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A CLASSIFICATION OF MONGE-AMPÈRE EQUATIONS

BY V. V. LYCHAGIN, V. N. RUBTSOV AND I. V. CHEKALOV

ABSTRACT. – A classical problem of a local classification of non-linear equation arising in S. Lie works is studied for the most natural class of Monge-Ampère equations (M.A.E.) on a smooth manifold M^n.

We solve this problem for a generic classical ($n=2$) case and give full proofs of S. Lie classical classification theorems.

For multidimensional generalizations of M.A.E. we reduce the problem to a problem in invariant theory which we solve for $n=3$ and give a partial description for $n\geqq 4$. C^∞-classification is obtained for $n=3$.

Our approach is based on the deep relations between M.A.E. on M^n and contact geometry of $J^1 M^n$ – the first jets space of M^n. This approach provides a possibility to apply symplectic and contact geometry methods in classical invariant theory and for calculation Spencer cohomologies as well.

In 1874 Sophus Lie raised the following problem: find the classes of local equivalence of non-linear 2nd order differential equations with respect to the group of contact diffeomorphisms. He formulated theorems on reducing the Monge-Ampère type differential equations to quasilinear and linear form [11]. As far as we know, a complete proof of this theorem had never been published.

The class of Monge-Ampère equations is a natural setting for a classification activity since the problem of local classification of arbitrary non-linear 2nd order equations contains as a subproblem the classification with respect to fractionally-linear transformations of all submanifolds in the space of quadrics.

In 1979 Morimoto [17] announced a number of statements on classification of Monge-Ampère equations of a special form based on the theory of G-structures.

Our approach to the classification problem is based on a relation between the differential forms on the manifold $J^1 M$ of 1-jets of smooth functions and Monge-Ampère equations [10]. We rely much upon [10].

Note also that the classification problem is in close connection with certain problems of the classical invariant theory which are, in our opinion, of an independent interest. One of them is the problem of description of orbits of the natural action of the symplectic group in the space of exterior forms. Notice that the analogy with differential equations enables us to understand better certain problems of the invariant theory for this group. In particular, this concerns the description of Sp-orbits of

3-forms on the 6-dimensional real symplectic space and also certain orbits of 4-forms on the 8-dimensional space.

The text consists of 6 sections (S).

In Section 0 we introduce basic concepts and constructions used below. The main source of definitions and ideas is paper [14].

In S.1 we consider the classical two-dimensional problem of S. Lie on local classification of Monge-Ampère equations. We start with the algebraic model of our problem and study a symplectic equivalence of 2-forms in $2n$-dimensional space. The normal forms are listed.

The non-degenerate two-dimensional Monge-Ampère equation determines an additional geometric structure on 4-dimensional symplectic manifold. The elliptic equations define an almost-complex structure and hyperbolic – an almost-product structure. The Newlander-Nirenberg theorem (in the elliptic case) and Frobenius theorem in a hyperbolic one give necessary and sufficient conditions of equivalence of our Monge-Ampère to a constant coefficients equation.

At the end of the section we prove two classic S. Lie theorems on the reduction of non-linear Monge-Ampère to a quazilinear one and on the normal forms of Monge-Ampère admitting an intermediate integral.

Sections 2-5 are devoted to the algebraic problems arising in a high-dimensional ($n \geqq 3$) classification.

In S.2 we solve the problem of symplectic classification of effective 3-forms on 6-dimensional real space. We list normal forms and indicate that even in the 3-dimensional case the Monge-Ampère equations corresponding to generic orbits are not linearizable even at a point.

Theorem 2.6 generalizes the corresponding results of J.-I. Igusa [8] and V. Popov [19] in the case of an algebraically non-closed field. Moreover we directly built an invariant of the classification problem.

In the next section (S.3) we give a short outline of the description procedure for normal forms in the dimension greater than 3 (Theorem 3.4). In theorem 3.5 the normal forms of effective 4-forms on 8-dimensional symplectic space are listed (under some natural conditions).

We establish a relation between the set of all transvections admitted by a given form and the symplectic classification of effective forms. After that the stabilizers of effective n-forms are described. We also calculate the stationary subalgebras of the most important types of effective forms.

In S.4 we make an algebraic digression and study the finiteness conditions on type of effective forms stabilizers. First we classify reductive subalgebras $\mathscr{J}$ in EndV with nontrivial first Cartan prolongation. If the representation $\mathscr{J} \to \mathrm{End}V$ is irreducible the results are known (Theorem 4.2.1). Theorem 4.3.1 solves this problem for reducible representations. Then we study stabilizers. Theorem 4.4.1 states the general result on finiteness of the stationary subalgebra of a regular element. We also give several reformulations and corollaries of this theorem for stabilizers of effective forms under a symplectic action. At the end of the section we explicitly calculate Cartan prolongations of the stabilizers of several important types of effective forms.

In S.5 we study involutiveness of the stabilizers of effective forms. The importance of these questions to the classification problem is explained in 6 where we identify the symbol of a Monge-Ampère equation corresponding to the homology equation of the classification problem with the stabilizor of the corresponding form. The involutiveness of the symbol is one of the conditions of the criterion for formal integrability.

S.6 is the central one from the classification problem viewpoint. Theorem 6.4.1 gives conditions for reducibility of an equation with analytic coefficients by an analytic symplectic diffeomorphism to an equation with constant coefficients in $\mathbb{R}^n$.

The finiteness of stabilizer condition enables us to strengthen this theorem and generalize it to C^∞-setting (Theorem 6.6.1).

At the end of the section Theorem 6.6.1 is applied to the classification of effective forms and the corresponding Monge-Ampère equations on the 3-dimensional manifolds. The main results were published in [5, 15, 16].

0. Formulation of the problem

0.1. Let (V, Ω) be a symplectic space over $\mathbb{R}$ with the structure form $\Omega \in \Lambda^2(V^*)$ and $\dim V = 2n$. Denote by $\Gamma : V \to V^*$ the isomorphism determined by the structure form Ω *i.e.* $\Gamma(X) = i_X(\Omega)$ and by $\Gamma_s : \Lambda^s(V) \to \Lambda^s(V^*)$ its exterior powers, $\Gamma_s = \Lambda^s(\Gamma)$. For every $\omega \in \Lambda^s(V^*)$ denote by $v_\omega \in \Lambda^s(V)$ the s-vector corresponding to ω, *i.e.* $\Gamma_s(v_\omega) = \omega$.

In the algebra of exterior forms $\Lambda^*(V^*) = \bigoplus_{s \geqq 0} \Lambda^s(V^*)$ introduce two operators $T : \Lambda^s(V^*) \to \Lambda^{s+2}(V^*)$, the operator of exterior multiplication by the structure form Ω $T(\omega) = \omega \wedge \Omega$, and $\perp : \Lambda^s(V^*) \to \Lambda^{s-2}(V^*)$, the operator of inner multiplication by the canonical bivector v_Ω, $\perp(\omega) = i_{v_\Omega}(\omega)$.

An exterior form $\omega \in \Lambda^k(V^*)$, $k \leqq n$ will be called effective if $\perp \omega = 0$ or, equivalently, $\omega \in \Lambda^k(V^*)$ is effective if and only if $T_{s+1}\omega = 0$ for $s = n - k$, $T_s = (1/s!)T^s$ [14].

0.2. Let M be a smooth manifold, $J^1(M)$ the manifold of 1-jets of smooth functions on M, $U \in \Lambda^1(J^1 M)$ the universal 1-form on $J^1 M$ which determines the contact structure [12]. At each point $x \in J^1 M$ the restriction of dU_x onto $C(x) = \operatorname{Ker} U_x$ determines a symplectic structure and therefore the operators $T : \Lambda^s(C^*(x)) \to \Lambda^{s+2}(C^*(x))$ and $\perp : \Lambda^s(C^*(x)) \to \Lambda^{s-2}(C^*(x))$. The tangent space $T_x(J^1 M)$ splits into the direct sum $T_x(J^1 M) = C(x) \oplus \mathbb{R} X_1$ where X_1 is the contact vector field with the generating function 1 [12]. Therefore, if $\Lambda^s(C^*)$ the space of differential s-forms on $J^1 M$ degenerated along X_1, $(\Lambda^s(C^*))_x$ is naturally identified with $\Lambda^s(C^*(x))$ and, besides, we have

$$\Lambda^s(J^1 M) = \Lambda^s(C^*) \oplus \Lambda^{s-1}(C^*).$$

We will say that $\omega \in \Lambda^s(C^*)$ is an effective form on $J^1 M$ if $\perp \omega = 0$. Denote by Λ^s_ε the set of all the effective s-forms.

0.3. For every differential n-form $\omega \in \Lambda^n(J^1 M)$ determine a non-linear differential operator Δ_ω acting via the formula $\Delta_\omega(h) = j_1(h)^*(\omega)$, $\Delta_\omega : C^\infty(M) \to \Lambda^n(M)$ where $j_1(h)$ is the section determined by a function $h \in C^\infty(M)$. Two differential forms ω_1, $\omega_2 \in \Lambda^n(J^1 M)$ determine the same operator if and only if $\omega_1 - \omega_2 \in I_C^n$ where I_C^n is the n-th homogeneous component of the ideal $I_C \subset \Lambda^*(J^1 M)$ formed by the elements of the form $\omega_1 \wedge U + \omega_2 \wedge dU$. Since $\Lambda_\varepsilon^s(J^1 M) = \Lambda^s(J^1 M)/I_C^s$ then $\Delta\omega$ is determined by the effective part of the projection of ω onto $\Lambda^s(C^*)$. In what follows we will assume that $\Delta\omega$ is given by an effective form ω. We will call the operators $\Delta_\omega : C^\infty(M) \to \Lambda^n(M)$ the Monge-Ampère operators.

0.4. Determine the action of the group $Ct(J^1 M)$ of contact transformations of $J^1 M$ onto the Monge-Ampère operators setting $\alpha(\Delta_\omega)\Delta_{\alpha^*(\omega)}$ for $\alpha \in Ct(J^1 M)$. Similarly define the action of the Lie algebra $ct(J^1 M)$ of contact vector fields on $J^1 M$ setting $X_f(\Delta_\omega) = L_{X_f}(\omega)$; here L_X is the Lie derivative along X and X_f is a contact vector field with a generating function $f \in C^\infty(J^1 M)$ [12].

0.5. We will be interested in the problem of local classification of Monge-Ampère operators (equations) with respect to the group $Ct(m)$ of the germs of contact diffeomorphisms preserving a point m.

Hereafter we will assume that in a neighbourhood of m there exists an infinitesimal contact symmetry X_f, where $f(m) \neq 0$, of Δ_ω. Then there exists a local contact diffeomorphism sending X_f to X_1 so that we may assume that $L_{X_1}(\omega) = 0$. This means that ω can be considered as a form on $T^* M$ and the classification problem for operators (equations) given by such forms as a classification problem of differential k-forms on $T^* M$ with respect to the group of symplectic diffeomorphisms.

1. Classification of Monge-Ampère equations on 2-dimensional manifolds. S. Lie's theorems

1.1. Let (V, Ω) be a symplectic space over a field $k = \mathbb{R}$ or $\mathbb{C}$, $\dim V = 2n$.

For any $\omega \in \Lambda^2(V^*)$ define its Pfaffian $\mathrm{Pf}(\omega) \in k$ from the formula $\omega^n = \mathrm{Pf}(\omega)\,\Omega^n$.

The coefficients $P_j(\omega)$ of the characteristic polynomial $P_\omega(\lambda) = \mathrm{Pf}(\omega - \lambda\Omega)$ are invariants of the natural $Sp(V)$-action on $\Lambda^2(V^*)$ and $P_\omega(\lambda) = \sum_{0 \leqq j \leqq n} P_j \lambda^j$, *i.e.* these invariants completely determine generic orbits. Making use of the symplectic structure we may associate with any 2-form ω a linear operator $A_\omega : V \to V$, where $\omega(X, Y) = \Omega(A_\omega X, Y)$ for any $X, Y \in V$. Since ω is antisymmetric, then A_ω is symmetric with respect to the structure form, *i.e*

$$\Omega(A_\omega X, Y) = \Omega(X, A_\omega Y) \qquad \text{for} \quad X, Y \in V.$$

Remark. – X and $A_\omega X$ are skew orthogonal, *i.e.* $\Omega(X, A_\omega X) = \omega(X, X) = 0$ for any $X \in V$. There is a relation among A_ω, its characteristic polynomial $P_{A_\omega}(\lambda) = \det \| A_\omega - \lambda E \|$ and P_ω; namely, $[\mathrm{Pf}(\omega)]^2 = \det A_\omega$, in particular $[P_\omega(\lambda)]^2 = P_{A_\omega}(\lambda)$.

1.2. Let us establish a normal form of a 2-form $\omega \in \Lambda^2(V^*)$ with respect to the $Sp(V)$-action similarly to the case of symmetric forms for $k=\mathbb{C}$. Let λ_1, λ_2 be two roots of $P_\omega(\lambda)=0$ and $K(\lambda_1)$, $K(\lambda_2)$ the subspaces of V formed by the vectors annihilated by a power of $A_\omega-\lambda_1$ or $A_\omega-\lambda_2$, respectively.

LEMMA 1.2.1. – *Let* $\lambda_1 \neq \lambda_2$. *Then* $K(\lambda_1)$ *and* $K(\lambda_2)$ *are skew orthogonal.*

Proof. – Let $C(\lambda)$ be the subspace of eigenvectors of A_ω corresponding to the eigenvalue λ, $C(\lambda) \subset K(\lambda)$. First let us show that $C(\lambda_1)$ and $C(\lambda_2)$ are skeworthogonal; if $X \in C(\lambda_1)$, $Y \in C(\lambda_2)$ then (1.1) implies that $(\lambda_1-\lambda_2)\Omega(X, Y)=0$ hence $\Omega(X, Y)=0$.

Hence, the lemma is valid for a semi-simple operator. As for an arbitrary symmetric operator A_ω its semi-simple part is symmetric too.

