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$C^{-\infty}$-WHITTAKER VECTORS FOR COMPLEX SEMISIMPLE LIE GROUPS, WAVE FRONT SETS, AND GOLDIE RANK POLYNOMIAL REPRESENTATIONS

BY HISAYOSI MATUMOTO (1)

ABSTRACT. — The existence condition (resp. the dimension of the space) of $C^{-\infty}$-Whittaker vectors seems to be governed by wave front sets (resp. Goldie rank polynomial representations). In this article, I should like to show this is indeed the case for representations of connected complex semisimple Lie groups with integral infinitesimal characters.

Dedicated to Professor Bertram Kostant on his sixtieth birthday.

0. Introduction

Let G be a connected (quasi-split) real semisimple linear Lie group and let $\bar{N}$ be the nilradical of a minimal parabolic subgroup $\bar{P}$ of G. We take a "generic" character ψ on $\bar{N}$, namely a one dimensional representation of $\bar{N}$, and consider the induced representation of G from ψ on $\bar{N}$. If an irreducible representation V of G is realized as a subrepresentation of such an induced representation, we call V has a Whittaker model. (This usage of "model" is different from that of Gelfand-Graev.) Such induced representations are considered first in [GG1, 2] and they suggest the possibility of usefulness of such induced representations for a classification of irreducible representations. After the pioneer work of Gelfand-Graev, Whittaker models of representations of real semisimple Lie groups have been studied from the viewpoint of number theory by many authors ([JL], [Ja], [Sc], [Sh], [Ha]), etc. Especially, the multiplicity one property of the above induced representation for a quadi-split group is established by [JL], [Sh], [Ko2], etc.

In [Ko2], Kostant proved that if a representation V of a quasi-split group G has a Whittaker model, then the annihilator of V in the universal enveloping algebra of the complexified Lie algebra of G is a minimal primitive ideal. [Casselman and Zuckerman

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proved this result for $G = SL(n, \mathbb{R})$.] This result strongly suggests the possibility of the description of the singularities of representations in terms of similar kind of representations. [Ha2], [V1, 2], [How] also support such a possibility. Lynch developed the theory of Whittaker vectors for non-split case in his thesis at MIT [Ly], and generalized some of the important results of Kostant.

Before [Ko2], Rodier [R] had pointed out the relation between the existence condition of Whittaker models and distribution characters for the p-adic case. Recently, Kawanaka [Kaw1, 2, 3] and Mœglin-Waldspurger [MW] constructed such an induced representation from each nilpotent orbit and described the relation to the singularities of irreducible representations for reductive algebraic groups over finite fields and p-adic fields, respectively. It is natural to ask whether a similar phenomenon exists in the case of a real semisimple Lie group. In [Mat4, 5] (also *see* [Kaw3] 2.5, [Y1], [Mat2]), we proposed the study of Whittaker models in the general sense.

In this article, we give an affirmative answer in some special case. Namely, we assume G is a connected complex semisimple Lie group and V has an integral infinitesimal character. We also put some assumptions on ψ and P.

We are going into more detail. Hereafter we assume $\bar{N}$ is the nilradical of a parabolic subgroup $\bar{P}$ and consider the induced representation of G from a "generic" character on N. Unfortunately, apparently, this induced representation is too large. Namely, in general, we cannot expect that an induced representation of G appears with finite multiplicity. However, interestingly enough, it is known that some irreducible representations appear in the induced representation with finite multiplicity. So, we can study the following problem.

PROBLEM. — *Classify an irreducible representation which appears in the induced representation of* G *from* ψ *with finite multiplicity. What is the multiplicity of such a representation?*

As a matter of fact, the above problem is quite obscure. In order to clarify the problem, we should define what is "generic", "representation", "the induced representation", and "appears with finite multiplicity". First, we choose the definition of the induced representation from G as follows:

$$C^\infty(G/\bar{N}; \psi) = \{ f \in C^\infty(G) \mid f(gu) = \psi(u)^{-1} f(g) \text{ for all } g \in G, u \in \bar{N} \}.$$

G acts on the above space by the left translation. We regard $C^\infty(G/\bar{N}; \psi)$ as a Frechet representation in a usual manner.

