

# ANNALES SCIENTIFIQUES DE L'É.N.S.

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*Annales scientifiques de l'É.N.S. 4<sup>e</sup> série*, tome 26, n° 5 (1993), p. 545-564

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# BIFURCATION DIAGRAMS AND FOMENKO'S SURGERY ON LIOUVILLE TORI OF THE KOLOSOFF POTENTIAL $U = \rho + (1/\rho) - k \cos \varphi$

BY LJUBOMIR GAVRILOV, MOHAMMED OUAZZANI-JAMIL AND REGIS CABOZ

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ABSTRACT. — By making use of the rich algebraic structure of the problem and Fomenko's theory of surgery on (bifurcations of) Liouville tori, we give a complete description of the topology and bifurcations of the invariant level sets of the Kolossoff system corresponding to the integrable potential  $U = \rho + (1/\rho) - k \cos \varphi$ .

## I. Introduction

Consider the motion of a particle of unit mass on the plane  $(x, y)$  in a potential field

$$U = a\rho + \frac{b}{\rho} + c \cos \varphi + d \sin \varphi, \quad a, b, c, d \in \mathbf{R}$$

where  $x = \rho \cos \varphi$ ,  $y = \rho \sin \varphi$ . Without loss of generality one may suppose (after a rotation and  $\mathbf{R}$ -linear change of  $\rho$  and  $U$ ) that

$$U(x, y) = \pm \rho \pm \frac{1}{\rho} - k \cos \varphi, k \in \mathbf{R}$$

The corresponding Hamiltonian function is:

$$H = \frac{1}{2}(p_x^2 + p_y^2) + U(x, y)$$

and the energy level sets  $\{H = h\} \subset \mathbf{R}^4$  are compact if  $U = \rho + (1/\rho) - k \cos \varphi$ . The Hamiltonian system

$$(1) \quad \begin{cases} x' = \frac{dH}{dp_x}, & p'_x = -\frac{dH}{dx} \\ y' = \frac{dH}{dp_y}, & p'_y = -\frac{dH}{dy} \end{cases} \quad (') = \frac{d}{dt}$$

where

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \rho + \frac{1}{\rho} - k \cos \varphi$$

is integrable and the second integral of motion reads:

$$F = -(k^2 + y^2)p_x^2 + 2y(x-k)p_x p_y - p_y^2(x-k)^2 - \frac{2k(x-k)(kx-1)}{\sqrt{x^2+y^2}}$$

The integrability of the system (1) was discovered by Kolossoff [8] who used it to linearize the celebrated Kovalevskaya top.

In the present paper we give a complete description of the topology of the level sets

$$\mathbf{A}_{\mathbf{R}} = \{ (x, y, p_x, p_y) \in \mathbf{R}^4 : H = h, F = f \} \subset \mathbf{R}^4.$$

For doing that we find first the bifurcation diagram  $\mathbf{B}$  of the problem (1), *i. e.* the set of critical values of the energy-momentum mapping

$$(x, y, p_x, p_y) \rightarrow (F, H).$$

It turns out (like in Hénon-Heiles system [5], Gorjatchev-Tchaplygin [4] and Kovalevskaya top [9], [10]) that  $\mathbf{B}$  is exactly the discriminant locus of a certain polynomial whose coefficients are functions in  $f, h, k$ . The latter is closely related to the algebraic structure of the complexified system (1). This structure is studied in section 2 where we prove that the complexified generic level set  $\{H = h, F = f\}$  is an affine part of an Abelian variety (Theorem 1). Contrary to the most of the known examples [1], the Hamiltonian flows corresponding to  $H$  and  $F$  do not linearize on this Abelian variety. Thus the system (1) is not algebraically completely integrable in the sense of Adler and van Moerbeke [1]. For non-critical values of  $F$  and  $H$  the level set  $\mathbf{A}_{\mathbf{R}}$  is, according to Liouville theorem, a finite union of two-dimensional tori. Their number is related to the number of ovals of an associated genus two Riemann surface and could be calculated by making use of the results of chapter 2 (*see* Theorem 2 of section 3). At last, in section 4, we describe the structure of singular level sets  $\mathbf{A}_{\mathbf{R}}$ . According to Fomenko's theory of surgery on (bifurcations of) Liouville tori they turn out to be homeomorphic to a finite list of two-dimensional complexes. To "guess" exactly which bifurcation takes place we use once again the reach algebraic structure of the problem. Namely, each bifurcation of Liouville tori is related to a bifurcation of ovals on a Riemann surface (the last being easily studied). Thus we find all generic bifurcations of Liouville tori as  $f$  and  $h$  pass through the bifurcation diagram  $\mathbf{B}$  (Theorem 3 and Theorem 4 of section 4).

## II. Algebraic structure

Denote by  $\mathbf{A}_{\mathbf{C}}$  the complex affine algebraic variety:

$$\mathbf{A}_{\mathbf{C}} = \{ (x, y, p_x, p_y, z) \in \mathbf{C}^5 : H = h, F = f, x^2 + y^2 = z^2, z \neq 0 \} \subset \mathbf{C}^5,$$

where

$$H(x, y, p_x, p_y, z) = \frac{1}{2}(p_x^2 + p_y^2) + z + \frac{1}{z} - k \frac{x}{z},$$

$$F(x, y, p_x, p_y, z) = -(k^2 + y^2)p_x^2 + 2y(x - k)p_x p_y - p_y^2(x - k)^2 - \frac{2k(x - k)(kx - 1)}{z}$$

The variety  $\mathbf{A}_{\mathbf{C}}$  is invariant under the (complex) flow of the (complexified) system (1). Consider also the polynomial

$$(2) \quad \varphi(u) = -2(u^3 - hu^2 + (1 - k^2)u - f/2)$$

and the corresponding hyperelliptic curve

$$(3) \quad \mathbf{K}: \{ w^2 = (u^2 - k^2)\varphi(u) \}.$$

*Remark.* —  $\mathbf{K}$  is precisely the curve used by Kovalevskaya [11] to integrate the Kovalevskaya top.

**THEOREM 1.** — *If the polynomial  $(u^2 - k^2)\varphi(u)$  has no double roots then the affine algebraic variety  $\mathbf{A}_{\mathbf{C}}$  is a smooth complex manifold which is biholomorphically equivalent to the complex manifold  $\tilde{\mathbf{A}}_{\mathbf{C}} \setminus \mathcal{D}$ , where  $\tilde{\mathbf{A}}_{\mathbf{C}}$  is a complex algebraic torus (Abelian variety) and  $\mathcal{D}$  is a divisor.  $\tilde{\mathbf{A}}_{\mathbf{C}}$  is a two-sheeted unramified covering of the Jacobi variety  $\text{Jac}(\mathbf{K})$  of the genus algebraic two curve  $\mathbf{K}$ . The trajectories of the Hamiltonian flow generated by  $H$  on  $\mathbf{A}_{\mathbf{C}}$  are straight lines on which, however, the motion is non-linear. The trajectories of the Hamiltonian flows generated by  $H + sF$ ,  $s \neq 0$  on  $\mathbf{A}_{\mathbf{C}}$  are not straight lines.*

Theorem 1 will be proved later in this section. We recall that the Hamilton-Jacobi equation corresponding to (1) separates in the following  $(\lambda, \mu)$  coordinates (see [8] for details):

$$(4) \quad \left\{ \begin{array}{l} x = \frac{\lambda\mu}{k} + k \\ y = \frac{1}{k} \sqrt{(\lambda^2 - k^2)(k^2 - \mu^2)} \end{array} \right.$$

The canonical variables  $(p_\lambda, p_\mu, \lambda, \mu)$  on  $\mathbf{T}^*\mathbf{R}^2$  are given by

$$(5) \quad \begin{cases} p_x = \frac{(\lambda^2 - k^2)\mu p_\lambda - (\mu^2 - k^2)\lambda p_\mu}{k(\lambda^2 - \mu^2)} \\ p_y = \frac{\sqrt{(\lambda^2 - k^2)(\mu^2 - k^2)}(\lambda p_\lambda - \mu p_\mu)}{k(\lambda^2 - \mu^2)} \end{cases}$$