LEMMA 1.2.2. – *Let* $\omega \in \Lambda^2(V^*)$ *be such that* A_ω *is nilpotent. Then there exists a basis* $f_1^i, \ldots f_{n_i}^i, g_1^i, \ldots, g_{n_i}^i$, $i=1, 2, \ldots$ *in* V, *where* $f_1^i, g_1^i \ldots$ *is a basis in* $\operatorname{Ker} A_\omega$ *such that*

$$A_\omega f_k^i=f_{k-1}^i, \qquad A_\omega g_k^i=g_{k-1}^i, \qquad \Omega(f_s^i, g_l^j)=\delta_{ij}\,.\,\delta_{s+l,\, n_i+1}.$$

Proof. – Let f_{n_1} be a vector of maximal height with respect to A_ω *i.e.* $A_\omega^{n_1} f_{n_1}=0$, $A_\omega^{n_1-1} f_{n_1} \neq 0$. Set $f_m=A_\omega^{n_1-m} g_{n_1}$ and select a vector g_{n_1} so that $\Omega(f_1, g_{n_1})=1$. Then g_{n_1} is also a vector of height n_1 since $\Omega(f_{n_1}, A_\omega^{n_1-1} g_{n_1})=\Omega(f_1, g_{n_1}) \neq 0$. Let $g_k=A_\omega^{n_1-m} g_{n_1}$. Let us show that we can modify $g_2, \ldots, g_{n_1}$ somewhat so as to preserve $A_\omega g_m=g_{m-1}$ but with $\Omega(f_{n_1}, g_s)=0$ for $3 \geqq 2$. For this replace g_2 by $\tilde{g}_2=g_2-a_2 g_1$, where a_2 is selected so as $\Omega(f_{n_1}, \tilde{g}_2)=\Omega(f_{n_1}, g_2)-a_2\Omega(f_{n_1}, g_1)$, *i.e.* $a_2=\Omega(f_{n_1}, g_2)$.

Suppose that $\tilde{g}_2, \ldots, \tilde{g}_{k_1}$ with this property are already constructed and set $\tilde{g}_m=g_m-a_2 g_{m-1}-\ldots-a_m g_1$ provided $\tilde{g}_{m-1}=g_{m-1}-a_2 g_{m-2}-\ldots-a_{m-1} g_1$. The coefficient a_m is uniquely determined by the equation $\Omega(f_{n_1}, \tilde{g}_m)=0$ provided $a_2, \ldots, a_{m-1}$ are determined; in other words

$$a_m=\Omega(f_{n_1}, g_m)-a_2\Omega(f_{n_1}, g_{m-1})-\ldots-a_{m-1}\Omega(f_{n_1}, g_2)$$

Assuming that $g_{n_1}, \ldots, g_1$ is a set of vectors such that $\Omega(f_{n_1}, g_s)=\delta_{s,\,1}$ for arbitrary s, l, we get

$$\Omega(f_l, g_s)=\delta_{l+s,\, n_1+1}$$

Denote by F or G the subspace spanned by $f_1, \ldots, f_{n_1}$ or $g_1, \ldots, g_{n_1}$, respectively. Thanks to Remark 1.1 F and G are isotropic subspaces. In fact, the vectors X and $A_\omega X$ are skeworthogonal. Let us show that moreover $\Omega(X, A_\omega^k X)=0$ for all $\Omega m \geqq 0$. If $m=2l$ then $\Omega(X, A_\omega^{2l}X)=\Omega(A_\omega^l X, A_\omega^l X)=0$ and if $m=2l+1$ then $\Omega(X, A_\omega^{2l+1}X)=\Omega(Y, A_\omega Y)=0$, where $Y=A_\omega^l X$. Therefore V represents in the form $V=V' \oplus (F \oplus G)$ where V' is invariant with respect to A_ω and $\Omega|V'$ is nondegenerate. The induction in $1/2 \dim V$ finishes the proof.

These lemmas imply the following

THEOREM 1.2.3. — *The exterior 2-forms over* $\mathbb{R}$ *or* $\mathbb{C}$ *are* $\mathrm{Sp}(\mathrm{V})$*-equivalent if and only if* A_{ω_1} *and* A_{ω_2} *are equivalent with respect to* $\mathrm{GL}(\mathrm{V})$.

1.3. This theorem directly implies the list of normal forms of $\mathrm{Sp}(\mathrm{V})$-action on $\Lambda^2(\mathrm{V}^*)$.

THEOREM 1.3.1. — *For any* $\omega \in \Lambda^2(\mathrm{V}^*)$ *over* $\mathbb{R}$ *there exists a decomposition of* V *into the skew orthogonal with respect to* Ω *direct sum of subspaces* $\mathrm{V} = \bigoplus_{j, \lambda} \mathrm{V}_j(\lambda, \bar{\lambda})$ *where* λ *runs the roots of* $\mathrm{P}_\omega(\lambda)$ *and* $1 \leqq j \leqq 1/2 \dim \mathrm{K}(\lambda)$ *so that* $\omega = \sum \omega_{\lambda, j}$ *and* $\omega_{\lambda, j} \in \Lambda^2(\mathrm{V}_j)$ *in the symplectic basis* $(e_1, \ldots, e_r, f_1 \ldots, f_r)$ of $\mathrm{V}_\lambda(\lambda, \bar{\lambda})$ *is of the following form*:

(1) λ *is real*, $\dim \mathrm{V}_\lambda(\lambda, \bar{\lambda})$ *is even, then*

$$\omega = \lambda \sum_{1 \leqq i \leqq r} e_i^* \wedge f_i^* + \sum_{1 \leqq i \leqq r-1} e_i^* \wedge f_{i+1}^*$$

(2) $\lambda = \sigma + i\tau$, $\dim \mathrm{V}_j$ *is a multiple of* 4 *then*

$$\omega_{\lambda, j} = \sigma \sum_{1 \leqq i \leqq r} e_i^* \wedge f_i^* + \tau\Big(\sum_{2i \leqq r} e_{2i}^* \wedge f_{2i-1}^* - \sum_{2i+2 \leqq r} e_{2i+1}^* \wedge f_{2i+2}^*\Big) + \sum_{2 \leqq i \leqq r} e_i^* \wedge f_{i-2}^*$$

COROLLARY 1. — *If all the roots of the characteristic equation* $\mathrm{P}_\omega(\lambda) = 0$, $\lambda_1, \lambda_2, \ldots, \lambda_n$ *are real and different then* $\omega = \sum_{1 \leqq i \leqq r} \lambda_i e_i^* \wedge f_i^*$ *in a symplectic basis.*

COROLLARY 2. — *Any effective form* $\omega \in \Lambda_\varepsilon^2(\mathrm{V}_4^*)$ *on a* 4*-dimensional symplectic space* V_4 *can be transformed by a symplectic transformation to one of the following forms*

(1) $\omega = \lambda(f_1^* \wedge e_2^* - f_2^* \wedge e_1^*)$

(2) $\omega = \lambda(f_1^* \wedge e_1^* - f_2^* \wedge e_2^*)$

(3) $\omega = f_1^* \wedge e_2^*$

COROLLARY 3. — *Any effective form* $\omega \in \Lambda_\varepsilon^2(\mathrm{V}_6^*)$ *on a* 6*-dimensional symplectic space can be transformed by a symplectic transformation to one of the following forms*

(1) $\omega = \lambda_1 e_1^* \wedge f_1^* + \lambda_2 e_2^* \wedge f_2^* + \lambda_3 e_3^* \wedge f_3^*$, $\lambda_1 + \lambda_2 + \lambda_3 = 0$

(2) $\omega = 2\lambda e_1^* \wedge f_1^* - \lambda(\mathrm{e}_2^* \wedge f_2^* + e_3^* \wedge f_3^*) + \nu(e_3^* \wedge f_2^* - e_2^* \wedge f_3^*)$

(3) $\omega = 2\lambda e_1^* \wedge f_1^* - \lambda(e_2^* \wedge f_2^* + e_3^* \wedge f_3^*) + e_2^* \wedge f_3^*$

(4) $\omega = e_2^* \wedge f_1^* + e_3^* \wedge f_2^*$

1.4. Let us return to the 2-dimensional case. For the effective forms $\omega \in \Lambda_\varepsilon^2(\mathrm{V}_4^*)$ we have $\mathrm{P}_\omega(\lambda) = \mathrm{Pf}(\omega) + \lambda^2$ and its $\mathrm{Sp}(4)$-orbit is completely determined by the values of the Pfaffian.

An effective form $\omega \in \Lambda_\varepsilon^2)\mathrm{V}_4^*)$ is called

elliptic if $\mathrm{Pf}(\omega) > 0$,

hyperbolic if $\mathrm{Pf}(\omega) < 0$,

parabolic if $\mathrm{Pf}(\omega) = 0$.

The Monge-Ampère equation $\mathscr{E}_\omega$ on a two-dimensional manifold M determined by an effective form $\omega \in \Lambda^2 (J^1 M)$ will be called elliptic, hyperbolic or parabolic at $m \in J^1 M$, if so is the exterior form ω_m on $\operatorname{Ker} U_{1, m}$.

THEOREM 1.5. – *The Monge-Ampère equation determined by a form* $\omega \in \Lambda^2 (T^* M)$, $\dim M = 2$, *in a neighbourhood of* $m \in T^* M$ *where* $\mathrm{Pf}(\omega_m) \neq 0$, *is symplectic equivalent to a Monge-Ampère equation with constant coefficients with respect to a symplectic coordinate system if and only if*

$$d\omega = 1/2\, d \ln/\mathrm{Pf}(\omega)| \wedge \omega \tag{1.5}$$

Proof. – Necessity. Let $\omega_0 \in \Lambda^2 (T^* M)$ be an effective 2-form with constant coefficients with respect to a canonical coordinate system and $|\mathrm{Pf}(\omega_0)| = 1$. Let also $F : T^* M \rightarrow T^* M$ be a symplectic diffeomorphism such that $F^* (\omega_0) = \lambda \omega_0$ for a function $\lambda \in C^\infty (T^* M)$, $\lambda \neq 0$. Then $\mathrm{Pf}(F^* (\omega)) = F^* (\mathrm{Pf}(\omega)) = \lambda^2$ and therefore $\omega_0 = F^* (|\mathrm{Pf}(\omega)|^{-1/2} \omega)$ hence $d(|\mathrm{Pf}(\omega)|^{-1/2} \omega) = 0$ implying (1.5).

Sufficiency. First consider an elliptic case, $\mathrm{Pf}(\omega) > 0$ normalizing the form $\omega \rightarrow (\mathrm{Pf}(\omega))^{-1/2} \omega$ assume that $\omega \in \Lambda^2_\varepsilon (T^* M)$ is an effective form such that $d\omega = 0$. Consider the family of operators $A_x : T_x (T^* M) \rightarrow T_x (T^* M)$ dual to ω with respect to the symplectic structure: $A_x = A_{\omega_x}$. The operators A_x satisfy the characteristic equation $P_\omega (A_x) = 0$, therefore $A_x^2 + 1 = 0$ and the field $x \rightarrow A_x$ determines an almost complex structure on $T^* M$. Since ω is closed, this structure is integrable.

Prove this statement. Remind that an almost complex structure determined by the field of operators $A = (A_x)$ is integrable if and only if the vector field

$$Z = [X, Y] + A[X, AY] + A[AX, Y] - [AX, AY]$$

vanishes for arbitrary vector fields X, Y.

The relations $i_{[X, Y]} = [L_X, i_Y]$ and $L_X = i_X d + d i_X$ imply that on the closed forms the following relation holds:

$$i_{[X, Y]} = i_X d i_Y - i_Y d i_X + d i_X i_Y$$

To prove that $Z = 0$ it suffices to show that $i_Z \Omega = 0$ we have

$$i_{[X, Y]} \Omega = i_X d i_Y \Omega - i_Y d i_X \Omega - d\Omega (X, Y),$$

$$\begin{aligned} -i_{A[X, AY]} \Omega = i_{[X, AY]} \omega = i_X d i_{AY} \omega - i_{AY} d i_X \omega \\ - d\omega (X, AY) = - i_X d i_Y \Omega - i_{AY} d i_X \omega + d\Omega (X, Y). \end{aligned}$$

$$i_{A[AX, Y]} \Omega = i_{AX} d i_Y \omega + i_Y d i_X \Omega + d\Omega (X, Y).$$

$$i_{[AX, AY]} \Omega = i_{AX} d i_{AY} \Omega - i_{AY} d i_{AX} \Omega - d\Omega (AX, AY) = i_{AX} d i_Y \omega - i_{AY} d i_X \omega + d\Omega (X, Y).$$

implying the desired.

Therefore by Newlander-Nirenberg's theorem A determines a complex structure on $T^* M$.

On T^*M consider the complex-valued closed form $\theta=\Omega-i\omega$. Then making use of (1.2) we get $\theta(A_x, v_1, v_2)=\theta(v_1, A_x v_2)=i\theta(v_1, v_2)$ for all $v_1, v_2\in T_x(T^*M)$.

Therefore by Darboux theorem $\theta=dz_1\wedge dz_2$ in a local complex coordinate system. Set $z_1=q_1-iq_2$, $z_2=p_1+ip_2$ for some functions q_1, q_2, p_1, p_2 on T^*M which form a local coordinate system. Separating the real and imaginary parts in the relation $\theta=dz_1\wedge dz_2$we get

$$\Omega=dq_1\wedge dp_1+dp_2\wedge dp_2, \qquad \omega=dp_1\wedge dq_2-dp_2\wedge dq_1.$$

Therefore (q, p) is a canonical coordinate system in which $\mathscr{E}_\omega$ is the Laplace equation.

Now consider the hyperbolic case assuming as above that $\mathrm{Pf}(\omega)=-1$, $d\omega=0$. Then $\omega^2+\Omega^2=0$ and since ω is effective, then $(\omega+\Omega)^2=0$ and $(\omega-\Omega)^2=0$.

Therefore $\omega+\Omega$ and $\omega-\Omega$ are primitive 2-forms. Let E_+ and E_- be distributions determined by these forms. Let us show that $E_\pm$ are completely integrable. In fact, the annihilator of, say E_+, is formed by the 1-forms $\gamma\in\Lambda^1(T^*M)$ such that $\gamma\wedge(\omega+\Omega)=0$ but then $d\gamma\wedge(\omega+\Omega)=0$. Select the integrals f_1, f_2 and g_1, g_2 of E_+ and E_-, respectively, so that $\omega+\Omega=df_1\wedge df_2$, $\omega-\Omega=dg_1\wedge dg_2$ in a neighbourhood of $m\in T^*M$. Then $\Omega=1/2(df_1\wedge df_2+dg_1\wedge dg_2)$ $\omega=1/2(df_1\wedge df_2-dg_1\wedge dg_2)$ and therefore in the canonical coordinate system $q_1=1/2 f_1$, $q_2=1/2 g_1$, $p_1=f_2$, $p_2=g_2$, $\mathscr{E}_\omega$ is the wave equation.

THEOREM (S. Lie) 1.6. – *Let* M *be an analytic manifold,* $\dim M=2$ *and* $\mathscr{E}_\omega$ *a Monge-Ampère equation with analytic coefficients where* $\mathrm{Pf}(\omega)\neq 0$, $\omega\in\Lambda^2(J^1M)$. *Then* $\mathscr{E}_\omega$ *is locally equivalent to a quasilinear equation in a neighbourhood of* $m\in J^1M$.

The quasilinearity of $\mathscr{E}_\omega$ is equivalent to the fact that the fibers of $\mathscr{F}_{1,0}: J^1(M)\to J^0(M)$ are integral manifolds. By Cauchy-Kovalevsky theorem there exists a 3-parameter analytic family of solutions h_s which defines a foliation in a neighbourhood of $m\in J^1M$. By Weinstein's theorem [13] there exists a canonical transformation preserving m and sending this foliation into $\mathscr{F}_{1,0}^{-1}(x)$, $x\in J^1(M)$. Then $F^*(\omega)$ defines a quasilinear equation.