Second, we fix a maximal compact subgroup K of G and we consider Harish-Chandra modules (*cf.* [Vo3], [W2]) in stead of "representations of G".

Third, let $\bar{\mathfrak{n}}$ be the complexified Lie algebra of $\bar{N}$ and we denote the complexified differential character of ψ on $\bar{\mathfrak{n}}$ by the same letter. ψ is regarded as an element of the complexified Lie algebra of G by the Killing form. We say ψ is admissible if ψ is contained in the Richardson orbit $\mathcal{O}_{\bar{P}}$ with respect to $\bar{P}$. We replace "generic" in the above problem by "admissible".

Last, we should give the definition of "multiplicity". Let $\mathfrak{g}$ be the complexified Lie algebra of G and let $U(\mathfrak{g})$ be its universal enveloping algebra. The most naive definition is the followings. For an irreducible Harish-Chandra module V, we define the multiplicity of V in $C^\infty(G/\bar{N};\psi)$ by the dimension of the space of $U(\mathfrak{g})$-homomorphisms of V to $C^\infty(G/\bar{N};\psi)$.

In order to define "appears" in another way, for a Harish-Chandra module V, we consider an admissible Hilbert G-representation H whose K-finite part coincides with V. H is not uniquely determined by V, but the space of C^∞-vectors V^∞ is unique as a Frechet G-representation [Ca3]. If we take notice of the topology of $C^\infty(G/\bar{N};\psi)$, then we can give another definition of "multiplicity". Namely, we define the multiplicity of V in $C^\infty(G/\bar{N};\psi)$ the dimension of the space of continuous G-homomorphisms from V_∞ to $C^\infty(G/\bar{N};\psi)$.

It is known that the above two definitions of the multiplicity actually different [GW]. The problem in the first definition was studied in [GW], [at1, 2, 4, 5] (also *see* [Ko2], [Ha2], [Ly]).

In this article, we consider the second definition and assume G is a complex semisimple Lie group. We define the space of $C^{-\infty}$-Whittaker vectors of an irreducible Harish-Chandra module V as follows.

$$\mathrm{Wh}_\psi^\infty(V)=\{v\in V'_\infty \mid \forall X\in\bar{\mathfrak{n}}\ Xv=\psi(X)v\}.$$

Here, V'_∞ denotes the continuous dual space of V_∞. Then, the space of continuous G-homomorphisms of V_∞ to $C^\infty(G/\bar{N};\psi^{-1})$ can be identified with $\mathrm{Wh}_\psi^\infty(V)$ as a usual manner. So we can rephrase the above problem in terms of $\mathrm{Wh}_\psi^\infty(V)$.

For an irreducible Harish-Chandra module V, we denote by WF(V) the wave front set of V (*cf.* [How], [BV1, 2, 3, 4]). Let $X=G/\bar{P}$ be the generalized flag variety and let ψ be an admissible character on $\bar{N}$. We assume the moment map $\mu: T^*X\to\bar{\mathcal{O}}_{\bar{P}}$ (*cf.* [BoBr], [BoBrM]) is birational.

Let $\mathcal{O}$ be a nilpotent orbit of the Lie algebra of G. For example, we assume that $G=SL(n,\mathbb{C})$ or that $\mathcal{O}$ is even. Then, there exists some $\bar{P}$ such that:

(1) $\mathcal{O}$ coincides with the Richardson orbit corresponding to $\bar{P}$.

(2) The moment map $\mu: T^*X\to\bar{\mathcal{O}}$ is birational.

(3) There exists an admissible character on $\bar{N}$.

One of the main results of this article is:

THEOREM A (Theorem 3.4.1). — *We assume the moment map* μ *is birational and* ψ *is admissible. Then, for any irreducible Harish-Chandra module* V *with an integral infinitesimal character, the followings are equivalent.*

(1) $\mathrm{Wh}_\psi^\infty(V)\neq 0$ *and* $\dim \mathrm{Wh}_\psi^\infty(V)<\infty$.

(2) $\mathrm{WF}(V)=i\bar{\mathcal{O}}_{\bar{P}}$.

