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SUBSPACES OF MODULI SPACES OF RANK ONE LOCAL SYSTEMS

BY CARLOS SIMPSON

ABSTRACT. — Suppose X is a smooth projective variety. The moduli space $M(X)$ of rank one local systems on X has three different structures of complex algebraic group (Betti, de Rham, and Dolbeault). A subgroup which is algebraic for all three structures, we call a *triple torus*. We show that any closed subspace S of $M(X)$ which is defined in a natural way, for example by looking at cohomology groups and related constructions, is a finite union of translates of triple tori by torsion points. This answers a conjecture of Beauville and Catanese. The proof that the translates are by torsion points rests on a result from transcendental number theory.

1. Introduction

Let X be a smooth complex projective variety. We will consider the moduli space $M(X)$ of rank one local systems on X . It is a group under tensor product. There are many natural ways of defining subspaces, for example if we fix i and k we can let

$$S = \{v \in M(X), \dim H^i(X, v) \geq k\}.$$

The moduli space is isomorphic to a product of a reductive torus by a finite group. We will show that any naturally defined subset S such as above (cf. § 4, § 6, 7) is a finite union of translates of subtori by torsion points.

This type of result is originally due to M. Green and R. Lazarsfeld [14], [15], and A. Beauville [2]. They proved that the intersection of S with the space of unitary local systems is a finite union of translates of subtori. The question of whether the translations are by torsion points was posed by F. Catanese ([5], Problem 1.25) and Beauville [3]. They conjectured that the answer was yes, and Beauville has proved this in some cases [3].

We indicate three proofs that subspaces such as S are translates of tori, based on formal properties of the correspondences between rank one local systems (the Betti version), rank one vector bundles with integrable connection (the de Rham version), and rank one Higgs bundles with torsion Chern class (the Dolbeault version). A good reference for the correspondence between local systems and vector bundles with integrable connection is [7]. The references for the correspondence between local systems and Higgs bundles are [6], [11], [17], and [21]. However, in the rank one case the situation is much simpler, and all of the elements are present in the work of Green and

Lazarsfeld. The main part of the argument is Theorem 3.1, saying that subsets of M which are compatible with various of the natural structures which it has, must be translates of subtori. At the time of the final revision of this paper, I have included three arguments for this result, each using different hypotheses. The first is due to Arapura [1]. He proves that subsets of the moduli space which are complex analytic in the Betti version, and preserved by C^* in the Dolbeault space, are translates of subtori. The second was pointed out to me by Deligne: it says that subsets which are complex analytic in the Betti realization and complex algebraic in the Dolbeault realization are translates of subtori. The third was suggested recently by a question of Laumon: subsets which are algebraic in the Betti and de Rham versions are translates of subtori. These more precise versions of the statement used in the original version of this paper could permit a strengthening of the arguments presented in the last sections of the paper (this is left to the reader).

We prove that the translations are by torsion points. In fact this is not so much a proof as a remark based on a result from transcendental number theory. Suppose X is defined over $\bar{\mathbb{Q}}$. Then the moduli space M can be given structures of algebraic varieties M_B and M_{DR} . The first parametrizes representations of $\pi_1(X)$, and the second parametrizes line bundles with integrable algebraic connection on X . Both M_B and M_{DR} are defined over $\bar{\mathbb{Q}}$. The result from transcendental number theory says that $M_B(\bar{\mathbb{Q}}) \cap M_{DR}(\bar{\mathbb{Q}})$ consists exactly of the torsion points. It is an application given by Waldschmidt [26] of the criterion of Schneider-Lang [18] (there is a discussion in [24]). The result for any X defined over \mathbb{C} follows from a specialization argument. This application of transcendence theory is similar to Brieskorn's proof of the monodromy theorem [4].

The same methods apply to subvarieties of $M(X)$ defined in many ways – for example, by looking at various direct images and inverse images, cup products, and so forth. My first attempt to formalize this was with a notion of *canonically defined locally closed subvariety*. This version was somewhat heuristic. I have included it in Section 4, as it remains a fairly good picture of the basic idea. The second version of this discussion – given in Section 6 and 7 – is much longer and more technical, and still somewhat abbreviated, but more mathematically precise. It is based on the observation that these subvarieties have certain invariance properties with respect to field automorphisms. We assume the axiom of choice, and make a definition of *absolute constructible subset* by considering the action of $\text{Aut}(\mathbb{C}/\bar{\mathbb{Q}})$ on everything, much as in Deligne's definition of absolute Hodge cycles [9]. Using the previous parts of the paper and the Baire category theorem we show that absolute constructible subsets of the moduli space of rank one local systems are obtained by finite unions, complements and intersections of torsion translates of triple tori.

In Section 7 we define absolute functors between categories of local systems, and absolute natural transformations. These give rise to absolute constructible subsets. Examples of absolute functors are cohomology groups, inverse images, higher direct images, tensor products, and duals. Cup products are absolute natural transformations. The composition of two absolute functors is again an absolute functor – this formalizes the idea mentioned in Section 4, that various operations can be

applied sequentially with the resulting functor used to define an absolute constructible subset. At the end of Section 7, we show that as local systems V_1, \dots, V_k run through all semisimple local systems of given rank, the cup-product algebras of cohomology groups with coefficients in tensor products of the V_i (truncated at a finite stage) occupy only finitely many isomorphism classes of algebras; and that the set of (V_1, \dots, V_k) corresponding to a given isomorphism class is an absolute constructible subset.

These properties are topological properties of smooth projective varieties – some more in a long line of properties due to Hodge theory. It is not clear exactly what type of properties they are; it would be interesting, for instance, to have some examples of topological spaces not satisfying them.

I would like to thank M. Green and R. Lazarsfeld for explaining their work a long time ago; A. Beauville for posing the question of rationality (whether the translates are by torsion points) a few months ago; and F. Catanese for pointing out this conjecture in his talk in Trento (September 1991). I would like to thank P. Deligne for some helpful comments on a preliminary version, and for indicating a new proof of Theorem 3.1 which allows the subtraction of one of the hypotheses. I would like to thank G. Levitt and J.-C. Sikorav for a helpful discussion about topology.

2. Preliminary definitions

Let X be a smooth projective variety defined over $K \subset \mathbb{C}$. Let $M = M(X)$ denote the moduli space of complex local systems of rank one over X . We consider M as a real analytic group under the operation of tensor product. There are three natural algebraic groups whose underlying real analytic groups are canonically isomorphic to M . The first is

$$M_B = M_B(X) \stackrel{\text{def}}{=} \text{Hom}(\pi_1(X), \mathbb{C}^*).$$

The second is $M_{\text{DR}} = M_{\text{DR}}(X)$, the moduli space of pairs (L, ∇) where L is a line bundle on X and ∇ is an integrable algebraic connection on L . Recall that an integrable connection is a morphism of sheaves

$$\nabla : L \rightarrow L \otimes \Omega_X^1$$

such that $\nabla(au) = d(a)u + a\nabla(u)$ and (when ∇ is extended to an operation on forms with coefficients in L), $\nabla^2 : L \rightarrow L \otimes \Omega_X^2$ is zero.

The third space is $M_{\text{Dol}} = M_{\text{Dol}}(X)$, the moduli space of rank one Higgs bundles with first Chern class vanishing in the cohomology with rational coefficients. A rank one Higgs bundle is (E, φ) where E is a line bundle and $\varphi \in H^0(X, \Omega_X^1)$. We require that $c_1(E)$ is a torsion class in $H^2(X, \mathbb{Z})$. Recall that $\text{Pic}^*(X)$ denotes the group of line bundles on X whose first Chern classes are torsion. We have an isomorphism

$$M_{\text{Dol}} \cong \text{Pic}^*(X) \times H^0(X, \Omega_X^1).$$

We have isomorphisms of real analytic groups $M \cong M_B \cong M_{DR} \cong M_{Dol}$. The morphism $M_{DR} \rightarrow M_B$ associates to each (L, ∇) the monodromy representation of the local system of analytic solutions of $\nabla(u)=0$. This is complex analytic. The morphism $M_{Dol} \rightarrow M_B$ (which is not complex analytic) may be defined as follows. If $(E, \varphi) \in M_{Dol}$, there is a flat unitary metric κ on E , with corresponding connection d_κ (which is uniquely determined by E). The local system corresponding to (E, φ) is that of sections e of E with $d_\kappa(e) + (\varphi + \bar{\varphi})(e) = 0$.

For any subset $S \subset M$, let S_B , S_{DR} , and S_{Dol} denote the corresponding subsets of M_B , M_{DR} , and M_{Dol} . Let $U \subset M$ be the subgroup of unitary local systems. We have $U_{Dol} \cong \text{Pic}^*(X)$, and

$$U_B \cong \text{Hom}(\pi_1(X), U(1)).$$

Let $R \subset M$ denote the subgroup of real orientable local systems. We have $R_{Dol} \cong H^0(X, \Omega_X^1)$, and

$$R_B \cong \text{Hom}(\pi_1(X), \mathbf{R}^{+*}).$$

The direct product decomposition $M = U \times R$ is compatible with the previous direct product decomposition for M_{Dol} . Both U_B and R_B are totally real subgroups of M_B whose real dimensions are equal to the complex dimension of M_B . On the other hand, this product decomposition does not have a good description in M_{DR} .

Let $T \subset H_1(X, \mathbf{Z})$ denote the subgroup of torsion. Let $T^\vee = \text{Hom}(T, \mathbf{C}^*)$ denote the dual abelian group. Let M^0 denote the connected component of M which contains the identity (trivial local system). We have an exact sequence

$$1 \rightarrow M^0 \rightarrow M \rightarrow T^\vee \rightarrow 0.$$

This exact sequence splits (since T is a direct summand of $H_1(X, \mathbf{Z})$). Since T^\vee is a finite group, the splitting is an algebraic morphism for all three algebraic groups. The isomorphisms are compatible with the exact sequence and the splitting.

Remark. — Let $\text{Pic}^0(X)$ denote the group of line bundles whose Chern classes vanish in $H^2(X, \mathbf{Z})$. We have

$$M_{Dol}^0 \cong \text{Pic}^0(X) \times H^0(X, \Omega_X^1).$$

From the universal coefficients theorem, the subgroup of torsion in $H^2(X, \mathbf{Z})$ is isomorphic to $\text{Ext}^1(T, \mathbf{Z})$, so $\text{Pic}^*(X)/\text{Pic}^0(X) \cong \text{Ext}^1(T, \mathbf{Z})$ and we obtain a natural morphism from M_{Dol} to $\text{Ext}^1(T, \mathbf{Z})$. Similarly, if a line bundle L admits an integrable connection, then its Chern class is torsion, so we get a morphism $M_{DR} \rightarrow \text{Ext}^1(T, \mathbf{Z})$. These two morphisms are the same, since the holomorphic line bundle associated to a unitary connection is the same as the Higgs bundle. There is a natural isomorphism $T^\vee \cong \text{Ext}^1(T, \mathbf{Z})$ obtained using the exponential exact sequence. This is compatible with the definition of first Chern class by the exponential exact sequence, so our map from $M(X)$ to $\text{Ext}^1(T, \mathbf{Z})$ agrees with our map from $M(X)$ to T^\vee via this identification.

A *triple torus* is a closed, connected real analytic subgroup $N \subset M$ such that N_B , N_{DR} , and N_{Dol} are algebraic subgroups defined over \mathbb{C} . We say that a closed real analytic subspace $S \subset M$ is a *translate of a triple torus* if there exists a triple torus $N \subset M$ and a point $v \in M$ such that $S = \{v \otimes w, w \in N\}$. Note that, in this case, any choice of $v \in S$ will do.

We say that a point $v \in M$ is *torsion* if there exists an integer $a > 0$ such that $v^{\otimes a} = 1$. Let M^{tor} denote the set of torsion points. Note that for a given integer a , there are only finitely many solutions of $v^{\otimes a} = 1$. Hence, the points of M_B^{tor} are defined over $\bar{\mathbb{Q}}$, and the points of M_{DR}^{tor} and M_{Dol}^{tor} are defined over $\bar{\mathbb{K}}$.

We say that a closed subspace S is a *torsion translate of a triple torus* if S is a translate of a triple torus N by an element $v \in M^{\text{tor}}$. This is equivalent to asking that S be a translate of a triple torus, and contain a torsion point.

Let A be the Albanese variety of X (which can be defined as $H^0(X, \Omega_X^1)^*/H_1(X, \mathbb{Z})$). Let $X \rightarrow A$ be the map from X into A given by integration (from a basepoint, which will be suppressed in the notation but assumed to be defined over $\bar{\mathbb{K}}$). Pullback of local systems gives a natural map from $M(A)$ to $M(X)$, which is an isomorphism

$$M(A) \cong M^0(X).$$

The Albanese variety A is defined over $\bar{\mathbb{K}}$.

LEMMA 2.1. — *Let $N \subset M$ be a closed connected subgroup such that $N_B \subset M_B$ is complex analytic and $N_{Dol} \subset M_{Dol}$ is an algebraic subgroup. Then there is a connected abelian subvariety $P \subset A$, defined over $\bar{\mathbb{K}}$, such that N is the image in M of $M(A/P)$. In particular, N is a triple torus in M .*

Proof. — The condition that N_{Dol} is an algebraic subgroup of $U_{Dol} \times R_{Dol}$, and the facts that U_{Dol} is an abelian variety and R_{Dol} is a vector space, imply that $N \cong (N \cap U) \times (N \cap R)$. To see this, we can divide by $(N \cap U) \times (N \cap R)$, so it suffices to show that if $N \cap U = N \cap R = 0$ then $N = 0$. Under those circumstances, N would provide the graph of an injective morphism of algebraic groups from the projection $p_2(N) \subset R_{Dol}$ to U_{Dol} — but there are no nonzero morphisms of algebraic groups from a vector space to an abelian variety, so we get the claimed statement.

Note that $(N \cap U)_{Dol}$ is an abelian subvariety of A (hence defined over $\bar{\mathbb{K}}$), and $(N \cap R)_{Dol}$ is a vector subspace of R_{Dol} . Let $\Lambda = H_1(X, \mathbb{Z})/T$; it is also equal to $\pi_1(A)$. Then complex analytically,

$$M_B^0 \cong \text{Hom}(\Lambda, \mathbb{C})/\text{Hom}(\Lambda, \mathbb{Z}).$$

The product decomposition of N implies that there is an exact sequence

$$0 \rightarrow \Phi \rightarrow \Lambda \rightarrow \Psi \rightarrow 0$$

where Ψ has no torsion, such that

$$N_B \cong \text{Hom}(\Psi, \mathbb{C})/\text{Hom}(\Psi, \mathbb{Z}).$$

Note that $U_{\text{Dol}} = \text{Pic}^0(X)$ is the abelian variety dual to A . Hence the abelian subvariety $(N \cap U)_{\text{Dol}} \subset U_{\text{Dol}}$ is dual to a quotient A/P of the Albanese by a connected abelian subvariety P . The fundamental group of $(N \cap U)$ is the subgroup $\text{Hom}(\Psi, \mathbb{Z})$ of $\pi_1(U^0) = \text{Hom}(\Lambda, \mathbb{Z})$. Hence $\pi_1(P) = \Phi$ and $\pi_1(A/P) = \Psi$. Thus $N = M(A/P)$. Note that, as there are at most countably many abelian subvarieties of a given A , P must be defined over \bar{K} .

COROLLARY 2.2. — *Suppose $N \subset M$ is a triple torus. Then N_B is defined over \mathbb{Q} , and N_{DR} and N_{Dol} are defined over \bar{K} .*

Proof. — Since A/P is defined over \bar{K} , N_{DR} and N_{Dol} are defined over \bar{K} . The subtorus N_B is defined over \mathbb{Q} , since it may be expressed as $N_B = \text{Hom}(\pi_1(A/P), \mathbb{C}^*)$.

There is an action of \mathbb{C}^* on M_{Dol} , given by

$$t(E, \varphi) = (E, t\varphi).$$

Let p denote the projection from M to the first factor of $U \times R$. The tangent space $T(M)_v$ at any point v is naturally identified (via translation) with the tangent space $T(M)_1$ at the identity. We denote all of these tangent spaces simply by \mathfrak{m} . Similarly, let \mathfrak{u} and \mathfrak{r} denote the tangent spaces of U and R . The exponential map is a natural identification $\exp: \mathfrak{r} \cong R$. This is compatible with the vector space structure of R_{Dol} . The complex structure i of M_B acts on \mathfrak{m} , and provides an isomorphism $i\mathfrak{u} = \mathfrak{r}$.

