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## COMPOSITUM OF GALOIS EXTENSIONS OF HILBERTIAN FIELDS

BY D. HARAN AND M. JARDEN <sup>(1)</sup>

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### Introduction

Hilbert [H] proved in 1892 that for given irreducible polynomials  $f_i(T_1, \dots, T_r, X)$ ,  $i = 1, \dots, m$ , and a nonzero polynomial  $g(T_1, \dots, T_r)$  with rational coefficients there exists  $(a_1, \dots, a_r) \in \mathbb{Q}^r$  such that  $f_1(\mathbf{a}, X), \dots, f_m(\mathbf{a}, X)$  are irreducible in  $\mathbb{Q}[X]$  and  $g(\mathbf{a}) \neq 0$ .

Numerous proofs of Hilbert's irreducibility theorem have since been given. Many of them apply to other fields. So, each field  $K$  which satisfies the theorem has been called **Hilbertian**. The sets of  $\mathbf{a} \in K^r$  whose substitution in the polynomials leaves them irreducible and nonzero have been called **Hilbert sets**.

The investigation of Hilbertian fields has been extended in the last 98 years since Hilbert's original paper in several directions:

(a) *Study of Hilbert subsets of Hilbertian fields* (e. g. Dörge [D], Geyer [G], Sprindžuk [S], and Fried [F]).

(b) *Search for arithmetical conditions on a field which make it Hilbertian*. Beyond the classical example of fields of rational function over any field (Inaba [I] and Franz [Fr]) two results stand out: "Each  $\omega$ -free PAC field is Hilbertian" (Roquette [FJ], Cor. 24.38) and "The field of formal power series in at least two variables over any field is Hilbertian" (Weissauer [FJ], Cor. 14.18).

(c) *Infinite algebraic extensions of Hilbertian fields*. The first result in this direction is due to Kuyk [K]: "Every abelian extension of Hilbertian field is Hilbertian" In particular the field  $\mathbb{Q}_{\text{cycl}}$  obtained from  $\mathbb{Q}$  by adjoining all roots of unity is Hilbertian. Uchida [U] extended a result of Kuyk and proved that if an algebraic extension  $L$  of a Hilbertian field  $K$  is contained in a nilpotent extension and if the supernatural number  $[L : K]$  (see [FJ], Section 20.9) is divisible by at least two prime numbers, then  $L$  is Hilbertian. The strongest result however in this direction, is again due to Weissauer: "Every finite proper

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extension of a Galois extension of a Hilbertian field is Hilbertian". (See [W], Satz 9.7, for a nonstandard proof and [FJ], Cor. 12.15 for a standard proof.) We make an extensive use of this result and refer to it as **Weissauer's theorem**.

(d) *Realization of finite groups over Hilbertian fields, especially over number fields via Riemann existence theorem (see Matzat's exposition [M]).*

(e) *Properties of almost all  $e$ -tuples  $(\sigma_1, \dots, \sigma_e)$  of elements of the absolute Galois group of a Hilbertian field  $K$ .* For example, the group generated by almost all  $(\sigma_1, \dots, \sigma_e)$  is a free profinite group [FJ], Thm. 16.13, and if  $K$  is countable, then the fixed field  $K_s(\sigma_1, \dots, \sigma_e)$  of  $\sigma_1, \dots, \sigma_e$  in the separable closure  $K_s$  of  $K$  is PAC [FJ], Thm. 16.18.

This note is a contribution to the study of infinite algebraic extensions of Hilbertian fields. Weissauer's theorem implies that the compositum of a Galois extension  $M_1$  of a Hilbertian field  $K$  and a finite extension  $M_2$  of  $K$  which is not contained in  $M_1$  is Hilbertian. So, it is natural to ask whether the compositum  $N$  of two (infinite) linearly disjoint proper Galois extensions  $M_1$  and  $M_2$  of  $K$  is Hilbertian. Indeed, this has been stated as Problem 12.18 of [FJ]. However, the question goes back at least to Kuyk [K] (see Remark 2.6) and Weissauer. Kuyk proved that  $N$  is Hilbertian if an extra condition holds: "For each finite Galois extension  $L$  of  $K$  which is contained in  $N$  we have  $L \cap M_1 \neq K$  or  $L \cap M_2 \neq K$ ." In particular this is the case if the degrees  $[N:M_1]$  and  $[N:M_2]$  are relatively prime. The main tool in Kuyk's proof is the possibility to realize wreath products over  $K$ . Zorn [Z] gave a clearer exposition of Kuyk's proof while strengthening Kuyk's extra condition to: "Each open normal subgroup of an open normal subgroup of  $\mathcal{G}(N/K)$  is the direct product  $\mathcal{G}(N/M'_1) \times \mathcal{G}(N/M'_2)$ , where  $M'_i$  is a finite extension of  $M_i$  contained in  $N$ ."