Remark. — It is known that $\mathrm{Wh}_\psi^\infty(V)\neq 0$ implies $i\bar{\mathcal{O}}_{\bar{P}}\subseteq \mathrm{WF}(V)$ ([Mat2], also *see* 3.4).

Let $\mathfrak{g}$ be the Lie algebra of G (So, the complexified Lie algebra is $\mathfrak{g}\times\mathfrak{g}$.) We fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. We denote by P (resp. W) the integral weight lattice (resp.

the Weyl group) of $(\mathfrak{g}, \mathfrak{h})$. We remark that $W \times W$ acts on the polynomial ring on $\mathfrak{h}^* \times \mathfrak{h}^*$. We assume λ, μ is regular and denote by $F_{\lambda, \mu}$ the set of irreducible Harish-Chandra modules V with the infinitesimal character (λ, μ) such that $WF(V) = i\bar{\mathcal{O}}_{\bar{P}}$. For the dimension of $Wh_{\Psi}^{\infty}(V)$, we have a result. Since it requires further terminologies to state the whole statement, I do not present our second main result precisely here (*see* Theorem 3.3.6). However, at least, it contains the following result.

THEOREM B (*cf.* Theorem 3.3.6). — *Let* V *be an irreducible Harish-Chandra module with a regular integral infinitesimal character such that* $WF(V) = i\bar{\mathcal{O}}_{\bar{P}}$ *and let* $\Theta_V(\lambda, \mu)$, $(\lambda, \mu \in P)$ *be the coherent family in which* V *is embedded. Then* $P \times (\ni (\lambda, \mu) \rightsquigarrow \dim Wh_{\Psi}^{\infty}(\Theta_V)(\lambda, \mu)$ *is well-defined and extend uniquely to a harmonic polynomial* (*say* $p_{\Psi}[V]$) *on* $\mathfrak{h}^* \times \mathfrak{h}^*$. *Fix regular* λ, $\mu \in P$. *If we consider the* $\mathbb{C}$-*linear space* E *which is spanned by*

$$\{ p_{\Psi}[V] \mid V \in F_{\lambda, \mu} \}$$

then E *is closed under the* $W \times W$ *action. Moreover* E *is irreducible as a* $W \times W$-*representation and written by* $\sigma \otimes \sigma$. *Here,* σ *is the Goldie rank polynomial representation* (*the Springer representation*) *associated with* $\mathcal{O}_{\bar{P}}$.

The classical multiplicity one theorem can be related to the fact the Springer representation associated with the regular nilpotent orbit is a trivial representation $\mathbb{C}.1$.

The points of our proof are as follows:

(1) The exactness of $V \rightsquigarrow Wh_{\Psi}^{\infty}(V)$ (for precise statement, *see* Proposition 3.2.1).

(2) Yamashita's multiplicity theorem on induced representations [Y1].

(3) Vogan's construction of harmonic polynomials from coherent families [Vo1].

(4) Deep analysis on double cell representations due to Joseph, Lusztig, and, especially, Barbasch-Vogan [BV2, 3, 4].

Using the above facts and applying a similar method to [D3], we prove Theorem B above. Theorem A is a corollary of Theorem B and results in [Mat2, 5] (*cf.* Lemma 3.4.2 below).

The most crucial part is (1) above. It was W. Casselman who proved the corresponding result for the nilradical of a minimal parabolic subgroup of a general real semisimple Lie group. The main ingredients of his proof are:

(1) The vanishing of higher twisted cohomology groups of principal series.

(2) Casselman's subrepresentation theorem.

Casselman proved the above (1) by a very ingenious method "the Bruhat filtration". (He sketched the proof in [Ca1].) We show that his method is also applicable to a proof of a generalization of his result, which we need, under some minor modifications. We also use an idea from [Y1].

To generalize the above (2) is much more difficult, I think. Casselman's subrepresentation theorem itself is a fairly deep result. However, if we consider Harish-Chandra modules with integral infinitesimal character for complex semisimple Lie groups, we get an embedding theorem (Theorem 2.4.1) using the deep results of Joseph [Jo12], Lusztig [Lu7], and Lusztig-Xi Nanhua [LuN].