1.7. We say that f, $g\in C^\infty(T^*M)$ define an intermediate integral for Δ_ω if $\omega=\lambda_1\, df\wedge dg+\lambda_2\Omega$, $\lambda_i\in C^\infty(M)$. Since we are interested in normal forms of equations we may assume that $\omega=df\wedge dg+\lambda\Omega$, $\lambda\in C^\infty(T^*M)$.

THEOREM. – (S. Lie) *Suppose a Monge-Ampère equation* $\mathscr{E}_\omega$ $\omega\in\Lambda^2(T^*M)$ *admits an intermediate integral* (f, g) *for which the subspace* $\mathscr{J}\subset C^\infty(T^*M)$ *generated by* f, g, 1 *over* $\mathbb{R}$ *forms a Lie algebra with respect to the Poisson bracket. Then* $\mathscr{E}_\omega$ *reduces to the one of the following normal forms:*

(1) *hyperbolic type:* $\omega=1/2(dp_1\wedge dq_1-dp_2\wedge dq_2)$

(2) *parabolic type:* $\omega=dp_1\wedge dq_2$.

Proof. – Consider $\mathscr{J}$, then 1 belongs to the center of $\mathscr{J}$ and therefore it suffices to define $\{f, g\}$. Let $\{f, g\}=C_1+C_2 f+C_3 g$, $C_i\in\mathbb{R}$ then up to isomorphism the 3 cases are possible:

$$(a)\ \{f, g\}=1; \qquad (b)\ \{f, g\}=0; \qquad (c)\ \{f, g\}=f.$$

By a contact diffeomorphism f, g can be transformed into p_1, q_1 in case (*a*), into p_1, q_2 in case (*b*) and in case (*c*), replacing g by g/f, we have $\{f, g/f\}=1$ and therefore are in case (*a*).

2. Symplectic classification of exterior effective 3-forms on 6-dimensional space

In this section we describe the Sp(V)-orbits in the space of effective forms $\Lambda^n_\varepsilon(V^*_{2u})$ for $n=3$.

2.1. Let V be a 6-dimensional real symplectic space with the structure form Ω and $\omega \in \Lambda^3_\varepsilon(V^*)$ an effective form. Set $\omega_X = i_X \omega \in \Lambda^2(V^*)$, $E_X = \operatorname{Ker} \omega_X$ for all $X \in V$. Let us construct an invariant of Sp-action, the quadratic form q_ω on V associated with an effective ω. Notice that $\omega_X \wedge \Omega = 0$ since the effectiveness of ω, $\omega \wedge \Omega = 0$, implies that $\omega_X \wedge \Omega = \omega \wedge \theta_x$ where $\theta_X = i_X \Omega$ and therefore $\omega_X \wedge \Omega^2 = \omega \wedge \theta_x \wedge \Omega = \omega \wedge \Omega \wedge \theta_X = 0$. Therefore the characteristic polynomial of ω_X is of the form (up to the volume form Ω^3):

$$(\omega_X - \lambda\Omega)^3 = -\lambda^3 \Omega^3 - 3\lambda\, \omega_X^2 \wedge \Omega,$$

with $\omega_X^3 = 0$ since ω_X is degenerate. Therefore the roots of $P_{\omega_X}(\lambda)$ are $\lambda_1 = 0$ and $\lambda_{2,3} = \pm\sqrt{-1/4 \perp^2 \omega_X^2}$. In fact, for a non-zero root λ we have $\lambda^2 \Omega^3 = -3\,T\,\omega_X^2$ and since $\perp_3 : \Lambda^6(V^*) \to \mathbb{R}$ is an isomorphism, then $\lambda^2 \perp_3 \Omega^3 = -3 \perp_3 T\omega_X^2$. But $\perp_3 T\,\omega_X^2 = \perp_2 \omega_X^2$, $\perp_3 \Omega^3 = 6$ and therefore $\lambda^2 = -1/4 \perp^2 \omega_X^2$.

Making use of these remarks we define an Sp(V)-invariant, the quadratic form $q_\omega \in S^2(V^*)$:

$$q_\omega(X) = -1/4 \perp^2 \omega_X^2 \tag{2.1}$$

If $\omega_2 = F^*(\omega_1)$, $F \in Sp(V)$ then $q_{\omega_2} = F^*(q_{\omega_1})$. Notice also that if $q_\omega(X) \neq 0$ then $\omega_X^2 \neq 0$ and since $\omega_X^3 = 0$, then ω_X is a form of rank 4 and $\dim E_X = 2$.

PROPOSITION 2.2. – *Let* $X, Y \in V$ *be such that* $Y \in E_X$. *Then* $\omega = \omega_Y \wedge \theta_X - \omega_X \wedge \theta_Y$ *if* $\Omega(X, Y) = 1$ *and* $\omega_X \wedge \theta_X = \omega_Y \wedge \theta_Y$ *if* $\Omega(X, Y) = 0$.

Proof. – The effectiveness of ω implies

$$i_X(\omega \wedge \Omega) = \omega_X \wedge \Omega - \omega \wedge \theta_X = 0$$

yielding

$$\begin{aligned} i_X i_Y(\omega \wedge \Omega) &= i_Y(\omega_X \wedge \Omega) - i_Y(\omega \wedge \theta_X) = (i_Y \omega_X) \wedge \Omega \\ &\quad + \omega_X \wedge \theta_Y - \omega_Y \wedge \theta_X + \omega . \theta_X(Y) = \omega . \Omega(X, Y) - \omega_Y \wedge \theta_X + \omega_X \wedge \theta_Y. \end{aligned}$$

PROPOSITION 2.3. – *If* $q_\omega(x) \neq 0$ *and* $E_X \subset V$ *is an isotropic space then* $\omega = e_1^* \wedge e_2^* \wedge e_3^*$ *in a canonical basis of* V.

Proof. – Thanks to the above remark, $\dim E_X=2$ and $\Omega|_{E_X}=0$ by hypotheses. Therefore, $\omega_X \wedge \theta_Y=\omega_X \wedge \theta_Y \wedge \theta_X=0$ and, similarly, $\omega_Y \wedge \theta_X \wedge \theta_Y=0$ implying $\omega \wedge \theta_X \wedge \theta_Y=0$. In fact, the intersection of every 5-dimensional subspace $W \subset V$ with E_X contains at least one generic vector of the form $aX+bY$ and therefore $\omega \wedge \theta_X \wedge \theta_Y|_W=0$. It follows, there exists a vector $Z \in V$ such that $\omega=\theta_2 \wedge \theta_X \wedge \theta_Y$.

Since ω is effective, then

$$0=\perp\omega=\perp(\theta_Z \wedge \theta_X \wedge \theta_Y)=\theta_Z \wedge \perp(\theta_X \wedge \theta_Y)-i_Z(\theta_X \wedge \theta_Y)=\Omega(Y, Z)\,\theta_X-\Omega(X, Z)\,\theta_Y$$

since $\perp(\theta_X \wedge \theta_Y)=0$. Thus X, Y, Z are in involution with respect to Ω and therefore can be completed up to a canonical basis.

PROPOSITION 2.4. – *If $q_\omega \equiv 0$ but $\omega \neq 0$ then in a canonical basis $\omega=e_1^* \wedge e_2^* \wedge e_3^*$.*

Proof. – Since $q_\omega(X)=0$ for all $X \in V$, then $\perp_2(\omega_X \wedge \omega_Y)=0$ for all $X, Y \in V$ or, equivalently, $T^3\perp_2(\omega_X \wedge \omega_Y)=0$ for all $X, Y \in V$ or, equivalently, $T(\omega_X \wedge \omega_Y)=\Omega \wedge \omega_X \wedge \omega_Y=0$. Since ω is effective, then $\omega \wedge \Omega=0$ and $\theta_X \wedge \omega+\Omega \wedge \omega_X=0$. Therefore, multiplying this identity by ω_Y we get $\theta_X \wedge \omega \wedge \omega_Y=0$ for all $X, Y \in V$ yielding $\omega \wedge \omega_Y=0$ and $\omega_Y^2=0$. Therefore, ω_Y is simple for all $Y \in V$. Select $Y \in V$ so that $\omega_Y \neq 0$. Then $\omega_Y=\alpha \wedge \beta$, $\alpha, \beta \neq 0$ and $\omega \wedge \omega_Y=\omega \wedge \alpha \wedge \beta=0$. Therefore $\omega=\alpha \wedge \beta \wedge \gamma$ is simple and making use of the effectiveness of ω as in 2.3 we get the desired decomposition.

2.5. Now consider the case when E_X is a non-isotropic subspace in V. Select $Y \in E_X$ so that $\Omega(X, Y)=1$. Then by Proposition 2.2 we have

$$\omega=\omega_Y \wedge \theta_X-\omega_X \wedge \theta_Y \tag{2.2}$$

Set $E_X^\perp=\operatorname{Ker}\theta_X \cap \operatorname{Ker}\theta_Y$, $X, Y \in V$. We have $V=E_X \oplus E_X^\perp$, $\dim E_X^\perp=4$ and $E_X^\perp$ is symplectic with respect to the restriction of Ω. Let prime denote the restriction of a form onto $E_X^\perp$.

PROPOSITION. – (1) ω'_X, ω'_Y *are effective on* $E_X^\perp$; (2) vectors $X \in V$, $Y \in E_X$ *can be selected so that* ω'_X *is non-degenerate and* $\omega'_X \wedge \omega'_Y=0$.

Proof. – (1) Since $\Omega \wedge \omega=0$, then $\omega_X \wedge \Omega-\omega \wedge \theta_X=0$, $\omega_Y \wedge \Omega-\omega \wedge \theta_Y=0$ and therefore $\omega'_X \wedge \Omega'=\omega_X \wedge \Omega|_{E_X^\perp}=\omega \wedge \theta_X|_{E_X^\perp}=0$ since $E_X^\perp=\operatorname{Ker}\theta_X \cap \operatorname{Ker}\theta_Y$. Similarly, $\omega'_Y \wedge \Omega'=0$.

(2) Select X so that $q_\omega(X) \neq 0$. Then the equation $q_\omega(X, Z+tX)=0$, $Z \in E_X$ with respect to t always has a solution and therefore a vector $Y \in E_X$ can be selected so that $q_\omega(X, Y)=-1/4\perp_2(\omega_X \wedge \omega_Y)$. This is equivalent in turn to the fact $T\perp_2(\omega_X \wedge \omega_Y)=\perp_2(T\omega_X \wedge \omega_Y)=0$ but $\perp_2: \Lambda^6(V^*) \rightarrow \Lambda^2(V^*)$ is a monomorphism and therefore $T\omega_X \wedge \omega_Y=0$ or $\Omega \wedge \omega_X \wedge \omega_Y=0$. Substituting X into the identity obtained we get

$$i_X(\Omega \wedge \omega_X \wedge \omega_Y)=\omega_X \wedge \omega_Y \wedge \theta_X=0$$

and similarly $\omega_X \wedge \omega_Y \wedge \theta_X=0$ implying $\omega'_X \wedge \omega'_Y=0$. Note that ω'_X is non-degenerate: if $Z \in E_X^\perp$ belongs to its kernel then $i_Z\omega_X=0$ *i.e.* $Z \in E_X$ yielding $Z=0$.

2.6. Let us return to normal forms. Select $X \in V$, $Y \in E_X$ according to Proposition 2.5. Then taking ω'_X for the symplectic form on $E_X^\perp$ select a basis $\bar e_1^*, \bar e_2^*, \bar f_1^*, \bar f_2^*$ in $E_X^\perp$ so that

$$\omega'_X = \bar e_1^* \wedge \bar f_1^* + \bar e_2^* \wedge \bar f_2^* \tag{2.3}$$

Then ω'_Y can be, due to 1.3, either hyperbolic, or elliptic, or parabolic.

(*a*) ω'_Y is hyperbolic. Then a canonical basis $\bar e_1^*, \bar e_2^*, \bar f_1^*, \bar f_2^*$ in $E_X^\perp$ can be selected so that

$$\omega'_Y = \lambda (\bar e_1^* \wedge \bar f_1^* - \bar e_2^* \wedge \bar f_2^*) \tag{2.4}$$

where $\lambda \in \mathbb{R}$, $\lambda \neq 0$.

Since ω'_X and ω'_Y are effective with respect to Ω' then $\lambda(\lambda\omega'_X + \omega'_Y) \wedge \Omega' = 0$ and therefore $\bar e_1^* \wedge \bar f_1^* \wedge \Omega' = 0$. Similarly, $\bar e_2^* \wedge \bar f_2^* \wedge \Omega' = 0$.

Further, Ω' is effective with respect to ω'_X, the forms $\bar e_1^* \wedge \bar e_2^*, \bar f_1^* \wedge \bar f_2^*, \bar e_1^* \wedge \bar f_2^*, \bar e_2^* \wedge \bar f_1^*$, $\bar e_1^* \wedge \bar f_1^* - \bar e_2^* \wedge \bar f_2^*$, constitute a basis in the space of effective with respect to ω'_X 2-forms on $E_X^\perp$, where $\dim E_X^\perp = 4$, therefore

$$\Omega' = \bar e_1^* \wedge (p\bar f_2^* + r\bar e_2^*) - \bar f_1^* \wedge (q\bar e_2^* - s\bar f_2^*) \tag{2.5}$$

where $ps - qr \neq 0$ since $\Omega'^2 \neq 0$.