3. The main theorems

THEOREM 3.1. — *Suppose that $S \subset M$ is a closed irreducible real analytic subset. Suppose that one of the following sets of hypotheses holds:*

- (a) S_B is complex analytic and S_{Dol} is preserved by the \mathbb{C}^* action;
- (b) S_B is complex analytic and S_{Dol} is complex algebraic; or
- (c) S_B and S_{DR} are complex algebraic.

Then S is a translate of a triple torus.

Proof. — Using hypothesis (a), the theorem is essentially due to D. Arapura. He proves that if S_{Dol} is preserved by the action of $(\mathbb{R}^+)^*$ then S_B is a translate of an algebraic subtorus [1]. We give some details. By translation we may assume that the origin 0 is a smooth point of S . Then $\exp^{-1}(S) \subset TM_0$ is smooth at the origin and invariant under real scaling in the second factor of the decomposition $TM_0 = TU_0 \oplus TR_0$. This implies that $TS_0 = T(S \cap U)_0 \oplus T(S \cap R)_0$ and that $\exp(T(S \cap R)_0) \subset S$. Let \mathbf{k} denote the complex structure of M_B . Since S_B is \mathbf{k} -complex analytic, we have

$$\exp(T(S \cap R)_0 \oplus \mathbf{k}T(S \cap R)_0) \subset S.$$

Furthermore, $k: T(S \cap R)_0 \cong T(S \cap U)_0$. Therefore $\exp(TS_0) \subset S$. Since S is closed and irreducible, this implies that S_B is a closed complex analytic subgroup of M_B . We also get $S = (S \cap U) \times (S \cap R)$, which implies that S_B is an algebraic subtorus.

To prove the first part of the theorem, we have to use the hypothesis that M_{Dol} is preserved by the full C^* to conclude that S_{Dol} is complex algebraic. Let j denote the complex structure of M_{Dol} . Together, j and k are two of the three complex structures of a quaternionic structure [17], [12]. In particular, $kj = -jk$. The action of $i \in C^*$ on TR_0 is equal to the action of j . Thus the hypothesis implies that $S_{\text{Dol}} \cap R_{\text{Dol}}$ is a complex vector subspace of R_{Dol} . On the other hand, $T(S \cap U)_0 = k T(S \cap R)_0$, so the commutation formula for j and k implies that $T(S \cap U)_0$ is preserved by j . Thus $S_{\text{Dol}} \cap U_{\text{Dol}}$ is a complex analytic, hence algebraic, subgroup of the abelian variety U_{Dol} . Now Lemma 2.1 implies that S is a triple torus.

The second statement, using hypothesis (b), was pointed out to me by P. Deligne, in response to an earlier version of this paper. Briefly, the proof goes as follows. The universal covering of M has a structure of vector space over the quaternions. Any smooth submanifold of a quaternionic vector space which is holomorphic for the complex structures j and k must be a linear subspace. This follows from the more general principle that quaternionic subspaces of quaternionic-Kähler manifolds are totally geodesic [25]. In this case, a simple proof can be had by noting (as Deligne) that the subspace is locally the graph of a function which is j and k -holomorphic. Such a function must be linear, as the second derivative, a quadratic form, must vanish:

$$jkQ(u, v) = jQ(u, kv) = Q(ju, kv) = kQ(ju, v) = kjQ(u, v) = -jkQ(u, v).$$

We conclude that S is a translate of a closed subgroup. The hypothesis that S_{Dol} is algebraic allows us to apply Lemma 2.1 to conclude that S is a translate of a triple torus.

The third statement of the theorem, using hypothesis (c), has been added in proof, in response to a question of G. Laumon. For the proof of this statement, suppose A is an abelian variety. Say that a closed real analytic subset $S \subset M(A)$ is *Betti-de Rham* if S_B and S_{DR} are algebraic. We have to show that an irreducible Betti-de Rham subset is a translate of a triple torus. We may suppose that this statement is known for abelian varieties of smaller dimension. Translations and intersections of Betti-de Rham subsets are again Betti-de Rham subsets.

Our first claim is that there are no irreducible Betti-de Rham subsets of codimension one. If S were such, then S_B , as a divisor in the affine variety $M_B(A)$, would be defined by a function g in the coordinate ring of $M_B(A)$. If we choose a basis $H_1(A, \mathbb{Z}) \cong \mathbb{Z}^{2g}$ then we can express $M_B(A) \cong (C^*)^{2g}$. The coordinate ring becomes $C[x_1, x_1^{-1}, \dots, x_{2g}, x_{2g}^{-1}]$, so we can write

$$g = \sum a_i x_1^{i_1} \dots x_{2g}^{i_{2g}}.$$

There are at least two nonzero terms in this expression if S_B is nonempty. Choose a generic holomorphic one-form $\alpha \in H^0(A, \Omega_A^1)$, and consider the family of vector bundles

with integrable connection $\{\mathcal{O}_A, d+t\alpha\}_{t \in \mathbb{C}}$. Let $\lambda_1, \dots, \lambda_{2g}$ be the integrals of α around the basis elements of $H_1(A, \mathbb{Z})$ chosen above. The line bundle with connection $(\mathcal{O}_A, d+t\alpha)$ corresponds to the point $y(t) = (y_1(t), \dots, y_{2g}(t)) \in (\mathbb{C}^*)^{2g} \cong M_B(A)$ with $y_i(t) = \exp(t\lambda_i)$. Our family of connections corresponds to an algebraic morphism $\mathbb{C} \rightarrow M_{DR}(A)$, so the hypothesis that S_{DR} is algebraic implies that the set of values of t such that $g(y(t)) = 0$ is either empty, finite, or all of \mathbb{C} . On the other hand, the condition that α is generic implies that the λ_i are linearly independent over \mathbb{Q} . We can write

$$g(y(t)) = \sum a_u e^{ut}$$

where the sum is over distinct complex numbers u , and there are at least two nonzero terms. The argument of [20], Chapter 12 shows that $g(y(t)) = 0$ for a countably infinite set of values. This contradiction shows that a Betti-de Rham subset cannot be a divisor.

Assume that A is an irreducible abelian variety, and suppose that there exists a proper irreducible Betti-de Rham subset of positive dimension. Choose one of minimal dimension, and translate so that it passes through the origin. Then for some k the map from the k -fold product

$$f: S \times \dots \times S \rightarrow M(A)$$

defined by the group law of $M(A)$, is surjective and generically finite. Let $C \subset M(A)$ denote the subset of points y such that f is not a covering space over a neighborhood of y . Then C itself is a Betti-de Rham subset. On the other hand, C is of pure codimension 1 (by purity of the branch locus, the fact that in the Betti realizations f is an affine map, and Hartog's theorem). The claim of the previous paragraph implies that C is empty. Thus f is a covering space. By passing to a finite covering of A we may assume that f is an isomorphism. Let p_i denote the projection on the i -th factor and j the inclusion of S in $M(A)$. Put $v_i = jp_i f^{-1}: M(A) \rightarrow M(A)$. The composition of v_i with the projection to $\text{Pic}^0(A)$ factors through a map $\text{Pic}^0(A) \rightarrow \text{Pic}^0(A)$. For some i this map is nonzero, hence S surjects onto $\text{Pic}^0(A)$. On the other hand every point of $M_{DR}(A)$ has a rational curve passing through it, and this curve has a nontrivial projection into one of the factors. Thus if $s \in S$ is a general point, then there is a rational curve in S_{DR} passing through s . These curves must project to points in $\text{Pic}^0(A)$, so we get $\dim(S) > \dim(A)$. This contradicts the possibility that f is an isomorphism ($k > 1$ since S is a proper subset). This contradiction shows that if A is an irreducible abelian variety, then $M(A)$ contains no proper Betti-de Rham subsets of positive dimension.

We finish with the proof for any abelian variety A . Note that if S is an irreducible Betti-de Rham subset which is contained in a translate of a triple subtorus, or which is translation invariant by a triple subtorus, then the inductive hypothesis implies that S is a translate of a triple subtorus.

Suppose $S \subset M(A)$ is an irreducible Betti-de Rham subset of positive dimension. Choose a projection $M(A) \rightarrow M(A_i)$ where A_i is one of the irreducible factors of A , such that the projection $g: S \rightarrow M(A_i)$ is nonconstant. From the result of the previous paragraph, g is surjective. The fibers of the projection $M(A) \rightarrow M(A_i)$ are also of the form $M(B)$, and the fibers of g are Betti-de Rham subsets of $M(B)$. The

Stein factorization of a projective completion of g has branch locus which is a Betti-de Rham divisor in $M(A_i)$, so this branch locus is trivial. Thus we may go to a finite cover and assume that g has connected general fiber. Since we are proceeding by induction on the dimension of A , we may assume that the general fibers of g are translates of triple subtori of $M(B)$. Since there are only countably many possibilities, the general fibers are translates of the same triple subtorus. Thus S is translation invariant by this triple subtorus. By taking the quotient we reduce to a lower dimensional case, unless the general fiber of g is finite. In that case, by the same argument as above, g is a finite covering. In particular, S_B is a covering of $M_B(A_i)$, so it is a product of multiplicative groups. The morphism from S_B into $M_B(A)$ must be a translate of a group homomorphism, so S is a translate of a closed subgroup. A subgroup which is Betti-de Rham is a triple subtorus. This completes the proof.

COROLLARY 3.2. — *Suppose $S \subset M$ is a closed real analytic subset satisfying one of hypotheses (a), (b) or (c) of the above theorem. If hypothesis (a) is used, assume that S_B is complex algebraic. Then S is a union of translates of triple tori.*

Proof. — Divide S into a union of irreducible components. This is possible since one of S_B or S_{Dol} is assumed to be complex algebraic. The condition of irreducibility can be described topologically, once it is known that the subset is complex algebraic in one of the realizations. Apply the theorem to each irreducible component.

THEOREM 3.3. — *Suppose X is defined over $K = \bar{\mathbb{Q}}$. Suppose $S \subset M$ is a translate of a triple torus such that S_B and S_{DR} are defined over $\bar{\mathbb{Q}}$. Then S is a torsion translate of a triple torus.*

The proof of this theorem depends mostly on the following result from transcendental number theory. This statement may be found in the proof of Theorem 1 in [24]. There, the statement is quoted directly from Waldschmidt ([26], Corollary 5.2.7 and remark on pp. 92-93). In turn, Waldschmidt's proof is essentially an application of the criterion of Schneider-Lang [18]. We will give a short discussion here as Waldschmidt's remarks are very brief.

PROPOSITION 3.4. — *Suppose A is an abelian variety defined over $\bar{\mathbb{Q}}$. Suppose $v \in M(A)$ is a point such that v_B is defined over $\bar{\mathbb{Q}}$ in $M_B(A)$ and v_{DR} is defined over $\bar{\mathbb{Q}}$ in $M_{\text{DR}}(A)$. Then v is a torsion point.*

Proof. — The criterion of Schneider-Lang, as stated in [26] Theorem 5.2.1, says: if G is a connected commutative algebraic group defined over $\bar{\mathbb{Q}}$, if $\psi: \mathbb{C}^n \rightarrow G$ is an analytic homomorphism normalised so that the differential at the origin is defined over $\bar{\mathbb{Q}}$, and if $\Gamma \subset \mathbb{C}^n$ is a subgroup containing n elements linearly independent over \mathbb{C} with $\psi(\Gamma) \subset G(\bar{\mathbb{Q}})$, then the dimension of the Zariski closure of $\psi(\mathbb{C}^n)$ is less than or equal to n . Our point v_{DR} corresponds to a line bundle L with integrable connection ∇ over A . Consider the extension of algebraic groups

$$1 \rightarrow G_m \rightarrow G \rightarrow A \rightarrow 1$$

where G is the set of pairs (τ, f) with $\tau: A \rightarrow A$ a translation and $f: \tau^*(L) \cong L$. The connection ∇ gives an analytic homomorphism $\psi: TA \rightarrow G$ from the tangent space of A

to G , lifting the exponential map $TA \rightarrow A$. If (L, ∇) is defined over $\bar{\mathbf{Q}}$ then G and the differential of ψ at the origin are defined over $\bar{\mathbf{Q}}$. Let $\Gamma \subset TA$ denote the period lattice. The condition that v_B is defined over $\bar{\mathbf{Q}}$ means that $\psi(\Gamma) \subset G_m(\bar{\mathbf{Q}}) \subset G(\bar{\mathbf{Q}})$. By the criterion of Schneider-Lang, there is an algebraic subgroup $G' \subset G$ of dimension $n = \dim(A)$ such that $\psi(TA) \subset G'$. But $G' \rightarrow A$ is surjective, hence finite, so $G' \cap G_m$ is a finite group. Thus the monodromy representation $\Gamma \rightarrow G' \cap G_m \subset G_m$ takes values in the group of roots of unity, so v is a torsion point.

Proof of Theorem 3.3. — First of all, note that any irreducible component of M contains a torsion point. This is because the exact sequence

$$0 \rightarrow M^0 \rightarrow M \rightarrow T^\vee \rightarrow 0$$

splits. Let u_1 be a torsion point in the same component as S . Let $S_1 = u_1^{-1} \otimes S$. If we can show that S_1 is a torsion translate of a triple torus, then it follows that S will be. Hence we reduce to the case where $S \subset M^0$. In particular we may replace X by its Albanese A .

Let N be the triple torus such that S is a translate of N . Apply Lemma 2.1, to find a connected abelian subvariety $P \subset A$ such that $N = M(A/P)$. The natural morphism $M(A) \rightarrow M(P)$ identifies $M(P)$ with the quotient M/N . Let $v \in M(P)$ denote the point corresponding to the image of S in M/N . Then v_B and v_{DR} are defined over $\bar{\mathbf{Q}}$, so we may apply Proposition 3.4 to conclude that v is a torsion point. Finally, note that there is an exact sequence

$$0 \rightarrow M^{\text{tor}}(A/P) \rightarrow M^{\text{tor}}(A) \rightarrow M^{\text{tor}}(P) \rightarrow 0$$

(this can be seen by identifying $M_{\text{tor}}(A) \cong H^1(A, \mathbf{Q})/H^1(A, \mathbf{Z})$). Thus v may be lifted to a point $w \in M^{\text{tor}}$; then $w \in S$, and $S = w \otimes N$. This proves the theorem.

COROLLARY 3.5. — *Suppose X is defined over $K = \bar{\mathbf{Q}}$. Suppose $S \subset M(X)$ is a closed subset such that S_B and S_{DR} are algebraic subvarieties of M_B and M_{DR} , defined over $\bar{\mathbf{Q}}$. Suppose that S_{Dol} is a complex analytic subvariety of M_{Dol} , preserved by the action of \mathbf{C}^* . Then S is a finite union of torsion translates of triple tori.*

Proof. — By Corollary 3.2, S is a union of translates of triple tori. It has finitely many irreducible components, because S_B is an algebraic subvariety. Each of these components is defined over $\bar{\mathbf{Q}}$ in M_B and M_{DR} . By Theorem 3.3, each of these components is a torsion translate of a triple torus.

Remark. — With the argument of Theorem 3.1 part (c) (which was added in proof), we can remove the hypotheses in the third sentence of this corollary.

4. Canonically defined subvarieties

In this section we will indicate the existence of a large number of ways of defining subvarieties $S \subset M(X)$ so as to satisfy the hypotheses of Corollary 3.5. We call these

canonically defined subvarieties. This notion does not have a precise definition, but we state a theorem about it anyway. It will be replaced in Sections 6, 7 by the more precise notion of an *absolute constructible set*. There we will sketch a method of proving the things which we indicate heuristically here. It seemed like a good idea to keep this section, which was a preliminary version, because it serves to explain the idea more intuitively.

Fix a smooth projective variety X defined over a field $K \subset \mathbb{C}$. Let $M = M(X)$. We will say that a closed subset $S \subset M$ is *canonically defined* if the property $v \in S$ can be characterized by looking at the following types of things. For any morphism $f: Y \rightarrow X$ such that Y is a smooth projective variety, and any smooth morphism $g: Y \rightarrow Z$, we can consider the local systems $R^i g_* (f^* V)$. We can take tensor products, direct sums and duals (including also the trivial local systems). We may consider natural morphisms between these local systems, such as morphisms given by cup product, and take kernels and cokernels. We may iterate these types of operations to obtain a class of local systems W on smooth projective varieties Z , depending on one or several local systems V . Finally, given the data of all of these local systems, we may consider the ranks; and also their decompositions into irreducible or isotypic pieces. We may consider whether various of the local systems obtained are isomorphic. Furthermore, we may consider the invariant theory of the various multilinear forms on cohomology groups obtained by cup products. Any closed subset S of local systems V obtained by characterizing some set of such data, will be called *canonically defined*.