We extend here Kuyk's result to a complete affirmative solution of Problem 12.18 of [FJ]. Our proof is an elaboration of Zorn's in the case where  $[N:M_1]$  and  $[N:M_2]$  are relatively prime. For the case where the degrees are not relatively prime we generalize a lemma of Chatzidakis on normalizers of elements in wreath products [FJ], Lemma 52. Then we apply the setup used in the first case to conclude the proof in the second case.

An application of Weissauer's theorem gives even a sharper result:

**THEOREM.** — *The composition of two Galois extensions of a Hilbertian field, neither of which is contained in the other, is Hilbertian.*

Of course, the solution of Problem 12.18 of [FJ] immediately supplies an affirmative solution to Problem 12.19 of [FJ]:

**COROLLARY.** — *The separable closure of a Hilbertian field  $K$  cannot be presented as the compositum of two Galois extensions of  $K$ , neither of which is contained in the other.*

## 1. Wreath products

Recall that the wreath product  $H = A \text{ wr } G$  of finite groups  $A$  and  $G$  is the semidirect product  $G \ltimes A^G$ , where  $A^G$  is the group of all functions  $f: G \rightarrow A$  with the canonical

multiplication rule, and  $G$  acts on  $A^G$  by the formula  $f^\tau(\sigma) = f(\tau\sigma)$ . Thus each element of  $H$  is a pair  $(\sigma, f)$  with  $\sigma \in G$  and  $f \in A^G$ . The product and the inverse in  $H$  are given by

$$(1) \quad (\sigma, f)(\tau, g) = (\sigma\tau, f^\tau g) \quad \text{and} \quad (\sigma, f)^{-1} = (\sigma^{-1}, f^{-\sigma^{-1}}).$$

Let  $\pi: H \rightarrow G$  be the canonical projection. Embed  $A$  in  $A^G$  by identifying each  $a \in A$  with the function which maps 1 to  $a$  and  $\sigma$  to 1 for each  $\sigma \neq 1$ . Then  $A^G$  may also be considered as a direct product,  $A^G = \prod_{\sigma \in G} A^\sigma$ , and each element of  $A^\sigma$  has the form  $a^\sigma$  with  $a \in A$ .

Our first result generalizes a lemma of Chatzidakis [FJ], Lemma 24.52.

LEMMA 1.1. — *Let  $G$  and  $A$  be finite groups. For  $\sigma_1, \dots, \sigma_e \in G$  and  $1 \neq a \in A$  let  $G_0 = \langle \sigma_1, \dots, \sigma_e \rangle$  and  $H_0 = \langle (\sigma_1, a), \dots, (\sigma_e, a) \rangle$ . Then  $\pi$  maps the normalizer  $N = N_H(H_0)$  of  $H_0$  in  $H$  onto  $G_0$ .*

*Proof.* — Since  $\pi(H_0) = G_0$  it suffices to prove that  $\pi(N) \leq G_0$ . Consider  $A^{G_0}$  as the subgroup of  $A^G$  consisting of all functions  $f: G \rightarrow A$  for which  $f(\tau) = 1$  for each  $\tau \in G - G_0$ . It follows from (1) that  $H_1 = \{(\sigma, f) \mid \sigma \in G_0 \text{ and } f \in A^{G_0}\}$  is a subgroup of  $H$ . The main point to be observed here is that if  $(\sigma, f), (\tau, g) \in H_1$  and  $\rho \in G - G_0$ , then  $\tau\rho, \sigma^{-1}\rho \notin G_0$  and therefore  $(f^\tau g)(\rho) = f(\tau\rho)g(\rho) = 1$  and  $f^{-\sigma^{-1}}(\rho) = f(\sigma^{-1}\rho)^{-1} = 1$ . As  $(\sigma_i, a) \in H_1, i = 1, \dots, e$ , we have  $H_0 \leq H_1$ . In other words

$$(2) \quad (\sigma, f) \in H_0 \quad \text{implies that} \quad \sigma \in G_0 \text{ and } f \in A^{G_0}.$$