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1. Notations and preliminaries

1.1. General notations. — In this article, we use the following notations and conventions.

As usual we denote the complex number field, the real number field, the rational number field, the ring of (rational) integers, and the set of non-negative integers by $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, and $\mathbb{N}$ respectively. We denote by 0 the empty set. For each set A, we denote by card A the cardinality of A. Sometimes "i" denotes the imaginary unit $\sqrt{-1}$.

For any (non commutative) $\mathbb{C}$-algebra R, "ideal" means "2-sided ideal", "R-module" means "left R-module", and sometimes we denote by 0 (resp. 1) the trivial R-module $\{0\}$ (resp. $\mathbb{C}$). For 1n R-module M of finite length, we denote by JH (M) the set of irreducible constituents of M including multiplicities and denote by l(M) the length of M.

For an abelian category $\mathscr{A}$, we denote by K ($\mathscr{A}$) the Grothendieck group of $\mathscr{A}$. We denote by [A] the canonical image of an object A of $\mathscr{A}$ in K ($\mathscr{A}$). If $\mathscr{A}$ is a full subcategory of the category of R-modules of finite length, then [A] = [B] if and only if JH (A) = JH (B) for all objects A and B of $\mathscr{A}$.

Often, we identify a (small) category and the set of its objects.

Hereafter "dim" means the dimension as a complex vector space, and "$\otimes$" (resp. Hom) means the tensor product over $\mathbb{C}$ (resp. the space of $\mathbb{C}$-linear mappings), unless we specify.

For a complex vector space V, we denote by V^* the dual vector space and we denote by $S(V)$ the symmetric algebra of V. Sometimes, we identify $S(V)$ and the polynomial ring over V^*, if V is finite-dimensional. For any subspace W of V, put $W^\perp = \{ f \in V^* \mid f \mid W \equiv 0 \}$.

For real analytic manifold X, we denote by $C^\infty(X)$ the space of C^∞-functions on X. For a subset U of X, we denote by $\bar{U}$ the closure of U.

1.2. Notations for semisimple Lie algebras. — In this article, we fix the following notations. Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, and Δ the root system with respect to $(\mathfrak{g}, \mathfrak{h})$. We fix some positive root system Δ^+ and let Π be the set of simple roots. Let W be the Weyl group of the pair $(\mathfrak{g}, \mathfrak{h})$ and let $\langle\, , \rangle$ be the Killing form of $\mathfrak{g}$. We also denote the inner product on $\mathfrak{h}^*$ which is induced from the Killing form by the same symbols $\langle\, , \rangle$. For $\alpha \in \Delta$, we denote by s_α the reflection in W with respect to α. We denote by $l(w)$ the length of $w \in W$ and denote by w_0 the longest element of W.

For $\alpha \in \Delta$, we define the coroot $\check{\alpha}$ by $\check{\alpha} = 2\alpha/\langle \alpha, \alpha \rangle$, as usual.

We call $\lambda \in \mathfrak{h}^*$ is dominant (resp. anti-dominant), if $\langle \lambda, \check{\alpha} \rangle$ is not a negative (resp. positive) integer, for each $\alpha \in \Delta^+$. We call $\lambda \in \mathfrak{h}^*$ regular, if $\langle \lambda, \alpha \rangle \neq 0$, for each $\alpha \in \Delta$. We denote by P the integral weight lattice, namely

$$P = \{ \lambda \in \mathfrak{h}^* \mid \langle \lambda, \check{\alpha} \rangle \in \mathbb{Z} \text{ for all } \alpha \in \Delta \}.$$

If $\lambda \in \mathfrak{h}^*$ is contained in P, we call λ an integral weight. We denote by P^{--} (resp. P^{++}) the set of anti-dominant (resp. dominant) regular integral weights in $\mathfrak{h}^*$. We also denote by P^- (resp. P^+) the set of anti-dominant (resp. dominant) integral weights in $\mathfrak{h}^*$. We define $\rho \in P$ by $\rho = 1/2 \sum_{\alpha \in \Delta^+} \alpha$.

