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TAMELY RAMIFIED SUPERCUSPIDAL REPRESENTATIONS OF CLASSICAL GROUPS II: REPRESENTATION THEORY

BY LAWRENCE MORRIS (1)

Introduction

In Part I of this paper, henceforth referred to as I, we showed how to construct parahoric subgroups and filtrations associated to compact maximal tori in classical groups, when the tori are tamely ramified. In this paper we use these constructions to associate irreducible supercuspidal representations of a classical group G to appropriate inducing data, provided no wild ramification is present.

Briefly, the construction proceeds as follows. Firstly, if the torus T is unramified one can use the constructions of I, and [M] to construct a suitable inducing representation from an open compact subgroup. There is nothing really new or surprising here, and the construction is carried out in Section 5 of this paper.

Now suppose the torus contains a ramified part. One then uses the constructions of I Section 3, to produce a parahoric subgroup P, and a filtration $\{P_n\}_{n \geqq 0}$ which reflects the arithmethic properties of a prespecified ramified part of T. Next, one uses a "cuspidal datum" to produce an inducing representation on a certain open compact subgroup of G; in a sense this representation can be viewed as concentrated around the given ramified "block". One then uses more of the cuspidal datum to proceed on the other blocks. It is crucial for this construction that one use the latter part of Section 3 of I. On each block one proceeds somewhat as in the construction of Howe [H] for the case of $\mathbf{GL}_n$, but care must be taken to ensure that the pieces fit together in a coherent way. [It is interesting to note that the blocks used in our construction are intersections with G of Levi components of the $\mathbf{GL}(V)$ into which G embeds]. Eventually one is left with an unramified part (which may be trivial), which is taken care of by Section 5. This "block" part of the construction is completed in Section 6. For more details, we refer the reader to section 6.3, where a (brief) guide and motivation are provided.

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In Section 7 we show that the compactly induced representation (to G) is irreducible and supercuspidal. The method is similar in spirit to that carried out in Section 5 of [M].

Section 4 of this paper is concerned with certain semi-simple elements in Lie (G), and their centralizers. Such elements give rise to characters on the groups P_n, by virtue of the fact that the filtrations of I Section 3 have good duality properties. In particular we examine the relation between such a centralizer, and the filtration above (*cf.* Proposition 4.10, Lemma 4.3); we also provide an example to show that proposition 4.10 need not hold in general.

Proposition 4.10 implies several structural results (4.13-4.15) which are reminiscent in spirit at least, of [H] and [M]. The remainder of Section 4 is concerned with duality (4.17-4.20, 4.24), intertwining/conjugacy properties (4.21, 4.23), and isotropy subgroups (4.24, 4.25). This section is rather long, and technical; perhaps the reader should refer to it as necessary.

The representations of Sections 6 and 7 do not exhaust the supercuspidal spectrum of G. In an afterword (Section 8) we sketch how one can produce more supercuspidal representations by enlarging the filtrations of I, and adapting the results of [M] on principal tori.

It is worth emphasizing the differences between this paper and [M]. Originally the author had hoped to understand the general situation by writing a compact maximal torus as a product of principal tori (this can always be done). Our motivation in Sections 6 and 7 has been to proceed inductively through the various "ramified parts" of T; we return to the original idea in Section 8. We remark that if one starts with a principal torus T, the filtration used in [M] need not be the one used in this paper [*see* example (3.9) *b* (ii) of I and example 8.6]. Moreover, one obtains supercuspidal representations from principal tori for symplectic groups in this paper which were not constructed in [M]. The constructions here and in I also take care of forms with an anisotropic part. Thus the constructions of [M] which are perhaps easier to understand, and more intuitive, are much cruder than the methods used here.

As in [M] we have refrained from defining *admissible characters*, and we do not treat the problem of equivalences. We remark that characters of tori cannot give all supercuspidal representations in our situation: the reason is the existence of unipotent cuspidal representations for the finite classical groups, which can be pasted on to other data to give representations not parametrized by characters of compact maximal tori (*see* Section 6). Nonetheless, the influence of [H] is quite pervasive in this paper.

One might hope that the constructions we give yield (almost) all supercuspidal representations of G. The problem of exhaustion leads one to a suitable notion of *fundamental* G-*stratum* (*cf.* [B2], [H-M] for the case of $\mathbf{GL}_n$). As mentioned in the introduction to I, constructions similar to those of Section 2 of I play a role here.

As a final technical note, we remark that the numbering of this paper is a continuation of that of I, as is the notation.

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4. Semi-simple elements and characters

4.1. We adhere to the notation and conventions of Section 3.1, and we begin, as in 3.3, by supposing that $A=E_1=E$. One then has the lattice chain $\{\mathscr{P}_E^n\}_{n\in\mathbf{Z}}$, and the associated hereditary order $\mathscr{A}$, with Jacobson radical $\mathscr{B}$, filtration $\{\mathscr{B}^n\}_{n\in\mathbf{Z}}$, and open normal subgroups $\{P_n\}_{n\in\mathbf{N}}$ of the parahoric $P\subset G$.

Suppose that E is a tamely ramified field extension of k. Let $C=C_E$ be the group generated by roots of unity of order prime to p that are in E, and a given uniformizer π_E. Then (*cf.* [H]) $E^*=C.U_E$ where $U_E=U_E^1=1+\mathscr{P}_E$. If $x\in E$ one can find a unique $c=c_x\in C$ such $c^{-1}x\in U_E$, *i.e.*, $\mathrm{ord}_E(c-x)\geqq\mathrm{ord}_E(x)$, or again $|c-x|_E\leqq|x|_E$. We assume that $\pi_E^e=\pi$ where e = ramification degree of E/F.

The group C is preserved by $\mathrm{Aut}(E/k)$; if $\tau\in\mathrm{Aut}(E/k)$ then either $\tau c=c$ or $|\tau c-c|_E=|c|_E$, all $c\in C$. Moreover if $E\supset E'\supset k$, then $N_{E/E'}(C_E)\subset C_{E'}$, and $C_{E'}\subset C_E$.

Let $X\in C=C_E\subset E\subset\mathrm{End}_k(V)=\mathrm{End}_k(E)$. We write $\mathscr{C}[X]$ for the commutator of X in $\mathrm{End}_k(E)$; in other words

$$\mathscr{C}[X]=\{Y\in\mathrm{End}_k(E)\,|\,\mathrm{ad}(Y)X=0\}$$

where $\mathrm{ad}(Y)X=YX-XY$.

Let $\mathrm{tr}(Y)=\mathrm{trace}_{\mathrm{End}_k(E)}(Y)$ if $Y\in\mathrm{End}_k(E)$. We then have

$$\mathrm{End}_k(E)=\mathscr{C}[X]\oplus\mathscr{C}[X]^\perp$$

where

$$\mathscr{C}[X]^\perp=\{Y\in\mathrm{End}_k(E)\,|\,\mathrm{tr}(YZ)=0,\ \text{all } Z\in\mathscr{C}[X]\}$$

Lemma 3 in [H] says that in fact

$$\mathscr{B}^n=\mathscr{C}[X]\cap\mathscr{B}^n\oplus\mathscr{C}[X]^\perp\cap\mathscr{B}^n,\quad \text{all } n\in\mathbf{Z}.$$

Let $\mathscr{B}_-^n=(\mathscr{B}^n)_-=\{Y\in\mathscr{B}^n\,|\,Y+\sigma Y=0\}$. If $X+\sigma X=0$ we then find that there is a decomposition of k_0-vector spaces

$$\mathscr{B}_-^n=\mathscr{C}[X]\cap\mathscr{B}_-^n\oplus\mathscr{C}[X]^\perp\cap\mathscr{B}_-^n \tag{4.1.1}$$

(*cf.* [M], proof of Lemma 2.20).

Set

$$\mathscr{B}_X^n=\mathscr{B}^n\cap\mathscr{C}[X],$$
$$\mathscr{B}_X^{n\perp}=\mathscr{B}^n\cap\mathscr{C}[X]^\perp$$

Suppose that $\mathrm{ord}_E(X)=m$. Then lemma 4 of [H] says that

$$\mathrm{ad}(X)\mathscr{B}_X^{n\perp}=\mathscr{B}_X^{n+m\perp},\quad \text{all } n\in\mathbf{Z}$$

If $Y \in (\mathscr{B}_X^{n\perp})_-$, then

$$\begin{aligned}\sigma(XY - YX) &= \sigma Y \sigma X - \sigma X \sigma Y \\ &= YX - XY \qquad \text{if} \quad X + \sigma X = 0\end{aligned} \tag{4.1.2}$$
$$\text{whence } \operatorname{ad}(X)(\mathscr{B}_X^{n\perp}) = \mathscr{B}_X^{n+m\perp}.$$

We shall commence this section by proving analogues of (4.1.1) and (4.1.2) for the filtrations we have constructed in Section 3, and appropriately chosen elements X. We then derive some consequences of these results which will play an important role in the representation theory.

4.2. We begin with the unramified filtration of Section 3.4. Thus $A = \bigoplus_{i=1}^{r} E_i$ where each E_i is unramified over k. It was shown there that one obtained a unique lattice chain $\mathscr{L}_u = \mathscr{L}$ by the summation process of Section 2; the associated hereditary order $\mathscr{A}$ was given the filtration by powers of the Jacobson radical. We remind the reader of Lemma 2.17: $\mathscr{A}$ is either principal of period 1, or $\mathscr{A}$ has period 2 and is the intersection of two (canonically chosen) principal orders Λ, $\Lambda^{\#} = \sigma \Lambda$.

Suppose that $X \in A$. Then Section 1.18 of [M] says that

$$\mathscr{C}[X] \simeq \operatorname{End}_{A_X}(V) \simeq \bigoplus_{j=1}^{r} \operatorname{End}_{F_j}(E_j)$$

where, if $X = (X_1, \ldots, X_r)$, F_j is the field generated by X_j, and $A_X = \bigoplus_{j=1}^{r} F_j$. Put $\mathscr{O}_X = \oplus \mathscr{O}_j$ where $\mathscr{O}_j$ is the ring of integers in F_j, $\mathscr{P}_X = \oplus \mathscr{P}_j$. We have

$$\mathscr{P}_X = \pi \mathscr{O}_X \tag{4.2.1}$$

since each F_j is unramified over k.

We can now apply the proof of Lemmas 2.17 and 2.18 of [M] to deduce that

$$\Lambda = \mathscr{C}[X] \cap \Lambda \oplus (\mathscr{C}[X]^{\perp} \cap \Lambda)$$

(If $\mathscr{A}$ is principal, we can replace Λ by $\mathscr{A} = \Lambda = \Lambda^{\#}$.) The key observation is that those lemmas can be applied because of 4.2.1 and the fact that Λ is principal.

4.3. Lemma. — $\mathscr{A} = \mathscr{A} \cap \mathscr{C}[X] \oplus \mathscr{A} \cap \mathscr{C}[X]^{\perp}$.

Proof. — This is really Lemma 2.20 of [M] which we give to avoid repeatedly referring to that paper. We have

$$\Lambda = \Lambda \cap \mathscr{C}[X] \oplus \Lambda \cap \mathscr{C}[X]^{\perp}$$

and similarly for $\Lambda^{\#} = \sigma \Lambda$. Let $y = \sigma z \in \Lambda \cap \sigma \Lambda$. Then

$$y = y_X + y_X^{\perp}, \qquad y_X \in \Lambda \cap \mathscr{C}[X], \qquad \text{etc.}$$

Also

$$y=\sigma z=\sigma z_X+\sigma z_X^{\perp}, \qquad \sigma z_X\in\sigma\Lambda\cap\mathscr{C}[X], \qquad \text{etc.}$$

Then $y_X-\sigma z_X=\sigma z_X^{\perp}-y_X^{\perp}\in\mathscr{C}[X]\cap\mathscr{C}[X]^{\perp}$, so that $y_X=\sigma z_X$, $y_X^{\perp}=\sigma z_X^{\perp}$, and we are done. Note that we have used the following lemma, in proving this.

4.4. Lemma. — *If* $X+\sigma X=0$, *then* $\mathscr{C}[X]$, $\mathscr{C}[X]^{\perp}$ *are* σ-*stable.*

Proof. – Let $Y\in\mathscr{C}[X]$. Then $(\sigma Y)X-X(\sigma Y)=\sigma\{(\sigma X)Y-Y(\sigma X)\}$. But $\sigma X=-X$, so this last is simply $\sigma\{YX-XY\}=0$. Let $Y\in\mathscr{C}[X]^{\perp}$. Then

$$\operatorname{tr}(X\sigma Y)=\operatorname{tr}(\sigma(Y\sigma X))=-\operatorname{tr}(\sigma(YX)=-\sigma_0\operatorname{tr}(YX)=0.$$

(The penultimate equality follows from the known structure of involutions σ associated to $\sigma_0-\varepsilon$ sesquilinear forms, for example.)

4.5. Proposition. — *With the assumptions of* 4.2, *let* $\mathscr{B}$ *be the Jacobson radical of* $\mathscr{A}$. *Then for any integer n,*

$$\mathscr{B}^n=\mathscr{B}^n\cap C[X]\oplus\mathscr{B}^n\cap C[X]^{\perp}$$

Proof. – We imitate the proof of Proposition 2.21 of [M], and note that we have just proved the assertion in case $n=0$. Consider the case $n=1$; we observe that $\mathscr{B}=\pi\mathscr{A}^*$, so we may replace $\mathscr{B}$ by $\mathscr{A}^*$, and do so. Let $x\in\mathscr{A}^*$, so that $x\in\operatorname{End}_k(V)=\mathscr{C}[X]\oplus\mathscr{C}[X]^{\perp}$: we set $x=s+t$, where $s\in\mathscr{C}[X]$, $t\in\mathscr{C}[X]^{\perp}$. Now

$$\mathscr{O}\supset\operatorname{tr}(x\mathscr{A})=\operatorname{tr}((s+t)(\mathscr{A}_X\oplus A_X^{\perp}))$$

where

$$\mathscr{A}_X=\mathscr{C}[X]\cap\mathscr{A}, \qquad \mathscr{A}_X^{\perp}=\mathscr{C}[X]^{\perp}\cap\mathscr{A}.$$

It follows that

$$\mathscr{O}\supset\operatorname{tr}(x\mathscr{A})=\operatorname{tr}(s\mathscr{A}_X)+\operatorname{tr}(t\mathscr{A}_X^{\perp})$$

hence

$$\operatorname{tr}(s\mathscr{A}_X)\subseteqq\mathscr{O}, \qquad \operatorname{tr}(t\mathscr{A}_X^{\perp})\subseteqq\mathscr{O}.$$

In particular, $s\in\mathscr{A}_X^*$, where "*" is with respect to $\operatorname{trace}_{\operatorname{End}_{A_X(V)}}$. Now

$$\mathscr{A}_X=\mathscr{A}\cap\mathscr{C}[X]$$

is a sum of hereditary orders determined by the lattice chain for $\mathscr{A}$ (for each "E_i-component" of the summed chain is stable by $\mathscr{O}_{F_i}$), so that

$$\mathscr{A}_X^*=\pi^{-1}\mathscr{B}_X=\pi^{-1}(\mathscr{B}\cap\mathscr{C}[X]).$$

(Note that $\mathscr{B}\cap\mathscr{C}[X]$ is the Jacobson radical for $\mathscr{A}\cap\mathscr{C}[X]$ by repeating the various definitions.) Thus $\mathscr{A}_X^* = (\pi^{-1}\mathscr{B})\cap\mathscr{C}[X]=\mathscr{A}^*\cap\mathscr{C}[X]$. Also $\operatorname{tr}(t\mathscr{A}_X)=0$, and $\operatorname{tr}(t\mathscr{A}_X^\perp)\subset\mathcal{O}$, so $\operatorname{tr}(t\mathscr{A})\subseteqq\mathcal{O}$ *i.e.*, $t\in\mathscr{A}^*\cap\mathscr{C}[X]^\perp$. This proves the assertion when $n=1$.

We have already noted that if $\mathscr{B}$ is not principal, then $\mathscr{B}^2$ is principal: $\mathscr{B}^2=\pi\mathscr{A}$ in this case. The assertion then follows from that for $n=0$. Again, if $\mathscr{B}$ is not principal (in which case everything follows from the case $n=0$), we can write $\mathscr{B}^3=\mathscr{B}^2.\mathscr{B}=b\mathscr{B}$, and b can be chosen to be in $\mathcal{O}_A\subset A$. Thus we obtain a decomposition from that for $\mathscr{B}$, and the general case follows easily, for $n>0$. The case $n<0$ is then obtained by periodicity.

4.6. Next, we turn to the analogue of 4.1.2, in the case that all the field extensions are unramified over k. First, let $X\in A_{2j}$ if $e(\mathscr{A})=2$, otherwise $X\in A_m$ some m (*cf.* Lemma 3.4). In other words, if $e(\mathscr{A})=2$, we let $X\in A_m$, m an *even* integer. Next, we let $C_i=C_{E_i}$ denote the group corresponding to the field E_i in Section 4.1. We write $X=\pi^m b$, where b is a product of roots of unity of order prime to p: if we put $C_A=\langle\pi_A\rangle.\prod_i B_i$ where B_i is the group generated by roots of unity of order prime to p in E_i^*, and $\pi_A=(\pi_1,\ldots,\pi_n)$ π_i a given uniformizer for E_i, then we have chosen $X\in C_A$.

It follows that if we set $\mathscr{B}_X^{n\perp}=\mathscr{B}^n\cap\mathscr{C}[X]^\perp=\mathscr{A}_{X,n}^\perp$ then $\operatorname{ad}(X)\mathscr{A}_{X,n}^\perp\subseteqq\pi_A^m\mathscr{A}_{X,n}^\perp$.

This last is equal to

$$\begin{cases}\mathscr{A}_{X,n+2m}^\perp & \text{if } \mathscr{B}^2 \text{ is principal}\\ \mathscr{A}_{X,n+m}^\perp & \text{if not}\end{cases}$$

Set $l=2$ if $\mathscr{A}$ is not principal, 1 otherwise.

LEMMA. — $\operatorname{ad}(X)(\mathscr{A}_{X,n}^\perp)^-=\pi_A^m(\mathscr{A}_{X,n}^\perp)^-=(\mathscr{A}_{X,n+lm}^\perp)^-$.

Proof. — Exactly the same as the proof of Lemma 2.24 of [M].

4.7. Suppose $X'=X+B\in X+\mathscr{B}_{2m+1,X}^-$. If $b\in\pi_A^m(\mathscr{A}_{X,n}^\perp)^-$, the above says $\operatorname{ad}(X)a=b$, $a\in(\mathscr{A}_{X,n}^\perp)^-$, so that $\operatorname{ad}(X')a=b+\operatorname{ad}(B)a$, $\operatorname{ad}(B)a\in(\mathscr{B}_{X,n+lm+1}^\perp)^-$ where $l=2$ if $\mathscr{A}$ is not principal, 1 otherwise. Then we can find $a_1\in(\mathscr{A}_{X,n+1}^\perp)^-$ so that $\operatorname{ad}(X)a_1=Ba_1$, and $\operatorname{ad}(X')(a+a_1)=b-Ba_1$. Continuing we obtain an element a_0 so that $\operatorname{ad}X'a_0=b$. This implies the following

COROLLARY. — *Let* $X\in C_A\cap\mathscr{B}_{lm}^-$, $X'\in X+\mathscr{B}_{lm,X}^-$. *Then*

$$\operatorname{ad}(X')(\mathscr{A}_{X,n}^\perp)^-=(\mathscr{A}_{X,n+lm}^\perp)^-.$$

4.8. We now turn our attention to the situation where at least one of the E_i occurring in the sum for A is ramified over k [*i.e.*, $e(E_i)=e_i>1$]. We remind the reader of the framework in Sections 3.5-3.7. In particular we write

$$A=A_u\oplus E_{l+1}\oplus\ldots\oplus E_r$$

and the lattice chain associated to A is given inductively by $\mathscr{L}_0=\mathscr{L}_u$, $\mathscr{L}_j=\mathscr{L}_{j-1}\oplus\mathscr{M}_j$, if $j>1$

$$\mathscr{L}=\mathscr{L}_r=\mathscr{L}_{r-1}\oplus\mathscr{M}_r$$

where $\mathscr{M}_j$ is the self dual lattice chain $\{\mathscr{P}_j^n\}_{n\in\mathbf{Z}}$. We remark (again) that this construction depends on the ordering $(E_{l+1}, \ldots, E_n)$.