In these new variables the integrals of motion take the form

$$\begin{aligned} H &= \frac{(\lambda^2 - k^2)p_\lambda^2 - (\mu^2 - k^2)p_\mu^2 + 2(1 - k^2)(\lambda - \mu) + 2(\lambda^3 - \mu^3)}{2(\lambda^2 - \mu^2)}, \\ F &= \frac{-\mu^2(\lambda^2 - k^2)p_\lambda^2 + \lambda^2(\mu^2 - k^2)p_\mu^2 - 2\lambda\mu(\lambda\mu + k^2 + k^2 - 1)(\lambda - \mu)}{(\lambda^2 - \mu^2)} \end{aligned}$$

and hence on each level set  $\mathbf{A}_c$  holds

$$(6) \quad p_\lambda = \sqrt{\frac{\varphi(\lambda)}{\lambda^2 - k^2}}, \quad p_\mu = \sqrt{\frac{\varphi(\mu)}{\mu^2 - k^2}}.$$

For a further use we note also the relation

$$(7) \quad F = p_\mu^2(\mu^2 - k^2) + 2\mu^3 - 2\mu^2 H + 2\mu(1 - k^2).$$

Denote by  $d/dt_s$  the time derivative along the Hamiltonian flow of the function  $H_s = H + sF$ . By making use of the equations

$$\frac{d\lambda}{dt_s} = \frac{\partial H}{\partial p_\lambda}, \quad \frac{d\mu}{dt_s} = \frac{\partial H_s}{\partial p_\mu}$$

and (6) one obtains

$$(8) \quad \begin{cases} \frac{d\lambda}{\sqrt{\varphi(\lambda)(\lambda^2 - k^2)}} + \frac{d\mu}{\sqrt{\varphi(\mu)(\mu^2 - k^2)}} = -2s dt_s \\ \frac{\lambda^2 d\lambda}{\sqrt{\varphi(\lambda)(\lambda^2 - k^2)}} + \frac{\mu^2 d\mu}{\sqrt{\varphi(\mu)(\mu^2 - k^2)}} = dt_s \end{cases}$$

The system (8) can be also written in the following equivalent form

$$(9) \quad \begin{cases} \frac{d\lambda}{\sqrt{\varphi(\lambda)(\lambda^2 - k^2)}} + \frac{d\mu}{\sqrt{\varphi(\mu)(\mu^2 - k^2)}} = -2s dt_s \\ \frac{\lambda d\lambda}{\sqrt{\varphi(\lambda)(\lambda^2 - k^2)}} + \frac{\mu d\mu}{\sqrt{\varphi(\mu)(\mu^2 - k^2)}} = \frac{1 - 2s\lambda\mu}{\lambda + \mu} dt_s \end{cases}$$

The flow of Kolosoff system (1) corresponds to  $s=0$ , and obviously  $t_s|_{s=0} = t$ . The system (9) implies, roughly speaking, that our initial system linearizes on an Jacobian

variety after using a “new time”

$$(10) \quad d\tau = \frac{dt}{\lambda + \mu}.$$

The time  $\tau$  will play an important role and it is exactly the “Kovalevskaya time” (see [8] for details).

Define now the Abel-Jacobi map

$$\zeta : S^2 K \rightarrow \text{Jac}(K) : (P_1, P_2) \rightarrow \left( \int_{P_\infty}^{P_1} \omega_1 + \int_{P_\infty}^{P_2} \omega_1, \int_{P_\infty}^{P_1} \omega_2 + \int_{P_\infty}^{P_2} \omega_2 \right)$$

where

$$\omega_1 = \frac{du}{\sqrt{\varphi(u)(u^2 - k^2)}}, \quad \omega_2 = \frac{u du}{\sqrt{\varphi(u)(u^2 - k^2)}}$$

$P_1, P_2 \in K \setminus P_\infty$  is the “infinite” point on  $K$  and  $S^2 K$  is the second symmetric product of  $K$ .

Solving the Jacobi inversion problem (9), we obtain the explicit solutions of our initial problem (1) [2]. Thus  $x, y, p_x, p_y, z = \sqrt{x^2 + y^2}$  can be expressed in terms of genus two theta functions living on the Jacobi variety  $\text{Jac}(K)$ . These functions however are not single-valued as it can be seen from (4). Indeed to each point on the symmetric product  $S^2 K$  of the curve  $K$  (which is birational to  $\text{Jac}(K)$  according to Jacobi theorem) correspond two values of  $(x, y, p_x, p_y)$ . On the other hand these functions do not have branch points on  $\text{Jac}(K)$  and hence they are root functions (Wurzelfunktionen [14]) on  $\text{Jac}(K)$ .

Consider the Abelian variety  $\tilde{A}_C = \mathbb{C}^2 / \mathbb{Z} \{e_1, e_2, e_3, 2e_4\}$  where

$$\text{Jac}(K) = \mathbb{C}^2 / \mathbb{Z} \{e_1, e_2, e_3, e_4\}.$$

If the basis  $(e_1, e_2, e_3, e_4)$  of the period lattice is chosen in a proper way then the function  $x, y, p_x, p_y, z$  become single-valued on  $\tilde{A}_C$ . Let us fix such a basis. The natural projection

$$(11) \quad \pi : \tilde{A}_C \rightarrow \text{Jac}(K)$$

corresponds to the involution

$$(12) \quad (x, y, p_x, p_y, z) \rightarrow (x, -y, p_x, -p_y, z)$$

on  $\tilde{A}_C$ . Consider the mapping

$$i : \mathbb{C}^5 \rightarrow \mathbb{CP}^7 : (x, y, z, p_x, p_y) \rightarrow [f_0, f_1, \dots, f_7]$$

where

$$(13) \quad \left\{ \begin{array}{l} f_0 = 1 \\ f_1 = x \\ f_2 = y \\ f_3 = z \\ f_4 = xp_y - yp_x \\ f_5 = f_4^2 \\ f_6 = f_3(f_4 - kp_y) \\ f_7 = (p_x^2 - p_y^2)y - 2p_x p_y x - 2f_2 f_3. \end{array} \right.$$

LEMMA 1. — *The functions  $f_i$ ,  $i=0, 1, \dots, 7$  considered as single-valued meromorphic functions on  $\tilde{A}_{\mathbf{C}}$  provide a smooth embedding of  $\tilde{A}_{\mathbf{C}}$  into  $\mathbf{CP}^7$ .*

*Proof of theorem 1 assuming the above lemma.* — As the functions  $f_0, f_1, \dots, f_7$  provide an embedding of  $\tilde{A}_{\mathbf{C}}$  into  $\mathbf{CP}^7$  (Lemma 1) then the closure  $\overline{i(\tilde{A}_{\mathbf{C}})}$  of  $i(\tilde{A}_{\mathbf{C}})$  in  $\mathbf{CP}^7$  is biholomorphically equivalent to  $\tilde{A}_{\mathbf{C}}$ . Consider the divisors  $\mathcal{D}_{\infty}$  and  $\mathcal{D}'_{2\infty}$  defined by

$$(\lambda\mu)_{\infty} = 2(\zeta(P_{\infty}) + \zeta(K)) = 2\mathcal{D}_{\infty}$$