Let  $(\tau, g) \in N$ . Then  $(\sigma, f) = (\tau, g)^{-1}(\sigma_1, a)(\tau, g) \in H_0$ . By (1) and (2),

$$\sigma = \tau^{-1}\sigma_1\tau \in G_0 \quad \text{and} \quad f = g^{-\sigma}a^\tau g \in A^{G_0}.$$

Let  $n = \text{ord}(\sigma)$  and act with the powers of  $\sigma$  on  $f$  to get

$$f = g^{-\sigma}a^\tau g, \quad f^\sigma = g^{-\sigma^2}a^{\tau\sigma}g^\sigma, \quad \dots, \quad f^{\sigma^{n-1}} = g^{-\sigma^n}a^{\tau\sigma^{n-1}}g^{\sigma^{n-1}}.$$

Hence

$$(3) \quad f^{\sigma^{n-1}} \dots f^\sigma f = (g^{-1}a^{\tau\sigma^{n-1}}g^{\sigma^{n-1}}) \dots (g^{-\sigma^2}a^{\tau\sigma}g^\sigma)(g^{-\sigma}a^\tau g) = g^{-1}a^{\tau\sigma^{n-1}} \dots a^{\tau\sigma}a^\tau g$$

As  $\sigma \in G_0$  and  $f \in A^{G_0}$ , the left hand side of (3) belongs to  $A^{G_0}$ . Therefore, so does the right hand side of (3). So if  $\tau \notin G_0$ , then the value of the right hand side of (3) at  $\tau^{-1}$  is 1. Thus

$$(4) \quad g(\tau^{-1})^{-1}a(\tau\sigma^{n-1}\tau^{-1}) \dots a(\tau\sigma\tau^{-1})a(1)g(\tau^{-1}) = 1.$$

Finally, note that for  $j$  between 1 and  $n-1$  we have  $\tau\sigma^j\tau^{-1} \neq 1$ . Hence (4) reduces to  $a = 1$ . This contradiction to the choice of  $a$  proves that  $\tau \in G_0$ , as desired. ■

As a result, a certain embedding problem for a direct product of profinite groups cannot be properly solved:

LEMMA 1.2. — *Let  $C_1, C_2$  be nontrivial profinite groups. Let  $G_1, G_2$  be nontrivial finite quotients of  $C_1, C_2$ , respectively, such that either*

- (a) *the orders  $G_1$  and  $G_2$  are not relatively prime, or*
- (b) *the orders of  $C_1$  and  $C_2$  are relatively prime.*

*Let  $G = G_1 \times G_2$  and let  $\rho: C_1 \times C_2 \rightarrow G$  be the product of the quotient maps.*

*Let  $A$  be a nontrivial finite group,  $H = A \text{ wr } G$ , and  $\pi: H \rightarrow G$  the canonical projection. Then there exists no epimorphism  $\theta: C_1 \times C_2 \rightarrow H$  such that  $\pi \circ \theta = \rho$ .*

*Proof.* — Assume that there exists an epimorphism  $\theta: C_1 \times C_2 \rightarrow H$  such that  $\pi \circ \theta = \rho$ . We derive a contradiction in each of the two cases.

*Case (a): There exists a prime  $p$  and elements  $\sigma_i \in G_i$ ,  $i = 1, 2$ , of order  $p$ . Then the order of  $\sigma = \sigma_1 \sigma_2$  is also  $p$ . Use Lemma 1.1 for  $e = 1$  to find  $h \in H$  such that  $\pi(h) = \sigma$  and  $\pi(N) = \langle \sigma \rangle$ , with  $N = N_H \langle h \rangle$ . Write  $h = h_1 h_2$ , with  $h_i = \theta(c_i)$  and  $c_i \in C_i$ . Then  $c_1$  commutes with  $c_2$  and therefore  $h_i \in N$ . Hence  $\pi(h_i) = \rho(c_i) \in \langle \sigma \rangle \cap G_i = 1$ . It follows that  $\sigma = \pi(h) = 1$ . This is a contradiction.*

*Case (b): The orders of  $C_1$  and  $C_2$  are relatively prime. For  $i = 1, 2$  put  $H_i = \theta(C_i)$ . Then  $H_i \triangleleft H$ ,  $\pi(H_i) = G_i$  and there exists  $h \in H_i$  such that  $\sigma = \pi(h) \neq 1$ . Thus  $h = (\sigma, f)$ , where  $f \in A^G$ . As  $A^\sigma \triangleleft A^G$ , we have  $A^h = (A^\sigma)^f = A^\sigma$ . It follows that  $A = (A^\sigma)^{h^{-1}} \leq H_i \cdot A^\sigma$ , and therefore*

$$A^G = A \cdot \prod_{\substack{\tau \in G \\ \tau \neq 1}} A^\tau \leq H_i \cdot \prod_{\substack{\tau \in G \\ \tau \neq 1}} A^\tau.$$