Consider an operator A which acts identically on the plane (e_1, f_1) and arbitrarily on the plane (e_2, f_2), *i.e.* $A = 1 \oplus B$ where $E_X^\perp = (\bar e_1, \bar f_1) \oplus (\bar e_2, \bar f_2)$ then if $B \in SL(2, \mathbb{R})$ *i.e.* preserves $\bar e_2^* \wedge \bar f_2^*$, then A preserves ω'_X and ω'_Y thanks to (2.1) and (2.2) and linearly permutes the second factors in (2.3). Select B so as

$$B^*(p\bar f_2^* + r\bar e_2^*) = \mu \bar f_2^*, \qquad B^*(q\bar e_2^* - s\bar f_2^*) = \pm \mu \bar e_2^*, \qquad \mu \neq 0.$$

Thus we may select a basis $\bar e_1, \bar e_2, \bar f_1, \bar f_2$ in $E_X^\perp$ so as ω'_X, ω'_Y were of the form (2.1), (2.2) and

$$\Omega' = \mu (\bar e_1^* \wedge \bar f_2^* \pm \bar e_2^* \wedge \bar f_1^*) \tag{2.6}$$

Now select a canonical basis e_1, e_2, e_3, f_1, f_2, f_3 in V so as $X = e_1$, $Y = f_1$ and the restrictions of $e_2^*, e_3^*, f_2^*, f_3^*$ onto $E_X^\perp$ would coincide with $\mu \bar e_1^*$, $\pm \mu \bar e_2^*, \bar f_2^*, \bar f_1^*$, respectively. Then by Proposition 2.2

$$\omega = 1/\mu (e_1^* \wedge e_2^* \wedge f_3^* \pm e_1^* \wedge e_3^* \wedge f_2^* + \lambda f_3^* \wedge f_1^* \wedge e_2^* \pm \lambda f_2^* \wedge f_1^* \wedge e_3^*) \tag{2.7}$$

(*b*) ω'_Y is elliptic. In a canonical basis $\bar e_1^*, \bar e_2^*, \bar f_1^*, \bar f_2^*$ we have

$$\omega'_Y = \lambda (\bar e_1^* \wedge \bar f_2^* + \bar f_1^* \wedge \bar e_2^*). \tag{2.8}$$

The condition $\Omega' \wedge \omega'_Y = \Omega' \wedge \omega'_X = 0$ implies that

$$\Omega' = p\bar e_1^* \wedge \bar e_2^* + q\bar f_1^* \wedge \bar f_2^* + r(\bar e_1^* \wedge \bar f_2^* - \bar f_1^* \wedge \bar e_2^*) + s(\bar e_1^* \wedge \bar f_1^* - \bar e_2^* \wedge \bar f_2^*) \tag{2.9}$$

Consider the 1-parameter transformation groups whose elements are of the form

$$A_t=\begin{bmatrix}1&0&0&t\\0&1&t&0\\0&0&1&0\\0&0&0&1\end{bmatrix},\qquad B_t=\begin{bmatrix}1&0&t&0\\0&1&0&-t\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

in the basis $\bar e_1, \bar e_2, \bar f_1, \bar f_2$. It is not difficult to see that these transformations preserve ω'_X and ω'_Y and their action on Ω' is given by the following formulas

$$(2.10)\qquad \left\{\begin{array}{l} A_t:(p, q, r, s)\to(p-qt^2+2st, q, r, s-tq),\\ B_t:(p, q, r, s)\to(p-qt^2-2st, q, r+tq, s),\end{array}\right.$$

where (p, q, r, s) are the coordinates of Ω' according to (2.9). Applying transformations of the form A_t, B_t consequently we see that Ω' can be transformed as follows:

$$(p, q, r, s)\to(0, q, r, s)\to(q, 0, r, s)\to(0, 0, r, s)\to(0, 0, 0, s)$$

and therefore reduced to the form

$$(2.11)\qquad \Omega'=s(\bar e_1^*\wedge\bar f_1^*-\bar e_2^*\wedge\bar f_2^*).$$

Select in V a canonical basis $e_1, e_2, f_1, f_2, e_3, f_3$ so as $e_1=X, f_1=Y$ and the restrictions of $e_2^*, f_2^*, e_3^*, f_3^*$ onto $E_X^\perp$ would coincide with $s\bar e_2^*, \bar f_1^*, -s\bar e_2^*, \bar f_2^*$ respectively; then by 2.2 ω is of the form

$$(2.12)\qquad \omega=1/s(e_1^*\wedge e_2^*\wedge f_2^*-e_1^*\wedge e_3^*\wedge f_3^*+\lambda e_2^*\wedge f_2^*\wedge f_1^*-\lambda f_2^*\wedge e_3^*\wedge e_1^*).$$

(*c*) ω'_Y is parabolic. Then $\omega'_Y=\bar e_1^*\wedge\bar f_2^*$ and

$$(2.13)\qquad \Omega'=p\bar e_1^*\wedge\bar e_2^*+q\bar e_1^*\wedge\bar f_2^*+r\bar f_1^*\wedge\bar f_2^*+s(\bar e_1^*\wedge\bar f_1^*-\bar e_2^*\wedge\bar f_2^*).$$

Consider two one-parameter groups of symplectic transformations preserving ω'_Y whose elements in the basis $\bar e_1, \bar e_2, \bar f_1, \bar f_2$ are of the form

$$A_t=\begin{bmatrix}1&0&0&0\\0&1&0&0\\t&0&1&0\\0&0&0&1\end{bmatrix},\qquad B_t=\begin{bmatrix}e^{-t}&0&0&0\\0&e^{-t}&0&0\\0&0&e^t&0\\0&0&0&e^t\end{bmatrix}$$

Representing Ω' as above in coordinates (p, q, r, s), where $pr+s^2\neq 0$ since $(\Omega')^2\neq 0$, we express the action of these groups on Ω' in the form

$$A_t:(p, q, r, s)\mapsto(p, q+2tr, r, s),$$
$$B_t:(p, q, r, s)\mapsto(e^{-2t}p, q, e^{2t}r, s).$$

Therefore Ω' reduces to the form

$$\Omega' = s(\bar{e}_1^* \wedge \bar{f}_1^* - \bar{e}_2^* \wedge \bar{f}_2^*), \qquad r \neq 0 \tag{2.14}$$

or to the form

$$\Omega' = \bar{e}_1^* \wedge \bar{e}_2^* + r\bar{f}_1^* \wedge \bar{f}_2^*, \qquad r \neq 0 \tag{2.15}$$

Respectively we get two normal forms for ω:

$$\omega = s^{-1}(e_1^* \wedge e_2^* \wedge f_2^* - e_1^* \wedge e_3^* \wedge f_3^* - e_2^* \wedge f_3^* \wedge f_1^*), \tag{2.16}$$

$$\omega = r^{-1}(e_1^* \wedge e_2^* \wedge f_3) + f_2^* \wedge f_3^* \wedge e_1 - f_1^* \wedge f_3^* \wedge e_2^*. \tag{2.17}$$

(*d*) $\omega'_Y \equiv 0$. In this case there are two normal forms corresponding to the cases.

(1) Ω' is elliptic, then

$$\omega = \lambda^{-1}(f_1^* \wedge f_2^* \wedge e_2^* - f_1^* \wedge f_3^* \wedge e_3^*). \tag{2.18}$$

(2) Ω' is hyperbolic then

$$\omega = -\mu^2 f_3^* \wedge e_1^* \wedge e_2^* + f_2^* \wedge e_1^* \wedge e_3^* \tag{2.19}$$

THEOREM. – *Any effective form* $\omega \in \Lambda_\varepsilon^3(V_6^*)$ *is* Sp(6; R)-*equivalent to one of those listed in Table* 1. (*Here* $e_1, e_2, e_3, f_1, f_2, f_3$ *is a canonical basis in* V.)

TABLE 1. – *Normal forms of effective* 3-*forms on* V_6

N°	The normal form	The invariant
1.	$e_1^* \wedge e_2^* \wedge e_3^* + \lambda f_1^* \wedge f_2^* \wedge f_3^*,\ \lambda \neq 0$	$2q_\omega = \lambda(e_1^* f_1^* + e_2^* f_2^* + e_3^* f_3^*);$
2.	$f_1^* \wedge e_2^* \wedge e_3^* + f_2^* \wedge e_1^* \wedge e_3^*$ $+ f_3^* \wedge e_1^* \wedge e_2^* + \nu^2 f_1^* \wedge f_2^* \wedge f_3^*,\ \nu \neq 0$	$2q_\omega = (e_1^*)^2 - (e_2^*)^2 - (e_3^*)^2$ $+ \nu^2[(f_1^*)^2 - (f_2^*)^2 - (f_3^*)^2];$
3.	$f_1^* \wedge e_2^* \wedge e_3^* + f_2^* \wedge e_1^* \wedge e_3^* + f_3^* \wedge e_1^* \wedge e_2^*$	$2q_\omega = (e_1^*)^2 - (e_2^*)^2 + (e_3^*)^2;$
4.	$f_1^* \wedge e_2^* \wedge e_3^* + f_2^* \wedge e_1^* \wedge e_3^* + f_3^* \wedge e_1^* \wedge e_2^*$	$2q_\omega = (e_1^*)^2 - (e_2^*)^2 - (e_3^*)^2;$
5.	$f_3^* \wedge e_1^* \wedge e_2^* - f_2^* \wedge e_1^* \wedge e_3^* - f_1^* \wedge e_2^* \wedge e_3^*$	$2q_\omega = (e_1^*)^2 + (e_2^*)^2 + (e_3^*)^2;$
6.	$f_2^* \wedge e_2^* \wedge e_1^* - f_3^* \wedge e_3^* \wedge e_1^*$	$2q_\omega = (e_1^*)^2;$
7.	$f_2^* \wedge e_1^* \wedge e_3^* - f_3^* \wedge e_1^* \wedge e_2^*$	$2q_\omega = -(e_1^*)^2;$
8.	$e_1^* \wedge e_2^* \wedge e_3^*$	$q_\omega = 0;$
9.	0	$q_\omega = 0.$

Proof. – Let us show how to simplify (2.7), (2.12), (2.16), (2.17), (2.19) with the help of gauge symplectic transformations to get normal forms 1-6. In (2.7) let us perform the following sequence of transformations:

(1) $e_1 \to -f_1,\ e_2 \to -f_2,\ e_3 \to -f_3, f_1 \to e_1, f_2 \to e_2, f_3 \to e_3$.

(2) $e_1 \to \mu^{-1}\lambda e_1,\ e_2 \to e_2,\ e_3 \to e_3, f \to \mu\lambda^{-1} f_1, f_2 \to f_2, f_3 \to f_3$.

(3) $e_1 + \lambda\mu^{-2} f_1 \to f_1,\ -e_1 + \lambda\mu^{-1} f_1 \mapsto -2\lambda\mu^2 e_1,\ e_2 \to e_2,\ e_3 \to e_3, f_3 \to f_3, f_2 \to f_2$.

(4) $e_1 \to -1/2\,\mu^2\lambda^2 e_1,\quad e_2 \to e_2,\quad e_3 \to e_3,\quad f_1 \to -2\lambda\mu^{-2} f_1,\quad f_2 \to f_2,\quad f_3 \to f_3,$ that reduce (2.7) to the normal form

$$\omega = e_1^* \wedge e_2^* \wedge e_3^* \pm 2\lambda\mu^{-2} f_1^* \wedge f_2^* \wedge f_3^*$$

corresponding to (1) for $\lambda = \pm 2\lambda\mu^{-2}$.

Similarly let us transform (2.12) to get the normal form (2). For other forms all the parameters are killed with the help of gauge transformations.

In (2.16) the transformation

$$e_1 \to s e_1, f_1 \to s^{-1} f_1, e_2 \to s^2 e_2, f_2 \to s^{-2} f_2, f_3 \to e_3, e_3 \to -f_3$$

transforms (2.16) into

$$\omega = -f_1^* \wedge e_2^* \wedge e_3^* + f_2^* \wedge e_1^* \wedge e_2^* + f_3^* \wedge e_3^* \wedge e_1^*.$$

Accordingly, for (2.17) there arise two normal forms without any parameter. The transformation

$$e_1 \to r^{1/2} e_1, f_1 \to r^{-1/2} f_1, e_2 \to r^{1/2} e_2, f_2 \to r^{-1/2} f_2, e_3 \to -f_3, f_3 \to e_3 \qquad \text{for} \quad r>0$$

or

$$e_1 \to (-r)^{1/2} e_1, f_1 \to (-r)^{1/2} f_1, e_2 \to (-r)^{1/2} e_2,$$
$$f_2 \to (-r)^{-1/2} f_2, e_3 \to -f_3, f_3 \to e_3 \qquad \text{for} \quad r<0$$

reduced to the forms

$$\omega = -f_3^* \wedge e_1^* \wedge e_2^* + f_2^* \wedge e_3^* \wedge e_1^* - f_1^* \wedge e_3^* \wedge e_2^*;$$
$$\omega = f_3^* \wedge e_1^* \wedge e_2^* + f_2^* \wedge e_3^* \wedge e_1^* - f_1^* \wedge e_3^* \wedge e_2^*.$$

In (2.18) the transformations

(1) $\qquad f_1 \to \lambda f_1, e_2 \to e_2, e_1 \to \lambda^{-1} e_1, f_2 \to f_2, e_3 \to e_3, f_3 \to f_3,$

(2) $\qquad f_1 \to e_1, e_1 \to -f_1, f_2 \to f_2, e_2 \to e_2, e_3 \to e_3, f_3 \to f_3,$

yield

$$\omega = f_2^* \wedge e_2^* \wedge e_1^* - f_3^* \wedge e_3^* \wedge e_1^*.$$

In (2.19) the transformation

$$e_1 \to \mu e_1, f_1 \to \mu^{-1} f_1, e_2 \to \mu e_2, f_2 \to \mu^{-1} f_2, e_3 \to e_3, f_3 \to f_3$$

yields

$$\omega = -f_3^* \wedge e_1^* \wedge e_2^* + f_2^* \wedge e_1^* \wedge e_3^*.$$

Further, the forms q_ω for 1-6 are $GL(6, \mathbb{R})$-different, therefore the corresponding representatives belong to different orbits of $Sp(6, \mathbb{R})$. It remains to show that different values λ and ν^2 correspond to different orbits. For this consider the operator $A_\omega : V \to V$ corresponding to the quadric $q_\omega(X, Y) = \Omega(A_\omega X, Y)$. Then

$$\lambda = \sqrt[6]{-\det A_\omega}, \qquad \nu = 1/2 \sqrt[6]{-\det A_\omega} \tag{2.20}$$

Remarks. — (1) One may derive a description of $Sp(6, \mathbb{C})$-orbits in the space of effective 3-forms on a complex 6-dimensional space from [8, 19].

(2) Theorem 2.6 shows that generic orbits correspond to "non-linearizable" Monge-Ampère equations. Therefore in the classification problem of Monge-Ampère equations on 3-dimensional manifolds to generic orbits also non-linearizable (even at a point) equations correspond.

3. Effective forms and their stabilizers

3.1. For $n \geqq 4$ generic orbits in $\Lambda^n_\varepsilon(V^*_{2n})$ have trivial stabilizer, therefore instead of the list of normal forms we may make use of versal families $\omega_\mu = \omega_0 + \sum_i \mu_i \omega_i$ where ω_0 is a representative of a generic orbit and $\{\omega_i\}$ a basis in Coker B_0, and $B_0 : S^2(V^*_{2n}) \to \Lambda^n_\varepsilon(V^*_{2n})$ is the operator, defined by the formula $B_0(h) = \delta(X_h \lrcorner \omega_0)$, where X_h is a linear Hamiltonian vector field. We identify Lie algebra $Sp(V)$ with $S^2(V^*)$ assigning to each symmetric tensor h the linear operator

$$X_h \in Sp(V) \subset V \otimes V^*, \qquad X_h = (1 \otimes \Gamma^{-1})(\delta h),$$

where $\Gamma : V \to V^*$ is an isomorphism, determined by Ω; and $\delta : S^{p+1}(V^*) \to S^p(V^*) \otimes V^*$ is the Spencer cohomology operator [7].

