tilde{B}_n. \\ q^4 + 1 & \text{for type } \tilde{F}_4. \end{cases}$$

where $e_1, \dots, e_n$ are the exponents of W_0 .

By the proposition and 3.10(a) we obtain the following

PROPOSITION 3.12. — *In H we have*

$$C_{w_0} C_{r_0 w_0} = C_{w_0} C_{x_1 w_1} = \sigma_{1_0} C_{w_0 x_1} + \frac{\sigma_{1_0}}{[e_n + 1]} [e_n] C_{w_0},$$

where e_n is the largest exponent of W_0 and $[i] = (q^i - q^{-i})/(q - q^{-1})$ for any $i \in \mathbb{N}$.

3.13. When W is of type $\tilde{A}_n$, the highest short weight is $x_1 x_n$.

$$\eta_{1_0} = [2]_{q_0} [3]_{q_0} \dots [n+1]_{q_0},$$

where $[i]_{q_0}$ is the specialization at $q_0 \in \mathbb{C}^*$ of $[i]$. By 3.12, in H_{q_0} we have

$$C_{w_0} C_{r_0 w_0} = [2]_{q_0} [3]_{q_0} \dots [n-1]_{q_0} (C_{w_0 x_1 x_n} + [n]_{q_0}^2 C_{w_0}).$$

Now suppose $[n]_{q_0} = 0$ but $[i]_{q_0} \neq 0$ for $i, 1 \leq i \leq n-1$, then $\Delta_{q_0} = \{\{1, n\}, \{2\}, \{3\}, \dots, \{n-1\}\}$. By 3.9 and 3.12 we see that $\alpha_1 = 0, I \in \Delta_{q_0}$ is equivalent to $\text{tr}(s, V(x_1 x_n)) = 0, \text{tr}(s, V(x_i)) = 0, 2 \leq i \leq n-1$. Note that $\text{tr}(s, V(x_1 x_n)) = \text{tr}(s, V(x_1)) \text{tr}(s, V(x_n)) - 1$, by 3.9, we know that $\alpha_1 = 0, I \in \Delta_{q_0}$ if and only if the eigenpolynomial of s has the form $\lambda^{n+1} - a\lambda^n + (-1)^n a^{-1} \lambda + (-1)^{n+1}, a \in \mathbb{C}^*$. In other words, if $[n]_{q_0} = 0, [i]_{q_0} \neq 0, 1 \leq i \leq n-1$, then $(\phi_{q_0})_{*, c_0}(E_s) = 0$ if and only if the eigenpolynomial of s has the form $\lambda^{n+1} - a\lambda^n + (-1)^n a^{-1} \lambda + (-1)^{n+1}, a \in \mathbb{C}^*$.

4. Examples

4.1. Type $\tilde{A}_1$. In this case $G = \text{SL}_2(\mathbb{C}), S = \{r_0, r_1\}, x_1 = r_0 \omega, \Omega = \{e, \omega\}, \eta_{1_0} = q_0 + q_0^{-1}, c_0 = \{w \in W \mid l(w) > 0\}$. Another two-sided cell c of W is Ω .

J_c has two irreducible modules F_0, F_1 . Both have dimension 1 and t_ω acts on F_i by scalar $(-1)^i, i = 0, 1$. Via, $\phi_{q_0, c}: H_{q_0} \rightarrow J_c, F_i$ becomes H_{q_0} -module F_{i, q_0} . T_ω acts on F_{i, q_0} by scalar $(-1)^i$ and T_{r_i} acts on F_{i, q_0} by scalar -1 . $(\phi_{q_0})_{*, c}$ is an isomorphism for any $q_0 \in \mathbb{C}^*$.