Hence, with  $n = |G|$ , the order of  $A^n$  divides  $|H_i| \cdot |A|^{n-1}$ , and therefore  $|A|$  divides  $|H_i|$ , for  $i = 1, 2$ . This is a contradiction, since  $|H_1|$  and  $|H_2|$  are relatively prime. ■

REMARK 1.3. — *Characterization of wreath products.* Although we shall not use it in the sequel it is interesting to note that wreath products can be characterized by less data than above:

Given an extension of finite groups

$$(5) \quad 1 \rightarrow B \rightarrow H \rightarrow G \rightarrow 1,$$

the lifting of elements of  $G$  to elements of  $H$  determines a homomorphism  $\psi: G \rightarrow \text{Aut}(B)/\text{In}(B)$ . The set of all congruence classes of extensions with the same  $\psi$  bijectively corresponds to the group  $H^2(G, Z(B))$  [Mc], p. 128. In particular let  $B = A^G$  and  $\psi$  be the homomorphism obtained from the natural action of  $G$  on  $B$ . Then the  $G$ -module  $Z(B) = Z(A)^G$  is the induced module  $\text{Ind}_1^G Z(A)$ . Hence  $H^2(G, Z(B))$  is trivial [R], p. 146. It follows that the only extension (5) such that  $\psi$  is induced by the natural action of  $G$  on  $B = A^G$  is the wreath product.

REMARK 1.4. — *Interpretation of wreath products in Galois theory.* Consider a tower of fields  $K \subseteq L \subseteq F \subseteq \hat{F}$  where  $L/K$ ,  $F/L$  and  $\hat{F}/K$  are finite Galois extensions. Let also  $K'$  be a field such that  $K' \cap L = K$  and  $LK' = \hat{F}$ . Put  $G = \mathcal{G}(L/K) \cong \mathcal{G}(\hat{F}/K')$  and  $A = \mathcal{G}(F/L)$ . Suppose that the fields  $F^\sigma$ ,  $\sigma \in \mathcal{G}(\hat{F}/K')$  are linearly disjoint over  $L$  and their compositum is  $\hat{F}$ . Then there exists an isomorphism  $\varphi: \mathcal{G}(\hat{F}/K) \rightarrow A \text{ wr } G$  which maps  $\mathcal{G}(\hat{F}/L)$  onto  $A^G$  and induces the identity maps  $\mathcal{G}(F/L) = A$  and  $\mathcal{G}(L/K) = G$ . We say in this set up that the fields  $L, F, \hat{F}$  **realize the wreath product  $A \text{ wr } G$  over  $K$** .

$K'_0 = \hat{F}_0 \cap K'$ , and  $A_0 = \mathcal{G}(F_0/L)$ . Then  $K'_0 \cap L = K$  and  $LK'_0 = \hat{F}_0$ . Hence  $L, F_0, \hat{F}_0$  realize  $A_0 \text{ wr } G$  over  $K$ , as above.

## 2. Main results

We take the crucial step toward the solution to Problem 12.18 of [FJ] in the following lemma. It involves a construction of wreath products over fields of rational functions as in [K], Prop. 1.

LEMMA 2.1. — *Let  $M_1, M_2$  be linearly disjoint infinite Galois extensions of a field  $K$ , and let  $N = M_1 M_2$ . Let  $f \in K[T, X]$  be an absolutely irreducible polynomial, monic in  $X$ , and Galois over  $K(T)$ . Then there exists a finite Galois extension  $L$  of  $K$  contained in  $N$  such that for every basis  $c_1, \dots, c_n$  of  $L$  over  $K$  there is a Hilbert subset  $B$  of  $K^n$  such that for each  $(b_1, \dots, b_n) \in B$  the polynomial  $f(b_1 c_1 + \dots + b_n c_n, X)$  is irreducible over  $N$ .*

*Proof.* — There are three parts in the proof.