4.9. Let $X=X_r$ be a non zero element of E_r; we put $F=F_r=F_{X_r}$ for the field generated by X. Suppose that $X_r+\sigma X_r=0$ where $\sigma_r=\sigma\,|\,E_r$; then σ_r acts non-trivially on F, while preserving it. Let $G_r=U(f_r, E_r)=U(f_r, V_2)$ (in the notation of 3.11), with Lie algebra given by

$$\{Y\in \mathrm{End}_k(V_2)\,|\,Y+\sigma_r Y=0\}$$

where "par abus de notation" we let σ_r also denote the restriction of σ to $\mathrm{End}_k(V_2)$. It follows that

$$T_r=\{x\in E_r\,|\,x\,\sigma_r\,x=1\}$$

is a compact maximal torus in G_r, with Lie algebra

$$\mathrm{Lie}(T_r)=\{Y\in E_r\,|\,X+\sigma_r X=0\}$$

In particular $X_r\in \mathrm{Lie}(T_r)\subset \mathrm{Lie}(G_r)$. The centralizer in G_r of X_r is given as follows (*cf.* [M], 1.19)

$$Z_{G_r}(X_r)=\{g\in \mathrm{End}_{F_r}(V_2)\,|\,g\,\sigma_r\,g=1\}$$

We remark that the involution $\sigma_r=\sigma\,|\,\mathrm{End}_{F_r}(V_2)$ corresponds to a $\varepsilon-\sigma_0$ sesquilinear form

$$\tilde{F}_r:\quad E_r\times E_r\to F_r$$

such that $\mathrm{trace}_{F_r/k}\circ\tilde{F}_r=f_r$ (*cf.* [M], 1.18). Note that since $X_r+\sigma X_r=0$, the group $Z_{G_r}(X_r)$ can be identified as the (genuine) unitary group of the (skew) hermitian form $\tilde{F}_r$.

4.10. We have

$$\mathrm{Lie}(Z_{G_r}(X_r))\subset \mathrm{End}_{F_r}(V)\subseteqq \mathrm{End}_k(E_1)\oplus\ldots\oplus \mathrm{End}_k(E_r)\subseteqq \mathrm{End}_k(V_1)\oplus \mathrm{End}_k(V_2)$$

where as usual $V_1=E_1\oplus\ldots\oplus E_{r-1}$.

The commuting algebra of X_r in $\mathrm{End}_k(V)$, which we denote by $\mathscr{C}[X]$ is then easily computed to be

$$\mathrm{End}_k(V_1)\oplus \mathrm{End}_{F_r}(V_2)$$

[$X_r\in F_r^*$, hence is invertible in $\mathrm{End}_k(V_2)$].

Let $\mathscr{A}=\mathscr{A}'\cap\mathscr{A}''$, $\{\mathscr{B}_i\}$ be the hereditary order and filtration by $(\mathscr{A}', \mathscr{A})$ bimodules given in 3.5-3.7.

PROPOSITION. – $\mathscr{B}_i=\mathscr{B}_i\cap\mathscr{C}[X]\oplus\mathscr{B}_i\cap\mathscr{C}[X]^{\perp}$.

Proof. – As usual set $V=V_1\oplus V_2$ where $V_2=E_r$. By Lemma 3.12 we can write

$$\mathscr{B}=\bigoplus_{1\leqq l,\ m\leqq 2}\mathscr{B}_i(l, m)$$

where $\mathscr{B}_i(l, m)=\mathscr{B}_i\cap\operatorname{Hom}_k(V_m, V_l)$. Now $\mathscr{B}_i(1, 2)$, $\mathscr{B}_i(2, 1)$ are both in $\mathscr{C}[X]^{\perp}$, and $\mathscr{B}_i(1, 1)$ is in $\mathscr{C}[X]$, by the remarks preceding the statement of the proposition. On the other hand we can apply (4.1.1) to $\mathscr{B}_i(2, 2)$, $\operatorname{End}_k(V_2)$, X_r to see that

$$\mathscr{B}_i(2, 2)=\mathscr{B}_i(2, 2)\cap\mathscr{C}_r[X_r]\oplus\mathscr{B}_i(2, 2)\cap\mathscr{C}[X_r]^{\perp}$$

where we set $\mathscr{C}_{r(X_r)}=\operatorname{End}_{F_r}(X_r)$, and "$\perp$" is with respect to $\operatorname{trace}_{\operatorname{End}_k(V_2)}$

Putting all this together, we see that the assertion of the proposition is true.

4.11. One might be led to believe that proposition 4.10 is the rule rather than the exception. Care must be exercised however. As an example, one can take a quartic totally ramified extension E_4, and a quadratic totally ramified extension E_2 such that E_2 is a subfield of E_4. The resulting algebra $E_2\oplus E_4$ gives a torus in Sp_6 via

$$\operatorname{trace}_{E_2}(\omega^2 x\,\sigma y)+\operatorname{trace}_{E_4}(\omega x'\,\sigma' y').$$

Here we have chosen $\omega^4=\pi$, $E_4=k[\omega]$, $E^2=k[\omega^2]$, $\sigma: \omega^2\mapsto -\omega^2$, $\sigma': \omega\mapsto -\omega$. One can then find X in $E_2\oplus E_4$ which provides a diagonal embedding of $E_2\subsetneq E_2\oplus E_4$ such that

$$\mathscr{A}\supsetneqq\mathscr{A}\cap\mathscr{C}[X]\oplus\mathscr{A}\cap\mathscr{C}[X]^{\perp}$$

where $\mathscr{A}$ is the order which stabilizes the entire chain $\mathscr{L}\oplus\mathscr{M}$ (it is an Iwahori order), or the order $\mathscr{A}'\cap\mathscr{A}''$ of 3.5-3.7.

4.12. We now assume that X_r is a member of the group $C_r=C_{E_r}$ introduced in 4.1, and that $X_r\in\mathscr{B}_{2m}$, $X_r\notin\mathscr{B}_{2m+1}$. For brevity we set

$$\mathscr{B}_{i,X}^-=\mathscr{B}_i^-\cap C[X], \mathscr{B}_{i,X}^{-\perp}=\mathscr{B}_i^-\cap\mathscr{C}[X]^{\perp}.$$

From 2.10 we see that

$$\operatorname{ad}(X)(\mathscr{B}_i^-)\subseteqq\mathscr{B}_{i+2m}^-$$

The kernel of this map is $\mathscr{B}_{i,X}^-$. Restricting our attention to $\mathscr{B}_{i,X}^{-\perp}$ we obtain a map

$$\operatorname{ad}(X):\quad \mathscr{B}_{i,X}^{-\perp}\to\mathscr{B}_{i+2m,X}^{-\perp}\quad \text{of } \mathscr{O}\text{-modules}$$

PROPOSITION. – $\operatorname{ad}(X)$: $\mathscr{B}_{i,X}^{-\perp}\to\mathscr{B}_{i+2m,X}^{-\perp}$ *is an isomorphism. Similarly, if we replace* $\operatorname{ad}(X)$ *by* $\operatorname{ad}(X')$ *where* $X'\in X+\mathscr{B}_{2m+1,X}^-$, *then* $\operatorname{ad}(X')$ *is an isomorphism.*

Proof. – It is sufficient to prove this for ad(X), since the second assertion can be proved by an approximation argument as in 4.7. Also we only need show surjectivity.

By proposition 4.10 and Lemma 3.12 we can write

$$\mathscr{B}_{i,\,X}^{\perp}=\mathscr{B}_i(1,\,2)\oplus\mathscr{B}_i(2,\,1)\oplus\mathscr{B}_i(1,\,1)\cap\mathscr{C}[X_r]^{\perp}$$

where the last "$\perp$" is with respect to $\text{trace}_{\text{End}_k(V_1)}$. Thus, if $b\in\mathscr{B}_{i,\,X}^{\perp}$ we write it in matrix form as

$$b=\begin{pmatrix}0 & b_{12}\\ b_{21} & b_{22}\end{pmatrix}$$

where $b_{12}\in\mathscr{B}_i(1,\,2)=\mathscr{B}_i\cap\text{Hom}(V_2,\,V_1)$ etc. Moreover, in matrix form we may write our form $f=f_1\oplus f_2$ as

$$\begin{pmatrix}J_1 & 0\\ 0 & J_2\end{pmatrix}$$

and X_r has the form $\begin{pmatrix}0 & 0\\ 0 & X_r\end{pmatrix}$.

Given $c=\begin{pmatrix}0 & c_{12}\\ c_{21} & c_{22}\end{pmatrix}\in\mathscr{B}_{i+2m,\,X}^{-\perp}$ we must find $b\in\mathscr{B}_{i,\,X}^{-\perp}$ such that $\text{ad}(X)\,b=c$. Now set $b=\begin{pmatrix}0 & b_{12}\\ b_{21} & b_{22}\end{pmatrix}$. The equation to be solved is then

$$\begin{pmatrix}0 & -b_{12}.X_r\\ X_r.b_{21} & \text{ad}(X_r)\,b_{12}\end{pmatrix}=\begin{pmatrix}0 & c_{12}\\ c_{21} & c_{22}\end{pmatrix}.$$

As an endomorphism of V_2, X_r is invertible, so that we find

$$b_{12}=-c_{12}.X_r^{-1},\qquad b_{21}=X_r^{-1}.c_{21}$$
$$\text{ad}(X_r)\,b_{22}=c_{22}$$

By proposition 3.17 (*b*) and (*c*) and (4.1.2) this last equation is also solvable. It remains to check that the element $b=\begin{pmatrix}0 & b_{12}\\ b_{21} & b_{22}\end{pmatrix}$ that we have produced, lies in $\mathscr{B}_{i,\,X}^{-\perp}$. Now the way that b_{22} was produced, implies that this element lies in $\mathscr{C}[X]^{\perp}$. Moreover, since c is skew, and b_{22} is skew by (4.1.2), one sees by a short computation using the matrix description of the form f given above, that our element b is also skew. Finally, using the block description for $\mathscr{B}_i$, $\mathscr{B}_{i+2m}$, and (2.10) again, we see that $b\in\mathscr{B}_i$. Namely, we have $\mathscr{B}_{2l}.\mathscr{B}_i\subseteq\mathscr{B}_{i+2l}$, all l, and

$$\text{Hom}_k(V_2,\,V_1)\circ\text{Hom}_k(V_2,\,V_2)\subseteq\text{Hom}_k(V_2,\,V_1),\qquad\text{etc.,}$$

and this implies for example that

$$(\text{Hom}_k(V_2,\,V_1)\cap\mathscr{B}_i).X_r^{-1}\subseteq\text{Hom}_k(V_2,\,V_1)\cap\mathscr{B}_{i-2m}$$

4.13. The next result in the analogue for the filtrations we have constructed, of [H], Lemma 6, [M], Lemma 2.28. To avoid stating and proving a proposition for each of the filtration types we have constructed (case where each E_i is unramified over k, case where at least one is ramified over k) we adopt the following conventions. If A is unramified with lattice chain of period 2 we set $\mathscr{B}_i = \mathscr{B}^i$; if the period is 1 we shall set $\mathscr{B}_{2i} = \mathscr{B}^i = \mathscr{B}_{2i-1}$. When quoting results for each case, we shall give those for the ramified case first, followed by the unramified case in parentheses. *We shall employ these conventions whenever it is appropriate, throughout the remainder of the paper.* Now, let X be as in 4.12 (4.6). In particular, $X \in \mathscr{B}_{2m} - \mathscr{B}_{2m+1}$.

PROPOSITION. — $X + \mathscr{B}_l^- = \mathrm{Ad}(P_{l-2m})(X + \mathscr{B}_{l,X}^-)$ *if* $l > 2m$.

Proof. – Let $T \in X + \mathscr{B}_l^-$. By proposition 4.10 (4.5) we may write

$$T = X + u + v, \qquad u \in \mathscr{B}_{l,X}^- \quad v \in \mathscr{B}_{l,X}^{-\perp}.$$

By Proposition 4.12 (4.6) we can find $z' \in \mathscr{B}_{l-2m,X}^{-\perp}$, such that $v = \mathrm{ad}(X)(z')$. From this point onward, the argument follows the proof of Lemma 2.8 of [M].

4.14. As shown in [M], Sections 2.30-2.34, Proposition 4.10 (4.5) has a number of group theoretic implications. The basis for such results is the existence of a filtration satisfying (a weaker form of) FI-F IV of [M1], the Cayley transform, and the decomposition provided in 4.10 (4.5).

We shall summarize these results, without providing proofs, which can be found in *loc. cit.* (with the appropriate changes).

First we remind the reader [*cf.* Theorem 2.13 (*c*)] that the Cayley map C is defined on $\mathrm{Lie}(G) \cap \{X \mid \det(1+x) \neq 0\}$ by the rule $C(x) = (1-x)(1+x)^{-1}$. The image of this map lies in G; if $x \in \mathscr{B}_1$ (*cf.* the conventions of 4.13) then $(1+x)^{-1}$ is given by the convergent power series $1 - x + x^2 - x^3 + \ldots$ and $C(x)$ exists. We then have a bijection [2.13 (*c*)]

$$C: \quad \mathscr{B}_i^- \to P_i, \qquad i > 0$$

with inverse given by $p \mapsto (1-p)(1+p)^{-1} : p - 1 \in \mathscr{B}_i$, and $2 \in \mathcal{O}^*$ so that $(1+p)^{-1}$ is given by a convergent power series.

Set $H_i = Z_G(X) \cap P_i$, and define

$$E(i,j) = \{p \in P_i \mid p - 1 \in \mathscr{B}_{i,X}^{-\perp} \bmod \mathscr{B}_j\}$$

where

$$2i \geqq j \geqq i \geqq 1 \tag{4.14.1}$$

LEMMA. — $E(i,j)$ *is a subgroup of* P_i, *which normalized by* $H = P \cap Z_G(X)$.

Proof. – That $E(i,j)$ is a group follows directly from the definitions and the fact that $C: \mathscr{B}_i^- \to P_i$ is a bijection [Theorem 2.13 (*c*)]. Moreover, in the unramified case one can show that H normalizes $E(i,j)$ just as in the proof of Lemma 2.33 of [M]. For the ramified case, we can also copy the proof of *loc. cit.* once we have shown that P normalizes $\mathcal{B}_i$. To see this, note that

$$P=\mathcal{A}'\cap\mathcal{A}''\cap G,\qquad \mathcal{B}_{2i}=\mathcal{B}'^{i}\cap\mathcal{B}''^{i}$$

hence P normalizes $\mathcal{B}_{2i}$ *i.e.*, $p\mathcal{B}_{2i}p^{-1}=\mathcal{B}_{2i}$, all $p\in P$. If i is odd, then $\mathcal{B}_i=\mathcal{B}_{2m+1}=(\mathcal{B}_{(-e-m)2})^*$, and for any j, $(x^{-1}\mathcal{B}_j x)^*=x\mathcal{B}_j^* x^{-1}$ which implies the result in this case as well.

4.15. The next result is proved using Proposition 4.10 (4.5), the Cayley map, and approximation arguments, *cf.* [M], 2.32-2.34.

PROPOSITION:

(*a*) $P_i=E(i,j)H_i$ (*semi-direct product*), *assuming* (4.14.1).

(*b*) $H_i\cap E(i,j)=H_j$, *assuming* (4.14.1).

4.16. Henceforth, unless otherwise stated, whenever we refer to P_i/P_j or $(E(i,j)$ it will be tacitly assumed that the conditions (4.14.1) hold.

4.17. The remainder of this section is concerned with the dual of P_i/P_j, and the isotropy subgroups of appropriate characters in this dual.

Recall that $\sigma\,|\,k=\sigma_0$, and that the fixed field of σ_0 is k_0, so that k is a Galois extension of k_0 of degree 1 or 2. We remind the reader that $\mathcal{O}_0$, $\mathcal{P}_0$, π_0, ... denote the obvious objects.

Let tr_0 denote the composition

$$\mathrm{End}_k(V)\xrightarrow{\text{Trace}} k \xrightarrow{\mathrm{tr}_{k/k_0}} k_0$$

For brevity we write tr for $\mathrm{Trace}_{\mathrm{End}_k(V)}$.

If L is an $\mathcal{O}$ lattice in $\mathrm{End}_k(V)$ [hence an $\mathcal{O}_0$-lattice in $\mathrm{End}_k(V)$] we define its complementary/dual lattice to be

$$L^\# = \{x\in\mathrm{End}_k(V)\,|\,\mathrm{tr}_0(xL)\subseteq\mathcal{O}_0\}$$

Then (*cf.* [M], 3.2), $L^\#$ is an $\mathcal{O}$-lattice as well. Recall that

$$L^*=\{x\in\mathrm{End}_k(V)\,|\,\mathrm{tr}(xL)\subseteq\mathcal{O}\}$$

The relationship between $L^\#$ and L^* is easily explained since $\mathrm{tr}_0=\mathrm{tr}_{k/k_0}\circ\mathrm{tr}$. Indeed, let e_0 be the ramification degree of k over k_0. Since $2\in\mathcal{O}^*$, k is tamely ramified over k_0, and the inverse different of k is just $\mathcal{P}^{1-e_0}$. Then

$$\begin{aligned}L^\# &= \{y\,|\,\mathrm{tr}_{k/k_0}(\mathrm{tr}(yL))\subseteq\mathcal{O}_0\}\\ &=\{y\,|\,\mathrm{tr}(yL)\subseteq\mathcal{P}^{1-e_0}\}\\ &=\mathcal{P}^{1-e_0}L^*\end{aligned}$$

We can apply these remarks to the filtration by $(\mathscr{A}, \mathscr{A})$-bimodules $\mathscr{B}_i$. It follows that

$$\begin{aligned}\mathscr{B}_i^{\#} = \mathscr{P}^{1-e_0} \mathscr{B}_i^{*} &= \mathscr{P}^{1-e_0} \mathscr{B}_{1-s_1-i} \\ &= \mathscr{B}_{1-s_1-i+s_1(1-e_0)}\end{aligned}$$

where s_1 is an integer given by the lemma below, which we shall refer to henceforth as the *modified period of* $\mathscr{A}$.

LEMMA:

$$s_1 = \begin{array}{l} 2e \text{ in the ramified case, where } e = \text{period of } \mathscr{L}' \text{ or } \mathscr{L}'' \\ 2e \text{ in the unramified case, of period } e=1 \\ e=2 \text{ in the unramified case, of period } e=2. \end{array}$$

Proof. — Straightforward exercise. We note that in the ramified case, $2e$ need not be the period of $\mathscr{A}' \cap \mathscr{A}''$.

4.18. The ideals $\mathscr{B}_i$, are σ-stable by construction, and we have

$$\mathscr{B}_i = \mathscr{B}_i^{+} \oplus \mathscr{B}_i^{-}$$

via

$$x = \frac{x+\sigma x}{2} + \frac{x-\sigma x}{2}, \qquad 2 \in \mathscr{O}^*$$

This is a sum of k_0-vector spaces, orthogonal with respect to tr_0, since

$$\mathrm{tr}_0 = \mathrm{tr}_{k/k_0} \circ \mathrm{tr}$$

and

$$\mathrm{tr}_{k/k_0}(\mathrm{tr}(xy)) = \mathrm{tr}_{k/k_0}(\sigma_0 \, \mathrm{tr}(\sigma(xy))) = \mathrm{tr}_{k/k_0}(\mathrm{tr}(\sigma(xy))) = -\mathrm{tr}_{k/k_0}(\mathrm{tr}(xy))$$

if $x = \sigma x$, $y = -\sigma y$, which implies $\mathrm{tr}_0(xy) = 0$ since $2 \in \mathscr{O}^*$.