A *constructible canonically defined subset* is one obtained by finite unions, intersections and complements of closed canonically defined subsets. We restrict the discussion here to closed subsets, as Sections 6 and 7 will treat constructible subsets.

Examples

The first example is given by looking at the dimensions of cohomology groups. Fix i and k ; then

$$S = \{ v \in M(X), \dim H^i(X, v) \geq k \}$$

is canonically defined. We may also do the same for varieties over X . If $f: Y \rightarrow X$ is a morphism, with Y smooth and projective, then

$$S = \{ v \in M(X), \dim H^i(Y, f^* v) \geq k \}$$

is canonically defined. Note that both of these cases fit into the framework described above, because the cohomology groups are the same as the higher direct images for the morphisms from X or Y to a point.

We may look at higher direct images to other varieties. For example, suppose $f: Y \rightarrow X$ is a morphism, and $g: Y \rightarrow Z$ is a smooth projective morphism (with Y and Z smooth projective). The set S of $v \in M(X)$ such that $R^i g_* (f^* v)$ either has rank $> k$, or has rank k and is reducible, is a closed canonically defined subset.

We may look at the rank of morphisms induced by cup product. Let $S_{k,l}$ be the subset of local systems $V \in M(X)$ such that $\dim H^i(X, V) = k$ and $\dim H^i(X, V^*) = l$. Its closure $\bar{S}_{k,l}$ is a closed canonically defined subset, as is the complement $D = \bar{S}_{k,l} - S_{k,l}$. Let $r = \dim H^{i+j}(X, \mathbb{C})$. Fix an isomorphism class of pairing

$$q: \mathbb{C}^k \otimes \mathbb{C}^l \rightarrow \mathbb{C}^r.$$

Let $S(q)$ be the subset of $v \in S_{k,l}$ such that the cup product

$$H^i(X, V) \otimes H^j(X, V^*) \rightarrow H^{i+j}(X, \mathbb{C})$$

is isomorphic to the pairing q for some choice of frames $H^i(X, V) \cong \mathbb{C}^k$, $H^j(X, V^*) \cong \mathbb{C}^l$, and $H^{i+j}(X, \mathbb{C}) \cong \mathbb{C}^r$. Then $S(q) \cup D$ is a closed canonically defined subset of $M(X)$.

Properties

THEOREM 4.1. — *Suppose $S \subset M$ is a closed, canonically defined subset. Let K be a common field of definition for X and all of the objects involved in the definition of S . Then S has the following properties:*

1. S_B is an algebraic subvariety of M_B defined over $\bar{\mathbb{Q}}$;
2. S_{DR} and S_{Dol} are algebraic subvarieties of M_{DR} and M_{Dol} , defined over \bar{K} ;
3. S_{Dol} is invariant under the action of \mathbb{C}^* ;
4. there exists a specialization, that is a variety X' defined over $\bar{\mathbb{Q}}$ with an isomorphism of topological spaces $\psi: X'^{top} \cong (X')^{top}$, and a closed subset $S' \subset M(X')$, such that S' has properties 1-3 above (with respect to $\bar{\mathbb{Q}}$ in property 2), and under the induced isomorphism $\psi^*: M(X') \cong M(X)$ we have $\psi^*(S') = S$.

Proof. — Inverse images and smooth direct images can be defined in the categories of local systems, vector bundles with integrable connection, and polystable Higgs bundles with vanishing Chern classes. For vector bundles with integrable connection, the smooth direct images are defined by the relative algebraic de Rham cohomology of Grothendieck [16]. For Higgs bundles, the direct image is defined similarly as follows: suppose $f: Y \rightarrow Z$ is a smooth morphism, and (E, φ) is a polystable Higgs bundle with vanishing Chern classes on Y . Let $\Omega_{Y/Z}^i(E, \varphi)$ be the complex whose terms are $\Omega_{Y/Z}^i \otimes E$ with differentials given by φ (the *relative Dolbeault complex*). Put

$$R_{Dol}^i f_* (E, \varphi) = R^i f_* (\Omega_{Y/Z}^i(E, \varphi)).$$

This has a Higgs field θ defined by using the connecting morphism in an exact sequence. In [23] it is shown that this direct image is compatible with the correspondence with harmonic bundles and local systems. Note that all local systems involved remain semisimple, and correspond to harmonic bundles. (The compatibility is probably also true for semistable Higgs bundles with vanishing Chern classes, and nonsemisimple local systems, but I haven't thought about this.)

Tensor products and cup product morphisms may also be defined in all three of these categories. The correspondences between local systems, vector bundles with integrable connection, and Higgs bundles are preserved by these operations. Furthermore, the sub-objects correspond. Hence there are algebraic subvarieties S_B , S_{DR} and S_{Dol} with the same underlying set of points $S \subset M$. For property 3, note that the action of C^* commutes with inverse images and direct images. For example, if f is a smooth morphism, then there are natural isomorphisms

$$\alpha_t : (R_{Dol}^i f_* (E, \varphi), t\theta) \cong (R_{Dol}^i f_* (E, t\varphi), \theta),$$

obtained from the isomorphism of complexes $\Omega_{Y/Z}(E, \varphi) \cong \Omega_{Y/Z}(E, t\varphi)$ which one makes by multiplying by t^i on the terms of degree i . These isomorphisms are compatible with tensor products and cup products. In particular, the invariants of the multilinear forms which one can obtain by using tensor products and cup products are unchanged by the action of C^* . This completes the verification of the first three properties.

For the fourth, note that we may find a subring $O \subset K$, finitely generated over \mathbb{Q} , such that all of the objects required in the definition of S are defined over O (since M_B is noetherian, we need only consider a finite set of data to define S). We may further assume that all of the varieties and morphisms which are required to be projective (resp. smooth), are projective (resp. smooth) over $\text{Spec}(O)$. Then all of the varieties are topologically fibrations over $\text{Spec}(O)$ $(C)^{top}$, and the local systems and morphisms of local systems which occur in the definition of S are locally constant over $\text{Spec}(O)$ $(C)^{top}$.

The inclusion $O \subset K \subset C$ corresponds to a point $\sigma \in \text{Spec}(O)(C)$. There exists a point $P \in \text{Spec}(O)(\bar{\mathbb{Q}})$, and a continuous path from σ to P in $\text{Spec}(O)(C)^{top}$. Let X' be the fiber of the scheme corresponding to X , over P . It is a smooth projective variety defined over $\bar{\mathbb{Q}}$. Transporting along the path gives an isomorphism $\psi : X^{top} \cong (X')^{top}$. We may also transport all of the topological data along this path, to obtain a canonically defined subset $S' \subset M(X')$ such that $S = \psi^*(S')$. The data defining S' are defined over $\bar{\mathbb{Q}}$, so S'_{DR} and S'_{Dol} are defined over $\bar{\mathbb{Q}}$. This gives property 4.

This theorem may now be combined with the previous theorems to give our main statement.

THEOREM 4.2. — *Suppose X is a smooth projective variety over C . Suppose $S \subset M(X)$ is a closed, canonically defined subset. Then S is a finite union of torsion translates of triple tori.*

Proof. — By properties 1-3 of the previous theorem, and Corollary 3.2, S is a finite union of translates of triple tori. Property 4 indicates that there is a specialization X' defined over $\bar{\mathbb{Q}}$, and a subvariety $S' \subset M(X')$, such that $S = \psi^*(S')$. We have S'_{DR} and S'_B defined over $\bar{\mathbb{Q}}$. By Corollary 3.5, this implies that S' is a union of torsion translates of triple tori. The fact that each of the irreducible components of S' contains a torsion point may be seen in $M_B(X')$; hence the fact that the isomorphism ψ^* between $M_B(X')$ and $M_B(X)$ takes S' to S implies that each irreducible component of S contains a torsion point. Hence S is a union of torsion translates of triple tori.

Remark. — If X is defined over a field K , then any canonically defined subvariety $S \subset M(X)$ is defined over \bar{K} (in the DR and Dol versions)—even if the data used to define S are not defined over K .

Finally, we note another principle. There may be morphisms from canonically defined locally closed subvarieties $S \subset M$ (which means that \bar{S} and $\bar{S} - S$ are canonically defined closed subvarieties), to some algebraic varieties Q , defined by considering the invariant theory of various multilinear forms obtained from cup products and tensor products of local systems associated to points of S . These morphisms are algebraic with respect to all three structures S_B , S_{DR} , and S_{Dol} . Any such morphism must be locally constant on S . Thus, the invariant theory of the multilinear forms cannot change continuously with the parameter of the local system (*cf.* Theorems 7.19-7.23 below).

5. Applications to coherent sheaf cohomology

The previous results imply the results of Green and Lazarsfeld (this was pointed out in [1]). Consider the stratification of M by canonically defined locally closed subvarieties made by considering the rank of the cohomology $H^i(X, v)$. Restrict this stratification to the abelian variety $U_{Dol} = \text{Pic}^r(X)$. We obtain a stratification $U_{Dol} = \bigcup S^k$. By the previous theorems, the closures of the S^k are torsion translates of abelian subvarieties of U_{Dol} .

The Higgs bundle associated to a point $v \in U_{Dol}$ is simply a line bundle E with torsion Chern class, and Higgs field $\phi = 0$. Hence the Dolbeault complex has differentials equal to zero, so the Dolbeault cohomology splits:

$$H_{Dol}^i(X, E) = \bigoplus_{p+q=i} H^q(X, E \otimes \Omega_X^p).$$

The dimensions of each of the pieces vary semicontinuously in E . It follows that the stratification made according to the total, or sum of the dimensions, is finer than the stratifications made by considering the dimensions $h^q(X, E \otimes \Omega_X^p)$. That is, the strata for the individual pieces are unions of strata for the total. Hence the closures of the strata for each $h^q(X, E \otimes \Omega_X^p)$ are finite unions of torsion translates of abelian subvarieties. We recover the results of Green and Lazarsfeld [14], and the rationality results of [3] as well as the more general statements conjectured by Beauville and Catanese.

6. Absolute constructible subsets

The reader may (justifiably) complain that the discussion in Section 4 was somewhat vague. In this section and the next, we try to formulate things more precisely. We explicitly assume the axiom of choice (although we may have used it already in a more standard way), so that $\text{Aut}(C/Q)$ has the properties one would expect—for example that the fixed field is Q . We replace the notion of “canonically defined subvariety” by a

notion of *absolute constructible subset*. Absoluteness concerns only the relationship between the algebraic de Rham, the Dolbeault, and the topological interpretations of the category of local systems, and the behaviour under the action of $\text{Aut}(\mathbf{C}/\mathbf{Q})$ (this idea of looking at the Galois action comes from Deligne's definition of absolute Hodge cycles [9]). In the next section, we will discuss absolute functors: these give ways of constructing absolute subsets; we will give a collection of lemmas and propositions which allow one to verify that any canonically defined subset, in the sense of Section 4, is an absolute subset. To give the motivation, we first show that a closed absolute subset of local systems of rank one is a finite union of torsion translates of triple tori—the absolute constructible subsets are then obtained by taking finite unions, complements, and intersections.

As we will be working with local systems of any rank, we change notation with respect to the previous portion of the paper. Suppose X is a smooth complex projective variety. If X is connected, choose a base point in X and let $R(X)$ denote the space of representations of $\pi_1(X)$. If X is not connected, let $R(X)$ denote the product of the spaces of representations for the connected components. The space $R(X)$ decomposes into a disjoint union of affine schemes of finite type

$$R(X) = \bigcup_{n \in H^0(X, \mathbf{N})} R_n(X),$$

where

$$R_n(X) = \prod_i \text{Hom}(\pi_1(X_i), \text{Gl}(n_i, \mathbf{C}))$$

(the product being taken over the connected components of X). It has a structure of scheme defined over $\bar{\mathbf{Q}}$, in other words there is a scheme $R_n(X)_{\bar{\mathbf{Q}}}$ over $\text{Spec}(\bar{\mathbf{Q}})$ with an isomorphism $R_n(X) \cong R_n(X)_{\bar{\mathbf{Q}}} \times_{\text{Spec}(\bar{\mathbf{Q}})} \text{Spec}(\mathbf{C})$. The group $\prod_i \text{Gl}(n_i, \mathbf{C})$ acts on the

components $R_n(X)$. Let $M_n(X)$ denote the universal categorical quotient by this action, and let $M(X)$ denote the disjoint union of the $M_n(X)$. Thus, what was referred to as $M(X)$ in the previous sections is now $M_1(X) = R_1(X)$.

Note that we are conserving subscripts by using the notations $M(X)$ and $R(X)$ for the algebraic varieties which would have been denoted $M_B(X)$ and $R_B(X)$ in previous sections (and the references).

Recall that there are varieties $R_{\text{DR}}(X)$, $M_{\text{DR}}(X)$, $R_{\text{Dol}}(X)$, and $M_{\text{Dol}}(X)$ ([22], [19]). They are, respectively: the moduli spaces of vector bundles with integrable connection framed at one point in each connected component; the union of universal categorical quotients of its components by the actions of $\prod_i \text{Gl}(n_i, \mathbf{C})$; the moduli space

of semistable Higgs bundles with Chern classes vanishing in rational cohomology, framed at one point in each connected component; and again the union of universal categorical quotients. We have isomorphisms of sets of points

$$\begin{aligned} \psi : R(X) &\cong R_{\text{DR}}(X) \\ \psi : M(X) &\cong M_{\text{DR}}(X) \end{aligned}$$

$$\varphi : R(X) \cong R_{\text{Dol}}(X)$$

$$\varphi : M(X) \cong M_{\text{Dol}}(X).$$

The first and second are isomorphisms of complex analytic spaces; the fourth is a homeomorphism of topological spaces. The third is, however, not even continuous.

Suppose X is a smooth projective variety defined over \mathbb{C} . For each $\sigma \in \text{Aut}(\mathbb{C}/\mathbb{Q})$ we obtain a smooth projective variety X^σ defined over \mathbb{C} . There are natural maps

$$p_\sigma : M_{\text{DR}}(X) \rightarrow M_{\text{DR}}(X^\sigma),$$

$$q_\sigma : M_{\text{Dol}}(X) \rightarrow M_{\text{Dol}}(X^\sigma),$$

and similarly for the representation spaces. These are the transport of structure maps obtained from the fact that the spaces concerned are moduli spaces for algebraic geometric objects.

A subset $S \subset M(X)$ is an *absolute constructible subset* if the following conditions are satisfied. First, $S_{\text{DR}} = \psi(S)$ and $S_{\text{Dol}} = \varphi(S)$ are constructible subsets of $M_{\text{DR}}(X)$ and $M_{\text{Dol}}(X)$ respectively. Second, for each $\sigma \in \text{Aut}(\mathbb{C}/\mathbb{Q})$ there exists a constructible set $S_\sigma \subset M(X^\sigma)$ defined over \mathbb{Q} , such that $p_\sigma(S_{\text{DR}}) = \psi(S_\sigma)$ and $q_\sigma(S_{\text{Dol}}) = \varphi(S_\sigma)$ in $M(X^\sigma)$. And third, S_{Dol} is preserved by the action of \mathbb{C}^* .

An *absolute closed* (resp. *locally closed*) *subset* is an absolute constructible subset which is closed (resp. which is an open subset of a closed absolute constructible subset).

We obtain the following theorem in the case of representations of rank 1.

THEOREM 6.1. — *Suppose X is a smooth projective variety over \mathbb{C} and $S \subset M_1(X)$ is an absolute closed subset. Then S is a finite union of torsion translates of triple tori.*

Proof. — There is a field K , finitely generated over \mathbb{Q} , such that X , S_{DR} and S_{Dol} are defined over K . There is a subring $A \subset K$, finitely generated over \mathbb{Q} and with K as field of fractions, such that $\text{Spec}(A)$ is smooth and connected. Note that there is a generic geometric point $\eta : \text{Spec}(\mathbb{C}) \rightarrow \text{Spec}(A)$ corresponding to the inclusion $\varphi_\eta : A \subset \mathbb{C}$. We may choose A so that there is a variety X^A smooth and projective over $\text{Spec}(A)$, such that $\eta^*(X^A) = X$. Then the relative moduli spaces $M_{\text{DR}}(X^A/A)$ and $M_{\text{Dol}}(X^A/A)$ may be constructed. If one wants to avoid referring to this construction, note that these varieties can be obtained — from the varieties M_{DR} and M_{Dol} defined over K — by choosing A appropriately. By choice of A we may also assume the existence of closed subvarieties $S_{\text{DR}}^A \subset M_{\text{DR}}(X^A/A)$ and $S_{\text{Dol}}^A \subset M_{\text{Dol}}(X^A/A)$ with $\eta^*(S_{\text{DR}}^A) = S_{\text{DR}}^A$ and $\eta^*(S_{\text{Dol}}^A) = S_{\text{Dol}}^A$. There are stratifications of S_{DR}^A and S_{Dol}^A with the property that any stratum which maps surjectively to $\text{Spec}(A)$ is smooth over $\text{Spec}(A)$, and by further choice of A we may assume that all strata are smooth over $\text{Spec}(A)$.