For c_0 , we have $J_{c_0} = M_{2 \times 2}(\mathbb{R}_G)$ and

$$\begin{aligned}\phi_{q_0, c_0}(C_{r_1}) &= \begin{pmatrix} \eta_{l_0} & V(x_1) \\ 0 & 0 \end{pmatrix} \\ \phi_{q_0, c_0}(C_{r_0}) &= \begin{pmatrix} 0 & 0 \\ V(x_1) & \eta_{l_0} \end{pmatrix} \\ \phi_{q_0, c_0}(C_\omega) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.\end{aligned}$$

Suppose that $\eta_{l_0} \neq 0$. Let s be the semisimple conjugacy class of G containing $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \in G$, then E_{s, q_0} is irreducible if and only if $\eta_{l_0} \neq \pm(t + t^{-1})$. When $\eta_{l_0} = \pm(t + t^{-1})$, $E_{s, q_0}/F_{i, q_0} \simeq M_{s, q_0}$, where $i=0$ if $\eta_{l_0} = -(t + t^{-1})$ and $i=1$ if $\eta_{l_0} = t + t^{-1}$. T_ω acts on M_{s, q_0} by scalar $(-1)^{i-1}$ and T_{r_i} acts on M_{s, q_0} by scalar q_0^2 . $(\phi_{q_0})_{*, c}(E_s) = E_{s, q_0}$ if $\eta_{l_0} \neq \pm(t + t^{-1})$, $(\phi_{q_0})_{*, c_0}(E_s) = M_{s, q_0}$ if $\eta_{l_0} = \pm(t + t^{-1})$. In particular, when $\eta_{l_0} \neq 0$, $(\phi_{q_0})_*$ is an isomorphism.

When $\eta_{l_0} = 0$, one verifies that E_{s, q_0} is irreducible if $t + t^{-1} \neq 0$ and $E_{s, q_0} = F_{0, q_0} \oplus F_{1, q_0}$ if $t + t^{-1} = 0$. In particular $\text{rank ker}(\phi_{q_0})_* = 1$.

4.2. Type $\tilde{A}_2$. In this case we have $G = \text{SL}_3(\mathbb{C})$, $S = \{r_0, r_1, r_2\}$, $\Omega = \{1, \omega, \omega^2\}$ and $\omega r_0 = r_1 \omega$, $\omega r_1 = r_2 \omega$, $\omega r_2 = r_0 \omega$, $x_1 = r_0 r_2 \omega$, $x_2 = r_0 r_1 \omega^2$. W has three two-sided cells: $c = \Omega$, c_0 , $c' = W - c \cup c_0$. c' is the two-sided cell of W containing r_0, r_1, r_2 .

It is obviously $(\phi_{q_0})_{*, c}$ is an isomorphism.

Now consider $J_{c'}$. Any element in c' has one of the following forms: $\omega^i r_1 x_1^a \omega^j$, $\omega^{i+1} x_1^a \omega^j$, $\omega^{i+2} r_2 x_2^a \omega^{j+1}$, $\omega^{i+1} x_2^a \omega^{j+1}$, $i, j = 0, 1, 2$. We define a $\mathbb{C}$ -linear map $\theta: J_{c'} \rightarrow M_{3 \times 3}(A)$, $A = \mathbb{C}[q, q^{-1}]$, by $\theta(t_w) = (\mathcal{M}_{ab}) \in M_{3 \times 3}(A)$, $w \in c'$. Assume that w is of one of the above forms, then $m_{ab} = 0$ except $(a, b) = (i+1, j+1)$ and

$$m_{i+1, j+1} = \begin{cases} q^{2a} & \text{if } w = \omega^i r_1 x_1^a \omega^j \\ q^{2a-1} & \text{if } w = \omega^{i+1} x_1^a \omega^j \\ q^{-2a} & \text{if } w = \omega^{i+2} r_2 x_2^a \omega^{j+1} \\ q^{-2a+1} & \text{if } w = \omega^{i+1} x_2^a \omega^{j+1}. \end{cases}$$

By [L 1, 3.8] we know that θ is a $\mathbb{C}$ -algebra isomorphism. We have

$$\begin{aligned}\theta \phi_{q_0, c'}(C_{r_1}) &= \begin{pmatrix} [2]_{q_0} & q^{-1} & q \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \theta \phi_{q_0, c'}(C_\omega) &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.\end{aligned}$$

Specialize q to $a \in \mathbb{C}^*$, we get a simple representation ψ_a of $J_{c'} = M_{3 \times 3}(A)$ and any simple representation of $J_{c'}$ is isomorphic to some ψ_a , $a \in \mathbb{C}^*$. Let E_a be a simple $J_{c'}$ -module providing ψ_a .