*Part A: Construction of  $L$ .* Let  $C_1 = \mathcal{G}(N/M_1)$  and  $C_2 = \mathcal{G}(N/M_2)$ . Then  $\mathcal{G}(N/M) = C_1 \times C_2$ . Choose nontrivial finite quotients  $G_1, G_2$  of  $C_1, C_2$ , respectively. If the orders of  $C_1$  and  $C_2$  are not relatively prime, choose  $G_1$  and  $G_2$  with orders having a common prime divisor. Let  $\rho: C_1 \times C_2 \rightarrow G_1 \times G_2$  be the product of the quotient maps. Consider the fixed field  $L$  of  $\text{Ker}(\rho)$  in  $N$ . Then  $G = \mathcal{G}(L/K) = G_1 \times G_2$ . By Lemma 1.2, for no nontrivial finite group  $A_0$  there exist fields  $L \subseteq E \subseteq \hat{E} \subseteq N$  such that  $L, E, \hat{E}$  realize  $A_0 \text{ wr } G$  over  $K$ .

*Part B: Construction of wreath product over a field of rational functions.* Choose a set  $\{u^\sigma \mid \sigma \in G\}$  of algebraically independent elements over  $K$ . For each  $\sigma \in G$  let  $x^\sigma$  be a root of  $f(u^\sigma, X)$ . As  $f$  is absolutely irreducible, the field  $K(u^\sigma, x^\sigma)$  is a regular extension of  $K$ . Hence  $L(u^\sigma, x^\sigma)$  is a regular extension of  $L$ . As these fields are algebraically independent over  $L$ , the field  $\hat{Q} = L(u^\sigma, x^\sigma \mid \sigma \in G)$  is a regular extension of  $L$  [FJ], p.112. Moreover, the field  $Q = L(u^\sigma \mid \sigma \in G)$  is linearly disjoint from  $K(u^\sigma, x^\sigma)$  over  $K(u^\sigma)$ . Hence  $Q(x^\sigma)/Q$  is a Galois extension with Galois group isomorphic to  $A = \mathcal{G}(f(T, X), K(T))$ . The set of all  $Q(x^\sigma)$  is linearly disjoint over  $Q$ . So,  $\mathcal{G}(\hat{Q}/Q) \cong A^G$ .

Put  $n = [L:K] = |G|$ , and let  $c_1, \dots, c_n$  be a basis for  $L/K$ . Let  $t_1, \dots, t_n$  be the unique solution of the following system of linear equations:

$$(1) \quad T_1 c_1^\sigma + \dots + T_n c_n^\sigma = u^\sigma, \quad \sigma \in G$$

As the matrix  $(c_i^\sigma)$  is invertible [L], p. 212,  $L(t_1, \dots, t_n) = L(u^\sigma \mid \sigma \in G) = Q$ . Since  $n$  is the transcendence degree of  $Q$  over  $L$ , the elements  $t_1, \dots, t_n$  are algebraically independent over  $L$  and hence also over  $K$ .

Extend the action of  $G$  on  $L$  to an action on  $\hat{Q}$  in a natural way:  $(u^\sigma)^\tau = u^{\sigma\tau}$  and  $(x^\sigma)^\tau = x^{\sigma\tau}$ . In particular  $\tau$  permutes the equations of the system (1). As  $(t_1^\tau, \dots, t_n^\tau)$  is also a solution of (1), it coincides with  $(t_1, \dots, t_n)$ . Thus  $\tau$  leaves each element of  $P = K(t_1, \dots, t_n)$  element wise fixed. So, the fixed field  $Q(G)$  of  $G$  in  $Q$  contains  $P$ . In particular  $n \leq [Q:P]$ . As  $LP = Q$ , this implies that  $P = Q(G)$  and that  $L \cap P = K$ .

The subgroup  $H$  of  $\text{Aut}(\hat{Q})$  generated by  $G$  and  $\mathcal{G}(\hat{Q}/Q)$  is contained in  $\text{Aut}(\hat{Q}/P)$ . As  $\hat{Q}/P$  is separable, the latter group is finite and therefore so is  $H$ . Since  $P$  is the fixed field of  $H$ , the field  $\hat{Q}$  is Galois over  $P$  and  $H = \mathcal{G}(\hat{Q}/P)$ .

Now consider the fixed field  $P' = \hat{Q}(G)$ . Its intersection with  $Q$  is  $P$  and their compositum is  $\hat{Q}$ . So,  $Q, Q(x), \hat{Q}$  realize  $A$  wr  $G$  over  $P$ .