3.2. Omitting routine calculations give an example of a representative of a generic $Sp(8, \mathbb{R})$-orbit on $\Lambda^4_\varepsilon(V^*_8)$:

$$\begin{aligned} \omega_0 = {} & e^*_1 \wedge e^*_2 \wedge e^*_3 \wedge e^*_4 + f^*_1 \wedge f^*_2 \wedge f^*_3 \wedge f^*_4 + f^*_1 \wedge f^*_2 \wedge e^*_3 \wedge e^*_4 \\ & - 2 f^*_1 \wedge f^*_3 \wedge e^*_2 \wedge e^*_4 + 3 f^*_1 \wedge f^*_4 \wedge e^*_2 \wedge e^*_3 - 4 f^*_2 \wedge f^*_4 \wedge e^*_1 \wedge e^*_3 \\ & + 5 f^*_3 \wedge f^*_4 \wedge e^*_1 \wedge e^*_2 + 6 f^*_2 \wedge f^*_3 \wedge e^*_1 \wedge e^*_4. \end{aligned}$$

It is known that every symplectic transformation may be represented as a composition of transvections, *i.e.* shift transformations along the trajectories of linear Hamiltonian X_h fields, where $h = \lambda^2$, $\lambda \in V^*$, $\lambda \neq 0$. The explicit form of the transvections is

$$V \to V' = V + 2t\lambda(V) X_\lambda \tag{3.1}$$

In what follows under a transvection we will also mean their infinitesimal generators, Hamiltonian fields of the form X_{λ^2}, $\lambda \in V^*$. A vector X_λ will be called a directrix of the transvection X_{λ^2}.

3.3. Let $\omega \in \Lambda^k_\varepsilon(V^*_{2n})$ be an effective k-form on V.

The stabilizer (Lie algebra of linear symplectic symmetries) of ω with respect to the natural Sp(V)-action on $\Lambda^k_\varepsilon(V^*_{2n})$ is

$$\mathcal{J}_\omega = \{ g \in \mathrm{Sp}(V) \mid g\omega = 0 \} = \{ h \in S^2(V^*) \mid \delta(X_h \lrcorner \omega) = 0 \}. \tag{3.2}$$

A transvection X_{λ^2}, $\lambda \in V^*$ is called a characteristic transvection of $\omega \in \Lambda^k_\varepsilon(V^*_{2n})$ if $\lambda^2 \in \mathcal{J}_\omega$.

Denote Char $\mathcal{J}_\omega$ the set of characteristic transvections of ω.

Then Char $\mathcal{J}_\omega \cup \{0\}$ is an algebraic (conic) subvariety in V^*. This subvariety is an Sp(V)-invariant of ω: if $\omega_2 = F^*(\omega_1)$, $F \in \mathrm{Sp}(V)$, then $F^*(\mathrm{Char}\, \mathcal{J}_{\omega_1}) = \mathrm{Char}\, \mathcal{J}_{\omega_2}$.

PROPOSITION. — X_{λ^2} *is characteristic for* $\omega \in \Lambda^k_\varepsilon(V^*_{2n})$ *if and only if* $\lambda \wedge (X_\lambda \lrcorner \omega) = 0$.

Proof. — Let $L_{X_{\lambda^2}}$ be a Lie derivative along X_{λ^2} then by (3.1) we have

$$L_{X_{\lambda^2}}(\omega)(V_1, \ldots, V_k) = 2 \sum_{1 \leq i \leq k} \lambda(V_i)\, \omega(V_1, \ldots, X^i_\lambda, \ldots, V_k).$$

We may assume (perhaps after a linear transformation of $V_1, \ldots, V_k$) that the first $k-1$ of these vectors belong to Ker λ and $\lambda \cap (V_k) \neq 0$ therefore $L_{\lambda^2}(\omega) = 0$ if and only if $X_\lambda \lrcorner \omega 0$ on Ker λ.

3.4. Fix $\lambda \in V^*$, $\lambda \neq 0$ and consider $\prod^k_\lambda = \{ \omega \in \Lambda^k_\varepsilon(V^*_{2n}),\ \lambda \wedge (X_\lambda \lrcorner \omega) = 0 \}$. This space consists of effective k-forms for which X_{λ^2} is characteristic.

An effective form $\omega \in \Lambda^k_\varepsilon(V^*_{2n})$ is called regular if its Sp(V)-orbit does not intersect with $\prod^k_\lambda$; otherwise ω is regular if it has no characteristic transvections.

THEOREM. — *Let* $\omega \in \Lambda^k_\varepsilon(V^*_{2n})$ *have a pair of characteristic transvections whose directrices are not in involution. Then there exists a decomposition* $V = V' \oplus k X_\theta \oplus k X_{\theta^+}$ *such that*

$$\begin{gathered} \Omega(X_\theta, X_{\theta^+}) = 1;\ \theta, \theta^+ \in V^*, \qquad V' = \mathrm{Ker}\,\theta \cap \mathrm{Ker}\,\theta^+, \\ \omega = \left(\theta \wedge \theta^+ - \frac{1}{n-k+1} \Omega' \right) \wedge \omega_0 + \omega_1, \end{gathered} \tag{3.3}$$

where $\Omega' = \Omega \mid V'$; ω_0, ω_1 are effective forms on V'.

Proof. — For θ, θ^+ take the directrices of characteristic transvections whose existence is assumed in Theorem. Making use of decomposition from the proof of Theorem 1.6 in [14] and Proposition 3.3 we get (3.3).

3.5. Theorem 3.4 and the results of Sections 1 and 2 allow us to refine the classification of effective forms from $\Lambda^n_\varepsilon(V^*_{2n})$. Namely if $\omega \in \Lambda^n_\varepsilon(V^*_{2n})$ has two characteristic transvections which are not in involution then (3.3) takes the form $\omega = (\theta \wedge \theta^+ - \Omega') \wedge \omega_0$ where $\omega_0 \in \Lambda^{n-2}_\varepsilon(V'^*_{2n-2})$ since $\omega_1 \in \Lambda^n_\varepsilon(V^{*\prime}_{2n-2}) = 0$. Therefore making use of the classification of 2-forms on a symplectic space we get the following result.

THEOREM. — *Any effective form* $\omega \in \Lambda^4_\varepsilon(V_8^*)$ *possessing a pair of characteristic transvection whose directrices are not in involution may be transformed by a symplectic transformation to one of the following forms:*

$$
\begin{aligned}
(1)\quad \omega = {} & \lambda_2 (e_1^* \wedge f_1^* \wedge e_2^* \wedge f_2^* + e_3^* \wedge f_3^* \wedge e_4^* \wedge f_4^*) \\
& + \lambda_3 (e_1^* \wedge f_1^* \wedge e_3^* \wedge f_3^* + e_2^* \wedge f_2^* \wedge e_4^* \wedge f_4^*) \\
& + \lambda_4 (e_1^* \wedge f_1^* \wedge e_4^* \wedge f_4^* + e_2^* \wedge f_2^* \wedge e_3^* \wedge f_3^*), \qquad \lambda_2 + \lambda_3 + \lambda_4 = 0
\end{aligned}
$$

$$
\begin{aligned}
(2)\quad \omega = {} & e_1^* \wedge f_1^* \wedge (2\lambda e_2^* \wedge f_2^* - \lambda e_3^* \wedge f_3^* - \lambda e_4^* \wedge f_4^*) \\
& + e_1^* \wedge f_1^* \wedge (\nu e_4^* \wedge f_3^* - \nu e_3^* \wedge f_4^*) - e_2^* \wedge f_2^* \wedge (\lambda e_3^* \wedge f_3^* \\
& + \lambda e_4^* \wedge f_4^* + \nu e_4^* \wedge f_3^* - \nu e_3^* \wedge f_4^*) + 2\lambda e_3^* \wedge f_3^* \wedge e_4^* \wedge f_4^*,
\end{aligned}
$$

$$
\begin{aligned}
(3)\quad \omega = {} & e_1^* \wedge f_1^* \wedge e_3^* \wedge f_4^* - e_2^* \wedge f_2^* \wedge e_3^* \wedge f_4^* + e_1^* \wedge f_1^* \wedge (2\lambda e_2^* \wedge f_2^* \\
& - \lambda e_3^* \wedge f_3^* - \lambda e_4^* \wedge f_4^*) - e_2^* \wedge f_2^* \wedge (\lambda e_3^* \wedge f_3^* + \lambda e_4^* \wedge f_4^*) \\
& + 2\lambda e_3^* \wedge f_3^* \wedge e_4^* \wedge f_4^*,
\end{aligned}
$$

$$
(4)\quad \omega = e_1^* \wedge e_3^* \wedge f_1^* \wedge f_4^* - e_1^* \wedge e_3^* \wedge f_1^* \wedge f_2^* + e_2^* \wedge e_4^* \wedge f_1^* \wedge f_3^* + e_1^* \wedge e_4^* \wedge f_2^* \wedge f_4^*.
$$

(Here e_1^*, e_2^*, e_3^*, e_4^*, f_1^*, f_2^*, f_3^*, f_4^* is a symplectic basis in V_8^*).

3.6. Before we start describing stabilizers of the most important types of effective forms let us make several remarks.

Let V be represented as the direct sum of Lagrangean subspaces $V = E + F$. Then making use of Ω we may identify F with E^* and under this identification Ω turns into the standard 2-form on $E \oplus E^*$ for which $\Omega(e, e^*) = e^*(e)$ with $e \in E$, $e^* \in E^*$. Every linear transformation $A : E \to E$ generates a symplectic transformation $A \oplus (A^*)^{-1}$ of V. If $e_1, \ldots, e_n, f_1, \ldots, f_n$ is a symplectic basis of V and E, F are linear spans of $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ respectively, then to infinitesimal analogues of the symplectic transformations above correspond tensors of the form $h = \sum_{i,j} a_{ij} e_i^* \otimes f_j^*$. Let us make use of (3.2) to find $\mathscr{I}_\omega$. Direct calculations show the validity of the following.

PROPOSITION. — *The stabilizers of effective forms* $\omega \in \Lambda^3_\varepsilon(V_6^*)$ *reducible to the normal forms* 1-5 *of Theorem* 2.6 *respectively are*

(1) $\mathscr{I}_\omega \simeq \mathrm{sl}(3, \mathbb{R})$ *generated by the tensors of the form* $h = \sum_{1 \leqq i, j \leqq 3} a_{ij} e_i^* . f_j^*$, *where* $\|a_{ij}\| \in \mathrm{SL}(3, \mathbb{R})$;

(2) $\mathscr{I}_\omega \simeq \mathrm{sl}(3, \mathbb{R})$ *generated by the tensors of the form*

$$
\begin{aligned}
h = {} & \sum_{1 \leqq i, j \leqq 3} a_{ij} e_i^* f_j^* + \sum_{1 \leqq i \leqq 3} \lambda_i (e_i^{*2} + \nu^2 f_i^{*2}) + \gamma_1 (e_2^* e_3^* - \nu^2 f_2^* f_3^*) \\
& + \gamma_2 (e_2^* e_3^* + \nu^2 f_1^* f_3^*) + \gamma_3 (e_1^* e_2^* - \nu^2 f_1^* f_3^*), \qquad \lambda_2 = \lambda_1 + \lambda_3,\ \lambda_i,\ \gamma_i \in \mathbb{R};
\end{aligned}
$$

(3) $\mathscr{I}_\omega$ *the semidirect sum of* $\mathrm{SO}(2, 1) = \mathrm{SO}(q_\omega|_E)$ *and* $\mathrm{H}_2(2; 1) \subset \mathrm{S}^2(\mathrm{E}^*)$ *generated by symmetric tensors* $h = \sum_{1 \leqq i, j \leqq 3} b_{ij} e_i^* e_j^*$ *such that* $b_{11} + b_{22} - b_{33} = 0$;

(4) $\mathcal{G}_\omega$ *is the semidirect sum of* $SO(1, 2)=SO(q_\omega|_E)$ *and the space* $H_2(1; 2) \subset S^2(E^*)$ *generated by the symmetric tensors* $h=\sum_{1 \leqq ij \leqq 3} b_{ij} e_i^* e_j^*$ *such that* $b_{11}-b_{22}-b_{33}=0$.

(5) $\mathcal{G}_\omega$ *the semidirect sum of* $SO(3)=SO(q_\omega|_E)$ *and the space* $H_2(3) \subset S^2(E^*)$ *of harmonic tensors of the form* $h=\sum_{1 \leqq i, j \leqq 3} b_{ij} e_i^* e_j^*$ *where* $b_{11}+b_{22}+b_{33}=0$.

3.7. Let $e_1, \ldots, e_n, f_1, \ldots, f_n$ be a canonical basis of the symplectic space V and $(\nu_1, \ldots, \nu_n)$ an arbitrary set of numbers of which p are positive and q are negative, $p+q=n$. Denote $H_2(p; q)$ the space of quasiharmonic tensors and $so(p, q)$ the Lie algebra preserving $\sum_{1 \leqq i \leqq n} \nu_i (e_i^*)^2$.

Proposition. – *Let in a canonical basis* $e_1, \ldots, e_n, f_1, \ldots, f_n$ *of* V *an effective form* $\omega \in \Lambda_\varepsilon^n (V_{2n}^*)$ *be reduced to one of the following forms*

(1) $\omega=f_1^* \wedge \ldots \wedge f_n^*+\lambda e_1^* \wedge \ldots \wedge e_n^*$, $\lambda \in \mathbb{R}$, $\lambda \neq 0$;

(2) $\omega=\sum_{1 \leqq i \leqq n} \nu_i e_1^* \wedge \ldots \wedge f_i^* \wedge \ldots \wedge e_n^*$, $\nu_i \in \mathbb{R}$

then in case (1) $\mathcal{G}_\omega=sl(n, \mathbb{R})$ is generated by the tensors of the form $\sum_{1 \leqq i, j \leqq n} a_{ij} e_i^* f_j^*$ where $\|a_{ij}\| \in sl(n, \mathbb{R})$.

In case (2) $\mathcal{G}_\omega=SO(p, q) \to H_2(p, q)$ is generated by the elements of the form $\sum_{i, j} a_{ij} e_i^* f_j^*+\sum_{i, j} b_{ij} e_i^* e_j^*$ where $\|a_{ij}\| \in SO(p, q)$ and $\sum_{i, j} b_{ij} e_i^* e_j^* \in H_2(p, q) \subset S^2(V_{2n}^*)$

4. Prolongations of reductive Lie algebras and stabilizers of effective forms.

4.1. Let V and W be finite dimensional vector spaces over field k (unless otherwise stated $k=\mathbb{C}$). The first Cartan prolongation $\mathcal{G}^{(1)}$ of a subspace $\mathcal{G} \subset \operatorname{Hom}(V, W)$ is defined as follows ([7]):

$$(4.0) \quad \mathcal{G}^{(1)}=\{T \in \operatorname{Hom}(V, \mathcal{G}) \mid T(u) v=T(v) u,\ u, v \in V\}=(W \otimes S^2(V^*)) \cap \mathcal{G} \otimes V^*$$

The k-th prolongation of $\mathcal{G}$ is defined inductively as $\mathcal{G}^{(k)}=(\mathcal{G}^{(k-1)})^{(1)}$ and coincides with $(W \otimes S^{k+1}(V^*)) \cap \mathcal{G} \otimes S^k(V^*)$.