Let T denote the usual topological manifold underlying $\text{Spec}(A)(\mathbb{C})$. For $t \in T$ let the subscript t denote the fiber over t in a variety over $\text{Spec}(A)$ (e.g. X_t). Note that η is a point in T , and $X_\eta = X$. Choose a simply connected usual neighborhood U of η . For values of t in this neighborhood, we may identify the topological spaces X_t^{top} in a

continuous fashion, and hence we may naturally identify $\beta_t: M(X_t) \cong M(X)$ via isomorphisms defined over $\bar{\mathbf{Q}}$. On the other hand, we have continuous families of varieties $M_{\text{DR}}(X_t)$ and $M_{\text{Dol}}(X_t)$. The compositions of the above identifications with the homeomorphisms $M(X_t) \cong M_{\text{DR}}(X_t)$ and $M(X_t) \cong M_{\text{Dol}}(X_t)$ provide continuous families of homeomorphisms

$$B_{\text{DR}, t}: M_{\text{DR}}(X_t) \cong M(X)$$

and

$$B_{\text{Dol}, t}: M_{\text{Dol}}(X_t) \cong M(X).$$

The continuous variation of the isomorphism $M_{\text{Dol}}(X_t) \cong M(X_t)$ when X_t varies with parameters is easy to see for the case of rank one local systems, as the isomorphism can then be described concretely. This is all we need for the proof of the theorem. The statement of continuous variation with parameters for any rank should be a relatively standard exercise in nonlinear partial differential equations—it will be included in the final version of [22].

For any closed subset $\Sigma \subset M(X)$, define $V(\Sigma) \subset U$ to be the set of points $t \in U$ such that $B_{\text{DR}, t}(S_{\text{DR}, t}) = \Sigma$ and $B_{\text{Dol}, t}(S_{\text{Dol}, t}) = \Sigma$. Note that $V(\Sigma)$ is a closed subset; this uses the fact that $\Sigma \times U \rightarrow U$, $S_{\text{DR}}^A|_U \rightarrow U$ and $S_{\text{Dol}}^A|_U \rightarrow U$ are open maps, in turn due to the existence of smooth stratifications. Define U^{gen} to be the set of generic geometric points in U , in other words the set of points $\text{Spec}(\mathbf{C}) \rightarrow \text{Spec}(\mathbf{A})$ such that the image is contained in U and the corresponding morphism $A \rightarrow \mathbf{C}$ is injective. The complement $U - U^{\text{gen}}$ is a countable union of closed subvarieties.

Suppose $t \in U^{\text{gen}}$. Let $\varphi_t: A \rightarrow \mathbf{C}$ denote the inclusion dual to the geometric point with image t . There is an automorphism $\sigma \in \text{Aut}(\mathbf{C}/\mathbf{Q})$ such that $\eta \circ \sigma = \varphi_t$ (by the axiom of choice). Then $X_t = X^\sigma$ and

$$p_\sigma(S_{\text{DR}}) = S_{\text{DR}, t} \subset M_{\text{DR}}(X_t)$$

$$q_\sigma(S_{\text{Dol}}) = S_{\text{Dol}, t} \subset M_{\text{Dol}}(X_t).$$

The condition that S is absolute implies (after transporting from $M(X_t)$ back to $M(X)$ by our identifications β_t) that there is a closed subvariety $\Sigma_t \subset M(X)$ defined over $\bar{\mathbf{Q}}$ with

$$B_{\text{DR}, t}(S_{\text{DR}, t}) = B_{\text{Dol}, t}(S_{\text{Dol}, t}) = \Sigma_t.$$

Thus $t \in V(\Sigma_t)$.

There are countably many closed subvarieties $\Sigma \subset M(X)$ defined over $\bar{\mathbf{Q}}$. The Baire category theorem states that U is not the union of countably many nowhere dense subsets. But $U - U^{\text{gen}}$ is a union of countably many closed subvarieties, each of which is nowhere dense. And we have seen above that U^{gen} is contained in a union of countably many closed subsets of the form $V(\Sigma)$ for $\Sigma \subset M(X)$ defined over $\bar{\mathbf{Q}}$. The Baire category theorem implies that there exists $\Sigma \subset M(X)$ defined over $\bar{\mathbf{Q}}$ such that the interior $V(\Sigma)^0$ is nonempty.

There exists a point $\xi \in V(\Sigma)^0$ algebraic over \mathbf{Q} . Then $S_{\text{DR}, \xi} \subset M_{\text{DR}}(X_\xi)$ and $S_{\text{Dol}, \xi} \subset M_{\text{Dol}}(X_\xi)$ are defined over $\bar{\mathbf{Q}}$. Furthermore, there is a subvariety $S_\xi = \beta_\xi^{-1}(\Sigma) \subset M(X_\xi)$ defined over $\bar{\mathbf{Q}}$, which is equal to $S_{\text{DR}, \xi}$ and $S_{\text{Dol}, \xi}$ in $M(X_\xi)$. By Corollary 3.5, S_ξ is a finite union of torsion translates of triple tori. Hence $\Sigma \subset M(X)$ is a finite union of torsion translates of triple tori. By looking instead at a generic geometric point in $V(\Sigma)^0$ we find that $p_\sigma(S_{\text{DR}})$ and $q_\sigma(S_{\text{Dol}})$ are finite unions of torsion translates of triple tori, for some $\sigma \in \text{Aut}(\mathbf{C}/\mathbf{Q})$. This implies that S_{DR} and S_{Dol} are, and hence that our original subset $S \subset M(X)$ is. This proves the theorem.

COROLLARY 6.2. — *An absolute locally closed subset in the moduli space of rank one local systems is the complement of a finite union of torsion translates of triple tori, within another finite union of torsion translates of triple tori. If S is an absolute constructible subset then it is a finite union of such absolute locally closed subsets.*

Proof. — If S is an absolute constructible subset then its closure \bar{S} is an absolute closed subset and the complement $\bar{S} - S$ is an absolute constructible subset. If S is locally closed then the complement is an absolute closed subset, whence the first statement. In the constructible case, $S' = \bar{S} - (\bar{S} - S)$ is an absolute locally closed subset contained in S , and the complement $S - S'$ is an absolute constructible subset of smaller dimension. By induction we can write S as a finite union of absolute locally closed subsets.

Remark. — We may obtain the same result without including the condition that S_{Dol} is preserved by \mathbf{C}^* , using Deligne's strengthening of 3.1. Similarly, throughout the following discussion we could leave out the conditions relating to the \mathbf{C}^* action.

A characterization of constructible subsets

Suppose X is a complex variety defined over a field $K \subset \mathbf{C}$. Let $i_\sigma: X \rightarrow X$ denote the conjugation maps for $\sigma \in \text{Aut}(\mathbf{C}/K)$.

We say that a subset $S \subset X$ is *topologically constructible* if there exists a complex algebraic variety Y with a constructible subset $T \subset Y$, and a homeomorphism of topological spaces $f: Y \cong X$ such that $f(T) = S$.

THEOREM 6.3. — *Suppose X is defined over a countable field $K \subset \mathbf{C}$, and $S \subset X$ is a subset such that as σ runs through $\text{Aut}(\mathbf{C}/K)$, $i_\sigma(S)$ runs through countably many subsets of X . Suppose that S is topologically constructible. Then S is a constructible subset of X .*

Proof. — We first reduce to the case where S is fixed by i_σ . Let $\{S_j\}$ denote the set of subsets which occur as $i_\sigma(S)$. For each pair (j, k) with $S_j \neq S_k$, choose a point x_{jk} in one of S_j or S_k but not the other. Then, among all the subsets, a given one is determined by the information of whether x_{jk} is in it or not for all j, k . On the other hand, there are countably many points x_{jk} so we may choose a countable field extension K'/K so that $i_\sigma(x_{jk}) = x_{jk}$ for $\sigma \in \text{Aut}(\mathbf{C}/K')$. Then $i_\sigma(S) = S$ for $\sigma \in \text{Aut}(\mathbf{C}/K')$. This completes the reduction: we may assume that S is fixed by the i_σ .

Suppose $x \in S$, and let V be the smallest closed algebraic subvariety of X , defined over K , containing x . Then V is irreducible, and x is a generic geometric point of V . For any other generic geometric point y , there is an automorphism $\sigma \in \text{Aut}(C/K)$ such that $i_\sigma(x) = y$. This implies that S contains the set of all generic geometric points of V . The closure \bar{S} therefore contains V .

To complete the proof, we will use the following facts from topology (cf. [25]). Suppose M is an m -dimensional manifold and $N \subset M$ is a subset. If N has a structure of n -dimensional manifold then $n \leq m$; and if $n = m$ then N is an open subset of M (the Brouwer theorem of invariance of domain). There is a topological notion of dimension: a space Z has dimension $\leq n$ if every open covering has a refinement whose nerve is a simplicial complex of dimension $\leq n$. This agrees with the usual notion for manifolds and simplicial complexes (hence with the real dimension of complex analytic varieties), and decreases for closed subsets. If Z contains an open subset which is a manifold of dimension n then the topological dimension of Z is at least n . If M is a connected manifold of dimension m and $Z \subset M$ is a closed subset such that M can be covered by sufficiently small relatively compact open sets M_i with $\dim(Z \cap \bar{M}_i) \leq m-2$ and $\dim(Z \cap \partial M_i) \leq m-3$, then $M-Z$ remains connected. This follows from Alexander duality: $H_0(M_i)$ is isomorphic to the Čech cohomology $\check{H}^{m-1}(\partial M_i)$, and the addition of $Z \cap M_i$ to ∂M_i doesn't change \check{H}^{m-1} so $H_0(M_i - Z \cap M_i) \cong H_0(M_i)$; an argument with the covering shows that $M-Z$ is connected.

Continue the previous argument. Since S is a topologically constructible subset, there exists a subset $S^{\text{reg}} \subset S$ such that S^{reg} is a dense open subset of \bar{S} , and which decomposes as a finite disjoint union

$$S^{\text{reg}} = \bigcup_j S_j^{\text{reg}}$$

where S_j^{reg} are connected manifolds of real dimension m_j . Suppose $x \in S_j^{\text{reg}}$, and let V be the smallest closed subvariety of X , defined over K , containing x . Then $V \cap S_j^{\text{reg}}$ is a closed subset of S_j^{reg} of dimension $\leq \dim(V)$. If $\dim(V) < m_j$ then $V \cap S_j^{\text{reg}}$ is nowhere dense (otherwise it would contain an open subset of dimension m_j). There are countably many subvarieties of X defined over K . The Baire category theorem implies that S_j^{reg} is not the union of subsets of the form $V \cap S_j^{\text{reg}}$ for $\dim(V) < m_j$. Hence there exists a point $x \in S_j^{\text{reg}}$ such that the corresponding variety has dimension $\geq m_j$. Since $V \subset \bar{S}$ and S_j^{reg} is an open subset of \bar{S} , $V \cap S_j^{\text{reg}}$ is an open subset of V . In particular it contains a smooth point of V , hence it contains a manifold of dimension $\dim(V)$. Thus $\dim(V) = m_j$. Let V^{reg} denote the set of smooth points of V ; it is the complement $V^{\text{reg}} = V - D$ where D is a closed subvariety of V . Now S_j^{reg} is the intersection of S with an open subset $W \subset X$. Then W can be covered by small relatively compact open subsets W_i such that $\dim(D \cap \bar{W}_i) \leq m_j - 2$ and $\dim(D \cap \partial W_i) \leq m_j - 3$. This restricts to a covering of S_j^{reg} with the same property, so the fact mentioned above implies that $U = S_j^{\text{reg}} - D \cap S_j^{\text{reg}}$. On the other hand $V \cap U = V^{\text{reg}} \cap S_j^{\text{reg}}$ is a closed subset of U which also has a structure of manifold of dimension m_j . By the invariance of domain $V \cap U$ is open in U , so $V \cap U = U$. But $D \cap S_j^{\text{reg}}$ is closed and nowhere dense in S_j^{reg} (by consideration of dimension), so its complement U is dense in S_j^{reg} . As V is a closed

subset of \bar{S} , we have $S_j^{\text{reg}} \subset V$. Do the same for each of the components S_j^{reg} , and take the union of the varieties; we get a closed subvariety V of X (no longer irreducible) such that $S^{\text{reg}} \subset V \subset \bar{S}$. But S^{reg} is dense in \bar{S} , so $\bar{S} = V$.

We have proved that if S is a topologically constructible set which is preserved by the action of $\text{Aut}(C/K)$, then its closure \bar{S} is a closed subvariety. In particular, the complement $\bar{S} - S$ is preserved by the action of $\text{Aut}(C/K)$. The definition of topologically constructible subset implies that $\bar{S} - S$ is a topologically constructible subset of strictly smaller dimension. We may argue by induction on the dimension: then at the present stage we will already know that $\bar{S} - S$ is a constructible subset. Thus S is the complement of a constructible subset within a closed subvariety, so S is constructible. This completes the proof of the theorem.

This theorem allows us to remove the conditions that S_{DR} and S_{Dol} are constructible subsets of M_{DR} and M_{Dol} from the definition of absolute constructible subset. From the maps p_σ and q_σ we obtain

$$P_\sigma = \psi^{-1} p_\sigma \psi : M(X) \rightarrow M(X^\sigma)$$

and

$$Q_\sigma = \phi^{-1} q_\sigma \phi : M(X) \rightarrow M(X^\sigma)$$

for $\sigma \in \text{Aut}(C/Q)$.

COROLLARY 6.4. — *Suppose $S \subset M(X)$ is a subset such that for every $\sigma \in \text{Aut}(C/Q)$, there exists an algebraic subset $S_\sigma \subset M(X^\sigma)$ defined over \bar{Q} with $P_\sigma(S) = S_\sigma$ and $Q_\sigma(S) = S_\sigma$. Then S_{DR} and S_{Dol} are constructible subsets of $M_{\text{DR}}(X)$ and $M_{\text{Dol}}(X)$ respectively.*

Proof. — We may choose a countable field of definition $K \subset C$ for X . This consists of a collection of isomorphisms $\alpha_\sigma : X \cong X^\sigma$ for $\sigma \in \text{Aut}(C/K)$, compatible with composition of σ . The induced maps $\alpha_\sigma^* : M(X^\sigma) \cong M(X)$ are defined over \bar{Q} . Let i_σ denote the conjugations giving the definitions of $M_{\text{DR}}(X)$ and $M_{\text{Dol}}(X)$ over K . The hypotheses of the corollary now imply that $i_\sigma(S_{\text{DR}}) = \psi(\alpha_\sigma^*(S_\sigma))$ and $i_\sigma(S_{\text{Dol}}) = \phi(\alpha_\sigma^*(S_\sigma))$. In particular, since ψ and ϕ are homeomorphisms of topological spaces, $i_\sigma(S_{\text{DR}})$ and $i_\sigma(S_{\text{Dol}})$ are topological constructible subsets. Furthermore, the $\alpha_\sigma^*(S_\sigma)$ are defined over \bar{Q} , so they run through countably many possibilities. The hypotheses of the theorem are satisfied, hence S_{DR} and S_{Dol} are constructible subsets.

COROLLARY 6.5. — *With the assumptions of the previous corollary, suppose also that S is preserved by the action of C^* . Then S is an absolute constructible subset.*

Proof. — This follows immediately from the previous corollary and the definition of absolute constructible subset.

7. Absolute functors

Denote by X, Y, Z , etc. various smooth complex projective varieties. Let $L(X)$ denote the category of local systems on X . A *saturated subcategory* is a full subcategory $D \subset L(X)$ such that if U is isomorphic to an object of D , then $U \in D$. A *partially defined functor* F from $L(X)$ to $L(Y)$ consists of a saturated subcategory $\text{Dom}(F) \subset L(X)$ and a functor $F: \text{Dom}(F) \rightarrow L(Y)$. Define the composition GF of two partially defined functors in the following way:

$$\text{Dom}(GF) = \{ U \in \text{Dom}(F) \text{ s. t. } F(U) \in \text{Dom}(G) \},$$

and GF is the composition of functors.