*Part C: Definition of  $B$  and conclusion of the proof.* Write  $\hat{Q}$  as  $P(z)$  with  $z$  integral over  $K[t_1, \dots, t_n]$  and let  $h(t_1, \dots, t_n, Z) = \text{irr}(z, P)$ . Then  $f(T_1 c_1 + \dots + T_n c_n, X)$  is irreducible over  $L$ . Use [FJ], Lemma 12.12 and Cor. 11.7, to find a Hilbert subset  $B$  of  $K^n$  such that for each  $\mathbf{b} \in B$  and for  $a = \sum_{i=1}^n b_i c_i$ .

$$(2a) \quad \mathcal{G}(h(\mathbf{b}, Z), K) \cong \mathcal{G}(h(\mathbf{t}, Z), P),$$

$$(2b) \quad f(a, X) \text{ is irreducible over } L,$$

and the specialization  $\mathbf{t} \mapsto \mathbf{b}$  extends to a place of  $\hat{Q}$  over  $K$  such that the residue fields of  $P, Q, Q(x^\sigma), P', \hat{Q}$ , respectively, are  $K, L, F^\sigma, K', \hat{F}$ , where  $F^\sigma$  is the splitting field of  $f(a^\sigma, X)$  over  $L$ , for  $\sigma \in G$ . In particular  $L, F, \hat{F}$  realize  $A$  wr  $G$  over  $K$  and  $[F:L] = \deg(f(a, X))$ .

Let  $\mathbf{b} \in B$ ,  $a = \sum_{i=1}^n b_i c_i$ , and assume that  $f(a, X)$  is reducible over  $N$ . Then  $E = N \cap F$  is a proper Galois extension of  $L$ . Extend each  $\sigma \in \mathcal{G}(\hat{F}/K')$  to an element  $\sigma$  of the absolute Galois group  $G(K)$  of  $K$  to observe that  $E^\sigma = N \cap F^\sigma$  is contained in  $N$ . Let  $A_0 = \mathcal{G}(E/L)$  and  $\hat{E} = \prod_{\sigma \in \mathcal{G}(F/K')} E^\sigma$ . Then  $\hat{E} \subseteq N$  and, by Remark 1.4,  $L, E, \hat{E}$  realize

$A_0$  wr  $G$  over  $K$ . This contradiction to Part A proves that  $f(a, X)$  is irreducible over  $N$ , as desired. ■

LEMMA 2.2. — *Let  $N$  be a field,  $N'$  a finite Galois extension of  $N$ ,  $f \in N[T, X]$  an irreducible polynomial, which is separable in  $X$ , and  $g \in N'[T, X]$  a factor of  $f$  which is irreducible over  $N'$ . Then, for almost all  $a \in N$ , if  $g(a, X)$  is irreducible over  $N'$ , then  $f(a, X)$  is irreducible over  $N$ .*

*Proof.* — The polynomial  $f$  decomposes over  $N'$  as  $f(T, X) = \prod_{i=1}^m g_i(T, X)$  where each  $g_i$  is conjugate to  $g$  over  $N$  and for  $i \neq j$ ,  $g_i$  is not a multiple of  $g_j$  by an element of  $N'(T)$ . Suppose that for  $a \in N$  and each  $i \neq j$ ,  $g_i(a, X)$  is not a multiple of  $g_j(a, X)$  by an element of  $N'$  (this happens for almost all  $a \in N$ ) and  $g(a, X)$  is irreducible over  $N'$ . Then  $f(a, X)$  is irreducible over  $N$ . Indeed, let  $f(a, X) = h_1(X)h_2(X)$  be a decomposition over  $N$ . Then  $h_1(X)h_2(X) = \prod_{i=1}^m g_i(a, X)$ . As  $g(a, X)$  is irreducible, it divides, say,  $h_1(X)$ . Since each  $g_i(a, X)$  is conjugate to  $g(a, X)$  over  $N$ , it also divides  $h_1(X)$ . As  $g_1(a, X), \dots, g_m(a, X)$  are relatively prime,  $f(a, X) = \prod_{i=1}^m g_i(a, X)$  divides  $h_1(X)$ . Conclude that  $f(a, X)$  is irreducible over  $N$ . ■

PROPOSITION 2.3. — *Let  $M_1$  and  $M_2$  be infinite Galois extensions of Hilbertian field  $K$  such that  $M_1 \cap M_2 = K$ . Then their compositum  $N = M_1 M_2$  is Hilbertian. Moreover, given an irreducible polynomial  $f \in N[T, X]$ , separable in  $X$ , there exist  $c_1, \dots, c_n \in N$  and a Hilbert subset  $B$  of  $K^n$  such that for each  $(b_1, \dots, b_n) \in B$ , and for  $a = \sum_{i=1}^n b_i c_i$ , the polynomial  $f(a, X)$  is irreducible over  $N$ .*

*Proof.* — Note that the second statement means that if  $K$  is only separably Hilbertian [FJ], p. 147, then so is  $N$ . If  $K$  is Hilbertian, as we suppose, then it is imperfect. Hence, the second statement implies in this case that  $N$  is Hilbertian [FJ], Prop. 11.16.