Put

$$\mathfrak{g}_\alpha = \{ X \in \mathfrak{g} \mid \forall H \in \mathfrak{h}\, [H, X] = \alpha(H) X \},$$
$$\mathfrak{u} = \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha,$$
$$\bar{\mathfrak{u}} = \sum_{-\alpha \in \Delta^+} \mathfrak{g}_\alpha.$$

Put

$$\mathfrak{b} = \mathfrak{h} + \mathfrak{u},$$
$$\bar{\mathfrak{b}} = \mathfrak{h} + \bar{\mathfrak{u}}.$$

Then $\mathfrak{b}$ and $\bar{\mathfrak{b}}$ are Borel subalgebras of $\mathfrak{g}$.

Nest, we fix notations for a parabolic subalgebra (which contains $\mathfrak{b}$). Hereafter, through this article we fix an arbitrary subset S of Π. Let $\bar{S}$ be the set of the elements of Δ which are written by linear combinations of elements of S over $\mathbb{Z}$. Put

$$\mathfrak{a}_S = \{ H \in \mathfrak{h} \mid \forall \alpha \in S\ \alpha(H) = 0 \},$$

$$\mathfrak{l}_S = \mathfrak{h} + \sum_{\alpha \in \bar{S}} \mathfrak{g}_\alpha,$$

$$\mathfrak{n}_S = \sum_{\alpha \in \Delta^+ - \bar{S}} \mathfrak{g}_\alpha,$$

$$\bar{\mathfrak{n}}_S = \sum_{-\alpha \in \Delta^+ - \bar{S}} \mathfrak{g}_\alpha,$$

$$\mathfrak{m}_S = \{ X \in \mathfrak{l}_S \mid \forall H \in \mathfrak{a}_S \langle X, H \rangle = 0 \},$$

$$\mathfrak{p}_S = \mathfrak{m}_S + \mathfrak{a}_S + \mathfrak{n}_S = \mathfrak{l}_S + \mathfrak{n}_S,$$

$$\bar{\mathfrak{p}}_S = \mathfrak{m}_S + \mathfrak{a}_S + \bar{\mathfrak{n}}_S = \mathfrak{l}_S + \bar{\mathfrak{n}}_S.$$

Then $\mathfrak{p}_S$ (resp. $\bar{\mathfrak{p}}_S$) is a parabolic subalgebra of $\mathfrak{g}$ which contains $\mathfrak{b}$ (resp. $\bar{\mathfrak{b}}$). Conversely, for an arbitrary parabolic subalgebra $\mathfrak{p} \supseteq \mathfrak{b}$, there exists some $S \subseteq \Pi$ such that $\mathfrak{p} = \mathfrak{p}_S$. We denote by W_S the Weyl group for $(\mathfrak{l}_S, \mathfrak{h})$. W_S is identified with a subgroup of W generated by $\{ s_\alpha \mid \alpha \in S \}$. We denote by w_S the longest element of W_S.

It is known that there is a unique nilpotent (adjoint) orbit (say $\mathcal{O}_S$) whose intersection with $\mathfrak{n}_S$ is Zariski dense in $\mathfrak{n}_S$. $\mathcal{O}_S$ is called the Richardson orbit with respect to $\mathfrak{p}_S$. Using the Killing form, we sometimes identify $\mathfrak{g}$ with $\mathfrak{g}^*$. So, sometimes we regard $\mathcal{O}_S$ as a coadjoint orbit.

We denote by B, $\bar{B}$, A_S, L_S, N_S, . . . the analytic subgroup of G corresponding to $\mathfrak{b}$, $\bar{\mathfrak{b}}$, $\mathfrak{a}_S$, $\mathfrak{l}_S$, $\mathfrak{n}_S$, . . . respectively. We denote by Ad the adjoint actions on Lie algebras.

We denote the anti-automorphism of $U(\mathfrak{g})$ generated by $X \rightsquigarrow -X$ $(X \in \mathfrak{g})$ as follows.

$$u \rightsquigarrow \check{u}, \qquad (u \in U(\mathfrak{g})).$$

For an ideal I in $U(\mathfrak{g})$, we define $I^\vee = \{ \check{u} \mid u \in I \}$. Then $I^\vee$ is also an ideal.