A little surprisingly, the homomorphism $(\phi_{q_0})_{*,c'}: K(J_{c'}) \rightarrow K(H_{q_0})_{c'}$, is an isomorphism for any $q_0 \in \mathbb{C}^*$. In fact, via $\phi_{q_0,c'}: H_{q_0} \rightarrow J \rightarrow J_{c'}$, E_a gives rise to an H_{q_0} -module E_{a,q_0} . One verifies that E_{a,q_0} has a unique quotient M_{a,q_0} such that the attached two-sided cell is c' and $(\phi_{q_0})_{*,c'}(E_a) = M_{a,q_0}$, moreover, M_{a,q_0} is not isomorphic to M_{b,q_0} whenever $a \neq b$.

When $\eta_{1_0} = [2]_{q_0} [3]_{q_0} \neq 0$, $(\phi_{q_0})_{*,c_0}$ is an isomorphism by 3.4. So $(\phi_{q_0})_*$ is an isomorphism. When $[3]_{q_0} = 0$, by 3.9 we see that $(\phi_{q_0})_{*,c_0}(E_s) = 0$ if and only if

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \in s,$$

here we regard ω as a 3-th primitive root of 1 in $\mathbb{C}$. When $[2]_{q_0} = q_0 + q_0^{-1} = 0$, by 3.13 we see that $(\phi_{q_0})_{*,c_0}(E_s) = 0$ if and only if the eigenpolynomial of s has the form $\lambda^3 - a\lambda^2 + a^{-1}\lambda - 1$, $a \in \mathbb{C}^*$.

REFERENCES

- [IH] N. IWAHORI and H. MATSUMOTO, *On Some Bruhat Decomposition and the structure of the Hecke Ring of p-Adic Chevalley Groups* (Publ. math. IHES, Vol. 25, 1965, pp. 237-280).
- [KL1] D. KAZHDAN and G. LUSZTIG, *Representations of Coxeter Groups and Hecke algebras* (Inventiones Math., Vol. 53, 1979, pp. 165-184).
- [KL2] D. KAZHDAN and G. LUSZTIG, *Proof of the Deligne-Langlands Conjecture for Hecke Algebra* (Inventiones Math., Vol. 87, 1987, pp. 153-215).
- [L 1] G. LUSZTIG, *Some Examples on Square Integrable Representations of Semisimple p-Adic Groups* (Trans. of the AMS, Vol. 277, 1983, pp. 623-653).
- [L 2] G. LUSZTIG, *Singularities, Character Formulas, and a q-analog of Weight Multiplicities*, in *Analyse et Topologie sur les Espaces Singuliers (II-III)* (Astérisque, Vol. 101-102, 1983, pp. 208-227).
- [L 3] G. LUSZTIG, *Cells in affine Weyl groups, I-IV*, in *Algebraic Groups and Related Topics*, pp. 255-287. *Adv. Studies in Pure Math.*, Vol. 6, North Holland, Amsterdam, 1985; *J. Algebra*, Vol. 109, 1987, pp. 536-548; *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, Vol. 34, 1987, pp. 223-243, Vol. 36, 1989, No. 2, pp. 297-328.
- [LX] G. LUSZTIG and N. XI, *Canonical Left Cells in Affine Weyl Groups* (Adv. in Math., Vol. 72, 1988, pp. 284-288).
- [S] J.-Y. SHI, *A Two-Sided Cell in an Affine Weyl Group, I, II* (J. London Math. Soc., (2), Vol. 36, 1987, pp. 407-420; Vol. 37, 1988, pp. 253-264).
- [X] N. XI, *The Based Ring of the Lowest Two-Sided Cell of an Affine Weyl Group* (J. Algebra, Vol. 134, 1990, pp. 356-368).

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