Define

$$(\mathscr{B}_i^{-})^{\#} = \{ x \in \mathrm{End}_k(\mathrm{V})^{-} = \mathrm{Lie}(\mathrm{G}) \mid \mathrm{tr}_0(x \mathscr{B}_i^{-}) \subseteqq \mathscr{O}_0 \}$$

The remarks above tell us that

$$(\mathscr{B}_i^{-})^{\#} = (\mathscr{B}_i^{\#})^{-}$$

We summarize all this in the following lemma.

LEMMA. — *There is a linear function*

$$\begin{aligned}\lambda:\ & \mathbf{Z} \to \mathbf{Z} \\ & i \mapsto 1 - s_1 e_0 - i\end{aligned}$$

such that

$$\mathscr{B}_i^{\#} = \mathscr{B}_{\lambda(i)}$$
$$(\mathscr{B}_i^-)^{\#} = \mathscr{B}_{\lambda(i)}^-$$

In particular, λ *is order reversing*, $\lambda(i+1)=\lambda(i)-1$, *and* $\lambda(i)$ *is even* (*resp. odd*) *if and only if* i *is odd* (*resp. even*).

4.19. Let Ω be a character of the additive group k_0, which has conductor $\mathcal{O}_0$. The map

$$\operatorname{End}_k(\mathrm{V})^- \to (\operatorname{End}_k(\mathrm{V})^-)^\wedge$$
$$x \mapsto \Omega(\operatorname{tr}_0(x_-))$$

is an isomorphism of abelian groups, where we denote by "$\wedge$" the Pontrjagin dual. Given an $\mathcal{O}_0$-lattice L in $\operatorname{End}_k(\mathrm{V})^-$ we set

$$\mathrm{L}_\wedge = \{\chi \in (\operatorname{End}_k(\mathrm{V})^-)^\wedge \mid \chi(\mathrm{L}) \equiv 1\}$$

The identification above enables us to identify $\mathrm{L}_\wedge$ with $\mathrm{L}^{\#}$; if $\mathrm{L}_1 \supset \mathrm{L}_2$ then

$$(\mathrm{L}_1/\mathrm{L}_2)^\wedge \simeq \mathrm{L}_{2\wedge}/\mathrm{L}_{1\wedge} \simeq \mathrm{L}_2^{\#}/\mathrm{L}_1^{\#}$$

Together with Lemma 4.18 and Theorem 2.13 (*d*), these observations imply the following lemma.

LEMMA. — *Under the conditions* (4.14.1) *there is a* P-*equivariant* (*via* Ad, Ad* *respectively*) *isomorphism of abelian groups*

$$\mu:\quad \mathscr{B}_{\lambda(j)}^-/\mathscr{B}_{\lambda(i)}^- \xrightarrow{\sim} (\mathrm{P}_i/\mathrm{P}_j)^\wedge$$
$$b+\mathscr{B}_{\lambda(i)}^- \mapsto (p \mapsto \Omega(\operatorname{tr}_0(b(p-1))))$$

4.20. *Conventions*. — (i) We shall frequently write $\mathscr{A}^-(j)$ in place of $\mathscr{B}_j^-$, and $\mathscr{A}_{\mathrm{X}}^-(j)$, etc. in place of $\mathscr{B}_{j,\mathrm{X}}^-$, etc.

(ii) If $\psi \in (\mathrm{P}_i/\mathrm{P}_j)^\wedge$ is such that $\psi=\mu(b+\mathscr{A}^-(\lambda(i)))$ we shall say that b *represents* ψ, and we write $\psi=\psi_b$.

4.21. Now let $c \in \mathrm{A}^-$. In the unramified case we suppose that c has been chosen as the X in 4.6 *i.e.*, $c \in \mathrm{C}_{\mathrm{A}}$. In the ramified case we assume c has the form $(0, 0, \ldots, c_r)$ where $c_r \in \mathrm{C}_{\mathrm{E}_r}$, $c_r+\sigma_r c_r=0$.

In all cases we fix a positive integer j, and we suppose that $c \in \mathscr{A}^-(\lambda(j)) - \mathscr{A}^-(\lambda(j-1))$. To avoid any confusion later, we remind the reader of what this means in the various cases:

Ramified case (4.12). — We have $\mathscr{B}_i^- = \mathscr{A}^-(i)$ [4.20(i)]. From Proposition 3.11 (*c*) we know that $\mathscr{Q}_2^i = \mathscr{A}(2i) \cap \operatorname{End}_k(\mathrm{V}_2) = \mathscr{A}(2i-1) \cap \operatorname{End}_k(\mathrm{V}_2)$, and 3.11 (*b*) says that $\mathscr{Q}_2^i$ is a (two-sided) principal ideal generated by π_r^i. It follows that in this case $\lambda(j)$ is even (j odd 4.18) and c represents a character ψ_c on $\mathscr{A}_{j-1}^-/\mathscr{A}_j^- \simeq \mathrm{P}_{j-1}/\mathrm{P}_j$. Note that

ψ_c is trivial on $\mathscr{A}_{c,j-1}^{-\perp}$, so that it can also be defined by taking the character ψ_{c_r} on $\mathscr{A}_{c_r}^{-}(j-1)$ and extending is trivially. In particular, ψ_c will be trivial on $\mathscr{A}_{c_r}^{-}(j-1)(1, 1)$, in the notation of 3.12.

Unramified case (period 2). – In this case $\mathscr{B}_i=\mathscr{A}(i)=\mathscr{B}^i$ by definition, where $\mathscr{B}$ denotes the Jacobson radical. From Lemma 3.4 we know that $\mathscr{B}^{2i}\cap \mathrm{A}=\mathscr{B}^{2i-1}\cap \mathrm{A}$ for any i, so that in this case as well, $\lambda(j)$ must be even, and c represents ψ_c on $\mathscr{A}_{j-1}^{-}/\mathscr{A}_j^{-}\simeq \mathrm{P}_{j-1}/\mathrm{P}_j$, and ψ_c will be trivial on $\mathscr{A}_c^{-\perp}(j-1)$.

Unramified case (period 1). – According to the conventions of 4.13 we have defined $\mathscr{B}_{2i}=\mathscr{B}_{2i-1}=\mathscr{B}^i$, whence again $\lambda(j)$ must be even, and the same remarks hold as in the case just discussed.

LEMMA. — *Let* $\mathrm{S}, \mathrm{T}\in(c+\mathscr{A}^{-}(\lambda(j-1)))\cap\mathscr{C}[c]$. *Suppose that* $g\in \mathrm{G}$ *satisfies* $\mathrm{Ad}(g)\mathrm{T}=\mathrm{S}$. *Then* $g\in \mathrm{Z}_{\mathrm{G}}(c)$.

Proof. – We consider the ramified case first. In block form (*cf.* 4.12) we can write $\mathrm{T}=\begin{pmatrix}\mathrm{T}_1 & 0\\ 0 & \mathrm{T}_2\end{pmatrix}$, $\mathrm{S}\in\begin{pmatrix}\mathrm{S}_1 & 0\\ 0 & \mathrm{S}_2\end{pmatrix}$, where T_2, S_2 lie in $\mathrm{End}_{\mathrm{F}_r}(\mathrm{V}_2)$,

$$\mathrm{F}_r=k[c_r],\qquad c=\begin{pmatrix}0 & 0\\ 0 & c_r\end{pmatrix}.$$

We first show that $g\in\mathrm{End}_k(\mathrm{V}_1)\oplus\mathrm{End}_k(\mathrm{V}_2)$. Let $f=\mathrm{g.c.d.}(\lambda(j)/2, e(\mathscr{A}'))$. Then $\mathrm{S}^{e/f}$, $\mathrm{T}^{e/f}\in\pi^{(\lambda(j)/2f)}\mathscr{A}$, and are again conjugate by g. We assume until the end of this paragraph therefore that $\mathrm{S}, \mathrm{T}\in\mathscr{A}$, and then $\mathscr{A}/\mathscr{B}\simeq\mathscr{A}_{11}/\mathscr{B}_{11}\oplus\mathscr{A}_{22}/\mathscr{B}_{22}$. We also know that $\mathscr{A}_{ii}$ is the stabilizer of a lattice chain with Jacobson radical $\mathscr{B}_{ii}$ (*see* Lemma 3.13). It follows that $\mathscr{A}_{ii}/\mathscr{B}_{ii}$ is the Levi component of a parabolic subalgebra of $\mathrm{End}_{\mathbf{F}_q}(\bar{\mathrm{V}}_i)$ where $\bar{\mathrm{V}}_i$ is a finite dimensional $\mathbf{F}_q$-vector space which can be taken to be $\mathrm{L}_i/\pi\mathrm{L}_i$ for some suitable lattice $\mathrm{L}_i\subset\mathrm{V}_i\,(i=1, 2)$. Our description of S, T implies that $\bar{\mathrm{S}}$, $\bar{\mathrm{T}}$ each have characteristic polynomials of the form $t^r\bar{p}(t)$ where $r=\dim \mathrm{L}_1/\pi\mathrm{L}_1$, and $\bar{p}(t)$ has degree $\dim\mathrm{V}-\dim\mathrm{V}_1=\dim\mathrm{V}-r$, $(t, \bar{p}(t))=1$. Then S, T each have characteristic polynomial of the form $q(t)p(t)$ where $\overline{q(t)}=t^r$, $p(t)$ lifts $\bar{p}(t)$ and of course $p(t), q(t)\in\mathcal{O}[t]$: this follows from Hensel's lemma, and the remarks above. Moreover $(p(t), q(t))=1$. Since $\mathrm{Ad}(g)\mathrm{T}=\mathrm{S}$, it follows that g takes the rational Jordan decomposition of V as a T-module into the corresponding one for S. Since $(p(t), q(t))=1$ we see that this implies g preserves V_1 and V_2.

Thus in block form we have $g=\begin{pmatrix}g_1 & 0\\ 0 & g_2\end{pmatrix}$ where $g_i\in\mathrm{Aut}_k(\mathrm{V}_i)\,(i=1, 2)$, and we must show that $g_2\in\mathrm{End}_{\mathrm{F}_r}(\mathrm{V}_2)$. To do this we shall argue as in [H] Lemma 8.

To save notational burden we drop the subscripts "r" and "2" in what follows. Then $c^{-1}\mathrm{T}$, $c^{-1}\mathrm{S}\in(1+\mathscr{A}(1))$ which implies that $(c^{-1}\mathrm{T})^m=c^{-m}\mathrm{T}^m\to 1$ as $m\to\infty$, and similarly, $(c^{-1}\mathrm{S})^m\to 1$.

Now $\mathrm{C}/\langle\pi\rangle=\mathrm{C}_{\mathrm{E}_r}/\langle\pi\rangle$ is a finite group consisting of elements of order prime to p. Let l be the order of c in this group, then there is some $p^r\,(r\geqq 0)$ and infinitely many elements p^{n_i} (in the sequence $\{p^m\}_{m\in\mathbf{N}}$ such that $p^{n_i}\equiv p^r \bmod l$. Without loss we may assume $n_i \geqq r$, and then $p^{n_i - r} \equiv 1 \bmod l$. Thus there is an infinite sequence $m_i \to \infty$ with $c^{1-p_i^m} = \pi^{v_i}$, for some sequence v_i. We then have $\pi^{v_i} S^{p_i^m}$, $\pi^{v_i} T^{p_i^m} \to C$, and since π is a scalar, $\mathrm{Ad}(g)(\pi^{v_i} T^{p_i^m}) = \pi^{v_i} S^{p_i^m}$. Thus $\mathrm{Ad}(g)\, c = c$, as desired, and we are done with the ramified case.

The unramified case can be proved by adapting the second part of the proof above, and we leave it to the reader.

4.22. Let φ be a character on P_i/P_j (conditions 4.14.1) which restricts to ψ_c on P_{j-1}/P_j. With the notation of Lemma 4.19 this means that if $\varphi = \varphi_T = \mu(T)$, then $T - c \in \mathscr{A}^-(\lambda(j-1))$.

We now state some results which pertain to this situation; rather than give the proofs in complete detail we shall simply say in each case how to adapt the corresponding proofs in [M] by using the appropriate results proved in this section.

LEMMA. — *Let φ be as above. Then*

(*a*) $\mathrm{Ad}^* P_1(\varphi)$ *contains an element* φ' *which is represented by an element of* $\mathscr{C}[c]^-$.

(*b*) *Suppose* φ *is represented by* $T \in \mathscr{C}[c]^-$. *The stabilizer of* φ *in* P *is in the subgroup* $H . P_{j-i}$ [*where* $H = P \cap Z_G(c)$, cf. (4.14)].

Proof. — Apply the proof of Lemma 3.11 in [M]. In place of Lemma 3.10 of [M], one uses Lemma 4.21 above. In place of Propositions 2.21, 2.28 of [M], one uses 4.5 (4.10) and 4.13 respectively. We remind the reader that $\lambda(j)$, $j-1$ are even in all the discussions that take place here and below.

4.23. Suppose now that φ_1, φ_2 are characters of P_i/P_j (conditions 4.14.1) with $\varphi_l = \varphi_{T_l}$ $(l = 1, 2)$, where $T_l \in (c + \mathscr{A}^-(\lambda(j-1))) \cap \mathscr{C}[c]$.

Recall that $g \in G$ *intertwines* φ_2 with φ_1 if ${}^g\varphi_2 = \varphi_1$ on $g P_i g^{-1} \cap P_i$. In terms of cosets this means that $T_1 - g T_2 g^{-1} \in \mathscr{A}^-(\lambda(i)) + g \mathscr{A}^-(\lambda(i)) g^{-1}$.

LEMMA. — *With the notation and assumptions as above, if g intertwines* φ_2 *with* φ_1 *then* $g \in P_{j-i} Z_G(c) P_{j-i}$.

Proof. — The proof is similar to that of Lemma 3.13 of [M], but it is relatively short, and since this lemma plays an important role in Section 7 below we give the proof.

The remarks above imply that $T_1 - {}^gT_2 \in \mathscr{A}^-(\lambda(i)) + {}^g\mathscr{A}^-(\lambda(i))$ so that we can write $T_1 - {}^gT_2 = S_1 - {}^gS_2$ where $S_l (l = 1, 2) \in \mathscr{A}^-(\lambda(i))$. Thus $T_1 - S_1 = {}^g(T_2 - S_2)$ where $T_1, T_2 \in c + \mathscr{A}^-(\lambda(j-1))$. Applying Proposition 4.13 to T_1, T_2, we see that $T_l - S_l = \mathrm{Ad}(k_l(T'_l))$ where $k_l \in P_{j-i} (l = 1, 2)$, and $T'_l \in T_l + \mathscr{A}_c^-(\lambda(i))$. But then we can apply Lemma 4.21 to conclude that $k_1^{-1} g k_2 \in Z_G(c)$.

4.24. Now set $H_2 = P \cap Z_{G_2}(c)$ in the ramified case, and set $H_2 = H = P \cap Z_G(c)$ otherwise. There is a commutative diagram (condition 4.14.1).

$$\begin{array}{ccc} \mathscr{B}_i^- / \mathscr{B}_j^- & \xrightarrow{\sim} & P_i/P_j \\ \uparrow & & \uparrow \\ \mathscr{Q}_i^- / \mathscr{Q}_j^- & \xrightarrow{\sim} & H_{2,i}/H_{2,j} \end{array}$$

where we have put $\mathscr{Q}_i = (\mathrm{End}_{F_r}(E_r) \cap \mathscr{A}(i)$ in the ramified case (*cf.* Proposition 3.11), $\mathscr{Q}_i = \mathrm{End}_{A_c}(V) \cap \mathscr{A}(i)$ in the unramified case. Note that we are assuming that j is odd (4.18, 4.21) and that $\mathscr{A}_2 = \mathrm{End}_k(E_r) \cap \mathscr{A}$ is a principal order by Proposition 3.11. *A fortiori*, $\mathscr{A} \cap \mathrm{End}_{F_r}(E_r)$ is a principal order as well.

We define

$$\mathscr{Q}_i^* = \{l \in \mathrm{End}_{F_r}(E_r) \mid \mathrm{Tr}_0(l\,\mathscr{Q}_i) \subseteqq \mathscr{O}_0\}$$
$$(\mathscr{Q}_i)_\wedge = \{x \in \mathrm{End}_{F_r}(E_r)^\wedge \mid \mathrm{X}(\mathscr{Q}_i) \equiv 1\}$$

in the ramified case, and

$$\mathscr{Q}_i^* = \{l \in \mathrm{End}_{A_c}(V) \mid \mathrm{Tr}_0(l\,\mathscr{Q}_i) \subseteqq \mathscr{O}_0\}$$
$$(\mathscr{Q}_i)_\wedge = \{\chi \in \mathrm{End}_{A_c}(V)^\wedge \mid \chi(\mathscr{Q}_i) \equiv 1\}$$

in the unramified case. In the former case we obtain a commutative diagram of abelian groups

$$\begin{array}{ccc}
\mathrm{End}_{F_r}(E_r) & \rightarrow & \mathrm{End}_{F_r}(E_r)^\wedge \\
\downarrow{\scriptstyle \varkappa} & & \uparrow{\scriptstyle \tau} \\
\mathrm{End}_k(V) & \underset{\Omega(\mathrm{Tr}_0(\ ,\ ))}{\overset{\sim}{\rightarrow}} & \mathrm{End}_k(V)^\wedge
\end{array}$$

where τ is restriction and κ is the composition

$$\begin{array}{ccc}
\mathrm{End}_{F_r}(E_r) \rightarrow \mathrm{End}_k(E_r) \rightarrow \mathrm{End}_k(V_i) & \oplus & \mathrm{End}_k(V_2) \\
 & \downarrow & \\
 & \mathrm{End}_k(V) &
\end{array}$$

$(E_r \equiv V_2$ by definition).

The composition $\tau_0\,\Omega(\mathrm{Tr}_0(\kappa_-,\ ))$ is none other than the map $\mathrm{End}_{F_r}(E_r) \rightarrow \mathrm{End}_{F_r}(E_r)^\wedge$ which is induced by Tr_0, so that via these identifications

$$(\mathrm{H}_{2,i}/\mathrm{H}_{2,j})^\wedge \simeq \begin{cases} \mathscr{B}^-(\lambda(j)) \cap \mathrm{End}_{F_r}(E_r)/\mathscr{B}^-(\lambda(i)) \cap \mathrm{End}_{F_r}(E_r) \\ \mathscr{B}^-(\lambda(j)) \cap \mathrm{End}_{A_c}(V)/\mathscr{B}^-(\lambda(i)) \cap \mathrm{End}_{A_c}(V) \end{cases}$$

in the ramified (resp. unramified) case [condition (4.14.1)].

Warning. — We remind the reader that while $\mathscr{A} \cap \mathrm{End}_{F_r}(E_r)$ is a principal order with radical $\mathscr{Q}_1 = \mathscr{Q}$, the two sided ideal $\mathscr{Q}_i \neq \mathscr{Q}^i$ in general: $\mathscr{Q}_{2i} = \mathscr{Q}^i = \mathscr{Q}_{2i-1}$. There are similar reservations in the unramified case.

4.25. Returning to the framework of 4.21, we know that c represents a character $\psi = \psi_c$ of P_{j-1}/P_j, and furthermore c represents a character φ_0 of P_i/P_j which lies over (restricts to) ψ on P_{j-1}/P_j. If we combine our discussion in 4.24 with the results in 4.21 and 4.22 we obtain the following proposition.

PROPOSITION:

(*a*) φ_0 *is the extension of a character* φ'_0 *of* $\mathrm{H}_{2,i}/\mathrm{H}_{2,j}$.

(*b*) φ'_0 *is represented by c.*

(*c*) *The isotropy group of* φ_0 *under* $\mathrm{Ad}^*(\mathrm{P})$ *is* $\mathrm{H}.\mathrm{P}_{j-i}$.