Next we fix the notations for highest weight modules.

Define

$$P_S^{++} = \{ \lambda \in \mathfrak{h}^* \mid \forall \alpha \in S \langle \lambda, \check{\alpha} \rangle \in \{ 1, 2, \ldots \} \}.$$

If $S = \Pi$ (resp. $S = \varnothing$), then $P_S^{++} = P^{++}$ (resp. $P_S^{++} = \mathfrak{h}^*$).

For $\mu \in \mathfrak{h}^*$ such that $\mu + \rho \in P_S^{++}$, we denote by $\sigma_S(\mu)$ the irreducible finite-dimensional $\mathfrak{l}_S$-representation whose highest weight is μ. Let $E_S(\mu)$ be the representation space of $\sigma_S(\mu)$.

We assume $\mu + \rho \in P_S^{++}$. We define a left action of $\mathfrak{n}_S(\mu)$ by $X.v = 0$ for all $X \in \mathfrak{n}_S$ and $v \in E_S(\mu)$. Then we can regard $E_S(\mu)$ as a $U(\mathfrak{p}_S)$-module.

For $\mu \in P_S^{++}$, we define the generalized Verma module (Lepowski [Le]) as follows.

$$M_S(\mu) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_S)} E_S(\mu - \rho).$$

For all $\mu \in \mathfrak{h}^*$, we define the Verma module by $M(\lambda) = M_\varphi(\lambda)$.

Let $L(\mu)$ be the unique highest weight $U(\mathfrak{g})$-module with the highest $\mu - \rho$. Namely, $L(\mu)$ is a unique irreducible quotient of $M(\mu)$. For $\mu \in P_S^{++}$, the canonical projection of $M(\lambda)$ to $L(\lambda)$ is factored by $M_S(\lambda)$.

For $\mu \in P^+$, we denote by E_μ the irreducible finite-dimensional $U(\mathfrak{g})$-module with the highest weight μ. Clearly $E_\mu = L(\mu + \rho)$ for all $\mu \in P^+$.

We denote by $Z(\mathfrak{g})$ the center of $U(\mathfrak{g})$. It is well-known that $Z(\mathfrak{g})$ acts on $M(\lambda)$ by the Harish-Chandra homomorphism $\chi_\lambda : Z(\mathfrak{g}) \to \mathbb{C}$ for all λ. $\chi_\lambda = \chi_\mu$ if and only if there exists some $w \in W$ such that $\lambda = w\mu$.

1.3. Associated variety, Gelfand-Kirillov dimensions, and multiplicities. — We recall some important invariants for finitely generated $U(\mathfrak{g})$-modules. For details, *see* [Vo1], [Vo4].

For a positive integer n, we denote by $U_n(\mathfrak{g})$ the subspace of $U(\mathfrak{g})$ spanned by products of at most n elements of $\mathfrak{g}$. We also put $U_0(\mathfrak{g}) = \mathbb{C} \subsetneq U(\mathfrak{g})$ and $U_{-1}(\mathfrak{g}) = 0$. Then the associated graded algebra $\operatorname{gr} U(\mathfrak{g}) = \bigoplus_{n \geq 0} U_n(\mathfrak{g})/U_{n-1}(\mathfrak{g})$ is naturally isomorphic to the symmetric algebra $S(\mathfrak{g})$ of $\mathfrak{g}$. Let M be a finitely generated $U(\mathfrak{g})$-module and $v_1, \ldots, v_h$ its generators. Put $M_n = \sum_{1 \leq i \leq h} U_n(\mathfrak{g}) v_i$ and consider the associated graded module over $S(\mathfrak{g}) : \operatorname{gr} M = \bigoplus_{n \geq 0} M_n/M_{n-1}$. Since we can identify $S(\mathfrak{g})$ and the polynomial ring over $\mathfrak{g}^*$, we can define the associated variety of M as follows.

$$\operatorname{Ass}(M) = \{ v \in \mathfrak{g}^* \mid f(v) = 0 \text{ for all } f \in \operatorname{Ann}_{S(\mathfrak{g})}(\operatorname{gr} M) \}.$$

Ass (M) is a Zariski closed set of $\mathfrak{g}^*$ and its definition does not depend on the choice of generators $v_1, \ldots, v_h$. Using the Killing form, we regard often Ass (M) as a closed subvariety of $\mathfrak{g}$. We call the dimension of Ass (M) the Gelfand-Kirillov dimension and we write Dim (M). We define $\operatorname{Dim}(0) = -\infty$, where 0 is the trivial module.