and

$$(z)_0 = (\lambda + \mu)_0 = \mathcal{D}'_{2\infty}$$

Obviously  $\mathcal{D}'_{2\infty} \sim 2\mathcal{D}_{\infty}$ . It is easily seen that  $A_{\mathbf{C}}$  is biholomorphically equivalent to  $\overline{i(A_{\mathbf{C}})} \setminus \{\mathcal{D}_{\infty} \cup \mathcal{D}'_{2\infty}\}$ . Indeed  $i$  is a biholomorphic mapping between some neighbourhood  $V_{A_{\mathbf{C}}}$  of  $A_{\mathbf{C}}$  in  $\mathbf{C}^5 \setminus \{z \neq 0\}$  and  $i(V_{A_{\mathbf{C}}}) \subset \mathbf{CP}^7$ . To check that it suffices to note that if  $(x, y, p_x, p_y, z) \in A_{\mathbf{C}}$  then

$$\det \left( \frac{\partial(f_1, f_2, f_3, f_4, f_6)}{\partial(x, y, p_x, p_y, z)} \right) = kyz$$

$$\det \left( \frac{\partial(f_1, f_2, f_3, f_5, f_7)}{\partial(x, y, p_x, p_y, z)} \right) = -4p_y(p_x x^2 y + p_x y^3 - p_y x^3 - p_y x y^2)$$

and hence  $\text{rank}(i) = 5$  (otherwise the equality  $y = p_y = 0$  implies  $\text{disc}((k^2 - u^2)\varphi(u)) = 0$ ). As  $i(A_{\mathbf{C}}) = \tilde{A}_{\mathbf{C}} \setminus \mathcal{D}_{\infty}$  is a smooth complex manifold, it is concluded that  $A_{\mathbf{C}}$  is also a smooth complex manifold.  $\triangle$

*Proof of Lemma 1.* — For an arbitrary divisor  $\mathcal{D} \subset \tilde{A}_{\mathbf{C}}$  we denote

$$\mathcal{L}(\mathcal{D}) = \{f \text{ meromorphic on } \tilde{A}_{\mathbf{C}}, (f) \geq -\mathcal{D}\}$$

As  $\zeta(K)$  defines (1, 1) polarization on  $\text{Jac}(K)$  then  $\mathcal{D}_{\infty} = \pi^{-1} \circ \zeta(K)$  defines (1, 2) polarization on  $A_{\mathbf{C}}$ . Thus  $2\mathcal{D}_{\infty}$  defines (2, 4) polarization on  $\tilde{A}_{\mathbf{C}}$  and  $\dim \mathcal{L}(2\mathcal{D}_{\infty}) = 2 \times 4 = 8$ , [7]. To prove lemma 1, it is enough to check that the functions  $f_0, f_1, \dots, f_7$  provide a basis of  $\mathcal{L}(2\mathcal{D}_{\infty})$ . First of all let us note that  $f_i$  blow up only along  $\mathcal{D}_{\infty}$ . Indeed in  $\lambda, \mu$

coordinates we have

$$\begin{aligned}
 f_1 &= 1 \\
 f_1 &= \frac{\lambda\mu}{k} + k \\
 f_2 &= \frac{1}{k} \sqrt{(\lambda^2 - k^2)(k^2 - \mu^2)} \\
 f_3 &= \lambda + \mu \\
 f_4 &= \frac{1}{(\lambda - \mu)} \{ \sqrt{(k^2 - \mu^2)} \sqrt{\varphi(\lambda)} - \sqrt{(\lambda^2 - k^2)} \sqrt{-\varphi(\mu)} \} \\
 f_5 &= f_4^2 \\
 f_6 &= \frac{1}{(\lambda - \mu)} \{ \mu \sqrt{(k^2 - \mu^2)} \sqrt{\varphi(\lambda)} - \lambda \sqrt{\varphi(\lambda)} - \sqrt{(\lambda^2 - k^2)} \sqrt{-\varphi(\mu)} \} \\
 f_7 &= \frac{1}{k(\lambda - \mu)} \{ 2(\lambda\mu - k^2) \sqrt{\varphi(\lambda)} \sqrt{-\varphi(\mu)} - \sqrt{(\lambda^2 - k^2)(k^2 - \mu^2)} (\varphi(\lambda) + \varphi(\mu)) \} - 2f_2f_3.
 \end{aligned}$$

To prove that  $f_i \in \mathcal{L}(2\mathcal{D}_\infty)$  we shall find, following [1], the asymptotic expansions of  $x, y, z$  as functions of the time  $\tau$  (10) in a neighbourhood of a generic point  $\tau^0 \in \mathcal{D}_\infty$ . Formulae (4) imply that  $\lambda + \mu = \sqrt{x^2 + y^2}$  and hence the changing of time in the system (1) is equivalent to multiplying each equation by  $z$ . According to (9) and (4) the variables  $x, y, z$  are meromorphic in  $\tau$  and the corresponding Laurent series are:

$$(14) \quad \left\{ \begin{aligned} x &= \sum_{j=0}^{\infty} x_j \tau^{j-2}, & p_x &= \sum_{j=0}^{\infty} p_{xj} \tau^{j-1} \\ y &= \sum_{j=0}^{\infty} y_j \tau^{j-2}, & p_y &= \sum_{j=0}^{\infty} p_{yj} \tau^{j-1} \\ z &= \sum_{j=0}^{\infty} z_j \tau^{j-2} \end{aligned} \right.$$

(here  $\tau$  stays for  $\tau - \tau_0$ ). After substituting the above series in the Kolossoff system (1) one obtains a recurrent system of linear equations for the coefficients  $x_j, y_j, z_j$ . The general solution (14) depends effectively upon three free parameters  $\alpha, \gamma, \delta$ :

$$(15) \quad \left\{ \begin{aligned} x &= \frac{\alpha}{\tau^2} + \frac{(k\beta^3 - 4\gamma\alpha)}{4\beta} + \delta\tau + \dots \\ y &= \frac{\beta}{\tau^2} - \frac{(k\alpha\beta + 4\gamma)}{4} - \frac{\alpha\delta}{\beta}\tau + \dots \\ z &= \frac{-2}{\tau^2} + \frac{2\gamma}{\beta} + \dots \end{aligned} \right.$$



where  $\alpha^2 + \beta^2 = 4$  (for details about the general procedure of finding the series (15) we refer the reader to [1] or [6, 15]). After substituting (15) in (14), we obtain

$$(16) \quad \left\{ \begin{array}{l} f_0 = 1 \\ f_1 = \frac{\alpha}{\tau^2} + \dots \\ f_2 = \frac{\beta}{\tau^2} + \dots \\ f_3 = -\frac{2}{\tau^2} + \dots \\ f_4 = \frac{k\beta}{\tau} + \dots \\ f_5 = \frac{k^2\beta^2}{\tau^2} + \dots \\ f_6 = \frac{12\delta}{\beta\tau^2} + \dots \\ f_7 = -2\frac{(k\alpha\beta + 6\gamma)}{\tau^2} + \dots \end{array} \right.$$

The complex constants  $\alpha$  (or  $\beta$  such that  $\alpha^2 + \beta^2 = 4$ ),  $\gamma, \delta$  parametrize the pole divisor  $\mathcal{D}_\infty$ . Indeed substituting (15) in  $\{H=h, F=f, z^2=x^2+y^2\}$  we obtain the genus three curve

$$(17) \quad \left\{ \begin{array}{l} \gamma = \frac{2h\beta - k\alpha\beta}{16}, \\ \delta^2 = \frac{\beta}{72}(k^3\alpha\beta^3 + 8k^2\gamma\beta^2 - 2k(1+k^2)\alpha\beta - 32k^2\gamma - 2f\beta), \\ \alpha^2 + \beta^2 = 4 \end{array} \right.$$

$\mathcal{D}_\infty$  is a double unramified covering of the genus two curve

$$(18) \quad \delta^2 = \frac{(\alpha^2 - 4)}{144}(k^3\alpha^3 + 2hk^2\alpha^2 + 4k(1-k^2)\alpha + 4f)$$

and obviously this curve (18) coincides with (3) after making the substitution

$$\alpha \rightarrow \frac{2u}{k}, \quad \delta \rightarrow \frac{w}{3k}.$$

Equations (16) and (18) imply that  $f_0, f_1, \dots, f_7$  are linearly independent on  $\tilde{A}_C$  which completes the proof of lemma 1.  $\triangle$

### III. Topology of Regular Level Sets

In this section we shall describe the topological type of  $A_R$  for all generic constants  $f, h, k \in \mathbb{R}$ . The system (1) will be considered as a real system of differential equations.