Let $L_{\text{DR}}(X)$ (resp. $L_{\text{Dol}}(X)$) denote the category of vector bundles with integrable connection (resp. semistable Higgs bundles with Chern classes vanishing in rational cohomology) on X . We have equivalences of categories

$$\psi: L(X) \rightarrow L_{\text{DR}}(X)$$

and

$$\varphi: L(X) \rightarrow L_{\text{Dol}}(X).$$

The first is the Riemann-Hilbert correspondence [7] and the second is the correspondence between Higgs bundles and local systems given in [21]. These functors have quasi-inverses which we denote by ψ^{-1} and φ^{-1} ; we pretend that they are strict inverses. We are using the same notation for these equivalences of categories as for the corresponding maps between moduli spaces.

The group of field automorphisms $\text{Aut}(\mathbf{C}/\mathbf{Q})$ acts in the following way. Suppose $\sigma \in \text{Aut}(\mathbf{C}/\mathbf{Q})$. Let X^σ denote the variety obtained by conjugating the equations of X by σ . The categories $L_{\text{DR}}(X)$ and $L_{\text{Dol}}(X)$ consist of algebraic geometric objects, so there are natural equivalences of categories

$$p_\sigma: L_{\text{DR}}(X) \rightarrow L_{\text{DR}}(X^\sigma),$$

and

$$q_\sigma: L_{\text{Dol}}(X) \rightarrow L_{\text{Dol}}(X^\sigma).$$

We obtain the following functors

$$P_\sigma = \psi^{-1} p_\sigma \psi: L(X) \rightarrow L(X^\sigma)$$

and

$$Q_\sigma = \varphi^{-1} q_\sigma \varphi: L(X) \rightarrow L(X^\sigma)$$

The group \mathbf{C}^* acts on $L_{\text{Dol}}(X)$ by functors

$$m_t: L_{\text{Dol}}(X) \rightarrow L_{\text{Dol}}(X), \quad m_t(E, \theta) = (E, t\theta);$$

denote the corresponding action on $L(X)$ with the same symbol,

$$m_t = \varphi^{-1} m_t \varphi : L(X) \rightarrow L(X)$$

We note some further actions. Suppose $K \subset \mathbb{C}$ is a countable field of definition for X . This means that we have a system of isomorphisms $d_\sigma : X \cong X^\sigma$ for $\sigma \in \text{Aut}(\mathbb{C}/K)$, compatible with composition of σ . Composing the P_σ and Q_σ defined above with d_σ we get functors

$$\tilde{P}_\sigma : L(X) \rightarrow L(X)$$

and

$$\tilde{Q}_\sigma : L(X) \rightarrow L(X),$$

for any $\sigma \in \text{Aut}(\mathbb{C}/K)$; these are again compatible with composition of σ .

We denote by the same letters the corresponding maps of sets of isomorphism classes of semisimple objects. These give the Galois action on $M(X)$ corresponding to the definitions of $M_{\text{Dol}}(X)$ and $M_{\text{DR}}(X)$ over K .

We need to discuss a notion of a partially defined functor which is algebraic and defined over $\bar{\mathbb{Q}}$ (in the “Betti” realization). There is a universal family of local systems W indexed by $R(X)$ (to be precise, W is a local system of $H^0(R(X), \mathcal{O}_{R(X)})$ -modules), so there is a scheme representing the sets of morphisms between objects. That is, there is a scheme $H(X)$ with a map $H(X) \rightarrow R(X) \times R(X)$ such that for any scheme T with maps $u, v : T \rightarrow R(X)$, the T -valued points of $H(X)$ lying over (u, v) correspond to the elements of $\text{Hom}(u^*W, v^*W)$. There is a map $H(X) \times_{R(X)} H(X) \rightarrow H(X)$ representing composition of morphisms, and a section $R(X) \rightarrow H(X)$ representing the identity morphisms. All of these things are defined over $\bar{\mathbb{Q}}$.

Suppose $f : U \rightarrow R(X)$ is a morphism of finite type, defined over $\bar{\mathbb{Q}}$. Suppose that U is reduced. Then we obtain a *little category of local systems* $L_U(X)$. The objects are the \mathbb{C} -valued points of U , and the morphisms are the \mathbb{C} -valued points of

$H_U \stackrel{\text{def}}{=} H(X) \times_{(R(X) \times R(X))} (U \times U)$. There is a natural fully faithful functor $j : L_U(X) \rightarrow L(X)$. Suppose $F_0 : U \rightarrow R(Y)$ and $F_1 : H_U \rightarrow H(Y)$ are morphisms of schemes, defined over $\bar{\mathbb{Q}}$, satisfying the conditions required for a functor (that is, F_1 preserves the composition laws and identities). We obtain a functor $(F_0, F_1) : L_U(X) \rightarrow L(Y)$. Call a functor obtained in this way a *little functor defined over $\bar{\mathbb{Q}}$* .

Here is a useful construction. Suppose $f : U \rightarrow R(X)$ and $g : V \rightarrow R(X)$ are two morphisms. Let

$$P(U, V) \subset (U \times V) \times_{R(X) \times R(X)} H(X)$$

denote the set of triples (u, v, h) where h is an isomorphism from u to v . There are two morphisms fp_1 and gp_2 from $P(U, V)$ to $R(X)$, so we obtain two little categories denoted

$L_{P(U, V), 1}(X)$ and $L_{P(U, V), 2}(X)$. There is a natural equivalence of categories

$$\theta: L_{P(U, V), 1}(X) \cong L_{P(U, V), 2}(X),$$

which is the identity on the set of objects and is on morphisms given by conjugation with h . There are natural functors

$$p_1: L_{P(U, V), 1}(X) \rightarrow L_U(X) \quad \text{and} \quad p_2: L_{P(U, V), 2}(X) \rightarrow L_V(X),$$

and a natural isomorphism $n: j_1 p_1 \cong j_2 p_2 \theta$, where $j_1: L_U(X) \rightarrow L(X)$ and $j_2: L_V(X) \rightarrow L(X)$ are as defined before.

Suppose $f: U \rightarrow R(X)$, and $F_0: U \rightarrow R(Y)$ and F_1 as above form a little functor $(F_0, F_1): L_U(X) \rightarrow L(Y)$. Suppose $g: V \rightarrow R(Y)$ is a morphism. Then, using the morphisms F_0 and g we can make a variety $P(U, V)$ mapping to $U \times V$. Using $F_0 p_1$ and $g p_2$ we can make the little categories $L_{P(U, V), 1}(Y)$ and $L_{P(U, V), 2}(Y)$. Furthermore, using the morphism $f p_1$ we can make a little category $L_{P(U, V)}(X)$. There is a natural functor $(I, F_1): L_{P(U, V)}(X) \rightarrow L_{P(U, V), 1}(Y)$. Let $q: L_{P(U, V)}(X) \rightarrow L_U(X)$ denote the projection. Let j_i be defined as above with respect to Y , and j with respect to X . There is a natural isomorphism between $j_1 p_1 (I, F_1)$ and $(F_0, F_1) q$. Hence there is a natural isomorphism between $j_2 p_2 \theta (I, F_1)$ and $(F_0, F_1) q$.

Suppose $(G_0, G_1): L_V(Y) \rightarrow L(Z)$ is a little functor. We can now define a composition. The functor $(G_0, G_1) p_2$ goes from $L_{P(U, V), 2}(Y)$ to $L(Z)$. Put

$$(G_0, G_1) \circ (F_0, F_1) \stackrel{\text{def}}{=} (G_0, G_1) p_2 \theta (I, F_1): L_{P(U, V)}(X) \rightarrow L(Z).$$

Note that there exist morphisms $E_0 = G_0 p_2$, and a corresponding E_1 , such that $(G_0, G_1) \circ (F_0, F_1) = (E_0, E_1)$.

An *algebraic functor defined over \bar{Q}* is a partially defined functor $F: L(X) \rightarrow L(Y)$ together with a little functor defined over \bar{Q} , $(F_0, F_1): L_U(X) \rightarrow L(Y)$ and a natural transformation η , such that the image of $L_U(X)$ is contained in $\text{Dom}(F)$, the functor $j: L_U(X) \rightarrow \text{Dom}(F)$ is an equivalence of categories, and η is a natural isomorphism from (F_0, F_1) to $F \circ j$.

Suppose F and G are algebraic functors defined over \bar{Q} , from $L(X)$ to $L(Y)$, with associated morphisms $f: U \rightarrow R(X)$ and $g: V \rightarrow R(X)$. Suppose that $\text{Dom}(F) = \text{Dom}(G)$. A natural transformation $\xi: F \rightarrow G$ is *algebraic, defined over \bar{Q}* if the corresponding natural transformation $(\xi j_2 p_2 \theta)(F_n)$ from $F j_1 p_1$ to $G j_2 p_2 \theta$ (which are functors from $L_{P(U, V)}(X)$ to $L(Y)$ naturally isomorphic to algebraic ones) has values in $H(Y)$ varying algebraically with the argument. In other words the values are given by a constructible algebraic function $P(U, V) \rightarrow H(Y)$ defined over \bar{Q} .

Suppose F is an algebraic functor defined over \bar{Q} from $L(X)$ to $L(Y)$, and G an algebraic functor defined over \bar{Q} from $L(Y)$ to $L(Z)$. We can define the *composition* $G \circ F$ to be the composition of functors GF , provided with the algebraic structure of the functor $(G_0, G_1) \circ (F_0, F_1)$ defined above, and the natural isomorphism between $GF_{j_q} \cong G(F_0, F_1) q \cong G j_2 p_2 \theta (I, F_1) \cong (G_0, G_1) p_2 \theta (I, F_1) = (G_0, G_1) \circ (F_0, F_1)$.

With these preliminary definitions done, we can give our main definition. An *absolute partially defined functor* $F: L(X) \rightarrow L(Y)$ is a collection of algebraic functors $F_\sigma: L(X^\sigma) \rightarrow L(Y^\sigma)$ defined over $\bar{\mathbf{Q}}$, F_1 being denoted by F , with the following data: natural isomorphisms of the underlying usual functors

$$u_\sigma: P_\sigma F \cong F_\sigma P_\sigma$$

and

$$v_\sigma: Q_\sigma F \cong F_\sigma Q_\sigma$$

for $\sigma \in \text{Aut}(\mathbf{C}/\mathbf{Q})$; a countable subfield $K \subset \mathbf{C}$ and definitions $\{\alpha_\sigma\}$ for X and Y over K , together with natural isomorphisms (algebraic, defined over $\bar{\mathbf{Q}}$)

$$d_\sigma: \alpha_\sigma^* F_\sigma \cong F \alpha_\sigma^*$$

for $\sigma \in \text{Aut}(\mathbf{C}/K)$; and natural isomorphisms $w_t: m_t F_{\text{Dol}} \cong F_{\text{Dol}} m_t$, such that some compatibility conditions hold. The compatibility conditions for d , u , and v say that if we put $\tilde{P}_\sigma = \alpha_\sigma^* P_\sigma$ and $\tilde{Q}_\sigma = \alpha_\sigma^* Q_\sigma$ then the systems of natural isomorphisms

$$\tilde{P}_\sigma F \cong F \tilde{P}_\sigma$$

and

$$\tilde{Q}_\sigma F \cong F \tilde{Q}_\sigma$$

are compatible with composition of σ ; the compatibility conditions for the w_t are that they are compatible with multiplication of $t \in \mathbf{C}^*$. Included in the existence of the natural isomorphisms is the condition that the domains of the functors are the same. Also put

$$F_{\text{DR}} = \psi^{-1} F \psi \quad \text{and} \quad F_{\text{Dol}} = \varphi^{-1} F \varphi.$$

We say that the field K which appears above is a *field of definition* (II) for F . The roman numeral (II) is meant to distinguish this notion from the definition of F and the F_σ over $\bar{\mathbf{Q}}$ in the Betti realization. Note that any bigger field is also a field of definition (II).

Suppose F and G are absolute functors from $L(X)$ to $L(Y)$. Suppose $\text{Dom}(F) = \text{Dom}(G)$ (note that this implies that $\text{Dom}(F_\sigma) = \text{Dom}(G_\sigma)$). Let K be a common field of definition (II) for F and G . An *absolute natural transformation* $\xi: F \rightarrow G$ with K as field of definition (II) is a natural transformation of the underlying usual functors together with a collection of algebraic natural transformations $\xi_\sigma: F_\sigma \rightarrow G_\sigma$ defined over $\bar{\mathbf{Q}}$, which are compatible with the natural transformations d_σ , u_σ , v_σ and w_t . This means that the following diagrams commute (the first one for all $\sigma \in \text{Aut}(\mathbf{C}/K)$, the next two for all $\sigma \in \text{Aut}(\mathbf{C}/\mathbf{Q})$, and the last for all $t \in \mathbf{C}^*$):

$$\begin{array}{ccccccc} \alpha_\sigma^* F \rightarrow F \alpha_\sigma^* & P_\sigma F \rightarrow F_\sigma P_\sigma & Q_\sigma F \rightarrow F_\sigma Q_\sigma & m_t F \rightarrow F m_t \\ \downarrow & \downarrow & \downarrow & \downarrow \\ \alpha_\sigma^* G \rightarrow G \alpha_\sigma^* & P_\sigma G \rightarrow G_\sigma P_\sigma & Q_\sigma G \rightarrow G_\sigma Q_\sigma & m_t G \rightarrow G m_t \end{array}$$

Note that the natural transformation ξ is determined by the corresponding natural transformation of the underlying usual functors.

The composition of two absolute functors is defined by $(G \circ F)_\sigma = G_\sigma \circ F_\sigma$, with underlying usual functors GF . The natural transformations are defined in the obvious way. The field of definition is the field generated by fields of definition for F and G .

Let $A \subset L(X)$ be a union of isomorphism classes of local systems. We say that A is *algebraic, defined over $\bar{\mathbb{Q}}$* , if the subset of points $r \in R(X)$ such that $W(r) \in A$ is a constructible subset defined over $\bar{\mathbb{Q}}$. We say that A is *absolute* if the following properties are satisfied: for each $\sigma \in \text{Aut}(\mathbb{C}/\bar{\mathbb{Q}})$, the subsets $P_\sigma(A)$ and $Q_\sigma(A)$ in $L(X^\sigma)$ are both equal to a subset A_σ which is algebraic, defined over $\bar{\mathbb{Q}}$; and $\varphi(A) \subset L_{\text{Dol}}(X)$ is preserved by the action of \mathbb{C}^* .

Complements and finite unions and intersections of absolute subsets remain absolute.

LEMMA 7.1. — *Suppose A is an absolute subset of local systems. Let $R_A(X) = \{r \in R(X) \text{ s. t. } W(r) \in A\}$, and let $M_A(X) \subset M(X)$ be the image of $R_A(X)$. Then $M_A(X)$ is an absolute constructible subset in the sense of Section 6.*

Proof. — For brevity, denote $S = M_A(X) \subset M(X)$ and let S_{DR} and S_{Dol} be the corresponding subsets. The definition of absolute subset of local systems implies that for each $\sigma \in \text{Aut}(\mathbb{C}/\bar{\mathbb{Q}})$ there is a subset $S_\sigma \subset M(X^\sigma)$, constructible and defined over $\bar{\mathbb{Q}}$, such that $\psi(S_\sigma) = p_\sigma(S_{\text{DR}})$ and $\varphi(S_\sigma) = q_\sigma(S_{\text{Dol}})$. Corollary 6.4 implies that S_{DR} and S_{Dol} are constructible algebraic subsets of M_{DR} and M_{Dol} . The set of points underlying S_{Dol} is invariant under the action of \mathbb{C}^* , so the variety S_{Dol} is. Thus S is an absolute constructible subset.

LEMMA 7.2. — *The domain of any absolute functor is an absolute subset. A subset A (union of isomorphism classes) is absolute if and only if there exists an absolute partially defined functor $I_A: L(X) \rightarrow L(X)$ such that $\text{Dom}(I_A) = A$ and I_A is the identity functor on A .*

Proof. — Suppose F is an absolute functor. Then its domain is an absolute subset. This is mostly clear; it suffices to note that the subset of points in $R(X^\sigma)$ corresponding to elements of $\text{Dom}(F_\sigma)$ is the image of the map $U \rightarrow R(X)$ used to define the corresponding little category and little functor. This image is a constructible set defined over $\bar{\mathbb{Q}}$.

Suppose A is an absolute subset. Then the subsets $R_A(X^\sigma)$ are constructible. We can choose morphisms $U_\sigma \rightarrow R(X^\sigma)$ with images equal to $R_A(X^\sigma)$. Use these, along with the appropriate liftings to the schemes of morphisms, to define the little functors associated to $I_{A,\sigma}$. Thus we get algebraic functors $I_{A,\sigma}$ defined over $\bar{\mathbb{Q}}$. The functors $I_{A,\text{DR}}$ and $I_{A,\text{Dol}}$ are defined in the obvious way, and the \mathbb{C}^* action on $I_{A,\text{Dol}}$ is given by the identity maps.