(*d*) φ'_0 *is stabilized by* $\mathrm{Ad}^*(\mathrm{H})$.

5. Inducing representations: unramified case

5.1. In this section, and the next, we shall define some data which provide finite dimensional representations of compact mod center subgroups of G. When induced to G, the resulting representations will be shown to be irreducible and supercuspidal. We shall construct these representations in the unramified case in this section. The assertions and proofs are quite similar to those in Section 4 of [M] to which we shall frequently refer the reader; thus, we shall be relatively brief in this section.

5.2. We begin by recalling the framework of Section 3.1. Thus A is a commutative semi-simple algebra with a non-degenerate (ε, σ_0) sesquilinear form

$$f_{\mathrm{A}}: \quad \mathrm{A}\times\mathrm{A}\to k.$$

We set $\mathrm{G}=\mathrm{U}(f_{\mathrm{A}}, \mathrm{A})=\mathrm{U}(f, \mathrm{V})$, and we suppose in this section that A is a direct sum of separable field extensions E_i over k, each of which is unramified over k.

In Section 3.4 we constructed a lattice chain $\mathscr{L}_u=\mathscr{L}$, and corresponding hereditary order $\mathscr{A}$ with Jacobson radical $\mathscr{B}$, parahoric subgroup P and congruence subgroups P_n. Recall that $e(\mathscr{A})=e(\mathscr{L}_u)=1$ or 2: if $e(\mathscr{A})=2$, then $\mathrm{A}_{2j-2}\supsetneq\mathrm{A}_{2j-1}=\mathrm{A}_{2j}$; if $e(\mathscr{A})=1$, then $\mathrm{A}_{2j-2}\supset\mathrm{A}_{2j-1}\supset\mathrm{A}_{2j}=\pi\,\mathrm{A}_{2j-1}$ where $\mathrm{A}_i=\mathscr{B}^i\cap\mathrm{A}$ (*cf.* [M], 3.9 *Remark*, and Lemma 3.4 above).

We remind the reader of the conventions introduced in Section 4.13: if $e(\mathscr{A})=2$, we set $\mathscr{B}_i=\mathscr{B}^i$, otherwise we set $\mathscr{B}_{2i}=\mathscr{B}^i=\mathscr{B}_{2i-1}$. It is appropriate at this point to also remind the reader of the duality of 4.17:

$$\mathscr{B}_i^{\sharp}=\mathscr{B}_{1-s_1-i+s_1(1-e_0)}$$

where $s_1=2$.

5.3. Next, we recall the notion of a cuspidal datum of rank n, introduced in Sections 3.18-3.19 of [M], in this framework. By a *cuspidal datum of rank n* (associated with T, $\mathscr{A}$, P) we mean the following set of objects:

(*a*) a sequence $f_1>f_2>\ldots>f_n$ of positive integers.

(*b*) if $f_n>1$, a sequence $c_1, c_2, \ldots, c_n\in\mathrm{C}_{\mathrm{A}}^-$ such that $k\subset\mathrm{A}_{c_1}\subset\mathrm{A}_{c_2}\subset\ldots\subset\mathrm{A}_{c_n}=\mathrm{A}$ and linear characters ψ_k of $\mathrm{Z}_{\mathrm{G}_{k-1}}(c_k)$ where $\mathrm{G}_l=\mathrm{Z}_{\mathrm{G}_{l-1}}(c_l)$, $\mathrm{G}_0=\mathrm{G}$, $1\leqq l\leqq n$ such that $\psi_k\,|\,\mathrm{T}_{f_k-1}=\Omega(\mathrm{tr}_0(c_k))$ and $c_k\in\mathscr{A}^-(\lambda(f_k)-\mathscr{A}^-(\lambda(f_k-1))$. Moreover, $\mathrm{Z}_{\mathrm{G}_{n-1}}(c_n)=\mathrm{T}$.

(c) In case $f_n=1$, we again take a sequence $c_1, \ldots, c_{n-1} \in C_A^-$ with the properties of $c_1, \ldots, c_{n-1}$ above. In addition, T is unramified, and we take $T \subset P(n-1)=P \cap G_{n-1}$ to fix a unique vertex in the affine building associated to G_{n-1} (*cf.* [T], 3.6, and the canonical construction of $\mathscr{L}=\mathscr{L}_u$ given in Section 3 above). Let $P(n-1)$ be the Levi component of $P(n-1)$; it is the group of $\mathbf{F}_q$-rational points of a reductive group defined over $\mathbf{F}_q$. We let $\bar{P}(n-1)^0$ be the group of rational points of the identity component of this reductive group. We take $\bar{T} \cap P(n-1)^0$ to be a minisotropic torus in $P(n-1)^0$ and an irreducible cuspidal representation σ of $\bar{P}(n-1)^0$ which is fixed by no element of $P(n-1)/P(n-1)^0$. Here $P(n-1)^0$ is the inverse image in $P(n-1)$ of $\bar{P}(n-1)^0$.]

(d) For each linear character ψ_k of $Z_{G_{k-1}}(c_k)$ as above, an element c'_k of A^- such that

$$\psi_k \mid P_{i'_k} \cap G_k = \Omega(\mathrm{Tr}_0(c'_k))$$

where $i'_k=[(f_k+1)/2]$. *Note that c'_k need not be c_k.*

Remarks:

(i) The groups G_k are products of "genuine" unitary groups over field extensions F_i of k, where each F_i is furnished with a non-trivial involution σ_i (this follows from the fact that $c_k \in C_A$, so none of its components are zero) *cf.* [M], 1.18; *i. e.* each of the component groups is a unitary group of an $\varepsilon-\sigma_i$ sesquilinear form where $\sigma_i \neq 1$. Such a unitary group $U(f_i, F_i)$ has a non-trivial determinant to the elements of norm 1 in F_i. (Here the norm refers to the fixed field F'_i of F_i with respect to the involution σ_i.) By a *linear character* we mean one which arises via factorization through determinants, followed by products of characters of the groups of elements of norm 1 in F_i.

(ii) As we have remarked above, G_l is a product of unitary groups (of skew hermitian forms): $G_l=Z_{G_{l-1}}(c_l)=Z_G(c_l)$ acts on the vector space V, and the lattice chain $\mathscr{L}_u$ is *a fortiori* an $\mathcal{O}_{c_l}$-chain, where $\mathcal{O}_{c_l}$ is the unique maximal order in A_{c_l}. One then sees that $P(l)=G_l \cap P$ inherits all the arithmetic structure that arises from T; we shall frequently use properties for $(G_l, T, \ldots)$ that have been proved for $(G, T, \ldots)$ without further comment.

5.4. Assume until further notice that $f_n>1$, and let $\Psi=(T, (\psi_1, c_1, c'_1, f_1), (\psi_2, c_2, c'_2, f_2), \ldots)$ be a cuspidal datum of rank n. As above, we set

$$G_k=Z_{G_{k-1}}(c_k), \qquad P(i)=P(i-1) \cap G_i=P \cap G_i$$

and $P(0)=P$. The open compact subgroup which will be of interest to us is constructed as follows. First let $i_k=[f_k/2]$ (note that $i_k+i'_k=f_k$, always). We start by taking $P_{i_1}=P_{i_1}(0)$. This is normal in P, so we can form $P_1(1)P_{i_1}$ and then $P_{i_2}(1)P_{i_1}(0)$. Suppose at stage k we have formed

$$P_{i_k}(k-1) \ldots P_{i_2}(1) P_{i_1}(0).$$

Then $P_{i_{k+1}}(k) \subseteqq P(k-1), \ldots, P(0)$ so normalizes the group above and we can form

$$L_{\Psi_{k+1}}=P_{i_{k+1}}(k) P_{i_k}(k-1) \ldots P_{i_2}(1) P_{i_1}(0).$$

At the nth stage we form

$$P_\Psi = P_{\Psi_0} = T_1 P_{i_n}(n-1) \dots P_{i_2}(1) P_{i_1}(0).$$

This is defined since $T \subset P(l)$, each l.

We also define $P_{\Psi_l} = T_1 P_{i_n}(n-1) \dots P_{i_{l+1}}(l)$, and set $L_{\Psi_0} = \{1\}$, so that $P_\Psi = P_{\Psi_0} = P_{\Psi_l} L_{\Psi_l}$ for $0 \leqq l \leqq n$, where we define $P_{\Psi_n} = T_1$.

Suppose f_l is even. Then $i_l = i'_l = f_l/2$, and on $P_{i_l}(l)$, ψ_l is represented by the element c'_l via $\psi_l(x) = \Omega(\mathrm{Tr}_0(c'_l(x-1)))$. On the other hand $i_l \leqq i_{l-1} \leqq \dots \leqq i_1$. It follows that ψ_l defines a character of L_{Ψ_l} via the definition $\psi_l(x) = \Omega(\mathrm{Tr}_0(c'_l(x-1)))$. We then obtain a character, also denoted by ψ_l, on $TP_\Psi = TP_{\Psi_l} L_{\Psi_l}$. (On $TP_{\Psi_l} \subset G(l)$, it is defined by ψ_l.)

LEMMA. — *If f_l is even, the character ψ_l extends to a characrer of* TP_Ψ. *On L_{Ψ_l} it is given by the rule*

$$x \to \Omega(\mathrm{Tr}_0(c_l(x-1)))$$

5.5. We are going to form a representation

$$\rho_\Psi = \rho_n \otimes \dots \otimes \rho_1 \quad \text{of } TP_\Psi = \tilde{P}_\Psi$$

If f_l is even, we define ρ_l to be the character ψ_l of Lemma 5.4.

The case of f_l odd is a little more involved. We shall describe the construction, and refer the reader to [M], Section 4 for the proofs, which are the same.

5.6. Suppose $f_l = 2\, i_l + 1$ so that $[f_l/2] = i_l$, $i'_l = i_l + 1$. Define a subgroup $P^*_{\Psi_{l-1}}$ as follows

$$\begin{aligned} P^*_{\Psi_{l-1}} &= T_1 P_{i_n}(n-1) \dots P_{i_{l+1}}(l) P_{i'_l}(l-1) \\ &= P_{\Psi_l} P_{i'_l}(l-1) \end{aligned}$$

On TP_{Ψ_l} we have the character ψ_l as before. We extend it to $P_{i'_l}(l-1)$ by the rule $x \to \Omega(\mathrm{Tr}_0(c'_1(x-1)))$. By restriction, this gives a character φ_l on $P^*_{\Psi_{l-1}}$.

LEMMA:

(*a*) ker φ_l *is normal in* $TP_{\Psi_{l-1}}$;

(*b*) $P^*_{\Psi_{l-1}}$, $P_{\Psi_{l-1}}$ *are normal in* $TP_{\Psi_{l-1}}$;

(*c*) $P^*_{\Psi_{l-1}}/\ker \varphi_l$ *is central in* $P_{\Psi_{l-1}}/\ker \varphi_l$.

Proof. – *See* Lemmas 4.5, 4.6, 4.7, respectively of [M].

5.7. Next consider the map induced by commutators

$$P_{i_l}(l-1) \times P_{i_l}(l-1) \to P_{2i_l}(l-1) = P_{f_{l-1}}(l-1)$$

Composing this with the character map φ_l as we may, we obtain a map $\langle\ ,\ \rangle : P_{i_l}(l-1) \times P_{i_l}(l-1) \to S^1$ (unit circle) with image the additive group $\mathbf{F}_p$.

Define $\mathscr{H}'_l = P_{\Psi_{l-1}}/\ker \varphi_l$, $\mathscr{Z}'_l = P^*_{\Psi_{l-1}}/\ker \varphi_l$.

Since the second group is central in the first, and $P^*_{\Psi_{l-1}} = P_{\Psi_l} P_{i'_l}(l-1)$, we obtain a map $\langle\ ,\ \rangle : \mathscr{H}'_l \times \mathscr{H}'_l \to S^1$ as well.

Following [M], 4.9, we can use the results of 4.15 (or rather, variants of them) to write

$$P_{\Psi_{l-1}} = P_{\Psi_l} E_l(i_l, f_l - 1)$$

where "E_l" means that the "E" in questions is for $P(l) \subset P(l-1)$. Let $\mathscr{H}_l$ be the image of $E_l(i_l, f_l - 1)$ in $\mathscr{H}'_l$, and let $\mathscr{Z}_l = \mathscr{Z}'_l \cap \mathscr{H}_l$.

LEMMA:

(*a*) $\langle\ ,\ \rangle$ *factors to a non degenerate skew symmetric* $\mathbf{F}_p$*-bilinear form on* $\mathscr{H}_l/\mathscr{Z}_l \times \mathscr{H}_l/\mathscr{Z}_l$ *with image isomorphic to* $\mathbf{F}_p$.

(*b*) *Every element of* $\mathscr{H}_l$ *has order p.*

Proof. – Lemmas 4.8, 4.9 of [M].

5.8. Lemmas 5.6, 5.7, imply that $\mathscr{H}_l$ is a Heisenberg group: a nilpotent group of exponent p such that $\mathscr{Z}_l = (\mathscr{H}_l, \mathscr{H}_l)$. The group $TP_{\Psi_{l-1}}$ acts on $\mathscr{H}_l$ by conjugation. Lemma 5.6 (*c*) implies that P_{Ψ_l} acts trivially, while Lemmas 5.6 (*a*), (*b*) imply that T preserves $\langle\ ,\ \rangle$, so acts through the symplectic group.

It follows from the theory of the oscillator representation that there is a unique representation δ_l of $\mathscr{H}_l$ which is parametrized by φ_l on $\mathscr{Z}_l$, and this representation extends in a unique way to an irreducible representation $\tilde{\delta}_l$ of $TP_{\Psi_{l-1}}$.

To complete the construction of ρ_l when f_l is odd, we extend $\tilde{\delta}_l$ as before to $L_{\Psi_{l-1}}$ via $x \mapsto \Omega(\mathrm{Tr}_0(c'_l(x-1)))$ and denote the resulting representation of TP_Ψ by ρ_l.

5.9. Suppose that $f_n = 1$. We define $i_n = 1$, and

$$P_{\Psi_l} = T_1 P_{i_n}(n-1) \ldots P_{i_{l+1}}(l)$$

By definition σ is an irreducible cuspidal representation of $P^0(n-1)/P_1(n-1)$ which induces to an irreducible representation of $P(n-1)/P_1(n-1)$ by part (*c*) of 5.3. This representation then inflates to one on $P(n-1)$. We extend it to $P(n-1) P_{\Psi_0}$ by defining it to be trivial on $L_{\Psi_{n-1}}$. This defines a representation ρ_n on $P(n-1) P_{\Psi_{l-1}}$.

Suppose that $f_l = 2 i_l + 1$ is odd. In place of $TP_{\Psi_{l-1}}$ we use $P(n-1) P_{\Psi_{l-1}}$, and $P^*_{\Psi_{l-1}}$ is defined just as before: the subgroups $P^*_{\Psi_{l-1}}$, $P_{\Psi_{l-1}}$ are normal in $P(n-1) P_{\Psi_{l-1}}$ [note that $i_n = 1$, so that these subgroups *contain* $P_1(n-1)$]. We can then define the character φ_l on $P^*_{\Psi_{l-1}}$, and obtain the Heisenberg group $\mathscr{H}_l$ on which $P(n-1) P_{\Psi_{l-1}}$ acts as a group of automorphisms. Proceeding as before we obtain an irreducible representation ρ_l of $P(n-1) P_\Psi = \tilde{P}_\Psi$. We should note that $P(n-1)$ acts via the symplectic group on $\mathscr{H}_l$ because it stabilizes φ_l, and moreover, under this action $P_1(n-1)$ acts trivially; this follows from lemmas corresponding to 5.6, 5.7.

The even case ($f_l = 2i_l$) is carried out just as before, but with the group

$$P(n-1)P_{\Psi_{l-1}}.$$

6. Inducing representations: general case

6.1. In this section we shall construct inducing representations in case T has a ramified part. As one might expect, the definition of the inducing data is rather involved. We first remind the reader of the framework. Next, we provide a heuristic picture for the definition that follows. Finally, using the definition, we construct the inducing representations.

6.2. We begin by reminding the reader of the framework that we have developed in earlier sections. To begin as in I 3.1, we have a commutative semi-simple algebra $A = E_1 \oplus \ldots \oplus E_l \oplus E_{l+1} \oplus \ldots \oplus E_r$ with the non degenerate form $f = f_A : A \times A \to k$, and $f = \sum_{i=1}^{r} f_i$ where each f_i is not degenerate on E_i. We have $G = U(f_A, A) = U(f, V)$ where $V = A$ as vector space.

Assume as in I 3.5 that $E_1, \ldots, E_l$ are unramified over k while $e(E_i/k) > 1$ if $i > l$. We put $V_u = A_u = \bigoplus_{i=1}^{l} E_i$, $f_u = \bigoplus_{i=1}^{l} f_i$. There is an embedding

$$U(f_u, V_u) \times \prod_{i=l+1}^{r} U(f_i, E_i) \to U(f, V).$$

In I 3.5 we constructed a self dual lattice chain

$$\mathscr{L} = \mathscr{L}_r = \mathscr{L}_{r-1} \oplus \mathscr{M}_r = (\mathscr{L}_{r-2} \oplus \mathscr{M}_{r-1}) \oplus \mathscr{M}_r;$$

inductively we had $\mathscr{L}_j = \mathscr{L}_{j-1} \oplus \mathscr{M}_j$, $j \geqq 1 + l$, $\mathscr{L}_l = \mathscr{L}_u$. Associated to $\mathscr{L}_r = \mathscr{L}$, we had the hereditary orders $\mathscr{A}^{(0)}$, $\mathscr{A}'^{(0)}$, $\mathscr{A}''^{(0)}$.

We remark (again) that this construction depends on proceeding through the $E_{l+1}, \ldots, E_r$ *in that order*.

Again, associated to $\mathscr{L}_{r-1}$ we have $\mathscr{A}'^{(1)}$, $\mathscr{A}''^{(1)}$, $\mathscr{A}^{(1)}$. We shall write $P^{(0)}$ for the parahoric subgroup associated to $\mathscr{A}^{(0)}$, and $P^{(1)}$ for that associated to $\mathscr{A}^{(1)}$. Proposition 3.14 tells us that

$$P^{(1)} \subseteqq P^{(0)} \cap U(f_1 \oplus \ldots \oplus f_{r-1}, E_1 \oplus \ldots \oplus E_{r-1}).$$

A little more generally, we have $P^{(i)}$ associated to $\mathscr{L}_{r-i}$ and then

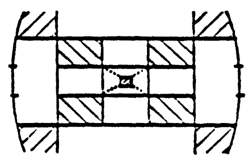
$$P^{(i)} \subseteqq P^{(i-1)} \cap U(f_1 \oplus \ldots \oplus f_{r-i}, E_1 \oplus \ldots \oplus E_{r-i}).$$

We shall write P_u, $\mathscr{A}_u$ for the objects associated to $\mathscr{L}_u$ (which do not depend on a chosen order in the same way that the $P^{(i)}$, $\mathscr{A}^{(i)}$ do).

6.3. It might be helpful to have a brief guide to the somewhat lengthy construction that follows. Suppose for example that $G = Sp(V)$, $V \simeq A = A_u \oplus E_{l+1} \oplus \ldots \oplus E_r$. In [M], 2.14 it is implicitly shown how to choose Witt bases for each of the pairs (A_u, f_u), $(E_i, f_i)\, l < i \leqq r$. (For simple examples, *see* I 3.9.) This enables one to realize the embedding

$$U(f_u, A_u) \times \ldots \times U(f_r, E_r) \to Sp(f, V)$$

in matrix form as blocks.



Here the four outer corners represent $U(f_r, E_r)$, the next four represent $U(f_{r-1}, E_{r-1}), \ldots$ The innermost square represents $U(f_u, A_u)$.