According to Theorem 1  $A_R$  is a smooth real manifold if the polynomial  $(k^2 - u^2)\varphi(u)$  has no double roots. Define the bifurcation set

$$(19) \quad B = \{(f, h, k) \in \mathbb{R}^3 : \text{disc}((u^2 - k^2)\varphi(u)) = 0\}.$$

It is clear that the topological type of  $A_R$  may change only as  $(f, h, k)$  passes through  $B$ . Thus in each connected component of the set  $\mathbb{R}^3 \setminus B$  the level set  $A_R$  has the same topological type. Note that the bifurcation set  $B \subset \mathbb{R}^3 \setminus \{f, h, k\}$  is invariant under the involution

$$(f, h, k) \rightarrow (f, h, -k)$$

and the topological type of the level set  $A_R$  is one and the same at the points  $(f, h, k)$  and  $(f, h, -k)$ . Thus it is enough to consider  $k \geq 0$ .

**THEOREM 2.** — *The set  $\{\mathbb{R}^3 \setminus B\} \cap \{k \geq 0\}$  consists of 12 connected components. The sections of these components with the plane  $\{k = \text{const.}\}$  are shown on figure 1. If  $(f, h, k) \in \mathbb{R}^3 \setminus B$  the level set  $A_R$  is (diffeomorphic to) a torus, to a disjoint union of two tori, or it is the empty set as it is shown in table I.*

*Remark.* — The notation 2T in table I means a disjoint union of 2 two-dimensional tori.

*Proof of Theorem 2.* — The complex conjugation

$$(20) \quad (x, y, z, p_x, p_y) \rightarrow (\bar{x}, \bar{y}, \bar{z}, \bar{p}_x, \bar{p}_y)$$

acts as an antiholomorphic involution on  $A_C$ . The set of its fixed points is the real part  $\Re(A_C)$  of  $A_C$  and  $A_R = \Re(A_C) \cap \{z > 0\}$ . Consider also the natural antiholomorphic involution  $\tau$  of the Kovalevskaya curve (3) given in  $(w, u)$  coordinates by:

$$\tau: (w, u) \rightarrow (\bar{w}, \bar{u}).$$

It induces an antiholomorphic involution on the symmetric product  $S^2 K$  and hence on  $\text{Jac}(K)$  and  $\tilde{A}_C$ . Formulae (4), (5), (6) imply that this involution coincides with the complex conjugation (20) on  $A_C$ . The upshot is that in order to describe  $A_R$  it is enough to study the projection

$$\pi: A_C \rightarrow \text{Jac}(K)$$

and the pair  $(K, \tau)$ .

*Remark.* — The pair  $(K, \tau)$  where  $K$  is a Riemann surface and  $\tau$  is an antiholomorphic involution on  $K$  is called Klein surface. For the theory of Klein surfaces we refer the reader to [12].

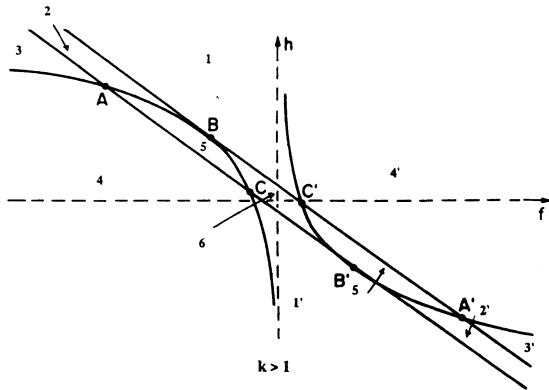


Fig. 1.1

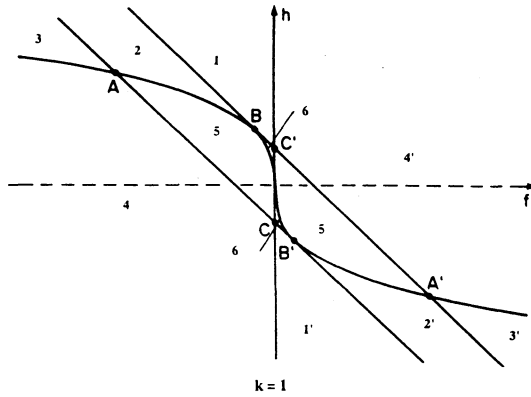


Fig. 1.2

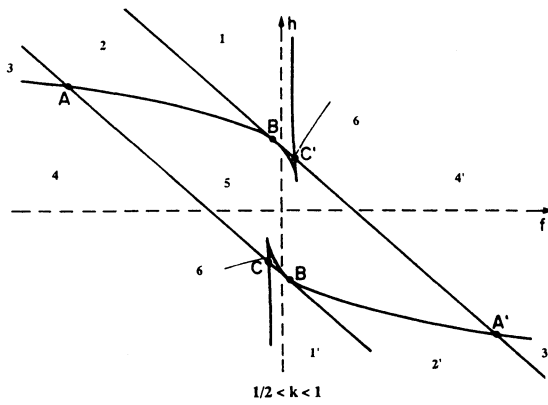


Fig. 1.3

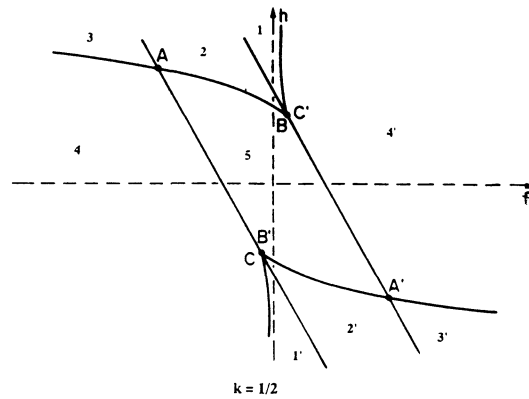


Fig. 1.4

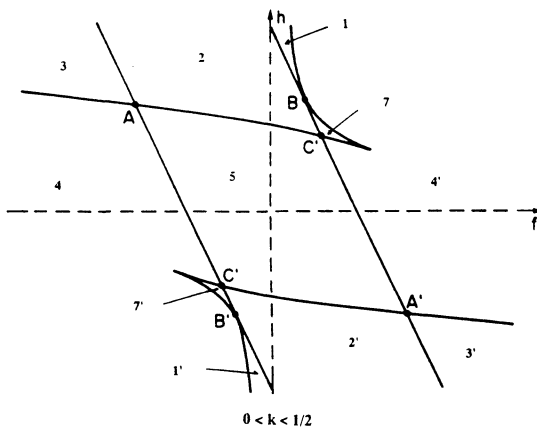


Fig. 1.5

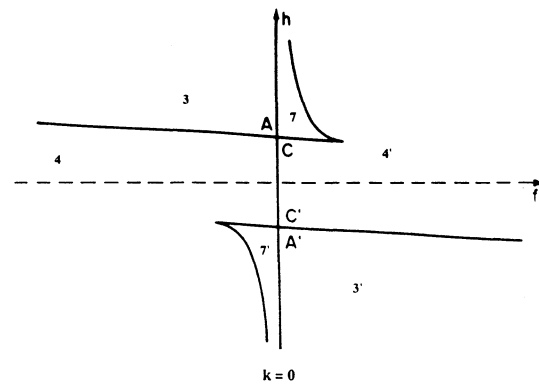


Fig. 1.6

Fig. 1. — The set  $\mathbf{B} \cap \{k = \text{const.}\}$  for  $k \geq 0$ .