COROLLARY 7.3. — *If F is an absolute functor from local systems on X to local systems on Y , and A is an absolute subset of local systems on Y , then*

$$F^{-1}(A) \stackrel{\text{def}}{=} \{ U \in \text{Dom}(F), F(U) \in A \}$$

is an absolute subset of local systems on X .

Proof. — It is the domain of the functor $I_A \circ F$.

LEMMA 7.4. — *If F is an absolute functor from local systems on X to local systems on Y , and A is an absolute subset of local systems on X , then*

$F(A) \stackrel{\text{def}}{=} \{ U \in L(Y), \exists V \in \text{Dom}(F) \cap A, U \cong F(V) \}$ *is an absolute subset of local systems on Y .*

Proof. — The image of an algebraic subset of local systems defined over $\bar{\mathbf{Q}}$, by an algebraic functor defined over $\bar{\mathbf{Q}}$, is again algebraic defined over $\bar{\mathbf{Q}}$. This is seen by noting that it is the image in $R(Y)$ of the fiber product over $R(X)$ of the variety U used to define the functor, and the variety U' used to define the subset of local systems. Here U' should be chosen so as to map surjectively to the union of orbits corresponding to the elements of the subset. Using this fact, the natural isomorphisms $F_\sigma P_\sigma \cong P_\sigma F$ and $F_\sigma Q_\sigma \cong Q_\sigma F$ and the conditions that $P_\sigma(A) = Q_\sigma(A) = A_\sigma$ are algebraic defined over $\bar{\mathbf{Q}}$ imply that $P_\sigma(F(A)) = Q_\sigma(F(A)) = F_\sigma(A_\sigma)$ are algebraic, defined over $\bar{\mathbf{Q}}$. Similarly, the natural isomorphisms $m_t F \cong F m_t$ and the condition that A is preserved by m_t imply that $F(A)$ is preserved by the action of \mathbf{C}^* . Thus $F(A)$ is absolute.

Suppose $A_0 \in L(X)$, and A is the isomorphism class containing A_0 . We say that A_0 is *absolute* if A is absolute.

We note some properties of an absolute semisimple object A_0 . The underlying local system is defined over $\bar{\mathbf{Q}}$, hence it has an automorphism group $\text{Aut}(A_0)$ defined over $\bar{\mathbf{Q}}$. Furthermore, suppose K is an algebraically closed field of definition for X . All of the local systems $\tilde{P}_\sigma(A_0) \cong \alpha_\sigma^* A_{0,\sigma}$ are defined over $\bar{\mathbf{Q}}$, so there are only countably many such. If the vector bundle with integrable connection $\psi^{-1}(A_0)$ were not defined over K , then it would have uncountably many different conjugates. Thus $\psi(A_0)$ and similarly $\phi(A_0)$ are defined over K . The local system A_0 is preserved up to isomorphism by the action of \mathbf{C}^* ; this implies that it underlies a variation of Hodge structure, and that $\phi(A_0)$ has a structure of system of Hodge bundles [21]. In particular, we can choose an action of \mathbf{C}^* , namely a compatible system of isomorphisms $A_0 \cong m_t A_0$ (although this choice is not unique). We may assume that these isomorphisms act algebraically on the Higgs bundle $\phi(A_0)$.

LEMMA 7.5. — *The trivial local system of rank n on X is absolute.*

Proof. — The images under the Galois action of the trivial vector bundle with integrable connection, or Higgs bundle, are themselves trivial. The orbit of the trivial local system in any $R(X^\circ)$ is algebraic, locally closed, and defined over $\bar{\mathbf{Q}}$. Finally, the trivial Higgs bundle is preserved by the \mathbf{C}^* action.

LEMMA 7.6. — *Suppose U is a rigid semisimple local system on X . Then U is absolute.*

Proof. — Let (N, ∇) be the associated vector bundle with integrable connection, and (E, θ) the associated Higgs bundle. The fact that U is rigid implies that it is defined over $\bar{\mathbf{Q}}$, and (E, θ) is preserved by the \mathbf{C}^* action. In particular, (E, θ) has a structure of system of Hodge bundles and U a structure of complex variation of Hodge structure (cf. [21]). Suppose $\sigma \in \text{Aut}(\mathbf{C}/\mathbf{Q})$. Then $p_\sigma(N, \nabla)$ and $q_\sigma(E, \theta)$ are rigid; thus $P_\sigma U$ and $Q_\sigma U$ are rigid, in particular they are defined over $\bar{\mathbf{Q}}$. We have to show that $P_\sigma U \cong Q_\sigma U$. For this, we make use of a result about moduli spaces which will appear elsewhere. It says that there is a quasiprojective moduli space $M_{\text{DR/Dol}}(X)$ with a morphism $\lambda: M_{\text{DR/Dol}}(X) \rightarrow \mathbf{A}^1$ and an action of \mathbf{C}^* covering the standard action on \mathbf{A}^1 , together with an identification $\lambda^{-1}(1) \cong M_{\text{DR}}(X)$ and $\lambda^{-1}(0) \cong M_{\text{Dol}}(X)$. The space $M_{\text{DR/Dol}}(X)$ is a coarse moduli space for triples (λ, V, ∇) where $\lambda \in \mathbf{C}$, V is a vector bundle with vanishing rational Chern classes, and ∇ is an operator satisfying Leibniz's rule with respect to λd , with (V, ∇) required to be semistable if $\lambda = 0$. The action of \mathbf{C}^* is given by $t(\lambda, V, \nabla) = (t\lambda, V, t\nabla)$.

We will show that the \mathbf{C}^* orbit of the point $(1, N, \nabla)$ corresponding to U has as its limit (for $t \rightarrow 0$) the point $(0, E, \theta)$. Write $N = \bigoplus N^p$ as a \mathbf{C}^∞ direct sum of its Hodge components. Decompose the flat connection D (equal to ∇ plus the holomorphic structure of N) into pieces in the usual way [21],

$$D = \partial + \bar{\partial} + \theta + \bar{\theta}.$$

Then the orbit of \mathbf{C}^* consists of the points

$$(t, (N, \bar{\partial} + \bar{\theta}), t\partial + t\theta).$$

Because of the Hodge types of the components of D , these objects are isomorphic, via isomorphisms obtained by multiplying by t^p on N^p , to the points

$$(t, (N, \bar{\partial} + t\bar{\theta}), t\partial + \theta).$$

These have as limit when $t \rightarrow 0$ the point

$$(0, (N, \bar{\partial}), \theta) \cong (0, E, \theta),$$

which is semistable. Hence the limit of the \mathbf{C}^* orbit is the corresponding Higgs bundle $(0, E, \theta)$. Note that this limit is unique since $M_{\text{DR/Dol}}(X)$ is separated. The construction of the moduli space (with its \mathbf{C}^* action) is an algebraic geometric operation compatible with field automorphisms. Also the process of taking the limit is algebraic (it consists of filling in the orbit in a unique way to a map from \mathbf{A}^1). Therefore the limit of the \mathbf{C}^* orbit of $(1, N^\sigma, \nabla^\sigma)$ in $M_{\text{DR/Dol}}(X^\sigma)$ is $(0, E^\sigma, \theta^\sigma)$. If we already know that $(N^\sigma, \nabla^\sigma)$ is a variation of Hodge structure (as in the present case by rigidity), the same argument shows that the limit of its \mathbf{C}^* orbit must be the corresponding Higgs bundle. Thus $(E^\sigma, \theta^\sigma)$ is the Higgs bundle corresponding to $(N^\sigma, \nabla^\sigma)$. This completes the proof that $P_\sigma U = Q_\sigma U$, so U is absolute.

Suppose V is a semisimple local system and X . Write $V = \bigoplus V_i \otimes W_i$ with V_i distinct irreducible local systems, and W_i vector spaces. Let $r_i = \text{rk}(V_i)$ and $d_i = \dim(W_i)$. Define the *type* of V to be the set of pairs (r_i, d_i) (considered with their multiplicities of occurrence). If we fix some irreducible local systems U_1, \dots, U_k , then the *type of V with respect to the U_j* is again the set of pairs (r_i, d_i) with multiplicities, but also with the pairs corresponding to each U_j singled out.

LEMMA 7.7. — *The set of semisimple local systems on X is absolute, as is the subset of those with a given type. If U_1, \dots, U_k are absolute local systems, then the set of local systems which has a given type relative to the U_j , is absolute.*

Proof. — The property of being semisimple, and the type, may be seen algebraically from the de Rham or Dolbeault realizations of a local system. In particular, these properties are preserved by the P_σ and Q_σ . They are defined over $\bar{\mathbb{Q}}$, and are also preserved by the \mathbb{C}^* action on the Dolbeault category. Furthermore, the dimension of the isotypic component corresponding to a given local system may be seen from the de Rham or Dolbeault realizations; and absolute local systems are preserved by the action of \mathbb{C}^* and the functors $Q_\sigma^{-1} P_\sigma$ (see below).

Suppose $\{x\}$ is a variety consisting of one point. The category $L(\{x\})$ is then the same as the category of vector spaces; we will denote this by Vect . Suppose X_1, \dots, X_k is a collection of smooth complex projective varieties. Let X be the disjoint union of the X_i . Then

$$L(X) = L(X_1) \times \dots \times L(X_k).$$

Via this isomorphism, we may use all of the above definitions for functors between products of categories of local systems on several varieties.

PROPOSITION 7.8. — *Suppose X and Y are smooth complex projective varieties, and $f: Y \rightarrow X$ is a morphism. There is an absolute functor $f^*: L(X) \rightarrow L(Y)$ whose domains is $L(X)$ and whose underlying usual functor is the inverse image.*

Proof. — It is easy to see that the Riemann-Hilbert correspondence is compatible with inverse image, and that inverse image of local systems has a structure of algebraic functor defined over $\bar{\mathbb{Q}}$. One can define the inverse image of a semistable Higgs bundle, and this is compatible with the inverse image of local systems (cf. [21]). The inverse image for Higgs bundles is compatible with the multiplication by \mathbb{C}^* . If K is a field of definition for X , Y , and f , then we obtain the natural isomorphisms d_σ by functoriality of the inverse image construction.

PROPOSITION 7.9. — *Suppose X is a smooth complex projective variety. For any $i \geq 0$ there is an absolute functor H^i from local systems on X to Vect , with $\text{Dom}(H^i) = L(X)$ and underlying functor given by $H^i(V) = H^i(X, V)$, the i -th cohomology with coefficients in V .*

Proof. — There is a constructible decomposition $R(X) = \bigcup R_j(X)$ into locally closed strata, defined over $\bar{\mathbb{Q}}$, with the following properties. For each stratum $R_j(X)$, there is a number m_j such that $\dim H^i(X, W(r)) = m_j$ for $r \in R_j(X)$. It follows that there is a

vector bundle of rank m_j on $R_j(X)$ whose fibers are the cohomology groups. These vector bundles are defined over $\bar{\mathbb{Q}}$. We may further assume that these vector bundles are trivial, and that trivializations are chosen (also defined over $\bar{\mathbb{Q}}$). Let U be the disjoint union of the strata, mapping to $R(X)$. The trivializations give a little functor $L_U(X) \rightarrow \text{Vect}$ defined over $\bar{\mathbb{Q}}$. In this way we obtain algebraic functors defined over $\bar{\mathbb{Q}}$, whose underlying functors are $V \mapsto H^i(X, V)$. Do the same for each X^σ .

One can define the algebraic de Rham cohomology with coefficients in a vector bundle with integrable connection, and so obtain $H_{\text{DR}}^i: L_{\text{DR}}(X) \rightarrow \text{Vect}$. For this (cf. [16]) one defines the cohomology of (N, ∇) to be the hypercohomology of the algebraic de Rham complex

$$\dots \xrightarrow{\nabla} N \otimes \Omega_X^i \xrightarrow{\nabla} \dots$$

There are natural transformations of compatibility u_σ , the natural isomorphisms between this hypercohomology (transported to X^σ) and the usual cohomology of the corresponding local systems of flat sections.

Similarly for the functor H_{Dol}^i , Dolbeault cohomology of a Higgs bundle (E, θ) is the hypercohomology of the complex

$$\dots \xrightarrow{\theta} E \otimes \Omega_X^i \xrightarrow{\theta} \dots$$

In case (E, θ) is a direct sum of stable Higgs bundles, the Dolbeault cohomology is shown in [21] to be naturally isomorphic to the cohomology of the corresponding local system.

We need the same statement for semistable Higgs bundles. This unfortunate omission from [21] actually follows easily from the discussion there. The equivalence of categories between semistable Higgs bundles with $c_i=0$ and local systems comes from a quasi-equivalence between the differential graded categories \mathcal{C}_{Dol} and \mathcal{C}_{DR} (in the notation of [21]). This gives, in particular, natural isomorphisms

$$\text{Ext}_{\mathcal{C}_{\text{Dol}}}^i(\mathbf{1}, E) \cong \text{Ext}_{\mathcal{C}_{\text{DR}}}^i(\mathbf{1}, V)$$

If E is the Higgs bundle corresponding to a local system V . But these Ext^i are, respectively, the Dolbeault cohomology of E and the \mathcal{C}^∞ de Rham cohomology of V . Hence we obtain a natural isomorphism between the Dolbeault cohomology of E and the cohomology of the local system V .

If K is a field of definition for X then we obtain the natural isomorphisms d_σ (by functoriality of the cohomology groups) needed to make K a field of definition (II).

Finally, note that multiplication by t^i in the i -th term of the Dolbeault complex provides the required natural isomorphism between $H_{\text{Dol}}^i(E)$ and $H_{\text{Dol}}^i(m_t E)$. This completes the definition of the absolute functors H^i .

PROPOSITION 7.10. — *Suppose X and Y are smooth complex projective varieties. Suppose $f: X \rightarrow Y$ is a smooth morphism. There exist absolute functors F^i from*

local systems on X to local systems on Y , with $\text{Dom}(F^i)$ equal to the set of semisimple local systems, and with underlying functors $F^i(V) = R^i f_* (V)$. The image of F^i is contained in the set of semisimple local systems.

Proof. — To define the algebraic direct images over \mathbf{Q} , use the same method as in the previous proof (using a stratification of the constructible subset in $R(X)$ corresponding to semisimple local systems). Similarly, one can define the direct image of a vector bundle with integrable connection, using the relative de Rham complex (one replaces Ω_X^i by $\Omega_{X/Y}^i$). One can also define the direct image of a Higgs bundle in this way. In [23] it is shown that the direct image of a Higgs bundle corresponds to the direct image of the corresponding local system, in the case where the local system is semisimple. Also shown at the same time is the fact that the direct image of a semisimple local system is semisimple. The natural isomorphisms d_σ again come from functoriality of the direct image construction.

Remark. — One expects a similar compatibility for direct images of non-semisimple local systems and the corresponding semistable Higgs bundles — then one could define the F^i with no restriction on the domain.

LEMMA 7.11. — *Tensor product $(U, V) \mapsto U \otimes V$ is an absolute functor from $L(X) \times L(X)$ to $L(X)$; dual $V \mapsto V^\vee$ is an absolute functor from $L(X)$ to $L(X)$; and similarly for the other linear algebra operations.*

Proof. — The correspondences between local systems, vector bundles with integrable connection, and semistable Higgs bundles with vanishing Chern classes are compatible with tensor product and dual (they have structures of tensor functor [21]). The conjugations P_σ and Q_σ are also compatible with tensor product and dual (as these are algebraic geometric operations on the corresponding vector bundles with integrable connections and Higgs bundles). It is easy to construct the required data to make tensor product or dual into an algebraic functor defined over $\bar{\mathbf{Q}}$; do the same for each X^σ . The d_σ are obtained by functoriality. The symmetric group acts by absolute natural transformations on multiple tensor products, from whence the other linear algebra operations are deduced.

PROPOSITION 7.12. — *Cup product gives absolute natural transformations $H^i(U) \otimes H^j(V) \rightarrow H^{i+j}(U \otimes V)$. If $f: X \rightarrow Y$ is smooth then for the functors F^i defined in Proposition 7.10, the relative cup product gives absolute natural transformations $F^i(U) \otimes F^j(V) \rightarrow F^{i+j}(U \otimes V)$.*

Proof. — Treat the case of the H^i first. It suffices to show the following things: that cup product is algebraic, defined over $\bar{\mathbf{Q}}$, and compatible with the d_σ ; that the isomorphisms between cohomology of a local system, de Rham cohomology of the vector bundle with integrable connection, and Dolbeault cohomology of the Higgs bundle, are compatible with cup product; and that cup product is compatible with the action of \mathbf{C}^* . The cup products in de Rham and Dolbeault cohomologies are defined in terms of algebraic geometry, so they are compatible with the action of field automorphisms. One then obtains, from the compatibility of these cup products with

the topological one, the required compatibilities of the cup product transformation with the u_σ and v_σ .