The most important case is the one when $W=V^*$ and $\mathcal{G} \subset \operatorname{End}(V)=V^* \otimes V$ is a Lie algebra. They say that $\mathcal{G}$ is of finite type k if $\mathcal{G}^{(k-1)} \neq 0$ and $\mathcal{G}^{(k)}=0$ (and therefore $\mathcal{G}^{(s)}=0$ for $s \geqq k$) and of infinite type otherwise.

Note also that the functors of complexification and prolongation commute. The algebra $\mathcal{G}^{(*)}=\mathcal{G}^{(-1)}+\mathcal{G}^{(0)}+\mathcal{G}^{(1)}+\ldots+\mathcal{G}^{(n)}+\ldots$ where $\mathcal{G}^{(1)}=V$, $\mathcal{G}^{(0)}=\mathcal{G}$ is a graded subalgebra of the Lie algebra of polynomial vector fields

$$\operatorname{Vect}(V)=V+V \otimes V^*+V \otimes S^2(V^*)+\ldots$$

i.e. $[\mathcal{G}^{(i)}, \mathcal{G}^{(j)}] \subset \mathcal{G}^{(i+j)}$ with respect to the bracket in Vect (V).

Clearly, $\mathscr{J}^{(*)}$ is finite dimensional if $\mathscr{J}$ is of finite type and infinite-dimensional otherwise. The complex irreducible in V Lie algebras $\mathscr{J} \subset \operatorname{End}(V)$ of infinite type were listed by E. Cartan [4], ct. [7].

We will be interested in classification of reductive subalgebras $\mathscr{J}$ given by a representation $\mathscr{S}$ in End V such that $(\mathscr{S}(\mathscr{J}))^{(1)} \neq 0$.

4.2. First consider the case when $\mathscr{S}: \mathscr{J} \to \operatorname{End} V$ is irreducible. The following theorem is a combination of the well-known results ([9]).

THEOREM. — *Let $\mathscr{J}$ be a reductive subalgebra in* End V *given by a faithful irreducible representation $\mathscr{S}$. Then $(\mathscr{S}(\mathscr{J}))^{(1)} \neq 0$ if and only if $(\mathscr{J}, \mathscr{S})$ is one of the pairs listed in Table* 2.

TABLE 2. — *Cartan prolongations of irreducible subalgebras*

No	$\mathscr{J}$	$\mathscr{J}$-module $\mathscr{J}^{(-1)}$	$\mathscr{J}^{(*)}$
1.	$A_k + A_l + \mathbb{C}$	$\circ-\circ-\cdots-\circ \otimes \circ-\circ-\cdots-\circ \otimes \mathbb{C}$ $k \qquad l$	A_{k+l+1}, $l \geqq 1$, $k \geqq 1$
2.	$A_n + \mathbb{C}$	$\circ-\circ-\circ-\cdots-\circ \otimes \mathbb{C}$ n	D_n, $n \geqq 4$
3.	$A_n + c$	$\circ-\circ-\cdots-\circ \otimes \mathbb{C}$ n	C_n, $n \geqq 4$
4.	$D_5 + \mathbb{C}$	$\circ-\circ-\circ \quad \otimes \mathbb{C}$	E_6
5.	$E_6 + \mathbb{C}$	$\circ-\circ-\circ-\circ-\circ \otimes \mathbb{C}$ $\vert$ $-$	E_7
6.	$B_n + \mathbb{C}$	$\overset{1}{\circ}-\circ-\cdots-\circ \Rightarrow \circ \otimes \mathbb{C}$	B_{n+1}, $n \geqq 2$
6^{I}.	$A_1 + \mathbb{C}$	$\overset{2}{\circ} \otimes \mathbb{C}$	B_2
7.	$D_n + \mathbb{C}$	$\circ-\circ-\cdots-\circ \qquad \otimes \mathbb{C}$	D_{n+1}, $n \geqq 4$
8.	$A_n + \mathbb{C}$	$\overset{1}{\circ}-\circ-\cdots-\circ \otimes \mathbb{C}$	W_{n+1}
9.	$C_n + \mathbb{C}$	$\overset{1}{\circ}-\circ-\cdots-\circ \Leftarrow \circ \otimes \mathbb{C}$	CH_n
10.	A_n	$\overset{1}{\circ}-\circ-\cdots-\circ$	S_{n+1}
11.	C_n	$\overset{1}{\circ}-\circ-\cdots-\circ \Leftarrow \circ$	H_n

Remarks. — (1) $\mathscr{S}$ is given up to an automorphism of $\mathscr{J}$.

(2) The cases (1)-(7) correspond to Hermitian spaces and in these cases $\mathscr{J}^{(*)} = V + \mathscr{J} + V^*$ whereas the cases (8)-(11) correspond to irreducible primitive Lie algebras of infinite type (Cartan series).

4.3. Now consider the case when $\mathscr{S}$ is reducible.

THEOREM. — *Let $\mathscr{J}$ be a reductive subalgebra in End*V *given by a faithful reducible representation $\mathscr{S}$. Then $(\mathscr{S}(\mathscr{J}))^{(1)} \neq 0$ if and only if $(\mathscr{J}, \mathscr{S}): \mathscr{J} = \mathscr{J}_0 \oplus \mathscr{G}$, $\mathscr{S} = \mathscr{S}_0 \oplus \psi$, $\mathscr{S}(\mathscr{J}) = \mathscr{S}_0(\mathscr{J}_0) \oplus \psi(\mathscr{G})$, $\mathscr{G} \subset$* Ker *$\mathscr{S}_0$, $\mathscr{J} \subset$* Ker *ψ and $(\mathscr{J}_0, \mathscr{S}_0)$ is one of the pairs listed in Table 2. Then $\mathscr{S}(\mathscr{J})$ is of infinite type if and only if so is $(\mathscr{J}_0, \mathscr{S}_0)$.*

Let us preface the proof of Theorem 4.3 *with two technical lemmas.*

LEMMA 4.3.1. – *Let* $\mathscr{J}$ *be a reductive algebra,* $\mathscr{S}:\mathscr{J}\to\operatorname{End}V$ *be such that* $\mathscr{S}=\sum_{1\leqq i\leqq s}\mathscr{S}_i$. *Let* $\mathscr{S}=\mathscr{S}_i+\psi_i$ *where* $\psi_i=\sum_{k\neq i}\mathscr{S}_k$ *and* $\mathscr{S}_i(\mathscr{J})\subset\operatorname{End}V_i$, $V=\sum_{1\leqq i\leqq s}V_i$. *If* $T:V\to\mathscr{S}(\mathscr{J})$ *is a linear map which enters the definition of the first prolongation* (4.1) *then* $T=\sum_{1\leqq i\leqq s}T_i$ *where* $T_i:V\to\mathscr{S}(\mathscr{J})$ *satisfies* (4.1) *for any i and* $T_i(V_i)\subset\mathscr{S}_i(\operatorname{Ker}\psi_i)$.

LEMMA 4.3.2. – *Under notations of Lemma* 4.3.1. *If* $T_i:V_i\to\mathscr{S}_i(\operatorname{Ker}\psi_i)$ *and* $(\psi(\mathscr{J}))^{(1)}\neq 0$ *then there exists* $\alpha\in\{1,\ldots,s\}$ *such that*

(*a*) $\operatorname{Ker}\psi_\alpha\cap\operatorname{Im}\mathscr{S}_\alpha=\operatorname{Im}\mathscr{S}_\alpha$;

(*b*) $(\mathscr{S}_\alpha(\operatorname{Ker}\psi_\alpha))^{(1)}\neq 0$.

Proof of Theorem 4.3. – Sufficiency. By Lemma 4.3.1 $T:V\to\mathscr{S}(\mathscr{J})$ is of the form $T=T_0+T_{\mathscr{G}}$ where $T_0:V_0\to\mathscr{S}_0(\mathscr{J}_0)$ and $T_{\mathscr{G}}:U\mapsto\psi(\mathscr{G})$ where $\mathscr{S}_0(\mathscr{J}_0)\subset\operatorname{End}(V_0)$, $\psi(\mathscr{G})\subset\operatorname{End}U$, $V=V_0\oplus U$.

By hypotheses there exists $T_0\neq 0$ satisfying (4.0) and therefore $\mathscr{J}^{(1)}\neq 0$, $\mathscr{J}^{(1)}=\mathscr{J}_0^{(1)}\oplus\mathscr{G}^{(1)}$. These assumptions imply $\mathscr{J}^{(k)}=\mathscr{J}_0^{(k)}\oplus\mathscr{G}^{(k)}$ and since $\mathscr{J}_0^{(k)}\neq 0$ for $k\geqq 1$ we get $\mathscr{J}^{(k)}\neq 0$ for $k\geqq 1$.

Necessity. Let $\mathscr{S}=\sum_{1\leqq i\leqq s}\mathscr{S}_i$ where $\mathscr{S}_i$ are irreducible. By Lemma 4.3.2. there exists a number α such that $\operatorname{Ker}\psi_\alpha\cap\operatorname{Im}\varphi_\alpha=\operatorname{Im}\varphi_\alpha$, $(\varphi_\alpha(\operatorname{Ker}\psi_\alpha))\neq 0$. This means that $\mathscr{J}=\mathscr{J}_0\oplus\mathscr{G}$. $\mathscr{S}(\mathscr{J})=\mathscr{S}_\alpha(\mathscr{J}_0)+\psi_\alpha(\mathscr{G})$ and $\mathscr{J}_0=\operatorname{Ker}\psi_\alpha$, *i.e.* $\psi_\alpha(\mathscr{J}_0)=0$. Since $\operatorname{Im}\psi_\alpha\cap\operatorname{Ker}\psi_\alpha=0$, $\operatorname{Im}\mathscr{S}_\alpha=\operatorname{Ker}\psi_\alpha$, then $\mathscr{S}_\alpha(\operatorname{Im}\psi_\alpha)=0$ *i.e.* $\psi_\alpha(\mathscr{J})=\psi_\alpha(\mathscr{G})$, where $\mathscr{G}=\operatorname{Im}\psi_\alpha$. Since $(\varphi_\alpha(\operatorname{Ker}\psi_\alpha))^{(1)}\neq 0$, then $(\mathscr{J}_0,\varphi_0)$ is contained among the pairs listed in Theorem 4.2. If $\mathscr{J}^{(k)}\neq 0$ for $k\geqq 1$ then $\mathscr{J}^{(1)}\neq 0$ and therefore $\mathscr{S}(\mathscr{J})=\mathscr{S}_1(\mathscr{J}_1)\oplus\ldots\oplus\mathscr{S}_p(\mathscr{J}_p)\oplus\psi(\mathscr{G})$ where $\mathscr{J}=\mathscr{J}_1\oplus\ldots\oplus\mathscr{J}_p\oplus\mathscr{G}$, $(\mathscr{S}_i(\mathscr{J}_i))^{(1)}\neq 0$, $(\psi(\mathscr{G}))^{(1)}=0$; consequently $(\varphi_\alpha(\mathscr{J}_\alpha))^{(k)}\neq 0$ for $k\geqq 1$ for a number $\alpha\in\{1,\ldots,p\}$, *i.e.* $\{\mathscr{J}_\alpha,\mathscr{S}_\alpha\}$ is one of the pairs (8)-(11) of Theorem 4.2.

COROLLARY. – *Let* $\mathscr{G}$ *be a reductive subalgebra in* sp (V). *Then either* $\mathscr{G}^{(1)}=0$ *or* $\mathscr{G}$ *is of infinite type. In the latter case* $\mathscr{G}$ *contains a simple ideal* sp (U), $U\subset V$ *which is a regular subalgebra in* sp (V), *where*

$$\operatorname{sp}(U)=\{f\in S^2(V^*)\mid X\lrcorner\,\delta f=0\quad\textit{for}\quad X\in U\}$$

4.4. Let us return to the study of stabilizors of effective forms and investigate when they are finite as subalgebras in sp (V). First formulate a general result on finiteness of stationary subalgebras.

Let V be a finite-dimensional vector space, G a semi-simple Lie group with Lie algebra $\mathscr{J}$ and $G\to\operatorname{Aut}V$ a faithful representation. Denote by $\{H_i\}$ the full set of pair-wise non-conjugate nilpotents in $\mathscr{J}$. A vector $v\in V$ is regular if the intersection of orbit v and $\{H_i\}$ $\mathcal{O}.v\cap H_i=\varnothing$ for any i.

THEOREM. – *The stationary subalgebra of a regular element is of finite type.*

Due to corollary of Serre theorem [20], [7] the stationary subalgebra of $v\in V$ is of finite type if and only if it has no elements of rank 1. Since $\mathscr{J}_v\subset\operatorname{End}V=V\otimes V^*$, the elements of rank 1 are of the form $\omega \otimes \omega^*$, $\omega \in \mathrm{V}$. Notice that we can confine ourselves to the study of nilpotent elements of this form only, since thanks to semisimplicity we may assume that the representation is given in $\mathrm{sl}(\mathrm{V})$ and therefore there are no non-zero diagonal elements. The regularity condition allows us to disregard nilpotents also.

Proposition 4.5. — (*k is algebraically closed*). *Let* $\omega \in \Lambda^k_\varepsilon(\mathrm{V}^*_{2n})$.

(1) $\mathcal{G}_\omega$ *is of finite type if and only if* ω *has no characteristic transvections.*

(2) *If* $\mathcal{G}_\omega$ *is of infinite type then* $\omega = \lambda \wedge \omega_0 + \omega_1$ *where* $\mathrm{X}_\lambda \lrcorner\, \omega_1 = 0$.

Proof. — (1) is a corollary of Theorem 4.1. (2) Follows from 3.3.

Corollary. — (*k is algebraically closed*). *The stabilizers of regular forms are of finite type.*

Remark. — The results of [1] imply that for $n=3$ the stabilizor of a generic $\omega \in \Lambda^k_\varepsilon(\mathrm{V}^*_{2n})$ is non-zero if $k=2$ or $k=3$.

4.6. Having in mind some future applications let us discuss the problem of finiteness of the type for stabilizers of effective forms of a special form.

Let $\omega \in \Lambda^k_\varepsilon(\mathrm{V}^*_{2n})$ and $\mathcal{G}_\omega = \mathcal{S}_\omega + \mathcal{K}_\omega$, where $\mathcal{S}_\omega$ is the Levi subalgebra.