To prove the second statement consider a transcendental element  $t$  over  $K$ . Let  $\hat{N}$  be the splitting field of  $f(t, X)$  over  $N(t)$ . Choose a primitive element  $y$  for  $\hat{N}$  over  $N(t)$  such that  $h = \text{irr}(y, N(t))$  has coefficients in  $N[t]$ . Then  $h$  is monic and Galois in  $X$ . If we find  $c_1, \dots, c_n \in N$  and a Hilbert subset  $B$  of  $K^n$  such that for each  $(b_1, \dots, b_n) \in B$  and with  $a = \sum_{i=1}^n b_i c_i$ , the polynomial  $h(a, X)$  is irreducible over  $N$ , then  $K^n$  has a Hilbert subset  $B_0$  of  $B$  such that for  $(b_1, \dots, b_n) \in B_0$  the polynomials  $f(a, X)$  is also irreducible over  $N$ . Indeed, the proof of [FJ], Lemma 12.12, shows that if  $a$  is not a zero of a certain nonzero polynomial with coefficients in  $N$  and  $h(a, X)$  is irreducible, then  $\mathcal{G}(f(a, X), N)$  and  $\mathcal{G}(f(t, X), N(t))$  are isomorphic as permutation groups of the roots. In particular the former group operates transitively on the roots of  $f(a, X)$ . This implies that  $f(a, X)$  is irreducible. Note that the exclusion of finitely many values  $a_1, \dots, a_k$  for  $a$  imposes the extra condition

$$\prod_{j=1}^k \left( \sum_{i=1}^n b_i c_i - a_j \right) \neq 0$$

on  $(b_1, \dots, b_n) \in B$ . This defines  $B_0$ . So, without loss, assume that  $f$  is monic and Galois in  $X$ .



Choose an absolutely irreducible factor  $g$  of  $f$ . Let  $K'_0$  be a finite Galois extension of  $K$  which contains the coefficients of  $g$ . Let  $K_1$  and  $K_2$  be finite Galois extensions of  $K$  contained in  $M_1$  and  $M_2$ , respectively, such that  $K'_0 \cap N \subseteq K_1 K_2$ . Then  $K' = K_1 K_2 K'_0$  satisfies  $N \cap K' = K_1 K_2$  and  $M_1 K_2 \cap M_2 K_1 = K_1 K_2$  (use the tower property of linear disjointness [FJ], Lemma 9.3).

Let  $M'_1 = M_1 K_2 K'$ ,  $M'_2 = M_2 K_1 K'$ ,  $N' = NK'$ . Then  $M'_1, M'_2$  are linearly disjoint Galois extensions of  $K'$  and  $N' = M'_1 M'_2$ . By Lemma 2.1 there is a finite Galois extension  $L'$  of  $K'$  contained in  $N'$  such that for every basis  $c_1, \dots, c_n$  of  $L'/K'$  there is a Hilbert subset  $B'$  of  $(K')^n$  such that for each  $b_1, \dots, b_n \in B'$  the polynomial  $g(b_1 c_1 + \dots + b_n c_n, X)$  is irreducible over  $N'$ . As  $\mathcal{G}(N'/K') = \mathcal{G}(N/K_1 K_2)$ , there is a finite Galois extension  $L$  of  $K_1 K_2$  in  $N$  such that  $L' = LK'$ . A basis  $c_1, \dots, c_n$  of  $L/K_1 K_2$  is also a basis of  $L'/K'$ . By Lemma 2.2 and by [FJ], Cor. 11.7,  $K^n$  has a Hilbert subset  $B \subseteq B'$  such that  $f(b_1 c_1 + \dots + b_n c_n, X)$  is irreducible over  $N$ , for every  $b_1, \dots, b_n \in B$ . ■

We are now ready to solve Problem 12.18 of [FJ] in a much stronger form:

**THEOREM 2.4.** — *Let  $M_1$  and  $M_2$  be Galois extensions of Hilbertian field  $K$  neither of which is contained in the other. Then their compositum  $N = M_1 M_2$  is Hilbertian.*