First, calculate the cohomology with coefficients in U by a triangulation, and that with coefficients in V by a transverse triangulation. Then the cup product is the dual of the intersection of simplicial chains. The chain complexes calculating cohomology depend algebraically (over \mathbf{Q}) on the monodromy representations of U and V . The morphism of complexes given by intersection of simplices (with values in a complex calculating the cohomology of $U \otimes V$ via a common refinement of the triangulations) is algebraic, defined over \mathbf{Q} . The cup product is natural with respect to the isomorphisms α_σ^* , so it is compatible with d_σ . The cup product defined in this way is the same as that obtained by wedging C^∞ de Rham representatives. On the other hand, the C^∞ de Rham complex is a resolution of the analytic de Rham complex, so this cup product is the same as the cup product in analytic de Rham cohomology. In turn, there is a natural inclusion of the algebraic de Rham complex in the analytic one, compatible with wedge product and giving the de Rham isomorphism on cohomology. Thus the de Rham isomorphism is compatible with cup product.

As for the Dolbeault isomorphism, we first indicate what happens when the local systems are semisimple (cf. [21]). The flat connection decomposes $D = D' + D''$ with $D'' = \bar{\partial} + \theta$ the operator giving the Higgs bundle structure. Let A^* denote the complex of forms with values in the vector bundle (either U , V , or $U \otimes V$). The complex (A^*, D) calculates the de Rham cohomology, while the complex (A^*, D'') calculates the Dolbeault cohomology. There is a subcomplex $(\ker(D'), D'')$ which maps quasiisomorphically into both of these complexes. This gives the Dolbeault isomorphism on cohomology. But this subcomplex is stable under the operation of wedging forms (with coefficients in the appropriate bundles). Hence there is a cup product in the cohomology of $(\ker(D'), D'')$ and the two quasiisomorphisms are compatible with this. Thus the Dolbeault isomorphism is compatible with cup product. To treat the case of nonsemisimple objects, we hide more deeply behind the notation of [21]. The fact that the differential graded categories \mathcal{C}_{DR} and \mathcal{C}_{Dol} are quasiisomorphic means that we have an isomorphism $\text{Ext}_{\mathcal{C}_{\text{DR}}}^i(\mathbf{1}, U) \cong \text{Ext}_{\mathcal{C}_{\text{Dol}}}^i(\mathbf{1}, U')$ where $U' \in \mathcal{C}_{\text{Dol}}$ is the object corresponding to $U \in \mathcal{C}_{\text{DR}}$. The fact that this quasiisomorphism extends to a quasiisomorphism of differential graded tensor categories means that these isomorphisms are compatible with the transformations (which are cup products)

$$\text{Ext}^i(\mathbf{1}, U) \otimes \text{Ext}^j(\mathbf{1}, V) \rightarrow \text{Ext}^{i+j}(\mathbf{1}, U \otimes V).$$

As noted before, the $\text{Ext}^i(\mathbf{1}, U)$ are the same as $H^i(U)$. Thus the Dolbeault isomorphism is compatible with cup products. Finally, suppose E is a Higgs bundle. The isomorphism between $H^*(E)$ and $H^*(m_t E)$ is given by multiplying by t^j in the component $E \otimes \Omega_X^j$ of the Dolbeault complex. This is compatible with wedging forms (and tensoring the coefficients). This completes the proof for the H^i .

Now we treat the relative case. The cup product morphism on the direct image local systems is determined by its values in the fiber over one point. Thus, the fact that it is algebraic and defined over \mathbf{Q} follows from the case of the H^i . The cup product morphisms on the direct image vector bundles with integrable connections, or Higgs bundles,

may be defined in terms of algebraic geometry (by a product on the level of the relative de Rham or Dolbeault complexes). These morphisms are determined by their values in all of the fibers considered separately. But in each fiber one is reduced to the case of the H^i considered above. Note that compatibility between direct image of a Higgs bundle and direct image of the local system is at present only established for semisimple local systems [23] (and this compatibility is itself compatible with inverse image, so when restricted to the fiber over one point it is the same as the quasiisomorphism referred to above). This completes the proof of the proposition.

LEMMA 7.13. — *Suppose $\xi: F \rightarrow G$ is an absolute natural transformation between two absolute functors (with $\text{Dom}(F) = \text{Dom}(G)$). Then there exist absolute functors K and C , with the same domain, whose underlying usual functors are $\ker(\xi)$ and $\text{coker}(\xi)$ respectively.*

Proof. — By using the construction $P(U, V)$ defined above, we may assume that the little functors corresponding to F and G have the same domain category $L_U(X)$. Define the algebraic functors K and C using the same type of argument as in Proposition 7.9: one can divide U into a union of locally closed strata defined over $\bar{\mathbb{Q}}$, such that the kernel and cokernel of ξ form trivial vector bundles over the strata. Let U' be the disjoint union of the strata, and define accordingly on $L_{U'}(X)$ the little functors corresponding to K and C . Do the same for each σ , to obtain the K_σ and C_σ . To complete the construction, the de Rham and Dolbeault functors are just the kernels and cokernels in the categories of vector bundles with integrable connection and semistable Higgs bundles with vanishing Chern classes respectively.

Using 7.2-7.13, and taking compositions of functors, one can create many absolute functors between categories of local systems, and many absolute natural transformations between them. We obtain several corollaries. For example, the following.

COROLLARY 7.14. — *The subset of local systems V on X such that $\dim H^i(V) = m$ is absolute.*

COROLLARY 7.15. — *Let $f: X \rightarrow Y$ be a smooth morphism of smooth complex projective varieties. Then $R^i f_*(C)$ is absolute.*

COROLLARY 7.16. — *Suppose $f: X \rightarrow Y$ is a smooth morphism of projective varieties, and suppose U_1, \dots, U_k are given absolute local systems on Y . Then the set of semisimple local systems V on X such that $R^i f_* V$ has a given type with respect to the U_j , is absolute.*

COROLLARY 7.17. — *Suppose $f: X \rightarrow Y$ is a smooth morphism of projective varieties. For semisimple local systems U and V on X , let $K^{ij}(U, V)$ be the kernel and $C^{ij}(U, V)$ be the cokernel of the cup product map*

$$R^i f_*(U) \otimes R^j f_*(V) \rightarrow R^{i+j} f_*(U \otimes V).$$

If U_1, \dots, U_k are fixed absolute local systems, then the subsets of (U, V) such that $K^{ij}(U, V)$ and $C^{ij}(U, V)$ have given type with respect to U_1, \dots, U_k , are absolute.

Many more statements may be obtained by composing functors, for example:

COROLLARY 7.18. — *Suppose $f_1, f_2: Y \rightarrow X$ are two morphisms. The set of local systems V on X such that $\dim H^i(Y, f_1^* V \otimes f_2^* V) = m$ is absolute.*

Hodge type

Our definition of absolute natural transformation includes, implicitly, a notion of Hodge type (0). (Here we use one index for Hodge type, corresponding to the first p of the usual (p, q) .) We may extend the notion of absolute natural transformation in the following way. Suppose U is an absolute local system, provided with an action of \mathbb{C}^* , $m_t U \cong U$ (giving an algebraic action on the Higgs bundle $\phi(U)$). This is equivalent to saying that U is provided with a structure of variation of Hodge structure. Then the functor $V \mapsto V \otimes U$ is an absolute functor. We may define one such local system on any X , the “Tate twist” $C_X(1)$, by taking the trivial local system with t acting by t (in other words the Hodge type is (1)). Set $C_X(p) = C_X(1)^{\otimes p}$ for $p \in \mathbb{Z}$. Suppose F and G are absolute functors from $L(X)$ to $L(Y)$. An *absolute natural transformation of Hodge type (p)* from F to G is a natural transformation

$$\xi(V): F(V) \rightarrow G(V) \otimes C_Y(p).$$

This is essentially the same as an absolute natural transformation from F to G , except that the condition of compatibility with the natural isomorphisms w_t is replaced by the requirement that the diagram commutes after multiplying one of the maps by t^p . Note that if $p=0$ we recover the previous notion of absolute natural transformation. All of our results about natural transformations can be carried over to natural transformations of Hodge type different from zero.

We obtain the following relationship with Deligne’s notion of absolute Hodge cycle [9]. Suppose

$$u \in H^{2p}(X, \mathbb{C})$$

is an absolute Hodge cycle (necessarily of Hodge type (p)) on X . Then cup product with u is an absolute natural transformation of Hodge type (p) from $H^i(V)$ to $H^{i+2p}(V)$. Similarly, if U is an absolute local system with action of \mathbb{C}^* , one could make a definition of absolute Hodge cycle of type (p) in $H^j(U)$. Then cup product with an absolute Hodge cycle of type (p) would be an absolute natural transformation of Hodge type (p) from $H^i(V)$ to $H^{i+j}(V \otimes U)$.

One is tempted to think that our results give some new topological conditions satisfied by absolute Hodge cycles (for example that the jump loci in $M_1(X)$ for kernels and cokernels of the cup product morphisms are torsion translates of triple tori). However, it is not clear that these conditions give any distinction vis-a-vis general cohomology classes: it looks like an argument similar to the one given below should show that all cohomology classes satisfy the same conditions. Of course this is worth looking at more closely.

Isomorphism classes of natural transformations

Suppose $A_0 \in L(Y)$ is an absolute object, and A is the absolute isomorphism class containing it. Suppose $F: L(X) \rightarrow L(Y)$ and $G, H: L(Y) \rightarrow L(Z)$ are absolute functors; and finally suppose $\xi: GF \rightarrow HF$ is an absolute natural transformation. The complex algebraic group $\text{Aut}(A_0)$ acts linearly on the vector space $\text{Hom}_{L(Z)}(G(A_0), H(A_0))$. Note however that $\text{Aut}(A_0)$ is defined over $\bar{\mathbb{Q}}$ and the vector space $\text{Hom}_{L(Z)}(G(A_0), H(A_0))$ has a $\bar{\mathbb{Q}}$ structure.

Let $B \subset L(X)$ be the inverse image $F^{-1}(A)$. For each $V \in B$, choose an isomorphism $i(V): F(V) \cong A_0$. Then put

$$\xi(V) = H(i(V)) \xi(V) G(i(V)^{-1}) \in \text{Hom}_{L(Z)}(G(A_0), H(A_0)).$$

THEOREM 7.19. — *Keep these notations. Suppose that $\text{Dom}(F)$ is contained in the set of semisimple objects. As V runs through the elements of B , the points $\xi(V)$ occupy a finite number of orbits O_j under the action of the group $\text{Aut}(A_0)$. These orbits are defined over $\bar{\mathbb{Q}}$.*

Proof. — Before beginning the proof, we streamline the notation. Let $\Gamma = \text{Aut}(A_0)$, and let $\mathbf{H} = \text{Hom}_{L(Z)}(G(A_0), H(A_0))$. The group Γ is an algebraic group defined over $\bar{\mathbb{Q}}$, and the vector space \mathbf{H} is a linear representation defined over $\bar{\mathbb{Q}}$. Let K be a countable algebraically closed field of definition for X, Y, Z , for the functors G and H , and for the objects $\psi(A_0)$ and $\varphi(A_0)$. Then the vector space \mathbf{H} has two structures of K -vector space as well as its structure of $\bar{\mathbb{Q}}$ -vector space. By enlarging K , assume that all three structures of K -vector space coincide. Let $\beta_\sigma: \mathbf{H} \rightarrow \mathbf{H}$ denote the resulting action of $\sigma \in \text{Aut}(C/K)$.

We have compatible systems of isomorphisms $\tilde{P}_\sigma A_0 \cong A_0$ and $\tilde{Q}_\sigma A_0 \cong A_0$. We then get four systems of isomorphisms, the first being $\tilde{P}_\sigma G A_0 \cong G A_0$ with the rest obtained by substituting Q and H . The conjugation β_σ is equal to the composition

$$\text{Hom}(G A_0, H A_0) \xrightarrow{\tilde{P}_\sigma} \text{Hom}(\tilde{P}_\sigma G A_0, \tilde{P}_\sigma H A_0) \cong \text{Hom}(G A_0, H A_0);$$

it is also equal to the same with P replaced by Q . (Note that the compatibility between \tilde{P}_σ or \tilde{Q}_σ and composition of σ implies that the β_σ satisfy the cocycle condition necessary to give a definition of the vector space \mathbf{H} over K .)

For each $V \in B$ we have a point $\xi(V) \in \mathbf{H}$. Up to the action of Γ , this point is independent of the choice of representative for an isomorphism class of objects V . The objects in B are by hypothesis semisimple, so their isomorphism classes are parametrized by points of an absolute constructible subset $S_B \subset M(X)$. Hence we obtain a map from S_B to the set of orbits of the action of Γ on \mathbf{H} . Let

$$\Lambda \subset M(X) \times \mathbf{H}$$

denote the graph of this correspondence. It is a constructible subset defined over $\bar{\mathbb{Q}}$ (since ξ is algebraic, defined over $\bar{\mathbb{Q}}$).

We claim that

$$(\tilde{P}_\sigma, \beta_\sigma)(\Lambda) = (\tilde{Q}_\sigma, \beta_\sigma)(\Lambda) = \Lambda.$$

The compatibility between ξ , ξ_σ and the d_σ and u_σ means that the diagrams

$$\begin{array}{ccc} \tilde{P}_\sigma GFV & \cong & GF\tilde{P}_\sigma \\ \downarrow & & \downarrow \\ \tilde{P}_\sigma HFV & \cong & HF\tilde{P}_\sigma V \end{array}$$

commute. We can use our original choice $FV \cong A_0$ to obtain choices of isomorphisms $F\tilde{P}_\sigma V \cong \tilde{P}_\sigma FV \cong \tilde{P}_\sigma A_0 \cong A_0$. With this choice, we obtain (after following the arrows in a big diagram) $\xi(\tilde{P}_\sigma V) = \beta_\sigma \xi(V)$. In particular, on the level of correspondences we get $(\tilde{P}_\sigma, \beta_\sigma)(\Lambda) = \Lambda$. The same argument holds with P replaced by Q , so this completes the proof of the claim.

Using this claim, Theorem 6.3 implies that Λ is a constructible set with respect to both the algebraic structures, that of M_{DR} and that of M_{Dol} .

Before completing the proof of the theorem we need a lemma.

LEMMA 7.20. — *There is a stratification $H = \bigcup H_\alpha$ into a finite number of disjoint locally closed Γ -invariant subsets such that for each α a quotient $H_\alpha \rightarrow J_\alpha$, which is a surjective Γ -invariant morphism such that the fibers are single orbits, exists.*

Proof. — We will show that if Z is a quasiprojective variety with the action of an algebraic group Γ , then there exists an open Γ -invariant subset Z_2 and a quotient $Z_2 \rightarrow J_2$. This statement, applied inductively, gives the desired stratification of H . Choose a projective closure \bar{Z} containing Z as an open subset. Let $R \subset Z \times Z$ be the graph of the relation of points lying in the same Γ -orbit. It is a constructible subset. Let \bar{R} denote its closure in $\bar{Z} \times \bar{Z}$. Let Γ act on $\bar{Z} \times \bar{Z}$ trivially on the first factor, and by its given action on the second factor. Then R and hence \bar{R} are Γ -invariant subsets. Let $p: \bar{R} \rightarrow \bar{Z}$ denote projection on the second factor, and let $Z_1 \subset \bar{Z}$ be the largest subset over which p is flat. This is a nonempty Γ -invariant open subset. Let $\bar{R}_1 = p^{-1}(Z_1)$. The flat subscheme $\bar{R}_1 \subset \bar{Z} \times Z_1$ gives a morphism Φ from Z_1 into the Hilbert scheme of subschemes of \bar{Z} . The universal property of the Hilbert scheme and the fact that \bar{R}_1 is Γ -invariant implies that Φ is Γ -invariant. Furthermore, if z and z' occupy different Γ -orbits then $p^{-1}(z)$ and $p^{-1}(z')$ have different intersections with $Z \subset \bar{Z}$, so $\Phi(z) \neq \Phi(z')$. Let J_1 be the image of Φ . It is a constructible subset of the Hilbert scheme, so we may divide it into locally closed subsets. This pulls back to a division of Z_1 into locally closed subsets. One of these will be an open subset Z_2 with the restriction of Φ giving a quotient $\Phi: Z_2 \rightarrow J_2$ (where J_2 is the corresponding locally closed subset of J_1). This proves the lemma.