TABLE I

Topological type of  $\mathbf{A}_{\mathbf{R}}$  and real roots of the polynomial  $(u^2 - k^2)\varphi(u)$   
for  $(f, h, k) \in \mathbf{R}^3 \setminus \mathbf{B}$  (see fig. 1).

Domain	Roots	Topological type
1 . . . . .	$-k < u_1 < u_2 < k < u_3$	T
1' . . . . .	$u_1 < -k < u_2 < u_3 < k$	$\emptyset$
2 . . . . .	$-k < u_1 < k < u_2 < u_3$	T
2' . . . . .	$u_1 < u_2 < -k < u_3 < k$	$\emptyset$
3 . . . . .	$u_1 < -k < k < u_2 < u_3$	2 T
3' . . . . .	$u_1 < u_2 < -k < k < u_3$	$\emptyset$
4 . . . . .	$u_1 < -k < k$	$\emptyset$
4' . . . . .	$-k < k < u_1$	$\emptyset$
5 . . . . .	$-k < u_1 < k$	$\emptyset$
6 . . . . .	$-k < u_1 < u_2 < u_3 < k$	$\emptyset$
7 . . . . .	$-k < k < u_1 < u_2 < u_3$	$\emptyset$
7' . . . . .	$u_1 < u_2 < u_3 < -k < k$	$\emptyset$

DEFINITION. — A connected component of the set of fixed points of  $\tau$  on  $K$  is called an oval.

To determine the ovals of  $K$  it suffices to study the real roots of the polynomial  $(u^2 - k^2)\varphi(u)$  for different values of  $f$ ,  $h$  and  $k$ . These roots are shown on table I. Using the formulae (4), (5), (6) and the condition  $(x, y, z, p_x, p_y) \in \mathbf{R}^5$  we obtain that  $\mathbf{A}_{\mathbf{R}} \neq \emptyset$  only if  $(f, h, k)$  belongs to domain 1, 2 or 3. There we find exactly two “admissible”

TABLE II

Domain	1	2	3
Projection of the “admissible” ovals on $z$ -plane . . . . .	$\begin{cases} \Delta_1 = [u_1, u_2] \\ \Delta_2 = [k, u_3] \end{cases}$	$\begin{cases} \Delta_1 = [u_1, k] \\ \Delta_2 = [u_2, u_3] \end{cases}$	$\begin{cases} \Delta_1 = [-k, k] \\ \Delta_2 = [u_2, u_3] \end{cases}$

ovals whose projections on the  $z$ -plane are given by the intervals  $\Delta_1$  and  $\Delta_2$  (see table II). The product of the “admissible” ovals in  $S^2 K$  [and hence in  $\text{Jac}(K)$ ] gives a Liouville torus. Thus we proved that  $\pi(\mathbf{A}_{\mathbf{R}})$  consists of a torus  $T$ . There are two possibilities for  $\mathbf{A}_{\mathbf{R}} = \pi^{-1}(T)$  (recall that  $\mathbf{A}_{\mathbf{C}}$  is a double covering of  $\text{Jac}(K) \setminus \mathcal{D}$  and the projection is given by the map (11)):

- $\mathbf{A}_{\mathbf{R}}$  is a disjoint union of two copies of  $T$ ;
- $\mathbf{A}_{\mathbf{R}}$  is homeomorphic to a torus two times “longer” than  $T$ .

To determine which case arises it suffices to note that when  $\lambda$  (respectively  $\mu$ ) makes one turn around the interval  $\Delta_1$  (respectively  $\Delta_2$ ) in a complex domain then the function  $y$  does not change in the first case, whereas in the second case it changes the sign [we recall that the projection  $\pi$  corresponds to the involution (20)]. Thus we find that in domain 1 and 2  $\mathbf{A}_{\mathbf{R}}$  is a torus and in domain 3 it is a disjoint union of two tori.  $\triangle$

At last we shall find the topological type of the regular energy-level surface  $\{H=h\}$

LEMMA 2. — *The bifurcation set  $\Sigma$  of the family of surfaces*

$$Q_{h,k} = \left\{ \frac{p_x^2 + p_y^2}{2} + \rho + \frac{1}{\rho} - k \cos \varphi = h \right\}$$

is given by the union of two lines  $\Sigma = \{h=2+k\} \cup \{h=2-k\} \subset \mathbf{R}^2\{h,k\}$ . The set  $\mathbf{R}^2 \setminus \Sigma$  consists of 4 components shown on figure 2. The topological type of  $Q_{h,k}$  in each of these domains is given in table 3.

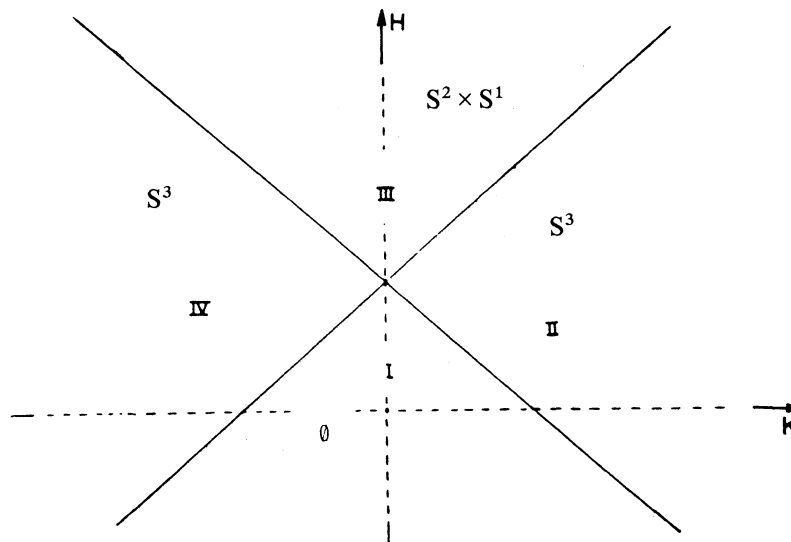


Fig. 2. — The set  $\Sigma$ .

*Remarks.* — We note that the three dimensional constant-energy surfaces most often met in mathematical physics and theoretical mechanics are:  $S^3$  (the sphere),  $\mathbf{R}P^3$  (the projective space),  $T^3$  (the torus) and  $S^2 \times S^1$  (the direct product), see [13] for details.

TABLE III

*Topological type of the energy level set*  
 $Q_{h,k} = \{H=h\}$  for  $(h,k) \in \mathbf{R}^2 \setminus \Sigma$  (see fig. 2).

Domain	1	2	3	4
Topological type . . . . .	$\emptyset$	$S^3$	$S^2 \times S^1$	$S^3$

*Proof of Lemma 2.* — The function  $H$  has exactly two critical points  $p_x = p_y = 0$ ,  $y = 0$ ,  $x = \pm 1$ , for  $k \neq 0$  and a critical variety  $\{p_x = p_y = 0, x^2 + y^2 = 1\}$  for  $k = 0$  with

corresponding critical values  $h = 2 \pm k$  ( $k \neq 0$ ) and  $h = 2$  ( $k = 0$ ). Let us compute the topological type of  $Q_{h,k}$ . If  $k = 0$  then

$$H = \frac{p_x^2 + p_y^2}{2} + \frac{(\rho - 1)^2}{\rho} + 2 \geq 2$$

and hence for  $h < 2$  we have  $Q_{h,k} = \emptyset$ . This implies that in domain 1  $Q_{h,k} = \emptyset$ . Suppose now that  $k = 0$ . On the surface  $H = 2 + \varepsilon$  where  $\varepsilon$  is small and positive,  $\rho - 1$  is small together with  $\varepsilon$ . As

$$\left\{ \frac{p_x^2 + p_y^2}{2} + \rho + \frac{1}{\rho} = 2 + \varepsilon \right\}$$

can be written as

$$\left\{ \frac{p_x^2 + p_y^2}{2} + \frac{(\rho - 1)^2}{(\rho - 1 + 1)} = \varepsilon \right\} \Leftrightarrow \left\{ \frac{p_x^2 + p_y^2}{2} + (\rho - 1)^2 - (\rho - 1)^3 + \dots = \varepsilon \right\} \sim S^2$$