Consider $P^{(0)} \cap U(f_r, E_r)$. According to Proposition 3.11 of I the filtration of $P^{(0)}$ corresponds to that given by a principal order in $\mathbf{GL}_k(E_r)$ (*i. e.*, it collapses to that). Moreover the norm 1 elements in E_r^* with respect to the given involution on E_r are the k_0-points of a compact maximal torus in $U(f_r, E_r)$. If we were just dealing with this group we could proceed as in [M], Section 4 to construct an irreducible supercuspidal representation by providing data which enables one to perform an analogue of the Howe construction for $\mathbf{GL}_k(E_r)$. The problem is that we must ultimately deal with G, not $U(f_r, E_r)$.

We begin by starting from data $\Psi^{(0)}$ which is the analogue on $U(f_r, E_r)$ of the Howe factorization for $\mathbf{GL}_k(E_r)$. Since the filtration on $P^{(0)}$ is so well behaved with respect to $U(f_r, E_r)$ we can use this data to construct (6.7-6.14 below) a finite dimensional representation $\rho_{\Psi^{(0)}}$ on a certain compact open subgroup $P_{\underline{\Psi}}$ (*see* 6.6 below) of G. We then move to $P^{(1)}$, $U(f_1 \oplus \ldots \oplus f_{r-1}, A_u \oplus E_1 \oplus \ldots \oplus E_{r-1})$ and analogous data to that above, but for $U(f_{r-1}, E_{r-1})$. Using this data (denoted $\Psi^{(1)}$), we construct a second representation $\rho_{\Psi^{(1)}}$ on $P_{\underline{\Psi}}$. If we continue this process we end up with a sequence of finite dimensional representations $\rho_{\Psi^{(0)}}, \rho_{\Psi^{(1)}}, \ldots, \rho_{\Psi_u}$ on $P_{\underline{\Psi}}$. We denote their tensor product by $\rho_{\underline{\Psi}}$; in Section 7 we show that $c-\mathrm{Ind}_{P_{\underline{\Psi}}}^{G}$ is an irreducible supercuspidal representation of G.

The totality of data that we use for the construction of $\rho_{\underline{\Psi}}$ we call a *cuspidal datum* $\underline{\Psi}$; its definition is given in Sections 6.4-6.5 below.

For the method that we have outlined above to work successfully, some compatibility conditions must be satisfied. First, the group $P_{\underline{\Psi}}$ is defined as a product of subgroups $\tilde{P}_{\Psi^{(0)}}, \tilde{P}_{\Psi^{(1)}}, \ldots \tilde{P}_{\Psi_u}$. This definition uses I 3.14 in an essential way (*see* 6.6 below). Secondly, the representation $\rho_{\Psi^{(0)}}$ mentioned above is itself a tensor product of representations $\rho_{n_0}^{(0)}, \ldots, \rho_1^{(0)}$ which *a priori* are defined on (subgroups of) $\tilde{P}_{\Psi^{(0)}}$. To extend them (*see* 6.8, 6.15 below) to P_Ψ, the condition 6.5 (*b*) is used. Similar remarks apply to the other $\rho_{\Psi^{(t)}}$, while condition 6.5 (*c*) is used in an analogous fashion for ρ_{Ψ_u}.

Condition 6.5 (*d*) is imposed to help guarantee that the induced representation will be irreducible; it is used in the proof of Lemma 7.4. It is a very mild condition; to assist the reader we examine it for the case of example I 3.9 (*b*) (i) (where $G = Sp_4$) in the remark following the enunciation of the condition.

We remark that if one performed this construction in a different order on the blocks, one would obtain in general different representations π of G. This corresponds to which congruence subgroups are "seen first" by the representation π, and is related to the notion of fundamental G-strata.

Finally, we note that at level 1, we allow ourselves cuspidal representations of finite reductive groups which do not necessarily come from T. This is unlike the construction for $\mathbf{GL}_n$.

6.4. We now begin the definition of cuspidal datum. Since it is defined inductively, we give it in two steps.

Let σ_r denote the (non-trivial) involution on E_r^* which is inherited from the involution on $\mathrm{End}_k(V)$ corresponding to f (*i. e.*, σ_r corresponds to f_r), and let

$$T^{(0)} = \{ x \in E_r^* \mid x \sigma_r x = 1 \}.$$

This is a compact maximal torus in $U(f_r, E_r)$, corresponding to the embedding of E_r^* in $\mathbf{GL}_k(E_r)$. In addition, we have $\mathscr{A}^{(0)}$, $P^{(0)}$ [*not* a subgroup of $U(f_r, E_r)$], which lie in $\mathrm{End}_k(V)$, G respectively.

According to Proposition I 3.11, $\mathscr{A}^{(0)} \cap \mathrm{End}_k(E_r)$ is a principal order, with Jacobson radical $\mathscr{Q}^{(0)}$ generated by π_r (a uniformizing element for $\mathcal{O}_r \subseteqq E_r$).

We shall write $C^{(0)}$ for the group denoted $C_r = C_{E_r}$ in Section 4.1. This is in keeping with our notation above.

Now suppose we are given the data below

(*a*) A sequence $f_1^{(0)} > f_2^{(0)} > \ldots > f_{n_0}^{(0)} \geqq 1$ of integers.

(*b*) If $f_{n_0}^{(0)} > 1$ a sequence $c_1^{(0)}, \ldots, c_{n_0}^{(0)} \in C^{(0)-}$. We define $F_1^{(0)} = k(c_1^{(0)})$, $F_i^{(0)} = F_{i-1}^{(0)}(c_i^{(0)})$, $n_0 \geqq i > 1$, and we assume $F_{n_0}^{(0)} = E_r$.

Define

$$\begin{aligned} G^{(0)}(0) &= U(f_r, E_r) \\ G^{(0)}(l) &= Z_{G^{(0)}(l-1)}(c_l^{(0)}), \qquad n_0 \geqq l \geqq 1 \end{aligned}$$

and suppose

$$G^{(0)}(n_0) = T^{(0)}$$

The group $G^{(0)}(l)$ is a unitary group on the vector space E_r viewed as a $F_l^{(0)}$-vector space. The field $F_l^{(0)}$ is furnished with a *non* trivial involution coming from σ_r (it contains $c_l^{(0)}$), with fixed field $F_l^{(0)\prime}$. As algebraic group, $G^{(0)}(l)$ is defined over $F_l^{(0)\prime}$.

There is a determinant map

$$\mathrm{G}^{(0)}(l) \to (\mathrm{F}_l^{(0)})^*$$

By a *linear character* on $\mathrm{G}^{(0)}(l)$, we shall mean a character $\mathrm{G}^{(0)}(l) \to \mathrm{S}^1$ (unit circle), which factors through the determinant map above [*cf.* 5.3 Remarks (i)].

We assume the existence of linear characters $\psi_l^{(0)}$ of $\mathrm{G}^{(0)}(l)$ such that

$$\psi_l^{(0)} \mid \mathrm{T}_{f_l-1}^{(0)} = \Omega(\mathrm{tr}_0(c_l^{(0)}))$$

where $c_l^{(0)} \in \mathscr{A}^{(0)-}(\lambda(f_l^{(0)})) - \mathscr{A}^{(0)-}(\lambda(f_l^{(0)}-1))$.

Here we write

$$\mathscr{A}^{(0)-}(i) \qquad \text{for} \quad \mathscr{A}^{(0)}(i) \cap \mathrm{End}_k(\mathrm{E}_r)^-,$$

cf. I 3.11 (*b*) and the notation introduced in 4.20. We remark that tr_0 is still with respect to k_0. We also remind the reader of Proposition I 3.11 (*b*) and (*c*), as well as the statements made in Section 4.18, 4.21.

Finally, for each linear $\psi_l^{(0)}$ as above we assume an element $c_l'^{(0)} \in \mathrm{E}_r^-$ (not necessarily in $\mathrm{C}^{(0)}$) such that

$$\psi_l^{(0)} \mid \mathrm{P}_{i_l(0)}^{(0)} \cap \mathrm{G}^{(0)}(l) = \Omega(\mathrm{tr}_0\, c_l'^{(0)})$$

where by definition $i_l'^{(0)} = [(f_l^{(0)}+1)/2]$, and $f_l^{(0)} = i_l^{(0)} + i_l'^{(0)}$.

(*c*) Suppose on the other hand that $f_{n_0}^{(0)} = 1$. We again take a sequence $c_1^{(0)}, \ldots, c_{n_0-1}^{(0)}, c_1'^{(0)}, \ldots, c_{n_0-1}'^{(0)}, \psi_1^{(0)}, \ldots, \psi_{n_0-1}^{(0)}$, just as above and satisfying the same conditions. We shall suppose that

$$\mathrm{P}^{(0)}(n_0-1) = \mathrm{P}^{(0)} \cap \mathrm{G}^{(0)}(n_0-1)$$

fixes a unique vertex in the affine building that is associated to $\mathrm{G}^{(0)}(n_0-1)$. [We note that $\mathrm{P}^{(0)}(l) = \mathrm{P}^{(0)} \cap \mathrm{G}^{(0)}(l)$ is always the stabilizer of an $\mathcal{O}_l^{(0)}$-lattice chain in $\mathrm{G}^0(l)$, namely the one defined by powers of the prime ideal in E_r.] Write $\bar{\mathrm{P}}^{(0)}(n_0-1)$ for its Levi component; this is the group of rational points of a reductive group defined over $\mathcal{O}_{n_0-1}^{(0)\prime}/\mathscr{P}_{n_0-1}^{(0)\prime}$. Write $\bar{\mathrm{P}}^{(0)\,0}(n_0-1)$ for its identity component, and $\mathrm{P}^{(0)\,0}(n_0-1)$ for its inverse image in $\mathrm{P}^{(0)}(n_0-1)$. Our final piece of data will be an irreducible cuspidal representation $\sigma_{n_0}^{(0)}$ of $\bar{\mathrm{P}}^{(0)\,0}(n_0-1)$ which is fixed by no element of $\mathrm{P}^{(0)}(n_0-1)/\mathrm{P}^{(0)\,0}(n_0-1)$.

Definition. — By a $\mathrm{T}^{(0)}$-*cuspidal datum* $\Psi^{(0)}$ (relative to $\mathrm{P}^{(0)}$, $\{\mathrm{P}_n^{(0)}\}_{n \geq 1}$) of rank n_0 we mean a set of objects as just described satisfying the conditions above. We shall denote it by

$$\{(f_1^{(0)}, \ldots, f_{n_0}^{(0)}), (c_1^{(0)}, \ldots, c_{n_0}'^{(0)}), (c_1'^{(0)}, \ldots, c_{n_0}'^{(0)}), (\psi_1^{(0)}, \ldots, \psi_{n_0}^{(0)})\}$$
$$= \{f_i^{(0)}, c_i^{(0)}, c_i'^{(0)}, \psi_i^{(0)}\}_{1 \leq i \leq n_0}$$

or by

$$(\{f_i^{(0)}, c_i^{(0)}, c_i'^{(0)}, \psi_i^{(0)}\}_{1 \leqq i \leqq n_0 - 1}, \sigma_{n_0}^{(0)}),$$

as the case may be.

6.5. We are now ready to define a cuspidal datum for T. We need one more piece of notation: write $T = {}^{(r-1)}T \times T^{(0)}$, so that ${}^{(r-1)}T$ is a compact maximal torus in the group $U\left(\bigoplus_{i=1}^{r-1} f_i, \bigoplus_{i=1}^{r-1} E_i\right)$.

DEFINITION. — By a T-*cuspidal datum* relative to $P^{(0)}$, $\{P_n^{(0)}\}_{n \geqq 1})$ we mean the following.

(*a*) A $T^{(0)}$-cuspidal datum $\Psi^{(0)}$ (relative to $P^{(0)}$, $\{P_n^{(0)}\}_{n \geqq 1})$ of rank n_0.

(*b*) If $r > l+1$, *a* ${}^{(r-1)}T$-cuspidal datum relative to $(P^{(1)}, \{P_n^{(1)}\}_{n \geqq 1})$ such that if we define $f_1^{(1)} = i_1^{(1)} + i_1'^{(1)}$ as usual [*cf.* 6.4 (*b*) above], then

(i) $\ker(\Omega(\mathrm{tr}_0(c_1'^{(1)}))) \supseteqq P^{(1)}_{f_1^{(1)}} \supseteqq P^{(0)}_{f_1^{(0)}-1} \cap U(f_1 \oplus \ldots \oplus f_{r-1})$;

(ii) $\mathscr{A}^{(1)-}_{i_1'^{(1)}} \supseteqq \mathscr{A}^{(0)-}_{i_1^{(0)}} \cap \mathrm{End}_k(E_1 \oplus \ldots \oplus E_{r-1})$.

(*c*) If $r = l+1$, a cuspidal datum of rank n_{r-1} for the torus $T_u \subseteqq U(f_u, A_u)$ as in definition 5.3 such that

(i) if $f_1^u > 1$, then the appropriate versions of conditions (i) and (ii) as in (*b*) above hold [with (1) replaced by "*u*" throughout];

(ii) if $f_1^u = 1$, then $P_{u, f_1^u} \supseteqq P^{(0)}_{i_1^{(0)}} \cap U(f_u, A_u)$.

[We remind the reader that $P^{(1)}$ is the parahoric subgroup constructed from the lattice chain $\mathscr{L}_{r-1}$, while $f_1^{(1)} > f_2^{(1)} > \ldots > f_{n_1}^{(1)}$ is the sequence corresponding to the $T^{(1)}$-cuspidal datum implicit in the definition in part (*b*). Similarly P_u is the parahoric subgroup constructed from $\mathscr{L}_u \ldots$]

There is one more condition that we require of a T-cuspidal datum. To describe it, we remark that explicitly, a T-cuspidal datum consists of an $(r-l+1)$-tuple $(\Psi^{(0)}, \Psi^{(1)}, \ldots, \Psi^{(r-l)} = \Psi_u)$ where each $\Psi^{(t)}$ is a $T^{(t)}$-cuspidal datum of rank n_t relative to $P^{(t)}$, $\{P_n^{(t)}\}_{n \geqq 1}$ satisfying conditions (*b*) (*c*) above [namely: replace (0), (1) by (*t*), (*t*+1), respectively in condition (*b*), unless $t+1 = r-l$, in which case replace (0) in condition (*c*) by (*t*)]. Conditions 6.5 (*b*), (*c*) can be viewed as saying that a certain conductor has to be "shallower" than its predecessor.

We require the following condition as well

(*d*) For a given $t < r-l$, or $t = r-l$ and $n_u = n_{r-l} = n_t > 1$

$$P^{(t)}_{i_1^{(t)}} \subseteqq P_1^{(s)} \cap U(f_1 \oplus \ldots \oplus f_{r-t})$$

for all $s < t$.

Remarks:

(i) Condition (*d*) is not covered by proposition I 3.14; indeed, that proposition is false if one replaces "$\mathscr{A}$" by "$\mathscr{B}$" (or by "$\mathscr{B}_2$") throughout.

Example. — To see what condition 6.5 (*d*) says, consider the example 3.9 (*b*) (i) again. Then $r=2$, $l=1$, and we take $s=0$, $t=1$.

The definition of $\mathrm{P}_n^{(0)}$ implies that after the obvious identifications, $\mathrm{P}_1^{(0)} \cap \mathrm{U}(f_1)$ consists of 2×2 matrices of the form

$$\begin{pmatrix} 1+\mathscr{P} & \mathscr{P} \\ \mathscr{P} & 1+\mathscr{P} \end{pmatrix}$$

which lie in $\mathrm{Sp}_2 \simeq \mathrm{U}(f_1)$.

On the other hand $\mathrm{P}^{(1)} = \mathrm{Sp}_2(\mathcal{O})$, and the filtration is the standard one arising from the lattice chain

$$\ldots \supset \mathcal{O}\, e_2 \oplus \mathcal{O}\, e_3 \supset \mathscr{P} e_2 \oplus \mathscr{P} e_3 \supset \ldots$$

It follows that $\mathrm{P}_i^{(1)} \subseteqq \mathrm{P}_1^{(0)} \cap \mathrm{U}(f_1)$, provided that $i \geqq 1$.

6.6. Next, we define the open compact subgroup that will play an important role shortly. Indeed let

$$\underline{\Psi} = (\Psi^{(0)}, \ldots, \Psi^{(r-l)} (=\Psi_u)))$$

be a T-cuspidal datum.

We begin by defining a sequence of subgroups $\tilde{\mathrm{P}}_{\Psi(t)}$ each of which has the property that $\tilde{\mathrm{P}}_{\Psi(t+1)}$ normalizes $\tilde{\mathrm{P}}_{\Psi(t)}$. In fact, $\tilde{\mathrm{P}}_{\Psi(t)}$ will be a subgroup of $\mathrm{P}^{(t)}$. By proposition I 3.14,

$$\mathrm{P}^{(t+1)} \subseteqq \mathrm{P}^{(t)} \cap \mathrm{U}(f_1 \oplus \ldots \oplus f_{r-(t+1)}, E_1 \oplus \ldots \oplus \mathrm{E}_{r-(t+1)})$$

so that $\tilde{\mathrm{P}}_{\Psi(t+1)}$ will normalize $\mathrm{P}^{(t)}$, and more generally $\tilde{\mathrm{P}}_{\Psi(t+1)}$ will normalize $\mathrm{P}_n^{(t)}$. Our construction will proceed by pasting smaller subgroups onto $\mathrm{P}_n^{(t)}$ for a suitable n.

Indeed let $f_1^{(t)} > \ldots > f_{n_t}^{(t)}$ be the sequence of integers associated to $\Psi^{(t)}$, and for each j put $i_j^{(t)} = [f_j^{(t)}/2]$. (If $i_j'^{(t)} = [(f_j^{(t)}+1)/2]$, then $i_j^{(t)} + i_j'^{(t)} = f_j^{(t)}$.)

Since we are going to work with a fixed superscript $t > r-l$ for the moment, we shall drop it when possible, to alleviate the notation. Thus we shall write Ψ in place of $\Psi^{(t)}$, etc., and in particular we have the family $\{f_i, c_i, c_i', \psi_i\}_{1 \leqq i \leqq n_t}$ or $(\{f_i, c_i, c_i', \psi_i\}_{1 \leqq i \leqq n_t - 1}, \sigma_{n_t})$ according to whether $f_{n_t} > 1$ or $f_{n_t} = 1$, respectively.

We define $\mathrm{G}(0) = \mathrm{U}(f_{r-t}, \mathrm{E}_{r-t})$, $\mathrm{P}(0) = \mathrm{P}^{(t)}(0) = \mathrm{P}^{(t)} = \mathrm{P}$, and inductively define

$$\mathrm{G}(m) = \mathrm{Z}_{\mathrm{G}(m-1)}(c_m) \qquad [cf.\ 6.4\,(b)],$$
$$\mathrm{P}^{(t)}(\mathrm{m}) = \mathrm{P}(m) = \mathrm{P}(m-1) \cap \mathrm{Z}_{\mathrm{G}(m-1)}(c_m) \qquad \text{for} \quad m \geqq 1.$$

Now suppose for the time being that $f_{n_t} > 1$, so that

$$\mathrm{T}^{(t)} = \mathrm{G}(n_t)$$

The group $\tilde{\mathrm{P}}_{\Psi(t)}$ is constructed as follows. To start, we take $\mathrm{P}_{i_1} = \mathrm{P}_{i_1^{(t)}}^{(t)}$ which is normal in $\mathrm{P} = \mathrm{P}^{(t)}$, so it is normalized by $\mathrm{P}_{i_2}(1)$ in particular, and we can form the group

$P_{i_2}(1) P_{i_1}$. Suppose we have constructed

$$L_{\Psi_k^{(t)}} = L_{\Psi_k} = P_{i_k}(k-1) \ldots P_{i_2}(1) P_{i_1}$$

Then $P_{i_{k+1}}(k) \subseteqq P(k) = P(k-1) \cap Z_{G(k-1)}(c_k) \subset \ldots \subset P$ so normalizes L_{Ψ_k}, and we can form

$$L_{\Psi_{k+1}} = P_{i_{k+1}}(k) L_{\Psi_k} \qquad (\text{and let } L_{\Psi_1} = P_{i_1})$$

[We remind the reader that the congruence subgroups $P_n(k)$ are defined to be $P_n(k-1) \cap Z_{G(k-1)}(c_k)$, *cf.* I, Proposition 3.11.]