*Proof.* — If  $N$  is a finite extension of  $M_1$  or of  $M_2$ , then it is Hilbertian, by Weissauer's theorem. So, assume that  $N$  is an infinite extension of both  $M_1$  and  $M_2$ . In particular  $K_1 = M_1 \cap M_2$  has a finite proper Galois extension  $K'$  which is contained in  $M_2$ . Let  $M'_1 = M_1 K'$ . By Weissauer's theorem,  $K'$  is Hilbertian. Also,  $M'_1$  and  $M_2$  are infinite extensions of  $K'$  whose intersection is  $K'$  and whose compositum is  $N$ . Conclude from Proposition 2.3 that  $N$  is Hilbertian. ■

One of the consequences of Theorem 2.4 is a solution of Problem 12.19 of [FJ]:

**COROLLARY 2.5.** — *The separable (resp. solvable,  $p$ -) closure  $K_s$  (resp.  $K_{\text{solv}}$ ,  $K^{(p)}$ ) of a Hilbertian field  $K$  is not the compositum of two Galois extensions of  $K$  neither of which is equal to  $K_s$  (resp.,  $K_{\text{solv}}$ ,  $K^{(p)}$ ).*

*Proof.* — None of the above fields is Hilbertian. So the corollary follows from Theorem 2.4.

Nevertheless, as the separable case was the subject of an open question we sketch a short cut in the above proof in this case.

Assume that  $M_1$  and  $M_2$  are Galois extensions of  $K$  which are not separably closed such that  $M_1 M_2 = K_s$ . Use Weissauer's theorem to replace  $M_1$ ,  $M_2$ , and  $K$ , if necessary, by algebraic extensions to assume that  $M_1$ ,  $M_2$  are Hilbertian and  $M_1 \cap M_2 = K$ . In particular  $M_i$  has a cyclic extension  $M'_i$  of degree  $p$ ,  $i = 1, 2$  [FJ], Thm. 24.48.

Let  $K_1 = M_1 \cap M'_2$ ,  $K_2 = M_2 \cap M'_1$  and  $L = K_1 K_2$ . Then

$$G = \mathcal{G}(L/K) = \mathcal{G}(L/K_1) \times \mathcal{G}(L/K_2) \cong \mathbf{Z}/p\mathbf{Z} \times \mathbf{Z}/p\mathbf{Z}.$$

By [FJ], Prop. 24.47, there exists a Galois extension  $F$  of  $K$  which contains  $L$  and there exists an isomorphism  $\varphi: (\mathbf{Z}/p\mathbf{Z}) \text{ wr } G \rightarrow \mathcal{G}(F/K)$  such that  $\text{res}_L \circ \varphi$  is the canonical projection of the wreath product on  $G$ .

Now choose a generator  $\sigma_i$  of  $\mathcal{G}(L/K_i)$ ,  $i = 1, 2$  and let  $\sigma = \sigma_1 \sigma_2$ . Chatzidakis' Lemma [FJ], Lemma 24.52, extends  $\sigma$  to an element  $\tau$  of  $\mathcal{G}(F/K)$  such that restriction to  $L$  maps the normalizer of  $\langle \tau \rangle$  onto  $\langle \sigma \rangle$ . This gives a group theoretic contradiction as in Lemma 1.2.

Note that this proof actually works for each normal extension  $N$  of  $K$  which admits no  $p$ -extensions. In particular it works also for  $K_{\text{solv}}$  and  $K^{(p)}$ . ■

REMARK 2.6. — Kuyk [K], p. 120, states, contrary to Theorem 2.4, that the compositum of linearly disjoint Galois extensions of a Hilbertian field need not be Hilbertian. He adjoins  $p$ -th roots of all elements of  $\mathbb{Q}$  to  $K = \mathbb{Q}(\zeta_p)$  to get a Galois extension  $\mathbb{Q}^{(p)}$  of  $K$ . Then  $\mathcal{G}(\mathbb{Q}^{(p)}/K)$  is isomorphic to the direct product of infinitely many cyclic extensions of order  $p$ . Kuyk claims, without a proof, that  $\mathbb{Q}^{(p)}$  is not Hilbertian. However, as  $\mathbb{Q}^{(p)}$  is the compositum of a linearly disjoint finite Galois extension and an infinite Galois extension, already Weissauer's theorem implies that  $\mathbb{Q}^{(p)}$  is Hilbertian, contrary to Kuyk's statement.

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