Then  $Q_{2+\varepsilon,0}$  is topologically equivalent to  $S^2 \times S^1$  and hence in domain 3 the topological type of  $Q_{h,k}$  is  $S^2 \times S^1$ . Consider at last  $Q_{2,\varepsilon}$ , for  $\varepsilon$  small and positive

$$Q_{2,\varepsilon} = \left\{ \frac{p_x^2 + p_y^2}{2} + \frac{(\rho - 1)^2}{\rho} = \varepsilon \cos \varphi \right\}.$$

The set  $Q_{2,\varepsilon} \cap \{\varphi = \text{const.}\}$  is topologically equivalent to  $S^2$  for  $\varphi \in ((-\pi/2), (\pi/2))$  and to a point for  $\varphi = \pm(\pi/2)$ . Hence  $Q_{2,\varepsilon}$  is topologically equivalent to  $S^3$ . This implies that in domain 2 (and 4 by a symmetry) the topological type of  $Q_{h,k}$  is  $S^3$ .  $\triangle$

#### IV. Topology of Singular Level Sets and Surgery on Liouville Tori

In this section we shall find the topological type of the level set  $A_{\mathbf{R}}$  for generic values  $(f, h, k) \in \mathbf{B}$  and thus we shall describe all generic bifurcations of Liouville tori (the non-generic ones are easily found by continuity). For doing that we shall use the Fomenko's classification theorem of bifurcations of (surgery on) Liouville tori [3].

In section 3 we found the topological type of level set  $A_{\mathbf{R}}$  far from the bifurcation diagram. Suppose now that the constants  $f, h, k$  are changed in such a way, that  $(f, h, k)$  passes through the bifurcation diagram  $\mathbf{B}$ . Then the topological type of  $A_{\mathbf{R}}$  may change

and bifurcations of (surgery on) Liouville tori takes place. Consider the following three types of bifurcations (see *fig. 3*).

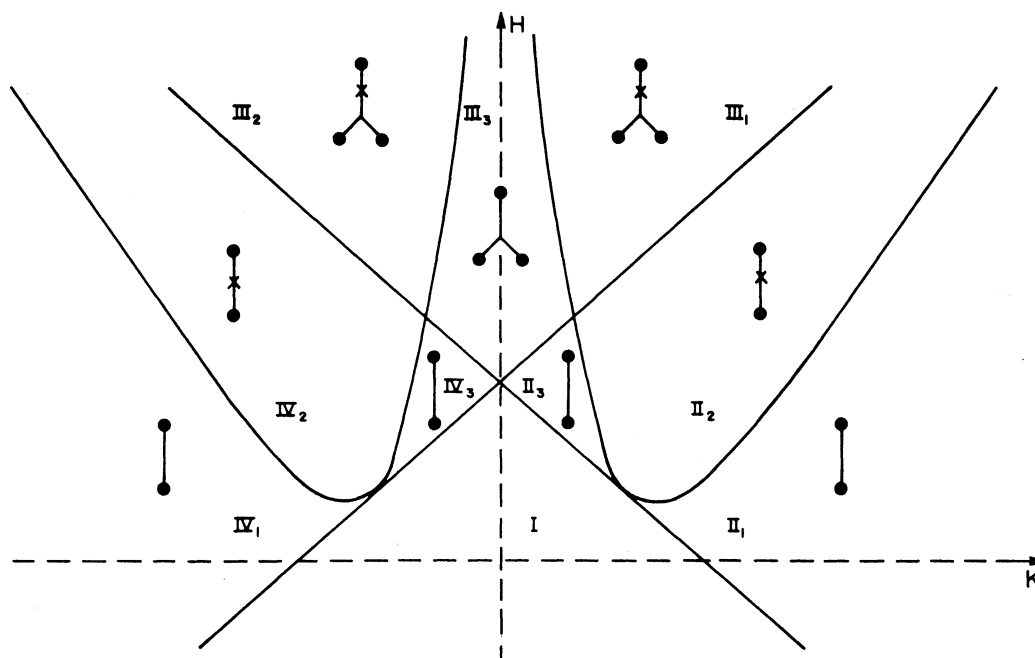


Fig. 3. — Bifurcations of two-dimensional invariant Liouville tori and the corresponding graphs.

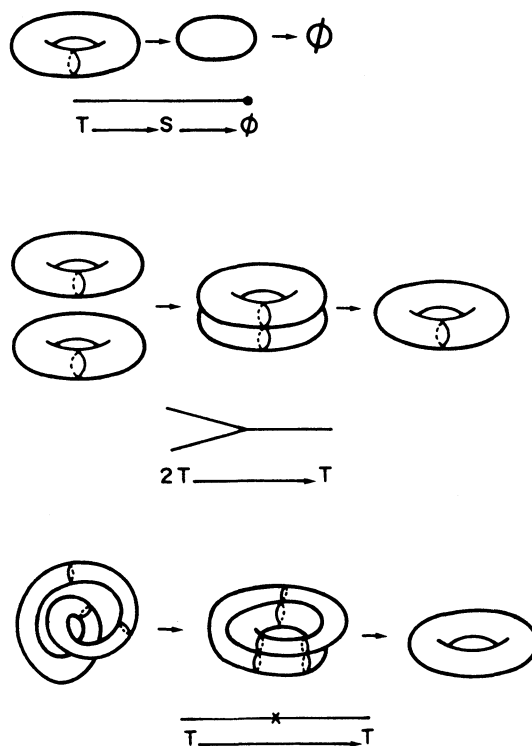
1) A (two-dimensional) torus  $T^2$  is contracted to the axial circle  $S^1$  and then vanishes. Denote this surgery as  $T \rightarrow S^1 \rightarrow \emptyset$ .

2) A torus  $T$  splits into two tori by passing through the complex  $S^1 \times \{S^1 \wedge S^1\}$  where  $S^1 \wedge S^1$  is a union of two circles having exactly one common point. Denote this bifurcation as  $T \rightarrow 2T$ .

3) A torus  $T$  becomes twice “shorter” as it spirals twice round a torus. The last complex is homeomorphic to a non-trivial section of the bundle  $S^1 \wedge S^1 \rightarrow S^1$ , and the corresponding bifurcation will be denoted as  $T \rightarrow T$ .

Following Fomenko [3] we present each of the above bifurcations by a graph shown on figure 3. An ordinary point denotes a non-singular Liouville torus. A black circle stands for a circle and a “branching” point (see *fig. 3*) stands for  $\{S^1 \wedge S^1\} \times S^1$ . At last asterisk denotes a set homeomorphic to a non-trivial section of the bundle  $S^1 \wedge S^1 \rightarrow S^1$ .

For fixed constants  $h$  and  $k$  let us consider the energy level surface  $Q_{h,k} = \{H=h\}$ . As  $f$  varies the Liouville tori contained in the level set  $\{F=f\}|_{Q_{h,k}}$  may change its topological type. Denote by  $\Gamma(Q_{h,k}, F)$  the graph describing the corresponding sequence of bifurcations of Liouville tori. The main result of this section is the following

Fig. 4. — The set  $\mathcal{D}$  and the graphs  $\Gamma(Q_{h,k}, F)$ .

THEOREM 3. — *If  $(h, k)$  belongs to one and the same connected component of the set*

$$\mathcal{D} = \{h \neq 2 \pm k\} \cap \left\{h \neq \pm \left(k + \frac{1}{2k}\right)\right\} \subset \mathbf{R}^2\{h, k\}$$

*then the graph  $\Gamma(Q_{h,k}, F)$  is the same and it is shown on figure 4.*

Theorem 3 also implies a description of all generic bifurcation of Liouville tori of our initial system (1). Namely, consider a parametrized smooth curve

$$\gamma(s) : s \rightarrow (f(s), h(s), k(s)) \in \mathbf{R}^3\{f, h, k\}$$

intersecting the bifurcation diagram  $\mathbf{B}$  at  $s = s_0$ .