At the n_t-stage form

$$L_{\Psi_{n_t}} = P_{i_{n_t}}(n_t - 1) \ldots P_{i_2}(1) P_{i_1}.$$

and put $P_{\Psi(t)} = T_1^{(t)} L_{\Psi_{n_t}}$, $\tilde{P}_{\Psi(t)} = T^{(t)} P_{\Psi(t)}$.

At each stage we also have the groups

$$P_{\Psi_k} = T_1^{(t)} P_{i_{n_t}}(n_t - 1) \ldots P_{i_{k+1}}(k) \subseteqq G(k)$$

and

$$P_{\Psi_{n_t}} = T_1^{(t)}, \qquad P_{\Psi_0} = T_1^{(t)} \ldots P_{i_1}$$

so that

$$P_{\Psi(t)} = P_{\Psi_k} . L_{\Psi_k}, \qquad 1 \leqq k \leqq n_t$$

This last definition also makes sense for $k=0$ if we define $L_{\Psi_0} = \{1\}$.

On the other hand suppose that $f_{n_t} = 1$. Now *define* $i_{n_t} = 1$, and define $P_{\Psi(t)}$, P_{Ψ_k}, L_{Ψ_k}, just as before. The group $\tilde{P}_{\Psi(t)}$ is defined as

$$\tilde{P}_{\Psi(t)} = P(n_t - 1) P_{\Psi(t)} = P^{(t)}(n_t - 1) P_{\Psi(t)}$$

We must show that $\tilde{P}_{\Psi(t+1)}$ normalizes $\tilde{P}_{\Psi(t)}$. By construction, each of these groups is a subgroup of $P^{(t+1)}$, $P^{(t)}$ respectively. Now we can write

$$P_{\Psi(t+1)} = P^{(t+1)}_{\Psi_1^{(t+1)}} L_{\Psi_1^{(t+1)}} = P^{(t+1)}_{\Psi_1^{(t+1)}} . P^{(t+1)}_{i_1^{(t+1)}}$$
$$P_{\Psi(t)} = \ldots = P^{(t)}_{\Psi_1^{(t)}} . P^{(t)}_{i_1^{(t)}}$$

Observe that

$$P^{(t+1)}_{\Psi_1^{(t+1)}}, \quad P^{(t+1)}_{i_1(t+1)} \quad \text{lie in } P^{(t+1)}$$

which by definition is a subgroup of $U(f_1 \oplus \ldots \oplus f_{r-(t+1)}, E_1 \oplus \ldots \oplus E_{r-t-1})$. On the other hand $P^{(t)}_{\Psi_1^{(t)}} \subseteqq U(f_{r-t}, E_{r-t})$, by definition, so that

$$P_{\Psi(t+1)} \quad \text{normalizes } P^{(t)}_{\Psi_1^{(t)}}$$

But $P^{(t)}_{i^{(t)}_1}$ is a normal subgroup of $P^{(t)}$, and $P^{(t+1)} \subseteqq P^{(t)}$ whence $P_{\Psi(t+1)}$ normalizes $P^{(t)}_{i^{(t)}_1}$.

The argument for $\tilde{P}_{\Psi(t+1)}$, $\tilde{P}_{\Psi(t)}$ is the same (modulo the definitions). We put $P_{\underline{\Psi}} = \tilde{P}_{\Psi_u} \dots \tilde{P}_{\Psi(1)} \tilde{P}_{\Psi(0)}$.

6.7. We now proceed to define the representations of interest to us. Let us start with the groups $P_{\Psi(0)}$, $\tilde{P}_{\Psi(0)}$. We have

$$P_{\Psi(0)} = T^{(0)}_1 P^{(0)}_{i^{(0)}_{n_0}}(n_0-1) \dots P^{(0)}_{i^{(0)}_k}(k-1) \dots P^{(0)}_{i^{(0)}_2}(1) P^{(0)}_{i^{(0)}_1}.$$

while

$$\tilde{P}_{\Psi(0)} = \left\{ \begin{array}{c} T^{(0)} P_{\Psi(0)} \ (f_{n_0} > 1) \\ P^{(0)}(f_{n_0}-1) P_{\Psi(0)} \ (f_{n_0} = 1) \end{array} \right\}.$$

We are going to construct a representation $\rho_{\Psi(0)} = \rho^{(0)}_{n_0} \otimes \dots \otimes \rho^{(0)}_1$ on $\tilde{P}_{\Psi(0)}$, which can be extended to $P_{\underline{\Psi}}$. To do this we shall proceed in a manner analogous to [M], § 4. Suppose until further notice that $f_{n_0} > 1$.

6.8. Suppose that $f^{(0)}_k = 2\, i^{(0)}_k$ is even. This implies $k > 1$: *see* the *Remark* in the proof of lemma 6.10 below. We can define a one dimensional representation $\rho^{(0)}_k$ on $\tilde{P}_{\Psi(0)}$ in the following way.

Write

$$\tilde{P}_{\Psi(0)} = T^{(0)} P_{\Psi(0)} = T^{(0)} P_{\Psi^{(0)}_k} L_{\Psi^{(0)}_k}$$

The character $\psi^{(0)}_k$ restricts to a character on

$$P^{(0)}_{\Psi^{(0)}_k} \subset G^{(0)}(k)$$

On $P^{(0)}_{i^{(0)}_k}(k)$, $\psi^{(0)}_k$ is represented by $\Omega(\mathrm{tr}_0(c'^{(0)}_k))$ by definition. (Note that $i^{(0)}_k \geqq i^{(0)}_{k+1}$, so that $P^{(0)}_{i^{(0)}_k} \subseteqq P^{(0)}_{i^{(0)}_{k+1}}$.)

Since $i^{(0)}_1 \geqq i^{(0)}_2 \geqq \dots \geqq i^{(0)}_k$, we can extend $\psi^{(0)}_k$ to a character of $L_{\Psi^{(0)}_k}$ by the definition

$$x \mapsto \Omega(\mathrm{tr}_0(c'^{(0)}_k)(x-1)))$$

It follows that $\psi^{(0)}_k$ gives rise to a character on the group $\tilde{P}_{\Psi(0)}$ by the rules above, provided that $f^{(0)}_k$ is even. We take this character to be $\rho^{(0)}_k$ in this case. It remains to show that $\rho^{(0)}_k$ can be extended to the group $\tilde{P}_{\Psi} = \tilde{P}_{\Psi_u} \dots \tilde{P}_{\Psi(1)} \tilde{P}_{\Psi(0)}$. For this we must refer to some facts which were proved in I, Section 3.

Note that in any case, $\tilde{P}_{\Psi_u} \dots \tilde{P}_{\Psi(1)}$ is a subgroup of

$$U(f_1 \oplus \dots \oplus f_{r-1},\ E_1 \oplus \dots \oplus E_{r-1}),$$

so that

$$(\tilde{P}_{\Psi_u} \dots \tilde{P}_{\Psi(1)}) \cap \tilde{P}_{\Psi(0)} \subseteqq P_{i^{(0)}_1} \cap U(f_1 \oplus \dots \oplus f_{r-1},\ E_1 \oplus \dots \oplus E_{r-1})$$

On $P_{i_1^{(0)}}$, $p_k^{(0)}$ is defined via $\Omega(\mathrm{tr}_0(c_k'^{(0)}-))$. If we apply the block decomposition of I 3.12 we see that $\Omega(\mathrm{tr}_0(c_k'^{(0)}-))$ is trivial on

$$P_{i_1^{(0)}} \cap U(f_1 \oplus \ldots \oplus f_{r-1}, E_1 \oplus \ldots \oplus E_{r-1}).$$

In particular, we may extend $\rho_k^{(0)}$ by defining it to be trivial on $\tilde{P}_{\Psi^u} \ldots \tilde{P}_{\Psi^{(1)}}$.

6.9. Suppose now that $f_k^{(0)} = 2\, i_k^{(0)} + 1$ is odd, so that $i_k'^{(0)} = i_k^{(0)} + 1$. To construct $\rho_k^{(0)}$ we shall use the theory of the Heisenberg group. The construction itself is actually quite similar in spirit to that carried out in Section 5, and [M], Section 4, so we shall sketch it, and refer the reader to [M], Section 4 for the proofs, when appropriate. We shall concentrate on the necessary modifications, and the problem of extending $\rho_k^{(0)}$ to all of $P_{\underline{\Psi}}$. This will occupy sections 6.10-6.14 below.

6.10. First, define $P^*_{\Psi_{k-1}^{(0)}} = P_{\Psi_k^{(0)}} P_{i_k+1}(k-1) \subseteq P_{\Psi_{k-1}^{(0)}}$. (Note: we are writing P for $P^{(0)}$ throughout our discussion.) This is a subgroup of $P_{\Psi_{k-1}^{(0)}}$. As in 6.8 we can define a character on

$$T^{(0)} P_{\Psi_k^{(0)}}$$

and extend it to $P_{i_k+1}(k-1)$, via $\Omega(\mathrm{tr}_0(c_k'^{(0)}))$ again. By restriction, we obtain a character $\varphi_k^{(0)}$, on $P^*_{\Psi_{k-1}^{(0)}}$.

We now state and prove a lemma which has a number of useful corollaries.

LEMMA. — *The commutator*

$$(T^{(0)} P_{\Psi_{k-1}^{(0)}}, P^*_{\Psi_{k-1}^{(0)}}),$$

is contained in ker $\varphi_k^{(0)}$.

Proof. – We can write $T^{(0)} P_{\Psi_{k-1}^{(0)}} = T^{(0)} P_{\Psi_k^{(0)}} P_{i_k}^{(0)}(k-1)$. The first thing to note is that on $T^{(0)} P_{\Psi_k^{(0)}}$, $\varphi_k^{(0)}$ is the restriction of a linear character of

$$G^{(0)}(k) = G(k)$$

so it is trivial on commutators.

Secondly, if $k>1$, we can write

$$P_{\Psi_{k-1}^{(0)}} = P_{\Psi_k^{(0)}} P_{i_k^{(0)}}(k)\, E_k(i_k^{(0)}, f_k^{(0)}-1) = P_{\Psi_k^{(0)}} E_k(i_k^{(0)}, f_k^{(0)}-1)$$

(since $i_k^{(0)} \geqq i_{k+1}^{(0)}$) where we use "$E_k$" to denote that the "E" in question is for the group $P(k) \subseteq P(k-1)$. This assertion is essentially [M], 2.32.

On the other hand, if $k=1$, then 4.15 tells us that

$$P_{\Psi_0^{(0)}} = P_{\Psi_1^{(0)}} (P_{i_1^{(0)}} \cap (f_1 \oplus \ldots \oplus f_{r-1}))\, P_{i_1^{(0)}}(1)\, E_1(i_1^{(0)}, f_1^{(0)}-1).$$

Suppose that $k>1$. Then

$$P_{\Psi_k^{(0)}} \subseteq P^*_{\Psi_{k-1}^{(0)}} \cap G^{(0)}(k)$$

Thus if $p' \in P_{\Psi_k^{(0)}}$, $yp \in P_{\Psi_k^{(0)}} P_{i_k^{(0)}+1}(k-1)$ we have

$$\varphi_k^{(0)}(p'(yp)p'^{-1}) = \varphi_k^{(0)}(p' y p'^{-1}) \Omega(\mathrm{tr}_0(c_k'^{(0)} p'(p-1)p'^{-1}))$$

$$= \varphi_k^{(0)}(y) \Omega(\mathrm{tr}_0(c_k'^{(0)} p'(p-1)p'^{-1})),$$

by the first remark above

$$= \varphi_k^{(0)}(y) \Omega(\mathrm{tr}_0(c_k'^{(0)}(p-1))),$$

since p', $c_k'^{(0)}$ commute with each other,

$$= \varphi_k^{(0)}(yp)$$

If $k=1$, a similar result holds with respect to

$$p' \in P_{\Psi_1^{(0)}}^{(0)}(P_{i_1^{(0)}} \cap U(f_1 \oplus \ldots \oplus f_{r-1})) P_{i_1^{(0)}}(1).$$

To complete the proof we must show that the commutator of an element of $E_k(i_k^{(0)}, f_k^{(0)}-1)$ with an element of $P_{\Psi_k^{(0)}}$ lies in the kernel of $\varphi_k^{(0)}$.

We shall compute this when $k=1$; the case $k>1$ is similar but easier, and in fact has been done in principle in [M], Lemma 4.5; *i.e.* it is a calculation within a "block" and is covered by *loc. cit.*

Let $1+a+b \in E_1(i_1^{(0)}, f_1^{(0)}-1)$, $1+y \in P_{\Psi_1^{(0)}}$ where $a \in \mathscr{B}^{-\perp}_{i_1^{(0)}, c_1^{(0)}}$, $b \in \mathscr{B}^{-}_{f_1^{(0)}-1}$ (using the notation of 4.14).

Moreover, assumption 6.4(*a*), and the construction of $P_{\Psi_1^{(0)}}$, guarantee that $y \in \mathscr{B}_{c_1^{(0)}, 1}$, at least [we do not need 6.5(*d*) here, since we are climbing up within a "block"].

[*Remark.* — By the ramified construction (*cf.* 4.21), $f_1^{(t)}$ is odd, always. This implies first that whenever we cross a block, a Heisenberg construction will ensue. Secondly $f_1^{(0)}-1$ is even, in particular, so that $E_1(i_1^{(0)}, f_1^{(0)}-1)$ is defined (by the conditions 4.14.1).]

Now we compute the commutator

$$(1+a+b,\ 1+y).$$

To do this, set $a_1=a+b$, $(1+a_1)^{-1}=(1+a')$, and $(1+y)^{-1}=1+y'$.

Then

$$a_1+a'+a_1 a' = a_1+a'+a' a_1 = 0,$$

and

$$y+y'+yy' = y+y'+y' y = 0$$

and we find

$$\begin{aligned}(1+a_1)(1+y)(1+a')(1+y') &= 1+ya'+ya'y'+a'y'+a_1a'y'+a_1ya'+a_1ya'y'\\ &= 1+y(-a-b)+y((a+b)^2-(a+b)^3+\ldots)+ya'y'\\ &\quad +(-(a+b)(-y+y^3-\ldots)+(a+b)^2(-y+y^3\ldots)+\ldots)\\ &\quad +a_1a'y'+a_1ya'+a_1ya'y'\end{aligned}$$

We now see that the assumptions on y, $a_1=a+b$ imply that this commutator lies in $P^{(0)}_{i_1^{(0)}+1}$ so that we can apply $\Omega(\operatorname{tr}(c_1'^{(0)}))$ to compute it (observe also that $P^{(0)}_{i_1^{(0)}+1}\subseteq P^*_{\Psi_0^{(0)}}$). Moreover note that these same assumptions imply that $a_1ya'y'$, $a_1a'y'$, a, $ya'\in\mathscr{A}_{f_1^{(0)}}$ so that these terms contribute nothing when we compute the character. This leaves us to compute

$$\Omega(\operatorname{tr}_0(c_1'^{(0)}(ya'+a'y'+ya'y')))$$

In computing this we only need to compute those terms arising from the contribution of $-(a+b)$, $-y$ in the expansions for a', y' respectively (for the same reason as above). Thus we must compute

$$\Omega(\operatorname{tr}_0(c_1'^{(0)}(-y(a+b)+(a+b)y'+y(a+b)y)))$$

Since $b\in\mathscr{B}^-_{f_1^{(0)}-1}$, $y\in\mathscr{B}_1\cap\mathscr{C}[c]^-$ (at least), we may ignore the terms involving b, so we are left with

$$\Omega(\operatorname{tr}_0(c_1'^{(0)}(ay-ya+yay)))=\Omega(\operatorname{tr}_0(\{yc_1'^{(0)}-c_1'^{(0)}y+yc_1'^{(0)}y\}a))$$

The term in braces belongs to $\mathscr{C}[c]$, while $a\in\mathscr{C}[c]^\perp$; thus $\operatorname{tr}_0(\{\ldots\}a)=0$, and the character is trivial.

6.11. COROLLARY:

(*a*) $\ker\varphi_k^{(0)}$ *is a normal subgroup of* $T^{(0)}P_{\Psi_{k-1}^{(0)}}$.

(*b*) $P^*_{\Psi_{k-1}^{(0)}}$, $P_{\Psi_{k-1}^{(0)}}$, *are normal subgroups of* $T^{(0)}P_{\Psi_{k-1}^{(0)}}$.

(*c*) $P^*_{\Psi_{k-1}^{(0)}}/\ker\varphi_k^{(0)}$ *is central in* $P_{\Psi_{k-1}^{(0)}}/\ker\varphi_k^{(0)}$.

Proof. — Part (*a*), and the first assertion in (*b*) follow immediately from Lemma 6.10. The second assertion is proved by a calculation similar to the first part of the proof of Lemma 6.10 [*i.e.*, the part not involving $E_k(i_k^{(0)}, f_k^{(0)}-1)$: one only has to show that $T^{(0)}$ normalizes $P_{\Psi_{k-1}^{(0)}}$ and $T^{(0)}$ commutes with $c_1'^{(0)}$]. Part (*c*) is trivial, given part (*a*).

6.12. We may now imitate [M], 4.8 to form $\mathscr{H}'_k$ which by definition is $P_{\Psi_{k-1}}/\ker\varphi_k^{(0)}$, and $\mathscr{Z}'_k=P^*_{\Psi_{k-1}}/\ker\varphi_k^{(0)}$.

Let $\mathscr{H}_k$ denote the image of $E_k(i_k^{(0)}, f_k^{(0)}-1)$ in $\mathscr{H}'_k$, and let $\mathscr{Z}_k=\mathscr{Z}'_k\cap\mathscr{H}_k$. An argument similar to that in [M], 4.8-4.9 then implies the following result.

LEMMA. — $\mathscr{H}_k$ *is a Heisenberg group: every element has order p, and* $\mathscr{Z}_k=(\mathscr{H}_k, \mathscr{H}_k)$. *Furthermore* $T^{(0)}P_{\Psi_{k-1}^{(0)}}$ *acts on* $\mathscr{H}_k$ *via conjugation, and this action factors* *through the action of the symplectic group (defined over* $\mathbf{F}_p$*) that is associated to* $\mathcal{H}_k$*. Furthermore* $P_{\Psi_k^{(0)}}$ *acts trivially by conjugation.*

We remind the reader that the form on $\mathcal{H}_k/\mathcal{Z}_k$ is inherited from the map

$$P_{i_k^{(0)}}(k-1)\times P_{i_k^{(0)}}(k-1)\to P_{f_k^{(0)}-1}(k-1)\xrightarrow{\varphi_k^{(0)}} S^1$$

where the first map is the commutator map.

6.13. From the theory of the Weil representation, there is a unique irreducible representation $\delta_k^{(0)}$ of $\mathcal{H}_k$ which is parametrized by $\varphi_k^{(0)}$ on $\mathcal{Z}_k$, and $\delta_k^{(0)}$ extends in a unique way to an irreducible representation $\tilde{\delta}_k^{(0)}$ of $T^{(0)}P_{\Psi_{k-1}^{(0)}}$.

To complete the construction of $\rho_k^{(0)}$ on $\tilde{P}_{\Psi(0)}$ we extend it, via

$$x\to\Omega(\operatorname{tr}_0(c_k'^{(0)}(x-1))),$$

to $L_{\Psi_{k-1}^{(0)}}$ just as before. (Note that $f_{k-1}>f_k$, so $i_{k-1}\geqq i_k+1$.)