Proposition. — $\mathcal{S}^{(1)}_\omega \neq 0$ *if and only if* ω *has a pair of characteristic non-involutive transvections.*

Proof. — Let $\mathcal{S}^{(1)}_\omega \neq 0$. Then by Corollary 4.3. $\mathcal{S}_\omega \supseteqq \mathrm{sp}(\mathrm{U})$ is simple and therefore $\mathcal{S}_\omega \supseteqq \mathrm{sp}(1)$, where $\mathrm{sp}(1)$ is a regular subalgebra in $\mathrm{sp}(\mathrm{v})$ generated by $\{\mathrm{X}_{\alpha_n}, \mathrm{X}_{-\alpha_n}, \mathrm{H}_n\}$.

Here $\mathrm{X}_{\pm\alpha_n}$, H_n are the elements of a Chevalley basis $\{\mathrm{X}_{\alpha_i}, \mathrm{X}_{-\alpha_i}, \mathrm{H}_i\}$ of $\mathrm{sp}(\mathrm{V})$ so any effective polycovector $\omega \in \Lambda^k_\varepsilon(\mathrm{V}^*_{2n})$ is a linear combination of elements of the form $\mathrm{X}^{m_1}_{-\alpha_1} \ldots \mathrm{X}^{m_e}_{-\alpha_e}(e^*_1 \wedge \ldots \wedge e^*_k)$ where $e^*_1 \wedge \ldots \wedge e^*_k$ is a simple (primitive) element of $\Lambda^k_\varepsilon(\mathrm{V}^*_{2n})$, H_n the corresponding element of the Cartan subalgebra ([13]). There exists a symplectic basis $\{e, f\}$ in V such that these elements act on the vectors of the basis according to the rules

$$\begin{array}{llll} & \mathrm{X}_{\alpha_n}(e_i) = \mathrm{X}_{-\alpha_n}(e_i) = 0, \quad i \neq n; & \mathrm{H}_{\alpha_n}(e_i) = 0, & i \neq n; \\ & \mathrm{X}_{\alpha_n}(f_n) = e_n, \quad \mathrm{X}_{-\alpha_n}(e_n) = -f_n; & \mathrm{H}_{\alpha_n}(f_i) = 0, & i \neq n; \\ \mathrm{X}_{\alpha_n}(f_i) = 0, \quad i \neq n; & \mathrm{X}_{-\alpha_n}(f_i) = 0; & \mathrm{H}_{\alpha_n}(e_n) = e_n; & \mathrm{H}_{\alpha_n}(f_n) = f_n. \end{array}$$

Let us present ω as follows: $\omega = e_n \wedge \omega_1 + f_n \wedge \omega_2 + e_n \wedge f_n \wedge \omega_3 + \omega_4$ where ω_i for $i = 1, 2, 3, 4$ do not contain either e_n or f_n.

The condition $\mathcal{S}_\omega \supseteqq \{\mathrm{X}_{\alpha_n}, \mathrm{X}_{-\alpha_n}, \mathrm{H}_{\alpha_n}\}$ means that $\mathrm{X}_{\alpha_n}\omega = \mathrm{X}_{-\alpha_n}\omega = \mathrm{H}_{\alpha_n}\omega = 0$. Rewriting these equations in the form $e_n \wedge \omega_2 = -f_n \wedge \omega_1 = e_n \wedge \omega_1 - f_n \wedge \omega_2$ we get $\omega_1 = \omega_2 = 0$ *i.e.* $\omega = e_n \wedge f_n \wedge \omega_3 + \omega_4$.

If $\omega = e_n \wedge f_n \wedge \omega_3 + \omega_4$, where ω_3, ω_4 do not contain either e_n or f_n, then taking the above into the account we get

$$\mathcal{G}_\omega \supset \{\mathrm{X}_{\alpha_n}, \mathrm{X}_{-\alpha_n}, \mathrm{H}_n\} \simeq \mathrm{sp}(1)$$

and therefore $\mathscr{S}^{(1)}_{\omega} \neq 0$ by Corollary 4.3.

PROPOSITION 4.7. – (1) *For the forms* $\omega \in \Lambda^n_{\varepsilon}(V^*_{2n})$ *of the form* 3.7 (1) $\mathscr{G}^{(k)}_{\omega} = 0$ *for* $k \geqq 1$.
(2) *For the forms* $\omega \in \Lambda^n_{\varepsilon}(V^*_{2n})$ *of the form* 3.7 (2)

$$\mathscr{G}^{(k)}_{\omega} \simeq H_{k+2}(p, q), \qquad k \geqq 1.$$

Proof. – Clearly it suffices to prove the statements for $k=1$. Let us prove (1). Let $\theta \in \mathscr{G}^{(1)}_{\omega} \subset S^3(V^*)$ then

$$e_i \lrcorner \delta\theta = e_i(\lambda) = \sum_{a,b} \alpha^i_{a,b} e^*_a f^*_b$$

$$f_j \lrcorner \delta\theta = f_j(\lambda) = \sum_{a,b} \beta^j_{a,b} e^*_a f^*_b$$

where $\alpha_i = \| \alpha^i_{a,b} \|$, $\beta_j = \| \beta^j_{a,b} \|$ belong to $sl(n, \mathbb{R})$. Making use of the equalities $f_j(e_i(\theta)) - e_i(f_j(\theta)) = 0$ we get $\sum_{a,b} \alpha^i_{ab} e^*_a = \sum_{a,b} \beta^j_{a,b} f^*_b$ yielding $\alpha_i = \beta_j = 0$.

COROLLARY. – $\mathscr{G}^{(k)}_{\omega} = 0$ *for* $K \geqq 0$ *for the effective forms* $\omega \in \Lambda^3_{\varepsilon}(V^*_6)$ *of the form* 2.6 (1, 2).

5. Spencer cohomology of the stabilizers of effective forms.

In this section we study the involutivity of symbols of Monge-Ampère equations (= stabilizers of the corresponding effective forms = symmetry algebras of the corresponding operators). Notice that by functorial property all the results on the triviality of Cartan prolongation and δ-cohomology obtained for $k = \mathbb{C}$ hold for $\mathbb{R}$ also.

5.1. Let V be a vector space, $\mathscr{G}$ a Lie algebra such that $\mathscr{G} \subset \operatorname{End} V = V \otimes V^*$ and $\mathscr{G}^{(k)}$ the k-th prolongation. Then the sequence

$$(5.1) \qquad \ldots \to \mathscr{G}^{(p)} \otimes \Lambda^{q-1}(V^*) \xrightarrow{\delta} \mathscr{G}^{(p-1)} \otimes \Lambda^q(V^*) \xrightarrow{\delta} \ldots$$

where

$$\delta(g \otimes \psi) = \sum_{i=1}^{n} [g, e_i] \otimes \psi \wedge e^*_i \qquad \text{for} \quad g \otimes \psi \in \mathscr{G}^{(p)} \otimes \Lambda^{q-1}(V^*)$$

and a basis $\{e_i\}$, $i = 1, \ldots, n$ of V is a cochain complex. Its cohomology

$$H^{p,q}(\mathscr{G}) = \operatorname{Ker} \delta(\mathscr{G}^{(p)} \otimes \Lambda^q(V^*)) / \operatorname{Im} \delta(\mathscr{G}^{(p+1)} \otimes \Lambda^{q-1}(V^*))$$

is called the Spencer cohomology of $\mathscr{G}$, see [7].

Since $\mathscr{J}^{(p)} \subset V \otimes S^{p+1}(V^*)$, then on the elements of the form $g = v \otimes f$, where f is a homogeneous polynomial of degree $p+1$, we have $[g, u] = v \otimes D_u f$, where $u \in V$ and $D_u f$ is the derivative of f along u.

If (V, Ω) is a symplectic space and $\mathscr{J} = \mathrm{sp}(V)$ then $\mathscr{J}$ is identified with $S^2(V^*)$ and its prolongations with $S^{k+2}(V^*)$; if $\mathscr{K} \subset \mathscr{J}$ is a subalgebra then $\mathscr{K}^{(k)} \subset S^{k+2}(V^*)$.

5.2. In the space of homogeneous formal series of degree k consider the subspace $\mathscr{H}^k(V^*)$ of homogeneous harmonic polynomials of order k. Let $\Delta = \sum_{1 \leqq i \leqq n} \partial^2/\partial x_i^2$ be the formal Laplace operator, $v \in V$, $v = \sum_{1 \leqq i \leqq n} \alpha_i e_i$, $D_v = \sum_{1 \leqq i \leqq n} \alpha_i \partial/\partial x_i$. Then

$$(5.2) \qquad D_v(\mathscr{H}^k(V^*)) \subset \mathscr{H}^{k-1}(V^*), \qquad S^k(V^*) = \mathscr{H}^k(V^*) + r^2 S^{k-2}(V^*)$$

where $r^2 = x_1^2 + \ldots + x_n^2$. A direct calculation with (5.2) shows that the following statement holds.

LEMMA. — $(\mathscr{H}^2(V^*))^{(k)} = \mathscr{H}^{k+2}(V^*)$.

5.3 $\mathscr{J}^{(*)}$ is called s-involutive if $H^{p,\,q}(\mathscr{J}) = 0$, $p \geqq s$, $q \geqq 0$. A 1-involutive algebra is called involutive [7].

PROPOSITION. — *The algebras* $S = V^* + S^2(V^*) + S^3(V^*) + \ldots$ *and* $\mathscr{H} = V^* + \mathscr{H}^2(V^*) + \mathscr{H}^3(V^*) + \ldots$ *are involutive.*

THEOREM. — *Let* $\mathscr{J}$ *be a reductive subalgebra in* End V *then* $\mathscr{J}^{(*)}$ *is* 3-*involutive and if* $\mathscr{J} \subset \mathrm{sp}(V)$ *then* $\mathscr{J}^{(*)}$ *is* 2-*involutive.*

Proof. — If $\mathscr{J} \subset \mathrm{End}\, V$ is of type k then $\mathscr{J}^{(*)}$ is $(k+1)$-involutive since $H^{s,\,q}(\mathscr{J}) = 0$ for $s \geqq k+1$; indeed, $\mathscr{J}^{(k+1)} = 0$ and $H^{k,\,n}(\mathscr{J}) = \mathscr{J}^{(k)}$, where $n = \dim V$. Since all the reductive subalgebras in End V are of three different types ($\mathscr{J}$ is of 1-type if $\mathscr{J}^{(*)} = V + \mathscr{J}$; of 2-type if $\mathscr{J}^{(*)} = V + \mathscr{J} + V^*$, and of infinite type otherwise) and also since the subalgebras $\mathscr{J}$ of $\mathrm{sp}(v)$ are of two types (of 1-type or of infinite type), then making use of the results on prolongations from S.4 we get the desired.

5.4. Now let us investigate the involutivity of symbols of Monge-Ampère operators (stabilizers of effective forms which define the operators). Notice that $\mathscr{J}_\omega^{(1)} = 0$ for ω of the form 3.7 (1) implies $H^{p,\,q}(\mathscr{J}_\omega) = 0$, $p > 0$ for $q \geqq 0$; the 2-involutiveness of $\mathscr{J}_\omega^{(*)}$ for ω of the form 3.7 (2) follows easily from the theorem stated in Serre's letter [7].

6. Classification of Monge-Ampère equations

6.0. Let us start with a classification of Monge-Ampère operators. We say that the operators Δ_{ω_1}, Δ_{ω_2}, $\omega_1, \omega_2 \in \Lambda^n(J^1 M)$ are equivalent at $m \in J^1 M$ if there exists a contact diffeomorphism $F: V_1 \to V_2$, $F(m) = m$ for some neighbourhoods V_1, V_2 of $m \in J^1 M$ such that $F(\Delta_{\omega_1}) = \Delta_{\omega_2}$ in a neighbourhood of m.

To find conditions needed to express Δ_ω in a coordinate system in the simplest form (with constant coefficients) remind that we supposed that there exists a contact symmetry X_f of Δ_ω such that $f(m) \neq 0$. Then there exists a local contact diffeomorphism preserving m and sending X_f into X_1. Therefore, we may assume that $L_{X_1}(\Delta_\omega)=0$.

The latter condition together with effectiveness of ω allows us to consider ω as a differential form on $T^* M$ and the classification problem of Monge-Ampère operators given by these forms as the classification problem of effective forms on $T^* M$ with respect to the group of symplectic diffeomorphisms (section 0).

The latter problem can be in turn reduced to integration of an overdefinite differential Monge-Ampère equation.

6.1. Let $\omega_1, \omega_2 \in \Lambda^n(T^* M)$ be two effective n-forms, $\dim M = n$, *i.e.* $\omega_1 \wedge \Omega = \omega_2 \wedge \Omega = 0$ where $\Omega \in \Lambda^2(T^* M)$ is the 2-form on $T^* M$ defining the symplectic structure. Consider $T^* M \times T^* M$ as a symplectic manifold with the structure 2-form $\bar{\Omega} = \pi_1^*(\Omega) - \pi_2^*(\Omega)$ where $\pi_i : T^* M \times T^* M \to T^* M$ is the projection onto the i-th factor, $i = 1, 2$. Every symplectic diffeomorphism $F : T^* M \to T^* M$ determines a Lagrangean manifold: the graph $(F) = (x, F(x)) \subset T^* M \times T^* M$, $x \in T^* M$. Then F transforms ω_2 into ω_1 and $F^* \omega_2 = \omega_1$ if and only if $\omega|_{\mathrm{graph}\,(F)} = 0$ where $\omega = \pi_1^*(\omega_1) - \pi_2^*(\omega_2)$ is an effective n-form on $T^* M \times T^* M$.

Since we are only interested in local equivalence, we may assume that $T^* M \times T^* M$ is symplectic diffeomorphic to some manifold $T^* N$ in a neighbourhood of $m \times m$ and the projection of the image of F onto N is non-singular. In this case (locally) graph (F) is determined by the differential of a function on N (the generating function of F) and the condition $F^* \omega_2 = \omega_1$ turns into the Monge-Ampère equation determined by ω with respect to the generating function of F. Moreover, since our problem is local, we may assume that $M = \mathbb{R}^n$, $m = 0$, $N = \mathbb{R}^{2n}$ and therefore the existence of F is equivalent to the solvability of an overdefinite Monge-Ampère equation $\mathcal{E}_\omega \subset J^2(M)$ where $\omega \in \Lambda^n(T^* \mathbb{R}^{2n})$.

6.2. Let us discuss when this equation is solvable. First consider the formal integrability. By Goldschmidt criterion [6] $\mathcal{E}_\omega$ is formally integrable if $\pi_{3,2} : \mathcal{E}_\omega^{(1)} \to \mathcal{E}_\omega$ and $\pi_{2,1} : \mathcal{E}_\omega \to_{N}$ are smooth vector bundles and $\mathcal{E}_\omega$ is 2-acyclic.