DEFINITION. — A bifurcation of Liouville tori contained in the level set

$$\mathbf{A}_{\mathbf{R}} \equiv Q_{h(s), k(s)} \cap \{F = f(s)\}$$

as  $s$  passes through  $s_0$  is called generic, provided that  $\mathbf{B}$  is smooth in a neighbourhood of  $(f(s_0), h(s_0), k(s_0))$  and  $\gamma(s)$  intersects  $\mathbf{B}$  transversally.



THEOREM 4. — *All generic bifurcations of Liouville tori of the system (1) are given in table IV.*

TABLE IV

Generic bifurcations of the level set $\mathbf{A}_R$ .					
$1 \rightarrow 2$	$1 \rightarrow 6$	$1 \rightarrow 4'$	$2 \rightarrow 3$	$2 \rightarrow 5$	$3 \rightarrow 4$
$T \rightarrow T$	$T \rightarrow \emptyset$	$T \rightarrow \emptyset$	$T \rightarrow 2T$	$T \rightarrow \emptyset$	$T \rightarrow \emptyset$
					$T \rightarrow \emptyset$

Before proving Theorem 3 and Theorem 4 we shall formulate Fomenko's theorem [3] (adapted to our case).

DEFINITION. — A smooth function  $F$  on a manifold  $Q$  is a Bott function, provided that its critical points form nondegenerate critical smooth submanifolds. A critical submanifold of a smooth function  $F$  on a manifold  $Q$  is called nondegenerated, provided that the Hessian matrix  $d^2 F$  is nondegenerate in normal planes to the submanifold.

Now we may state the Fomenko's classification theorem of bifurcations of two-dimensional Liouville tori.

THEOREM (Fomenko [3]). — *Let  $F$  be a Bott integral on a non-singular constant energy surface  $Q^3$  of an integrable two-degrees of freedom Hamiltonian system. Suppose that each critical manifold of  $F$  on  $Q^3$  is a union of circles. Then each bifurcation of Liouville tori contained in the level set  $\{F=f\}$ , as  $f$  varies, is a composition of the three bifurcations  $T \rightarrow S^1 \rightarrow \emptyset$ ,  $T \rightarrow 2T$ , and  $T \rightarrow T$  described above.*

Remark. — The condition that each critical manifold of  $F$  is a union of circles does not seem to be very restrictive. To our knowledge all studied integrable mechanical systems fall into this case (it may be a conjecture).

In order to apply Fomenko's theorem we need to check that  $F$  is a Bott function when restricted on an energy level surface  $Q_{h,k}$ .

LEMMA 3. — *The second integral  $F$  is a Bott function on the non-singular energy level surface  $Q_{h,k} = \{H=h\}$  provided that  $h \neq \pm(k + (1/2)k)$ .*

Proof of Lemma 3. — Suppose that  $Q_{h,k}$  is a non-singular compact manifold, i.e.  $h \neq 2 \pm k$  (Lemma 2). If  $F$  has a critical value  $f$  on  $Q_{h,k}$  then the corresponding level surface  $\mathbf{A}_R = \{H=h, F=f\}$  is degenerated and hence the polynomial  $(u^2 - k^2)\varphi(u)$  has multiple zeros. The condition  $h \neq \pm(k + (1/2)k)$  means that  $(u^2 - k^2)\varphi(u)$  has no triple zeros on the boundary of the domains 1, 2 and 3 on figure 1, as the  $h$ -coordinates of the points A, B, C' are  $2+k$ ,  $k + (1/2)k$ ,  $2-k$  for  $k > 0$  and  $2-k$ ,  $-k(1/2)k$ ,  $2+k$  for  $k < 0$ . So let us suppose that the level set  $\mathbf{A}_R$  is degenerated and consider a degenerated connected component of it. Such a component is parametrized locally by  $(\lambda, \mu)$ , formulae (5), (6) and (7), at least for  $\lambda \neq \mu$ . If in addition  $\lambda$  and  $\mu$  are far from a double root of  $(u^2 - k^2)\varphi(u)$  then the equations (8) imply that the Hamiltonian flows of  $H$  and  $H + sF$  are linearly independent and hence  $dH$  and  $dF$  are linearly independent at such point.

Thus critical points of  $F|_{Q_{h,k}}$  correspond only to  $(\lambda, \mu)$  such that  $\lambda$  (or  $\mu$ ) is a double root of  $(u^2 - k^2)\varphi(u)$ . This is an one-dimensional analytical set and hence it is a disjoint union of circles. The last follows from the fact that the flow of  $H$  on  $Q_{h,k}$  has no stationary points and the critical set of  $F$  on  $Q_{h,k}$  is invariant under the action of this flow.

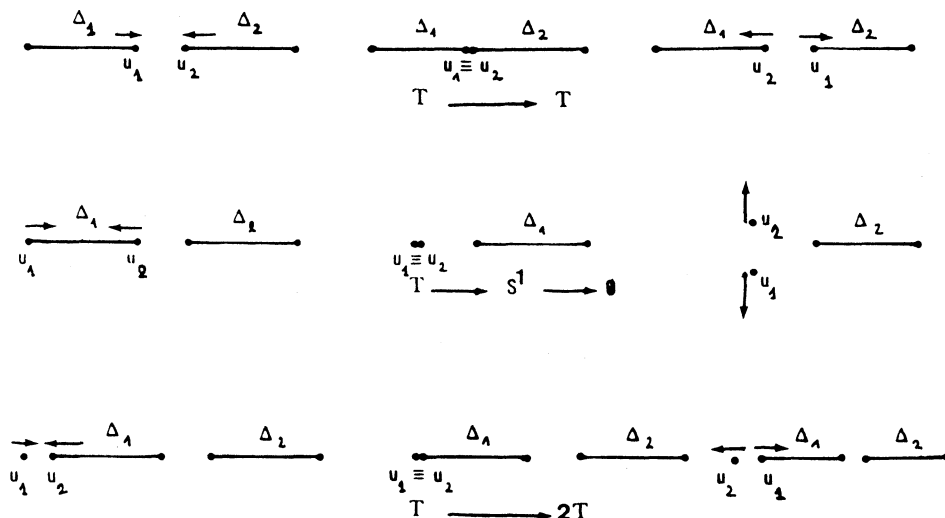


Fig. 5. — Correspondence between bifurcation of roots of the polynomial  $(u^2 - k^2)(u^3 - hu^2 + (1 - k^2)u - f/2)$  and bifurcations of invariant Liouville tori.