6.14. We want to extend $\rho_k^{(0)}$ ($f_k^{(0)}$ odd) to all of $P_{\underline{\Psi}}=\tilde{P}_{\Psi_u}\ldots\tilde{P}_{\Psi(1)}\tilde{P}_{\Psi(0)}$. Again, we have

$$(\tilde{P}_{\Psi_u}\ldots\tilde{P}_{\Psi(1)})\cap\tilde{P}_{\Psi(0)}\subseteqq P_{i_1^{(0)}}\cap U(f_1\oplus\ldots\oplus f_{r-1},\, E_1\oplus\ldots\oplus E_{r-1}).$$

Suppose that $k=1$. The Heisenberg construction above was constructed on a quotient of $E_1(i_1^{(0)}, f_1^{(0)}-1)$, and the intersection of this with $P\cap Z_G(c_1^{(0)})$ is $P_{f_1^{(0)}-1})\cap Z_G(c_1^{(0)})$ by 4.15 (*b*); the representation $\rho_k^{(0)}$ is given by the character $\Omega(\operatorname{tr}_0(c_1^{(0)}--))$ on this group. But this character is trivial on

$$(P_{f_1^{(0)}-1}\cap Z_G(c_1^{(0)}))\cap U(f_1\oplus\ldots\oplus f_{r-1},\, E,\oplus\ldots\oplus E_{r-1})$$

since $c_1^{(0)}\in \operatorname{Lie} U(f_r, E_r)$.
It follows that $\rho_k^{(0)}$ can be extended to $\tilde{P}_{\underline{\Psi}}$ in a trivial manner.

6.15. The construction of $\rho_k^{(t)}$ is similar to that of $\rho_k^{(0)}$, when $f_{n_t}^{(t)}>1$. The new problem that arises is to show that the representation $\rho_k^{(t)}$ extends on $\tilde{P}_{\Psi(t-1)}\ldots\tilde{P}_{\Psi(0)}$. For this we make use of definition 6.5 (*b*).

We know that

$$(\tilde{P}_{\Psi_u}\ldots\tilde{P}_{\Psi^{(t)}})\cap\tilde{P}_{\Psi^{(t-1)}}\subseteqq P^{(t-1)}_{i_1^{(t-1)}}\cap U(f_1\oplus\ldots\oplus f_{r-t},\, E_1\oplus\ldots\oplus E_{r-t})$$

By assumption, $\ker(\Omega(\operatorname{tr}_0(c_1'^{(t)})))\supseteqq P^{(t-1)}_{f_1^{(t-1)}-1}\cap U(f_1\oplus\ldots\oplus f_{r-t})$ [assumption 6.5 (*b*) (i)].

The ramified filtration on $P^{(t-1)}$ implies that $P^{(t-1)}_{i_1^{(t-1)}}/P^{(t-1)}_{f_1^{(t-1)}-1}$ is abelian. Now, on $P^{(t)}_{i_1^{(t)}}$, $\tilde{\rho}_1^{(t)}$ is defined by the character $\Omega(\operatorname{tr}_0(c_1'^{(t)}))$. By assumption 6.5 (*b*) (ii) and Lemma I 3.12, and the duality lemma 4.19 and theorem I 2.13 (iv), it follows that we can extend $\tilde{\rho}_1^{(t)}$ to $\tilde{P}_{\Psi(t-1)}$ via the character $\Omega(\operatorname{tr}_0(c_1'^{(t)}))$.

The extension to the groups $P_{\Psi^{(t-2)}} \ldots P_{\Psi(0)}$ now follows, because for example we have

$$(\tilde{P}_{\Psi_u} \ldots \tilde{P}_{\Psi^{(t-1)}}) \cap \tilde{P}_{\Psi^{(t-2)}} \subseteqq P_{i_1^{(t-2)}} \cap U(f_1 \oplus \ldots \oplus f_{r-(t+1)})$$

and the right hand group lies in $P_{i_1'^{(t-1)}} \subset P_{i_1^{(t-1)}}$ since $i_1^{(t-1)} \leqq i_1'^{(t-1)}$ always (since $f_1^{(t-1)}$ is odd, this inequality is strict), *cf.* 6.5 (*b*) (ii).

The extension $\tilde{\rho}_k^{(t)}$, $k>1$, is similar. Indeed by assumption

$$P^{(t)}_{f_k^{(t)}} \supseteqq P^{(t)}_{f_1^{(t)}}, \qquad \text{and} \qquad P^{(t)}_{i_k^{(t)}} \supseteqq P^{(t)}_{i_1^{(t)}} \supseteqq P^{(t)}_{i_1'^{(t)}}$$

so one can extend by the character $\Omega(\mathrm{tr}_0(c_k'^{(t)}))$, as usual.

6.16. So far we have assumed that $f_{n_t}^{(t)}>1$.

We now indicate how to modify the construction in case $f_{n_t}^{(t)}$ is equal to 1, when $t>r-l$.

First, we use $\tilde{P}_{\Psi}$ as constructed in 6.6: that is

$$P_{\underline{\Psi}} = \tilde{P}_{\Psi_u} \ldots \tilde{P}_{\Psi^{(0)}}$$

where

$$\tilde{P}_{\Psi^{(t)}} = P^{(t)}(n_t - 1) P_{\Psi^{(t)}}$$

To construct the representation $\rho_{n_t}^{(t)}$ we take the cuspidal representation $\sigma_{n_t}^{(t)}$ of $P^{(t)0}(n_t-1)$ which by definition is trivial on $P_1^{(t)}(n_t-1)$. The assumptions on the action of $P^{(t)}(n_t-1)/P^{(t)0}(n_t-1)$ on $\sigma_{n_t}^{(t)}$ ensure that this representation induces irreducibly to $P^{(1)}(n_t-1)/P_1^{(t)}(n_t-1)$ to that by inflation we obtain a representation of $P^{(t)}(n_t-1)$. This representation is trivial on $P_1^{(t)}(n_t-1)$ hence extends trivially to $P_{\Psi^{(t)}}$ by defining it to be trivial on $L_{\Psi^{(t)}_{n_t-1}}$. The extension to $P_{\Psi(t-1)} \ldots P_{\Psi(0)}$ is clear since the representation is trivial on anything below level 1.

The rest of the construction is similar to what has gone before. If $f_k^{(t)}$ is even, it is the same. If $f_k^{(t)}$ is odd one proceeds as before, but now one uses the fact that $P^{(t)*}_{\Psi^{(t)}_{k-1}}$, $P^{(t)}_{\Psi^{(t)}_{k-1}}$ are normal subgroups of $P^{(t)}(n_t-1)P^{(t)}_{\Psi^{(t)}_{k-1}}$, via an easy variant of Lemma 6.10.

This gives rise to a Heisenberg group $\mathscr{H}_k$ on which $P^{(t)}(n_t-1)P^{(t)}_{\Psi^{(t)}_{k-1}}$ acts as a group of automorphisms via conjugation. Under this action, $P_1^{(t)}(n_t-1)$ acts trivially, and $P^{(t)}(n_t-1)$ acts through the symplectic group since it stabilizes the character $\varphi_k^{(t)}$.

6.17. Suppose that $t=r-l$. If $n_u>1$ we simply take the construction given in Section 5 on P_{Ψ_u}, and extend the resulting representation to $P_{\underline{\Psi}}$ in the same way as 6.16 above.

On the other hand, if $n_u=1$ and $f_1^u=1$, we construct ρ_1^u just as in 6.16: it is trivial on $P_{u,1}$. By assumption 6.5 (*c*) (ii), we can extend this representation to $P_{\Psi^{(r-l-1)}} = P_{\Psi^{(t-1)}}$, and using 6.5 (*b*) (ii) as we already have earlier, we can extend ρ_1^u to all of P_{Ψ} : $\rho_1^u = \rho_{\Psi_u}$ in this case.

6.18. We have defined finite dimensional irreducible representations $\rho_{\Psi^{(i)}}$, ρ_{Ψ_u}, on $P_{\underline{\Psi}}$. We can take their tensor product $\rho_{\Psi_u}\otimes\ldots\otimes\rho_{\Psi^{(1)}}\otimes\rho_{\Psi^{(0)}}$ as well.

DEFINITION. — Given a T-cuspidal datum $\underline{\Psi}=(\Psi^{(0)}, \ldots \Psi^{(r-l)}=\Psi_u)$ we define the representation $\rho_{\underline{\Psi}}$ on $P_{\underline{\Psi}}$ by

$$\rho_{\underline{\Psi}}=\rho_{\Psi_u}\otimes\ldots\otimes\rho_{\Psi^{(1)}}\otimes\rho_{\Psi^{(0)}}.$$

6.19. The reader may wonder if the apparent asymmetry in our construction with respect to (non) ramification is necessary, The answer is that in the framework we have established, it is; the reason lies in I 2.14-2.15. Namely if we attempt to form $\mathscr{L}\oplus'\mathscr{M}$ then $\mathscr{L}\oplus\mathscr{M}$ in case $\mathscr{M}$ has period 1, we must take $\mathscr{L}\oplus'\mathscr{M}=\mathscr{L}\oplus\mathscr{M}$ and the resulting filtration must be the one arising from powers of the Jacobson radical. This filtration only has the kind of properties we want for our construction with respect to a compact maximal torus T, when T is unramified (*i.e.*, the extensions E_i are unramified over k). Otherwise one finds that if one attempts to use this filtration to look at an unramified part of a compact maximal torus first, it is quite unnatural. One can see this by looking at examples in $Sp_6(A=E_1\oplus E_2$ where E_1 is quartic ramified over k, E_2 is quadratic unramified over k).

We remark also that in the calculations in [Mo2] (reference in I) for GSp_4 one does not need to look at $E_1\oplus E_2$ (E_1 ramified, E_2 unramified) by examining E_2 first. Indeed the supercuspidal representations that arise from the associated tori can always be detected by means of a character on E_1 first.

7. The main theorem

7.1. In the preceding two sections, given a T-cuspidal datum $\underline{\Psi}$, we showed how to construct a finite dimensional representation $\rho_{\underline{\Psi}}$ of a certain open compact subgroup $P_{\underline{\Psi}}$. In this section we show if $\rho_{\underline{\Psi}}$ is suitably induced up to G, the result is an irreducible supercuspidal representation of G.

7.2. We begin by recalling some facts about representations of locally compact totally disconnected unimodular groups. If H is closed, unimodular, and ρ: $H\to V$ is a smooth representation of H, then we can define

$$c-\mathrm{Ind}_{H}^{G}(\rho)$$

to be the representation of G whose space is the space of functions

$$f:\quad G\to V$$

such that

(i) $f(hg)=\rho(h)f(g)$, all $h\in H$, $g\in G$.

(ii) there exists a compact open subgroup K_f such that $f(kg)=f(g)$, all $k\in K_f$, $g\in G$;

(iii) f has compact support modulo H.

We note that if H is open compact, then condition (ii) is implied by condition (i), while if G/H is compact, condition (iii) is satisfied.

In any case, the action of G on this space is via *right* translations, and the resulting representation is smooth.

Suppose that H is open, and compact mod centre. It is known (*see* [B1], theorem 1) that if $c-\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(\rho)$ is admissible, then it is a finite sum of irreducible supercuspidal representations of G. It is also known that if $c-\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(\rho)$ is irreducible, then it is admissible, and hence supercuspidal, since it plainly has compactly supported matrix coefficients. (For a proof of this, *see* [C], Section 1.)

7.3. We are going to show that $c-\mathrm{Ind}_{\mathrm{P}_{\underline{\Psi}}}^{\mathrm{G}}\,\rho_{\underline{\Psi}})$ is irreducible. For this it is enough to show that the only intertwining operators are scalars. (For a proof of this, see *loc. cit.*). The preceding remarks then imply *a fortiori* that it is supercuspidal.

Recall (*loc. cit.*, or [M1], 4.2) that

$$\mathrm{End}_{\mathrm{G}}(c-\mathrm{Ind}_{\mathrm{P}_{\underline{\Psi}}}^{\mathrm{G}}(\rho_{\underline{\Psi}}))\cong\bigoplus_{\mathrm{P}_{\underline{\Psi}}\diagdown \mathrm{G}/\mathrm{P}_{\underline{\Psi}}}\mathrm{Hom}_{\mathrm{P}_{\underline{\Psi}}\cap{}^{g}\mathrm{P}_{\underline{\Psi}}}(\rho_{\underline{\Psi}},{}^{g}\rho_{\underline{\Psi}})$$

Since $\underline{\Psi}$ is fixed throughout the discussion that follows we shall omit it as a subscript, to alleviate notation.

Suppose that S intertwines ρ, ${}^{g}\rho$ on $\mathrm{P}\cap{}^{g}\mathrm{P}$. Observe that

$$\rho=\rho_{\Psi_u}\otimes\ldots\otimes\rho_{\Psi(1)}\otimes\rho_{n_0}^{(0)}\otimes\ldots\otimes\rho_1^{(0)}.$$

Thus S intertwines ρ, ${}^{g}\rho$ on $\mathrm{P}_{i_1'(0)}^{(0)}\cap{}^{g}\mathrm{P}_{i_1'(0)}^{(0)}$ as well. The construction of Section 6 implies that $\rho_1'=\rho_1^{(0)}\,|\,\mathrm{P}_{i_1'(0)}^{(0)}$ is always a character, and moreover that on $\mathrm{P}_{i_1'(0)}^{(0)}\cap{}^{g}\mathrm{P}_{i_1'(0)}^{(0)}$ the representations ρ_1', ${}^{g}\rho_1'$ are multiples of $\Omega(\mathrm{tr}_0(c_1'^{(0)})).\chi$, ${}^{g}\Omega(\mathrm{tr}_0(c_1'^{(0)})).{}^{g}\chi$ respectively, where χ is a character represented by an element of $\mathscr{C}[c_1^{(0)}]^{-}$. Applying Lemma 4.23 we see that

$$g\in\mathrm{P}_{i_1^{(0)}}^{(0)}\,\mathrm{Z}_{\mathrm{G}}(c_1^{(0)})\,\mathrm{P}_{i_1^{(0)}}^{(0)}.$$

7.4 Assume for the time being that $f_1^u>1$. Consider the subgroup $\mathrm{P}'=\mathrm{P}_{u,\,i_1'(u)}.\mathrm{P}_{i_1'(l-r-1)}^{(r-l-1)}\ldots\mathrm{P}_{i_1'(1)}^{(1)}.\mathrm{P}_{i_1'(0)}^{(0)}$. (This is a subgroup, by the same sort of argument that was used in Section 6.6.) We can consider $\rho'=\rho_1^{(u)}\otimes\ldots\otimes\rho_1^{(1)}\otimes\rho_1^{(0)}$ restricted to this group; we note that ρ' is a character.

LEMMA. — $\mathrm{P}_{i_1^{(0)}}^{(0)}$ *stabilizes* ρ'.

Proof. — We show that the commutator of an element of P′ with an element k of $\mathrm{P}_{i_1^{(0)}}^{(0)}$, lies in the kernel of ρ'.

To start, consider an element y of $\mathrm{P}_{i_1'(0)}^{(0)}$. Then $(k, y)\in\mathrm{P}_{f_1^{(0)}}^{(0)}$. By construction, $\mathrm{P}_{f_1^{(0)}}^{(0)}$ is contained in the kernel of ρ'.

Next, consider an element of $P^{(1)}_{i'_1(1)}$, which we denote by $1+y$, where $y \in \mathscr{A}^{(1)-}_{i'_1}$. We can write, by 4.15,

$$P^{(0)}_{i^{(0)}_1} = (P^{(0)}_{i^{(0)}_1} \cap U(f_1 \oplus \ldots \oplus f_{r-1}))\, P^{(0)}_{i^{(0)}_1}(1)\, E_1(i^{(0)}_1, f^{(0)}_1 - 1)$$

since $Z_G(c^{(0)}_1) = U(f_1 \oplus \ldots \oplus f_{r-1}) \times Z_{U(f_r, E_r)}(c^{(0)}_1)$.

Now $P^{(0)}_{i^{(0)}_1}(1)$ acts trivially on $P^{(1)}_{i'_1(1)} \subset U(f_1 \otimes \ldots \oplus f_{r-1})$ and the left most group in the decomposition above is contained in $P^{(1)}_{i'_1(1)}$, by 6.5 (*b*).

Then regardless of whether $i'^{(1)}_1$ is even or odd, we have $(P^{(1)}_{i'_1(1)}, P^{(1)}_{i'_1(1)}) \subset P^{(1)}_{f^{(1)}_1}$ which is contained in the kernel of $\rho^{(u)}_1 \otimes \ldots \otimes \rho^{(1)}_1$. On the other hand $\rho^{(0)}_1$ acts on $P_{i'_1(1)}$ via a linear character, so it is trivial on commutators.

We are left with showing that the commutator of an element of $E_1(i^{(0)}_1, f^{(0)}_1 - 1)$ with $(1+y)$ lies in the kernel of ρ'. Let $1+a+b \in E_1(i^{(0)}_1, f^{(0)}_1 - 1)$ where $a \in \mathscr{C}[c^{(0)}_1]^{\perp} \cap \mathscr{B}^{-}_{i^{(0)}_1}$, $b \in \mathscr{B}^{-}_{f^{(0)}_1 - 1}$.

We may write $b = b_{11} + b_{12} + b_{21} + b_{22}$ (*cf.* I 3.12) where $b_{ij} \in \mathscr{B}^{-}_{f^{(0)}_1 - 1}(i, j)$ (notation of *loc. cit.*). In particular, b_{12}, $b_{21} \in \mathscr{C}[c^{(0)}_1]^{\perp}$, while $b_{11}\, y = y b_{11} = 0$.

Set $a_1 = a + b$, and let $(1+a')$, $(1+y')$ denote the inverses of $1+a_1$, $1+y$ respectively. We find

$$\begin{aligned}(1+a_1,\ 1+y) &= (1+a_1)(1+y)(1+a')(1+y')\\ &= (1+a_1+y+a_1 y)(1+a'+y+a' y')\\ &= 1 + ya' + ya' y' + a' y' + a_1 a' y' + a_1 ya' + a_1 ya' y'\end{aligned}$$

since $a_1 + a' + a_1 a' = 0$, $y + y' + yy' = 0$.

Before going any further we pause to take note of the fact that by assumption 6.5 (*d*), $P^{(1)}_{i'_1(1)} \subseteqq P^{(0)}_1 \cap U(f_1 \oplus \ldots \oplus f_{r-1})$. In general the same result holds if we replace (1) and (0) by (*s*), (*t*) respectively where $s > t$, and similarly if we replace P by $\mathscr{A}^{-}$ throughout.

Applying these remarks to the calculation above, we find that

$$(1+a')(1+y) \in P^{(0)}_{i'_1(0)},$$

at least which means that in terms of $\rho^{(0)}_1$ we can compute its kernel via $\Omega(\mathrm{tr}_0(c'^{(0)}_1))$. The same is true for the $\rho^{(t)}_1$, $t > 0$, because on $P^{(1)}_{i'_1(1)} P^{(0)}_{i'_1(0)}$ they are defined in terms of representatives which lie in $\mathscr{C}[c^{(0)}_1]^{-}$ as well.

Next, observe that b_{12}, b_{21} (in the decomposition above) lie in $\mathscr{C}[c^{(0)}_1]^{\perp}$, and also that $b_{11}\, y^n = y^n b_{11} = 0$, for any integer $n \geqq 1$.