We continue the proof of Theorem 7.19, using the stratification and quotients $H_\alpha \rightarrow J_\alpha$ given by the lemma. Let $S_\alpha \subset S_B$ be the subset of points v such that there exists a point $(v, w) \in \Lambda$ with $w \in H_\alpha$. Then S_α are disjoint constructible subsets whose union is S_B . The quotient of the corresponding piece of Λ by the action of Γ is the graph of a map $S_\alpha \rightarrow J_\alpha$. This map is constructible in terms of all three algebraic structures of S_α .

We may further divide up the S_α into a finite union of disjoint locally closed pieces $S_{\alpha,j}$, compatible with the algebraic structures of M_{DR} and M_{Dol} , such that: the map defined above restricts to a map $S_{\alpha,j} \rightarrow J_\alpha$ which is a morphism with respect to all three algebraic structures; a universal family of local systems, which is a direct sum of families of irreducible local systems (resp. vector bundles with integrable connection, Higgs bundles), exists over an étale covering of $S_{\alpha,j}$; and the isotypic decomposition is of constant type in this family.

Choose a regular function on an open set in J_α . We will show that it pulls back to a constant function on $S_{\alpha,j}$, by showing that the differential vanishes on the set of smooth points $S_{\alpha,j}^{\text{reg}}$. Suppose $v \in S_{\alpha,j}^{\text{reg}}$ is a point corresponding to a local system V which decomposes as a direct sum

$$V = \bigoplus V_i \otimes W_i$$

where V_i are distinct irreducible local systems and W_i are vector spaces. The real tangent space is a subspace

$$T_{\mathbf{R}}(S_{\alpha,j})_v \subset \bigoplus T_{\mathbf{R}}(M(X))_{V_i}.$$

For the purposes of this calculation we may reduce, by taking hyperplane sections, to the case where X is a curve. Then the moduli spaces $M(X)$ are smooth at the irreducible representations V_i . There are two complex structures on the real tangent space $T_{\mathbf{R}}(M_{n_i}(X))_{V_i}$. Let \mathbf{j} denote the complex structure corresponding to the topological and de Rham realizations, and let \mathbf{k} denote the complex structure corresponding to the Dolbeault realization. These form part of a quaternionic structure for the tangent space (Hitchin proved this in the rank 2 case [17]; Fujiki treats the general case [12]). The subspace $T_{\mathbf{R}}(S_{\alpha,j})_v$ is preserved by both complex structures, so \mathbf{j} and \mathbf{k} also induce a quaternionic structure there. The differential of our function is a complex-valued linear function on this space, which is complex-linear for both \mathbf{j} and \mathbf{k} . The real part is an element λ of the dual space $T_{\mathbf{R}}^*(S_{\alpha,j})_v$ with the property that $\mathbf{j}\lambda = \mathbf{k}\lambda$ is the imaginary part of the linear functional. But the quaternion equation $\mathbf{j}\mathbf{k} = -\mathbf{k}\mathbf{j}$ gives

$$\mathbf{j}\mathbf{k}\lambda = \mathbf{j}^2\lambda = -\lambda = -\mathbf{k}\mathbf{j}\lambda = -\mathbf{k}^2\lambda = \lambda,$$

whence $\lambda = 0$. This argument holds at every point $v \in S_{\alpha,j}^{\text{reg}}$, so the pullback of the holomorphic function is constant. This shows that the map $S_{\alpha,j} \rightarrow J_\alpha$ is constant. Since there are only finitely many strata $S_{\alpha,j}$, this shows that there are only finitely many orbits of Γ which are in the image of the correspondence Λ . As Λ is defined over $\bar{\mathbf{Q}}$, and there are only finitely many orbits in the image, each of the orbits is defined over $\bar{\mathbf{Q}}$. This completes the proof of the theorem.

Remark. — We would like to know that the subset B_j of objects V such that $\xi(V) \in O_j$ is absolute. The only problem is to know if $P_\sigma(O_j) = Q_\sigma(O_j)$. I don't see how to deduce this in general; so we include some additional hypotheses in the next lemmas. Another possible route to take would be to note that it is true for a subgroup of finite index in $\text{Aut}(\mathbf{C}/\mathbf{Q})$ (because there are only finitely many orbits O_j); one could

then proceed as above but using a definition of absoluteness over an extension of \mathbf{Q} . On the other hand I do not know of an example where $P_\sigma(O_j) \neq Q_\sigma(O_j)$.

LEMMA 7.21. — *Suppose, in the situation of the previous theorem, that O_j is a Γ -orbit in \mathbf{H} such that $P_\sigma(O_j) = Q_\sigma(O_j)$ for all $\sigma \in \text{Aut}(\mathbf{C}/\mathbf{Q})$. Then the subset B_j of objects $V \in B$ with $\xi(V) \in O_j$ is absolute.*

Proof. — Choose objects $A_{0,\sigma} \in L(Y^\sigma)$ with isomorphisms $a_\sigma: P_\sigma A_0 \cong A_{0,\sigma}$ and $a'_\sigma: Q_\sigma A_0 \cong A_{0,\sigma}$. Choose a point $\varphi \in \text{Hom}(GA_0, HA_0)$ representing the orbit O_j . We obtain isomorphisms which we denote (with some shorthand)

$$a_\sigma u_\sigma: P_\sigma GA_0 \cong G_\sigma A_{0,\sigma}$$

and

$$a'_\sigma v_\sigma: Q_\sigma GA_0 \cong G_\sigma A_{0,\sigma},$$

and similar ones with H . By hypothesis $\text{Ad}(a_\sigma u_\sigma) P_\sigma \varphi$ and $\text{Ad}(a'_\sigma v_\sigma) Q_\sigma \varphi$ both represent the same orbit $O_{j,\sigma}$ in $\text{Hom}(G_\sigma A_{0,\sigma}, H_\sigma A_{0,\sigma})$. Let

$$B_{j,\sigma} = \{ W \in F_\sigma^{-1}(A_\sigma) \subset L(X^\sigma) \text{ s.t. } \text{Ad}(w) \xi_0(W) \in O_{j,\sigma} \quad \text{for } w: F_\sigma W \cong A_{0,\sigma} \}.$$

If $W = P_\sigma V$ for $V \in B$, we may take as the isomorphism w the composition

$$F_\sigma P_\sigma V \cong P_\sigma FV \cong P_\sigma A_0 \cong A_{0,\sigma}.$$

We have a collection of isomorphisms, obtained from the above natural isomorphisms (and in some cases a choice of isomorphisms) — but which we denote by indexed letters i for convenience:

$$\begin{aligned} i_1: G_\sigma F_\sigma P_\sigma V &\cong G_\sigma P_\sigma FV, \\ i_2: G_\sigma P_\sigma FV &\cong G_\sigma P_\sigma A_0, \\ i_3: G_\sigma P_\sigma A_0 &\cong G_\sigma A_{0,\sigma}, \\ i_4: G_\sigma P_\sigma FV &\cong P_\sigma GFV, \\ i_5: P_\sigma GFV &\cong P_\sigma GA_0, \\ i_6: G_\sigma P_\sigma A_0 &\cong P_\sigma GA_0. \end{aligned}$$

Isomorphisms i_2 and i_5 result from the same choice of isomorphism $FV \cong A_0$. Isomorphisms i_1 , i_4 , and i_6 come from those denoted u_σ above, while i_3 comes from a_σ . There is a commutativity $i_6 i_2 = i_5 i_4$, while $i_3 i_6^{-1} = a_\sigma u_\sigma$.

We have the same collection of isomorphisms between objects obtained using H . For convenience we denote these by the same letter, and denote “conjugation” (involving the isomorphisms carrying the same name but related to G and H) by Ad .

We can choose $w: F_\sigma P_\sigma V \cong A_{0,\sigma}$ as above such that $G_\sigma w = i_3 i_2 i_1$. Then the condition that $P_\sigma V \in B_{j,\sigma}$ is by definition equivalent to the condition that

$$\text{Ad}(i_3 i_2 i_1) \xi_\sigma(P_\sigma V) \in O_{j,\sigma}.$$

By the definition of $O_{j,\sigma}$ this is equivalent to the condition that

$$\text{Ad}(i_6 i_2 i_1) \xi_\sigma(P_\sigma V) \sim P_\sigma \varphi$$

(i. e. these are equal up to an element of $\text{Aut}(A_0)$). The compatibility between ξ_σ and u_σ gives

$$\text{Ad}(i_4 i_1) \xi_\sigma(P_\sigma V) = P_\sigma \xi(V).$$

By the definition of $\tilde{\xi}(V)$, we have

$$\text{Ad}(i_5) P_\sigma \xi(V) \sim P_\sigma \tilde{\xi}(V),$$

hence (using $i_6 i_2 = i_5 i_4$)

$$\text{Ad}(i_6 i_2 i_1) \xi_\sigma(P_\sigma V) \sim P_\sigma \tilde{\xi}(V).$$

Hence $V \in B_{j,\sigma}$ if and only if $\tilde{\xi}(V) \sim \varphi$, which is to say $\tilde{\xi}(V) \in O_j$ or $V \in B_j$. Thus

$$P_\sigma(B_j) = B_{j,\sigma}.$$

A similar argument gives

$$Q_\sigma(B_j) = B_{j,\sigma}.$$

By the theorem above, B_j is algebraic, defined over $\bar{\mathbf{Q}}$. Similarly, by looking at the graph of the correspondence associated to ξ_σ one can see that the $B_{j,\sigma}$ are also algebraic, defined over $\bar{\mathbf{Q}}$. This almost completes the proof that B_j is absolute—we just need to note that it is preserved by the action of \mathbf{C}^* . This is because one can choose an action of \mathbf{C}^* on A_0 ; and via this, \mathbf{C}^* acts on \mathbf{H} and Γ , in a way compatible with the action of Γ on \mathbf{H} . Hence \mathbf{C}^* acts on the set of orbits. It preserves B , so it preserves the image of the correspondence Λ . But this image consists of a finite set of orbits, and \mathbf{C}^* cannot act nontrivially on a finite set (since it is a divisible group). Hence the orbit O_j is fixed, which implies that B_j is fixed.

COROLLARY 7.22. — *Suppose that Y and Z consist of finitely many points (i. e. $L(Y)$ and $L(Z)$ are products of Vect) and that G and H are standard linear algebra operations. Then for each of the finitely many orbits O_j given in Theorem 7.19, the set B_j of V such that $\tilde{\xi}(V) \in O_j$ is absolute.*

Proof. — Under these hypotheses, we have (for Y and Z) natural isomorphisms $P_\sigma \cong Q_\sigma$ commuting with the structural isomorphisms for G and H , in other words the two isomorphisms $P_\sigma G \cong G_\sigma Q_\sigma$ are equal, and similarly for H . Using reasoning similar to that of the previous proof, one can show that $P_\sigma(O_j) = Q_\sigma(O_j)$. Then the previous lemma applies to give the desired conclusion.

Example

Suppose $F = (F_0, \dots, F_k) : L(X) \rightarrow \text{Vect} \times \dots \times \text{Vect}$ is an absolute functor, and

$$\xi : F_1(V) \otimes \dots \otimes F_k(V) \rightarrow F_0(V)$$

is an absolute natural transformation. Let $A_0 = (C^{a_0}, \dots, C^{a_k}) \in \text{Vect} \times \dots \times \text{Vect}$. This is an absolute object (see Lemma 7.5). The group acting is

$$\text{Aut}(A_0) = \text{Gl}(a_0, C) \times \dots \times \text{Gl}(a_k, C).$$

Let B be the subset of objects $V \in L(X)$ such that $rk(F_i(V)) = a_i$. This is an absolute subset. Apply Theorem 7.19 and Corollary 7.22, with G being the tensor product of the last k factors, and H the projection on the first factor of $\text{Vect} \times \dots \times \text{Vect}$. The conclusion is as follows. The number of isomorphism classes of the pairing $\xi(V)$ (modulo $\text{Aut}(A_0)$) which occur is finite; each of these is defined over $\bar{\mathbb{Q}}$; and the subsets B_j of objects V with a given isomorphism class of pairing $\xi(V)$ are absolute.

An example of this situation is given by cup product of cohomology classes (Proposition 7.12). Fix a smooth complex projective variety X , and integers i_0, \dots, i_k with $i_0 = i_1 + \dots + i_k$. Cup product gives an absolute natural transformation

$$\xi(V_1, \dots, V_k) : H^{i_1}(V_1) \otimes \dots \otimes H^{i_k}(V_k) \rightarrow H^{i_0}(V_1 \otimes \dots \otimes V_k).$$

Once the ranks of the cohomology groups are fixed, there are only finitely many isomorphism classes of this pairing, and the subsets $B_j \subset L(X) \times \dots \times L(X)$ of (V_1, \dots, V_k) such that $\xi(V_1, \dots, V_k)$ has a fixed isomorphism class, are absolute. One can make more elaborate examples in the same direction. The finiteness statements for the isomorphism classes of the pairings give some concrete results even in the case of local systems of higher rank.

We can put these together in the following way. Let $C(V_1, \dots, V_k)$ be the cohomology ring for twisted coefficients,

$$C(V_1, \dots, V_k) = \bigoplus_{I=(i_1, \dots, i_k), j} H^j(X, V_1^{\otimes i_1} \otimes \dots \otimes V_k^{\otimes i_k})$$

with the sum taken over $i_j \geq 0$. For $N = (n_1, \dots, n_k)$ let $C_N(V_1, \dots, V_k)$ be the quotient ring given by taking the sum over $0 \leq i_j \leq n_j$.

THEOREM 7.23. — *Fix N and an integer b . As the local systems V_1, \dots, V_k run through the set of all semisimple local systems of rank less than b , the twisted cohomology rings $C_N(V_1, \dots, V_k)$ fall into finitely many isomorphism classes of rings graded by I, j . The set of k -uples (V_1, \dots, V_k) corresponding to a given isomorphism class is an absolute constructible subset of $M(X) \times \dots \times M(X)$.*

Proof. — The ring C_N is determined by its structure of a collection of vector spaces indexed by I and j , together with the map

$$C_N \otimes^{(N)} C_N \rightarrow C_N,$$

where $\otimes^{(N)}$ is a truncated tensor product involving only the terms with the total $i_j \leq n_j$. Choose Y and Z to be collections of points indexed by I and j . Let X' be the disjoint union of k copies of X . Then the twisted cohomology ring is a functor

$$C_N: L(X') \rightarrow L(Y)$$

(with domain put equal to the set of semisimple local systems of rank $\leq b$, absolute by Lemma 7.7). Let $G(C) = C \otimes^{(N)} C$ be the truncated tensor product, and $H(C) = C$ the identity. By Proposition 7.9 and Lemma 7.11, C_N , G and H are absolute functors; and by Proposition 7.12, cup product is an absolute natural transformation $\xi: GC_N \rightarrow HC_N$. The set of possible ranks of $H^j(X, V_1^{\otimes i_1} \otimes \dots \otimes V_k^{\otimes i_k})$ is finite (this can be seen by calculating the cohomology using a finite triangulation of X). Hence there is a finite set of absolute objects A_r in $L(Y)$ which can be isomorphic (as collections of vector spaces) to $C_N(V_1, \dots, V_k)$ —an A_r is given by specifying the ranks of the components of indices I, j . For each A_r , Theorem 7.19 implies that as (V_1, \dots, V_k) run through the set of objects with $C_N V_1, \dots, V_k \cong A_r$, the cup product transformations $\xi(V_1, \dots, V_k)$ occupy a finite set of isomorphism classes modulo $\text{Aut}(A_r)$. In other words, the set of ring structures which arise is, up to isomorphism, finite. As there are finitely many A_r , this shows that there are finitely many rings $C_N(V_1, \dots, V_k)$. Furthermore, Corollary 7.22 implies that the set of (V_1, \dots, V_k) such that $C_N(V_1, \dots, V_k)$ is isomorphic to a given ring, is absolute.

Question. — Can we use this theorem to deduce a similar finiteness statement for the set of twisted cohomology rings obtained by taking coherent sheaf cohomology (as in §5)?

Conclusion

We close by applying Corollary 6.2 and Lemma 7.1. Whenever one obtains an absolute subset $A \subset L(X)$ by one of the above constructions, it follows that the set $M_{A,1}(X) = M_A(X) \cap M_1(X)$, of points in the moduli space of rank one local systems corresponding to elements of A , decomposes as a finite union

$$M_{A,1}(X) = \bigcup S_i$$

where S_i are locally closed subsets with \bar{S}_i a torsion translate of a triple torus, and $\bar{S}_i - S_i$ a finite union of torsion translates of triple tori.

Question. — Is there a similar classification of absolute constructible subsets in the higher rank case?

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