In our case the fibre of the equation $\mathcal{E}_\omega$ over $(x, y) \in T^* N = T^* M \times T^* M$ is formed by the linear symplectic transformations $A : T_x(T^* M) \to T_y(T^* M)$ sending $\omega_{2,y}$ into $\omega_{1,x}$. Therefore the map $\mathcal{E}_\omega \to T^* N$ where ω_1 has constant coefficients on $T^* M = T^* \mathbb{R}^n$, is surjective if and only if the $\mathrm{Sp}(2n)$-orbit of ω_2 is constant and the symbol of $\mathcal{E}_\omega$ at the point $z \in \mathcal{E}_\omega$ projecting to (x, y) is isomorphic to the stabilizer of the effective form $\omega_{1,x}$ in $\mathrm{sp}(2n) = \mathrm{sp}(T_x(T^* M))$. Summarizing we get the following

PROPOSITION. – *Let* ω_1 *be an effective form with constant coefficients. Then in a neighbourhood of* $0 \times 0 \in T^* \mathbb{R}^n \times T^* \mathbb{R}^n$ *the map* $\mathcal{E}_\omega \to T^* \mathbb{R}^{2n}$ *is a vector bundle if and only if* $\omega_{2,y}$ *belong to the same orbit of* $\mathrm{Sp}(2n)$ *and for all sufficiently small* $y \in T^* \mathbb{R}^n$ *and the symbol of* $\mathcal{E}_\omega$ *is isomorphic to the stabilizer or an effective form* $\omega_{1,0}$, *where* $0 \in T^* \mathbb{R}^n$.

6.3. To derive the conditions for surjectiveness of the map $\mathcal{E}_\omega^{(1)} \to \mathcal{E}_\omega$ consider the following more general situation. Let $\pi_{i+3,i+2} : \mathcal{E}_\omega^{(i)} \to \mathcal{E}_\omega^{(i-1)}$ be vector bundles for all $i \leqq k$ and derive the conditions for surjectiveness of $\pi_{k+3,k+2} : \mathcal{E}_\omega^{(k+1)} \to \mathcal{E}_\omega^{(k)}$. If $[F]_x^k$ is the k-jet of F at x then $[F]_x^{k+1} \in \mathcal{E}_\omega^{(k)}$ if and only if $[\omega_1 - F^*(\omega_2)]_x^k = 0$. Let us find out when we can jiggle F so as to preserve the k-jet of F at x but get $[\omega_1 - \eta^* \circ F^*(\omega_2)]_x^{k+1} = 0$ where η is a symplectic diffeomorphism such that $[\eta]_x^{k+1} = 1$. Let $\varepsilon = \omega_1 - F^*(\omega_2)$ then by the choice of F we get $[\varepsilon]_x^{k+1} \in S^{k+1}(T_x^*(\Phi)) \otimes \Lambda_\varepsilon^n(T_x^*(\Phi))$ where $\Phi = T^*\mathbb{R}^n$.

Compare $[\varepsilon_\eta]_x^{k+1}$ with $[\varepsilon]_x^{k+1}$ where $\varepsilon_\eta = \omega_1^* - \eta^* \circ F^*(\omega_2)$. We get

$$(6.1)\quad \begin{aligned}[\varepsilon_\eta]_x^{k+1} &= [\omega_1^* - \eta^* \circ F^*(\omega_2)]_x^{k+1} = [\omega_1 - \eta^*(\omega_1 - \varepsilon)]_x^{k+1} \\ &= [(1-\eta^*)\,\omega_1]_x^{k+1} + [\eta^*(\varepsilon)]_x^{k+1} = [L_h(\omega_1)]_x^{k+1} + [\varepsilon]_x^{k+1}\end{aligned}$$

where L_h is the Lie derivative along X_h and $[h]_x^{k+2} = 0$ and the shift by $t = 1$ along X_h coincides with η.

Therefore, for surjectivity of $\pi_{k+3,\,k+2} : \mathcal{E}_\omega^{(k+1)} \to \mathcal{E}_\omega^{(k)}$ it is necessary that $\mathcal{E}_k(\omega_1, \omega_2) := [\varepsilon]_x^{k+1} \bmod \mathcal{I}\mathrm{m}\, C_{\omega_1}^{(k)} = 0$. Here

$$(6.2)\qquad C_{\omega_1}^{(k)} : S^{k+3}(T_x^*\Phi) \to S^{k+1}(T_x^*\Phi) \otimes \Lambda_\varepsilon^n(T_x^*\Phi)$$

acts as follows:

$$(6.3)\qquad C_{\omega_1}^{(k)}([h]_x^{k+3}) = [L_h(\omega_1)]_x^{k+1}, \qquad \text{where} \quad h \in C^\infty(\Phi), \qquad [h]_x^{k+2} = 0.$$

Thus $\mathcal{E}_k(\omega_1, \omega_2)$ considered as a section over $\mathcal{E}_\omega^{(k)}$ of the bundle with the fibre $\operatorname{Coker} C_{\omega_1,\,x}^{(k)}$ at $[F]_x^{k+1}$ is the obstruction to surjectivity of $\pi_{k+2,\,k+1} : \mathcal{E}_\omega^{(k+1)} \to \mathcal{E}_\omega^{(k)}$.

Taking into account that $\operatorname{Ker} C_{\omega_1,\,x}^{(k)}$ is the k-th prolongation of the stabilizer of $\omega_{x,\,1}$ and therefore does not depend on x and making use of the Goldschmidt's criterion of formal integrability we get the following

PROPOSITION. — *The Monge-Ampère equation* $\mathcal{E}_\omega \subset J^2 M$ *corresponding to a pair of effective forms* $\omega_1, \omega_2 \in \Lambda^n(T^*\mathbb{R}^n)$ *of which* ω_1 *is a form with constant coefficients is formally integrable if*

(1) $\pi_{i+3,\,i+2} : \mathcal{E}_\omega^{(i+1)} \to \mathcal{E}_\omega^{(i)}$ *are bundles for all* $0 \leqq i \leqq k$ *and* $\mathcal{E}_{(\omega_1,\,\omega_2)} = 0$;

(2) *the orbit of* ω_2 *with respect to* $\mathrm{Sp}(2n)$ *does not depend on* y:

(3) $H^{2,\,j}(\mathcal{I}_{\omega_1,\,0}) = 0$ *for all* $j \geqq k$.

THEOREM 6.4. — *Let* $\omega_1 \in \Lambda^n(T^*\mathbb{R}^n)$ *be an effective form with analytic coefficients satisfying on a neighbourhood of* $0 \in T^*\mathbb{R}^n$ *the conditions of Proposition* 6.3. *Then there exists an analytic symplectic diffeomorphism* $F : T^*\mathbb{R}^n \to T^*\mathbb{R}^n$, $F(0) = 0$ *for which* $F^*(\omega_2)$ *is a form with constant coefficients.*

Theorem follows from Proposition 6.3.1. and Cartan-Kahler's theorem [10].

6.5. If $\mathcal{I}_{\omega_1}$ is of finite type Theorem 6.4. can be generalized onto the smooth case. This transition is based on the following observation which allows us to replace Cartan-Kahler's theorem by Frobenius theorem.

Let $\pi : E \to M$ be a smooth bundle and $f : M \to E$, $f \in \Gamma(\pi)$, is a smooth section of π. Then the $(k+1)$-jet $[f]_x^{k+1} = x_{k+1}$ determines the subspace $L(x_{k+1}) \subset T_{x_k}(J^k(\pi))$ tangent to the graph $j_k(f)$ at $x_k = [f]_x^k$.

Let $C(x_k)$ be the linear span of the union of $L(x_k)$ over all $x_{k+1}=[h]_x^{k+1}$ such that $x_k=[h]_x^k$. The distribution $x_n \to C(x_n)$ on $J^k(\pi)$ is called the Cartan distribution.

PROPOSITION. – *Let $\mathscr{E} \subset J^k(\pi)$ be a system of differential equations with symbol $\mathscr{J}(x_k)$ of finite type r at every $x_k \in \mathscr{E}$. Then if $\pi_{k+i,k+i-1}: \mathscr{E}^{(i)} \to \mathscr{E}^{(i-1)}$ is a bundle for all $1 \leqq i \leqq r+1$ then the restriction of Cartan's distribution onto $\mathscr{E}^{(r)}$ is completely integrable.*

Proof. – In fact, the map $\pi_{k+r+1,k+r}: \mathscr{E}^{(r+1)} \to \mathscr{E}^{(r)}$ is a (local) diffeomorphism and $C(x_{k+r}) \cap T_{x_{k+r}}(\mathscr{E}^{(r)}) = L(x_{k+r+1})$, where $x_{k+r+1} \in \mathscr{E}^{(r+1)}$ is the preimage of $x_{k+r} \in \mathscr{E}^{(r)}$. It remains to notice that $L(x_{k+r+1})$ is the tangent to the graph of the jet $j_k(f)$ of $f \in \Gamma(\pi)$ and therefore the condition of the Frobenius theorem is automatically verified.

6.6. The following result is a corollary of Proposition 6.5.

THEOREM. – *An effective form $\omega_2 \in \Lambda^n(T^* \mathbb{R}^n)$ whose $\mathrm{Sp}(2n)$-orbit is constant and the stabilizor $\mathscr{J}_{\omega_2, 0}$ is of finite type can be reduced by a (smooth) symplectic diffeomorphism in a neighbourhood of $0 \in T^* \mathbb{R}^n$ to a form with constant coefficients if and only if $\varepsilon_0(\omega_1, \omega_2) = \ldots = \varepsilon_r(\omega_1, \omega_2) = 0$, where $\varepsilon_i(\omega_1, \omega_2)$ is the obstruction constructed in 6.3. and r the type of the stabilizor.*

6.7. Let us apply this theorem to classification of Monge-Ampère operators on 3-dimensional manifolds. Let us start with generic orbits. Then $\mathscr{J}_\omega^{(1)} = 0$ and therefore we are to verify whether $\varepsilon_0(\omega_1, \omega_2)$ and $\varepsilon_1(\omega_1, \omega_2)$ vanish or not. The geometric meaning of ε_0, ε_1 is the following one. If $\varepsilon_0 = 0$ then $\mathscr{E}_\omega^{(1)} \to \mathscr{E}_\omega$ is the surjection which by triviality of $\mathscr{J}_\omega^{(1)}$ is a local diffeomorphism. We can reformulate this fact: at every point $x_2 \in \mathscr{E}_\omega$ there is chosen a subspace $L(x_3)$ such that $x_3 \in \mathscr{E}_\omega^{(1)}$ and $\pi_{3,2}(x_3) = x_2$ smoothly depends on x_2 and is transversal to the fibre of the projection $\mathscr{E}_\omega \to T^* N$. Denote the obtained distribution by $\mathscr{L}_0$. The fibres of the projection $\mathscr{E}_\omega \to T^* N$ are identified with the stabilizor $G.\omega_1$ of the form $\omega_{1,0}$ in $\mathrm{Sp}(2n)$. Denote by θ the projection $T_{x_2}(\mathscr{E}_\omega) \to \mathscr{J}_{\omega_1}$ onto the tangent space to the fibre (identified with $\mathscr{J}_{\omega_1}$). Let us consider θ as a $\mathscr{J}_{\omega_1}$-valued 1-form on $\mathscr{E}_\omega$. In these terms the map $\mathscr{E}_\omega^{(2)} \to \mathscr{E}_\omega^{(1)}$ is surjective if and only if $\mathscr{L}_0$ is completely integrable, *i.e.* $d\theta = 0$ on $\mathscr{L}_0$.

THEOREM. – (1) *An effective form $\omega_2 \in \Lambda^3(T^* \mathbb{R}^3)$ such that $\det A_{\omega_2} = \mathrm{const} \neq 0$ and $\varepsilon_0(\omega_2, \omega_1) = \varepsilon_1(\omega_2, \omega_1) = 0$, reduces by a local symplectic diffeomorphism to one of the following forms*:

$$(1) \qquad \omega = dp_1 \wedge dp_2 \wedge dp_3 + \lambda\, dq_1 \wedge dq_2 \wedge dq_3, \qquad \lambda\text{-const} \neq 0.$$

$$(2) \quad \omega = dq_1 \wedge dq_2 \wedge dp_3 + dp_1 \wedge dq_2 \wedge dq_3 - dq_1 \wedge dp_2 \wedge dq_3 + \nu^2\, dp_1 \wedge dp_2 \wedge dp_3, \qquad \nu\text{-const} \neq 0.$$

(2) An effective form $\omega_2 \in \Lambda^3(T^* \mathbb{R}^3)$ with analytic coefficients such that $\det A_{\omega_2} = 0$, $\varepsilon_0(\omega_2, \omega_1) = \varepsilon_1(\omega_2, \omega_1) = 0$ and which belongs to one of the $\mathrm{Sp}(6)$-orbits 4.6 (3)-(5) reduces by a local analytic symplectic diffeomorphism to one of the following forms:

(3) $\omega = dp_1 \wedge dq_2 \wedge dq_3 - dq_1 \wedge dp_2 \wedge dq_3 + dq_1 \wedge dq_2 \wedge dp_3$;

(4) $\omega = dp_1 \wedge dq_2 \wedge dq_3 - dq_1 \wedge dp_2 \wedge dq_3 - dq_1 \wedge dq_2 \wedge dp_3$;

(5) $\omega = dp_1 \wedge dq_2 \wedge dq_3 + dq_1 \wedge dp_3 \wedge dq_3 + dq_1 \wedge dq_2 \wedge dp_3$.

6.8. For Monge-Ampère equations on 3-dimensional manifolds the corresponding result takes the following form.

THEOREM. — *Let a Monge-Ampère equation determined by an effective form* $\omega \in \Lambda^3 (T^* \mathbb{R}^3)$ *satisfy* $\varepsilon_0 (\omega_1, p\,\omega_2) = \varepsilon_1\, \omega_1, p\,\omega_2)$ *where* $p^{-1} = \sqrt{|\det A_{\omega 2}|}$, $\det A_{\omega_2} \neq 0$. *Then by a symplectic diffeomorphism it can be locally reduced to one of the following forms*:

$$\text{Hess}\, h + \lambda = 0, \qquad \lambda \neq 0; \tag{1}$$

$$\frac{\partial^2 h}{\partial q_1^2} - \frac{\partial^2 h}{\partial q_2^2} + \frac{\partial^2 h}{\partial q_3^2} + \nu^2\, \text{Hess}\, h = 0, \qquad \nu \neq 0 \tag{2}$$

where $q = (q_1, q_2, q_3)$ *is a coordinate system in* $\mathbb{R}^3$, $h = h(q)$ *and* $\text{Hess}\, h = \det \left\| \dfrac{\partial^2 h}{\partial q_i\, \partial q_j} \right\|$ *is the Hessian.*

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