At last let us prove that the hessian matrix of  $F|_{Q_{h,k}}$  is non-degenerated of the normal planes to these circles. Let  $\mu = \mu_0$  be a double root of  $(u^2 - k^2)\varphi(u)$ . According to (7) we have

$$F|_{Q_{h,k}} = (\mu^2 - k^2)p_\mu^2 + 2\mu^3 - 2\mu^2 h + 2\mu(1 - k^2)$$

and a critical circle of the level set  $\{F|_{Q_{h,k}}\} = f$  is given by  $\mu = \mu_0$ ,  $p_\mu = 0$ . The normal directions to this circle are given by derivations with respect to  $\mu$  and  $p_\mu$ . We have

$$\frac{\partial^2 F}{\partial \mu \partial p_\mu} = \begin{pmatrix} 2(\mu_0^2 - k^2) & 0 \\ 0 & -\varphi''(\mu_0) \end{pmatrix}$$

and as  $\mu_0 \neq \pm k$  then  $\text{rank}(d^2(F|_{Q_{h,k}})) \geq 2$ . On the other hand the Hessian  $d^2(F|_{Q_{h,k}})$  is degenerated on tangent lines to the critical circle and hence  $\text{rank}(d^2(F|_{Q_{h,k}})) = 2$  which completes the proof of lemma 3.  $\triangle$

*Proof of Theorem 3.* — Let us fix a regular energy level set  $Q_{h,k}$  with a Bott integral  $F$  on it, and let us consider the corresponding line  $h = \text{const.}$  on figure 1 (plane  $k = \text{const.}$ ,  $h = \text{const.}$  in the space  $\mathbf{R}^3\{f, h, k\}$ ). As  $f$  vary the topological type of  $A_{\mathbf{R}} = \{Q_{h,k}\} \cap \{F = f\}$  may change. Using Theorem 2 and the Fomenko's classification theorem we identify several possible bifurcations. For example passing from domain 3

(where  $A_R \sim 2T$ ) to domain 2 where ( $A_R \sim T$ ) on figure 1 we may have the following surgeries:  $2T \rightarrow T$ , or composition of  $T \rightarrow T$  and  $T \rightarrow \emptyset$ . To make the difference between the two possibilities it suffices to look at the bifurcations of roots of the polynomial  $(u^2 - k^2)\varphi(u)$ , and more specifically the four ends of the "admissible" ovals  $\Delta_1$  and  $\Delta_2$ . The correspondence between bifurcation of roots and tori is shown on figure 5. As the bifurcations of real roots of the polynomial  $(u^2 - k^2)\varphi(u)$  are easily described on table 1 then we obtain a description of the bifurcations of invariant Liouville tori of our initial system (1). By making use of figure 1 we note that if  $(h, k)$  is fixed and belongs to one and the same connected component of the set

$$\mathcal{D} = \{h \neq 2 \pm k\} \cap \left\{h \neq \pm \left(k + \frac{1}{2k}\right)\right\},$$

then changing  $f$  the same bifurcations of roots of the polynomial  $(u^2 - k^2)\varphi(u)$  take place. This implies that if  $(h, k)$  belongs to one and the same connected component of the set  $\mathcal{D}$  the corresponding Fomenko's graph  $\Gamma(Q_{h,k}, F)$  is the same and it is shown on figure 4. This completes the proof of theorem 3.  $\triangle$

**DEFINITION.** — The straight line  $l \subset \mathbb{R}^3 \{f, h, k\}$  is generic provided that it intersects  $\mathbf{B}$  transversally.

To prove Theorem 4 we note that instead of a generic smooth curve  $l \subset \mathbb{R}^3 \{f, h, k\}$  it suffice to consider a generic straight line

$$\{c_1 h + c_2 f + c_3 = 0, k = \text{const.}\} \subset \mathbb{R}^3 \{f, h, k\}.$$

Then Theorem 4 follows from the following

**LEMMA 4.** — Let  $\{c_1 h + c_2 f + c_3 = 0, k = \text{const.}\}$  be a generic straight line in  $\mathbb{R}^3 \{f, h, k\}$ . Then  $\{c_1 H + c_2 F + c_3 = 0\} \subset \mathbb{R}^4 \{x, y, p_x, p_y\}$  is a smooth surface, and  $F$  is a Bott integral on it.

Indeed, instead of  $H$  we may take for a Hamiltonian of (1) the function  $c_1 H + c_2 F$ . The same arguments as in the proof of Theorem 3 imply the desirable result (table IV).

To the end of the paper we shall prove Lemma 4 (which generalizes Lemma 2 and Lemma 3).

Let  $k = k_0$  be fixed,  $(f_0, h_0, k_0) \in \mathbf{B}$  be a generic point (i.e. in a neighbourhood of it  $\mathbf{B}$  is a smooth manifold), and let  $q = (x^0, y^0, p_x^0, p_y^0)$  be a point on the level set  $\{H = h_0, F = f_0\}$ . We shall prove that if

$$(21) \quad c_1 \text{grad}(H)|_q + c_2 \text{grad}(F)|_q = 0$$

then the straight line  $\{c_1 h + c_2 f + c_3 = 0\}$  is tangent to  $\mathbf{B}$  (and hence it is not generic). As the equation of a straight line tangent to  $\mathbf{B}$  at the point  $(f_0, h_0, k_0)$  is given by

$$\{u_0^3 - hu_0^2 + (1 - k^2)u_0 - f/2 = 0\} \subset \mathbb{R}^2 \{f, h\}$$

where  $u_0$  is the double root of the polynomial  $P(u) = (u^2 - k^2)\varphi(u)$  then it is enough to prove that  $c_1/c_2 = 2u_0^2$ . In  $(\lambda, \mu, p_\lambda, p_\mu)$  coordinates defined by (4), (5) we have the identity

(7)

$$F = p_\mu^2 (\mu^2 - k^2) + 2\mu^3 - 2\mu^2 H + 2\mu(1 - k^2).$$

Then, at least far from the locus we have

$$(22) \quad \{ \lambda = \mu \} \cup \{ (\lambda^2 - k^2)(\mu^2 - k^2) = 0 \}$$

$$(23) \quad \left\{ \begin{array}{l} \frac{\partial F}{\partial \mu} = 2\mu p_\mu^2 - \varphi'(\mu) - 2\mu^2 \frac{\partial H}{\partial \mu} \\ \frac{\partial F}{\partial p_\mu} = 2(\mu^2 - k^2)p_\mu - 2\mu^2 \frac{\partial H}{\partial p_\mu} \\ \frac{\partial F}{\partial \lambda} = -2\mu^2 \frac{\partial H}{\partial \lambda} \\ \frac{\partial F}{\partial p_\lambda} = -2\mu^2 \frac{\partial H}{\partial p_\lambda} \end{array} \right.$$

As  $\text{grad}(H)$  and  $\text{grad}(F)$  are colinear according to (21), then

$$(24) \quad \begin{cases} 2(\mu^2 - k^2)p_\mu = 0 \\ 2\mu p_\mu^2 - \varphi'(\mu) = 0 \end{cases}$$

and hence  $p_\mu = 0$ ,  $\varphi'(\mu) = 0$ . Now (6) implies that  $\varphi(\mu) = \varphi'(\mu) = 0$  and hence  $\mu$  is a double root of the polynomial  $(\mu^2 - k^2)\varphi(\mu)$ . Suppose now that  $\lambda^0, \mu^0, p_\lambda^0, p_\mu^0$  belongs to the locus (22) and let  $(\lambda, \mu, p_\lambda, p_\mu)$  tends to  $(\lambda^0, \mu^0, p_\lambda^0, p_\mu^0)$ . The vectors  $\text{grad}(H)$  and  $\text{grad}(F)$  tend to some vectors  $\text{grad}(H)^0$  and  $\text{grad}(F)^0$  and let us suppose that these vectors are colinear. Using (6), (23) and (24) we conclude that

$$(\mu^2 - k^2)\varphi(\mu) \rightarrow 0 \quad \text{and} \quad 2\mu \frac{\varphi(\mu)}{\mu^2 - k^2} - \varphi'(\mu) \rightarrow 0$$

and hence  $\mu^0$  is a double root of the polynomial  $(\mu^2 - k^2)\varphi(\mu)$ . The upshot is that if  $c_1 \text{grad}(H) + c_2 \text{grad}(F) = 0$  then  $c_1/c_2 = 2\mu_0^2$ , where  $\mu_0$  is the double root of the polynomial  $(\mu^2 - k^2)\varphi(\mu)$ , and hence the straight line

$$\{ c_1 h + c_2 f + c_3 = 0, k = \text{const.} \}$$

is tangent to **B**. This completes the proof of Lemma 4.  $\triangle$

### Acknowledgements

Acknowledgements are due to J.-P. Codaccioni and to P. Vanhaecke for the valuable discussions.

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(Manuscript received February 4, 1992;  
revised June 11, 1992.)

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