We have $a_1 yy' \in \mathscr{A}^{(0)}_{f^{(0)}_1}$, by the assumption remarked upon above, and similarly with $a_1 ya'$. This leaves us with $1 + ya' + ya' y' + a_1 y' + a_1 yy'$. Now

$$a' = -(a+b) + (a+b)^2 - (a+b)^3 + \ldots$$

$$a_1 = a + b = a + b_{12} + b_{21} + b_{11} + b_{22} = a_2 + b_{11} + b_{22}.$$

By the remark above concerning b_{11}, we can drop the term in b_{11}, so that we can replace a' by $-(a_2+b_{22})+(a_2+b_{22})^2\ldots$ and a_1 by a_2+b_{22}. Then we find

$$1+ya'+ya'y'+a_2y'+a_1ya'$$

is equal to

$$\begin{aligned}1+(-ya_2-yb_{22}+y(a_2+b_{22})^2)\ldots)+y(-a_2-b_{22}+(a_2+b_{22})^2\ldots)y'\\ +(a_2+b_{22})(y')+(a_2+b_{22})yy'\\ =1-ya_2+a_2y'-ya_2y'+a_2yy'-yb_{22}-yb_{22}y'+b_{22}y'+b_{22}y'y \bmod \mathscr{A}^{(0)}_{f_1^{(0)}}\end{aligned}$$

Subtracting 1, and applying $\operatorname{tr}_0(c_1'^{(0)})$ we see that all terms involving a_2 will vanish, since $a_2\in\mathscr{C}[c_1^{(0)}]^\perp$ and $c_1'^{(0)}$, y, $y'\in\mathscr{C}[c_1^{(0)}]$. On the other hand, all terms involving b_{22} will lie in $\mathscr{A}^{(0)}_{f_1^{(0)}}$ since

$$y\in\mathscr{A}^{(1)-}_{i_1'(1)}\subseteqq\mathscr{A}_1^{(0)-}\qquad[\text{by } (6.5(d)]$$

Thus we have shown that the said commutator lies in $\ker\rho_1^{(0)}$. It also lies in $\ker\rho_1^{(t)}$ for $t\geqq 1$ since (i) by assumption 6.5 (*b*) $\mathrm{P}^{(0)}_{f_1^{(0)}}\subseteqq\ker\rho_1^{(t)}$ (and the construction in Section 6), and (ii) on $\mathrm{P}^{(1)}_{i_1'(1)}\mathrm{P}^{(0)}_{i_1'(0)}$, $\rho_1^{(t)}$ is defined by a representative from $\mathscr{C}[c_1^{(0)}]^-$, so that exactly the same reasoning as above applies to $\rho_1^{(t)}$ as well.

Passing to $\mathrm{P}^{(2)}_{i_1'(2)}$, we again write

$$\mathrm{P}^{(0)}_{i_1^{(0)}}=(\mathrm{P}^{(0)}_{i_1^{(0)}}\cap\mathrm{U}(f_1\oplus\ldots\oplus f_{r-1}))\,\mathrm{P}^{(0)}_{i_1^{(0)}}(1)\,\mathrm{E}_1(i_1^{(0)},f_1^{(0)}-1)$$

As before, we can ignore $\mathrm{P}^{(0)}_{i_1^{(0)}}(1)$. The same calculation as before works for $\mathrm{E}_1(i_1^{(0)},f_1^{(0)}-1)$. As for the left most group in the decomposition above, by assumption 6.5 (*b*) it is contained in $\mathrm{P}^{(1)}_{i_1'(1)}$, so we may compute as above, replacing $\mathrm{P}^{(0)}_{i_1^{(0)}}$, by $\mathrm{P}^{(1)}_{i_1'(1)}$. The result follows in this case. The general case is now clear, and we leave it to the reader.

7.5. Recall the situation in 7.3: we have:

$$g\in\mathrm{P}^{(0)}_{i_1^{(0)}}\mathrm{Z}_{\mathrm{G}}(c_1^{(0)})\,\mathrm{P}^{(0)}_{i_1^{(0)}}$$

We also know that $\mathrm{P}^{(0)}_{i_1^{(0)}}$ stabilizes ρ', that is $\mathrm{P}^{(0)}_{i_1^{(0)}}$ normalizes $\mathrm{P}'/\ker\rho'$ and stabilizes ρ' under this action. If we write $g=p_1zp_2$ where $z\in\mathrm{Z}_{\mathrm{G}}(c_1^{(0)})$, $p_i\in\mathrm{P}^{(0)}_{i_1^{(0)}}(i=1, 2)$, this implies that z also interwines ρ'.

We can write

$$z\in\mathrm{U}(f_1\oplus\ldots\oplus f_{r-1})\times\mathrm{Z}_{\mathrm{U}(f_r,\mathrm{E}_r)}(c_1^{(0)})=\mathrm{Z}_{\mathrm{G}}(c_1^{(0)})$$

Consider the representation ρ'' within the block $\mathrm{U}(f_r,\mathrm{E}_r)$ given by $\rho^{(0)}_{n_0}\otimes\ldots\otimes\rho_2^{(0)}$ restricted to the group $\mathrm{P}_{i'_{n_0}(0)}(n_0-1)\ldots\mathrm{P}_{i'_2(0)}(1)$ (where again we assume $f_{n_0}>1$).

We always have $i_1^{(0)}\geqq i_2^{(0)}\geqq\ldots\geqq i_{n_0}^{(0)}$ so that the commutator of an element of $\mathrm{P}^{(0)}_{i_1^{(0)}}$ with an element of $\mathrm{P}^{(0)}_{i_1'(0)}(l-1)$ for $(l\geqq 2)$ always lies in $\mathrm{P}^{(0)}_{f_l^{(0)}}$. By construction (6.8, 6.13),

$P^{(0)}_{f_l^{(0)}}$ lies in the kernel of $\rho_l^{(0)}$ restricted to $P^{(0)}_{i'_{n_0}(0)}(n_0-1)\dots P^{(0)}_{i'_l(0)}(l-1)$, while on $P^{(0)}_{i'_{l-1}(0)}(l-2)\dots P^{(0)}_{i'_2(0)}(1)$, $\rho_l^{(0)}$ is the composition with a determinant which is trivial on commutators. It follows that $P^{(0)}_{i'_1(0)}$ stabilizes ρ''.

Thus $z=(g_1, g_2)$ intertwines this, and g_1 acts trivially. We can then argue as in [M], Section 5 to conclude that $g_2 \in T^{(r)}$, modulo (double cosets of) congruence subgroups. If $f_{n_0}=1$, we can adapt the argument of [M], 5 to show that $g_2 \in P(n_0-1)$ modulo other congruence subgroups.

As for g_1, we can now treat it the same way g was treated, to reduce it further within $U(f_1 \oplus \dots \oplus f_{r-1})$. We conclude that if $S \in \mathrm{Hom}_{\underline{P}_\Psi \cap {}^g\underline{P}_\Psi}(\rho, {}^g\rho)$ then $g \in \underline{P}_\Psi$, hence S is a scalar.

7.6. To complete our discussion, we must deal with the case where $f_1^u=1$.

If T is unramified, one applies [M], 5.3 directly. If not, consider $\rho_{\Psi_u} \otimes \rho_1^{(r-l-1)} \otimes \dots \otimes \rho_1^{(0)}$ restricted to the group $P^{(u)} P^{(r-l-1)}_{i'_1(r-l-1)} \dots P^{(1)}_{i'_1(1)} P^{(0)}_{i'_1(0)}$. We call this representation ρ' and write it as $\rho_{\Psi_u} \otimes \rho'_1$. Suppose then that S intertwines ρ, ${}^g\rho$ (as in 7.3). The same argument as in 7.3 implies that $g \in P^{(0)}_{i_1(0)} Z_G(c_1^{(0)}) P^{(0)}_{i'_1(0)}$.

Now consider $P^{(0)}_{i_1(0)}$ and $\rho_{\Psi_u} \otimes \rho'_1$. A variant of Lemma 7.4 implies that $P^{(0)}_{i_1(0)}$ stabilizes ρ'_1. It also stabilizes ρ_{Ψ_u}; this follows from the definition of the extension of $\rho_{u,1}^{(r-l)}$ to ρ_{Ψ_u} [and implicitly, the assumption 6.5 (*c*) (ii)]. This means that we may assume $g \in Z_G(c_1^{(0)})$, just as before. Thus g can be written as $g=(g_1, g_2)$ where $g_1 \in U(f_u, A_u)$, $g_2 \in U(f_l \oplus \dots \oplus f_r, E_{l+1}, \oplus \dots \oplus E_r)$. As before, one deduces that g_2 lies in a product of compact tori times parahoric subgroups (of centralizers). Since g_2 acts trivially on $U(f_u, A_u)$ we can confine our attention to g_1 and argue as in [M], 5.3 to deduce that $g_1 \in P_u$, where P_u is the parahoric subgroup in $U(f_u, A_u)$ that stabilizes the lattice chain constructed in I 3.4.

8. Afterword

8.1. The representations that we have constructed are not the most general that can be produced using the ideas in this, and the previous paper I. In this section we shall indicate how to modify the earlier constructions and definitions to produce other supercuspidal representations; for simplicity we have avoided incorporating this into the main body of the paper. We shall simply indicate the ideas, and the changes that must be made to the key definitions; the interested reader can either await further details, or supply them.

The idea is quite simple. Suppose that T is a compact maximal torus associated to $A=E_1 \oplus \dots \oplus E_l \oplus E_{l+1} \oplus \dots \oplus E_r$, $\sum_i \mathrm{trace}_{E_i/k}(\mu_i x_i \sigma_i y_i)$ as in I 3.1. Suppose that say $E_m, \dots, E_r$ have the same ramification degree $e>1$, and that

$$\max\{\mathrm{ord}_{E_i}(\mu_i)\} \leqq \min_i\{\mathrm{ord}_{E_i}(\mu_i)\}+1; \qquad m \leqq i \leqq r.$$

Then $E_m \oplus \ldots \oplus E_r$ behaves sufficiently like a field that we can imitate the constructions in this paper and I, on replacing E_r by $E_m \oplus \ldots \oplus E_r$.

8.2. We begin with the filtrations associated to compact maximal tori described in I, Section 3; in particular we use the notation of that paper freely. Suppose then that T is the compact maximal torus associated with $A = E_1 \oplus \ldots \oplus E_l \oplus E_{l+1} \oplus \ldots \oplus E_r$, $\sum_i \mathrm{trace}_{E_i/k}(\mu_i x_i \sigma_i y_i)$.

Assume for the time being that $l=0$ (no unramified extensions). We denote by $\mathcal{O}_i \ldots$ the ring of integers... associated to E_i. Let v_i be the E_i-valuation of μ_i, e_i the ramification degree of E_i over k. Suppose that $e_1 = \ldots = e_r = e$, say.

Let $\mathcal{O}_A = \bigoplus_i \mathcal{O}_i, \mathcal{P}_A = \bigoplus_i \mathcal{P}_i$. Then $\mathcal{P}_A^e = \pi \mathcal{O}_A$, so that $\{\mathcal{P}_A^n\}_{n \in \mathbf{Z}}$ defines an $\mathcal{O}$-lattice chain $\mathcal{L}$ in A (henceforth identified with V).The dual lattice chain $\mathcal{L}^\#$ has been computed in [M], 2.3; it is simply the chain $\{\mu^{-1} \mathcal{P}_A^{1-e-n}\}_{n \in \mathbf{Z}}$ where we write $\mu = (\mu_1, \ldots, \mu_r)$.

LEMMA. — *$\mathcal{L} \cup \mathcal{L}^\#$ is a lattice chain if and only if the following condition is satisfied: Let $v = \max\{v_1, \ldots, v_r\}$. Then for each i,*

$$v \geqq v_i \geqq v-1.$$

Proof. — Suppose $v = v_1$, then $1-n-e-v_1 = \min\{1-n-e-v_i\}$ for a given n. It follows that

$$\mathcal{P}_A^{1-n-e-v_1} \supseteqq \mu^{-1} \mathcal{P}_A^{1-e-n}.$$

Now $\mathcal{P}_A^{1-n-e-v_1} \in \mathcal{L}$; if $\mathcal{L} \cup \mathcal{L}^\#$ is a lattice chain we must have

$$\mathcal{P}_A^{1-n-e-v_1} \supseteqq \mu^{-1} \mathcal{P}_A^{1-e-n} \supseteqq \mathcal{P}_A^{2-n-e-v_1}$$

which is equivalent to the condition in the statement of the lemma. The converse is easy.

8.3. In [M], we began by considering the case where $v_i \in \{0, 1\}$ in 8.2. But the essential feature for many of the arithmetic constructions of Sections 2 and 3 of [M], is the existence of a principal hereditary order Λ associated to a lattice chain $\mathcal{L}$, such that $\mathcal{A} = \Lambda \cap \sigma \Lambda$ is a hereditary order as well (with lattice chain $\mathcal{L} \cup \mathcal{L}^\#$), whose Jacobson radical $\mathcal{B}$ has the property that either $\mathcal{B}$ or $\mathcal{B}^2$ is principal.

If we take the lattice chain $\mathcal{L}$ in 8.1, the associated hereditary order Λ is principal, for its Jacobson radical is generated by $\pi_A = (\pi_1, \ldots, \pi_r)$ (*see* e.g., [M], 2.3). Thus, if the condition of Lemma 8.2 is satisfied, then $\mathcal{L} \cup \mathcal{L}^\#$ is a self dual lattice chain with associated order $\mathcal{A} = \Lambda \cap \sigma \Lambda$ where Λ is a principal order.

Furthermore $\mathcal{B} = \mathcal{B}_{\mathcal{A}}$ is principal or $\mathcal{B}_{\mathcal{A}}^2$ is principal. Indeed, in Lemma 8.2, let $r_n = \min_i \{1-e-n-v_i\} = 1-e-n-\max_i v_i = 1-e-n-v$. We see that $r_{n+1} = r_n - 1$ always, and it follows immediately that if $\mathcal{B}$ is not principal, then $\mathcal{B}_{\mathcal{A}}^2$ is principal, and it is generated, by $\pi_{\mathcal{A}}$.

Following [M], 2.7 we set $P = \mathscr{A} \cap G$, $P = \{x \in P \mid x \equiv 1 \bmod \mathscr{B}^n\}$ when n is a positive integer. One sees immediately that Propositions 2.11 (approximation theorem) and 2.12, together with their proofs, hold in this framework.

8.4. The other key feature in [M], is the notion of *principal element;* this too, is easily defined in our context. Namely let $X \in A$, and write A_X for the algebra generated by X; it is a direct sum of subfields of the E_i, and we let $\mathcal{O}_X$ be the obvious order in A_X, with radical $\mathscr{P}_X$. Now assume that there is an integer $e_X \geqq 1$ such that $\mathscr{P}_X^{e_X} = \pi \mathcal{O}_X$. Using this property, and the fact that Λ is principal one sees that Proposition 2.21 of [M] (the analogue of 4.10) holds. Let C_A be the group defined in [M], 2.23 (*cf.* 4.1).

DEFINITION. — We say that $X \in A$ is a *principal element* if the following conditions hold:

(i) $X + \sigma X = 0$;

(ii) $\mathscr{P}_X^{e_X} = \pi \mathcal{O}_X$;

(iii) $X \in C_A$.

For such elements one finds that the results in [M], 2.24-2.33 are valid. (The proofs are exactly the same.) In particular, 2.24, 2.28, 2.32 of [M], are the analogues of 4.12, 4.13, 4.14, respectively, in this framework.

8.5. We return to the situation of an arbitrary compact maximal torus T arising from $A = E_1 \oplus \ldots \oplus E_l \oplus E_{l+1} \oplus \ldots \oplus E_r, \sum_i \mathrm{trace}_{E_i/k}(\mu_i x_i \sigma_i y_i)$. Suppose that we can write $A = A_u \oplus A_{l+1} \oplus \ldots \oplus A_s$, where $A_u = E_1 \oplus \ldots \oplus E_l$ and if $l+1 \leqq j \leqq s$,

$$A_j = E_{j,1} \oplus \ldots \oplus E_{i,i_j}$$

where $E_{j,1}, \ldots, E_{j,i_j}$ have the same ramification degree, and (with the obvious notation) the elements $v_{j,1}, \ldots, v_{j,i_j}$ satisfy the property of Lemma 8.2. *We warn the reader that* A_j *need not consist of all fields with the same ramification degree* (*and satisfying condition* 8.2).

Associated to $A_u, A_{l+1}, \ldots, A_s$ we have the lattice chains $\mathscr{L}_u, \mathscr{M}_{l+1}, \ldots, \mathscr{M}_s$. Here $\mathscr{L}_u$ is the same as I 3.5, and $\mathscr{M}_{l+1}, \ldots, \mathscr{M}_s$ are the chains associated to $A_{l+1}, \ldots, A_s$ by 8.2 above. We can now form the chains $\mathscr{L}_u = \mathscr{L}_0 = \ldots = \mathscr{L}_l$; $\mathscr{L}_{l+1} = \mathscr{L}_u \oplus \mathscr{M}_{l+1}$, $\mathscr{L}_{l+2} = \mathscr{L}_{l+1} \oplus \mathscr{M}_{l+2}, \ldots \mathscr{L}_s = \mathscr{L}_{s-1} \oplus \mathscr{M}_s$ by the summing procedure of I Section 2. We note in passing that I considered the case where each A_j $(j > l)$ consisted of precisely one field.

It is routine to check that Proposition 3.11 remains true except that $\mathscr{Q}_2$ is not necessarily principal, generated by π_{A_s} in this situation. Lemmas 3.12, 3.13 and 3.14 remain true (and their proofs remain the same) in this more general situation.

8.6. The definitions imply that if $\mathscr{Q}_2$ is not principal then $\mathscr{Q}_2^2$ is principal, and is generated π_{A_s}. (One simply checks the description of $\mathscr{M}_s$ as given by 8.1-8.2.) If $c_s \in A_s$ is principal, then $c_s = \pi_{A_s}^m b$, where $b = (b_{s,1}, \ldots, b_{s,i_s})$ and each b_{s,i_j} is a root of unity of order prime to p (*see* [M], 2.23). This means that if $\mathcal{Q}_2$ is not principal then

$$\begin{aligned} c_s \in \mathcal{Q}_2^{2m} &= \mathcal{B}'^{2m} \cap \mathcal{B}''^{2m} \cap \operatorname{End}_k(\mathrm{V}_2) \\ &= \mathcal{B}_{4m} \cap \operatorname{End}_k(\mathrm{V}_2) \end{aligned}$$

If $\mathcal{Q}_2$ is principal, then as before $c_s \in \mathcal{Q}_2^m = \mathcal{B}_{2m} \cap \operatorname{End}_k(\mathrm{V}_2)$.

With these remarks made, one sees that the appropriate variant of Section 4 remains valid, when one replaces c_r by c_s throughout, given the last assertion of 8.4.

This brings us to the definition of cuspidal datum. For the unramified case (Section 5) this is just as before. For the ramified case (6.4-6.5) we proceed as follows. First, in lieu of definition 6.4 we use definition 3.18 of [M]. Secondly, with this done, we proceed as before with definition 6.5, with the proviso that any $c_m^{(n)}$ that appears (implicitly or explicitly) is to be a principal element, unless it is associated to A_u, in which case it has already been defined in Section 5.

One can then carry through the constructions and results of Sections 6 and 7.

8.7. *Example.* — We return to the example 3.9 (*b*) (ii) of I. In that example we saw that the torus T embedded in the parahoric subgroup whose associated order consisted of matrices of the form

$$\begin{pmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \pi & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \pi & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \pi & \pi & \pi & \mathcal{O} \end{pmatrix}$$

and the filtration was the same as that of 3.9 (*b*) (i).

On the other hand a glance at the definitions tells us that for this torus (associated to $\mathrm{A} = \mathrm{E}_1 \oplus \mathrm{E}_2$ in the notation of the example) the condition 8.1 is satisfied. Carrying out the definitions we see that T embeds in the parahoric subgroup whose associated order has the 2×2 block form

$$\begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \pi & \mathcal{O} \end{pmatrix}$$

and the filtration is the period 2 filtration arising from powers of the Jacobson radical.

Both of these parahorics, and the corresponding filtrations, occur in the construction of supercuspidal representations for Sp_4 arising from T. Indeed, one can see why by considering a character $\chi = \chi_1 \times \chi_2$ on $\mathrm{T}_1 \times \mathrm{T}_2$. If χ_1 and χ_2 have conductors of differing levels, one would be led to the constructions of Sections 6 and 7. If they have conductors of the same level, one is led to the constructions sketched in this section.

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