Next we introduce another important invariant, the multiplicity. A classical theorem of Hilbert-Serre implies that there exists some polynomial $\chi(x)$ in one variable over $\mathbb{Q}$ such that $\dim_{\mathbb{C}} M_n = \chi(n)$ for sufficiently large n. We can also see the Gelfand-Kirillov dimension of M is the degree of $\chi(x)$. For $d \in \mathbb{N}$, we define $c_d(M)$ by

$$c_d(M) = \begin{cases} \text{the coefficient of } x^{\operatorname{Dim}(M)} \text{ in } d!\,\chi(x) & \text{if } d = \operatorname{Dim}(M) \\ 0 & \text{if } d > \operatorname{Dim}(M). \\ \infty & \text{if } d < \operatorname{Dim}(M). \end{cases}$$

If $d = \operatorname{Dim}(M)$, we call $c_d(M)$ the multiplicity of M. The multiplicity is always a non-negative integer and its definition does not depends on the choice of generator $v_1, \ldots, v_h$.

1.4. NOTATIONS PRELIMINARIES FOR COMPLEX SEMISIMPLE LIE GROUPS. — Here, we introduce some notations on representations of complex semisimple Lie groups and review some fundamental results. First, we introduce some notations. For details, *see* [D1].

Hereafter G will denote a connected simply-connected complex semisimple linear Lie group whose Lie algebra is $\mathfrak{g}$. Indeed, for our purpose, there is no harm in supposing G is simply-connected.

We can regard $\mathfrak{g}$ as a real Lie algebra as well as a complex Lie algebra. So, we want to consider its complexification.

First, we fix a (complex) Cartan subalgebra $\mathfrak{h}$ and a triangle decomposition $\mathfrak{g} = \bar{\mathfrak{u}} + \mathfrak{h} + \mathfrak{u}$ as above. We denote by $\mathfrak{g}_0$ the normal real form of $\mathfrak{g}$ which is compatible with the above decomposition and denote by $X \rightsquigarrow \bar{X}$ the complex conjugaison with respect to $\mathfrak{g}_0$. Then there is an anti-automorphism $X \rightsquigarrow {}^t X$ of $\mathfrak{g}$ which satisfies the following (1)-(3).

(1) ${}^t\mathfrak{g}_0 = \mathfrak{g}_0$.

(2) ${}^t\mathfrak{u} = \bar{\mathfrak{u}}$, ${}^t\bar{\mathfrak{u}} = \mathfrak{u}$.

(3) ${}^t X = X \, (X \in \mathfrak{h})$.

We extend $X \rightsquigarrow {}^t X$ to an anti-automorphism on $U(\mathfrak{g})$.

We define a homomorphism of real Lie algebra $\mathfrak{g} \to \mathfrak{g} \times \mathfrak{g}$ by $X \rightsquigarrow (X, X)$ for $X \in \mathfrak{g}$. Then the image of this homomorphism is a real form of $\mathfrak{g} \times \mathfrak{g}$. Hence, we can regard $\mathfrak{g} \times \mathfrak{g}$ as the complexification $\mathfrak{g}_c$ of $\mathfrak{g}$. $\mathfrak{k}_c = \{ (X, -{}^t X) \mid X \in \mathfrak{g} \}$ is identified with the complexification of a compact form of $\mathfrak{g}$. $\mathfrak{k}_c$ is also identified with $\mathfrak{g}$ by an isomorphism $X \rightsquigarrow (X, -{}^t X)$ as complex Lie algebras. So, sometimes we regard $E_\mu \, (\mu \in P^+)$ as a $U(\mathfrak{k}_c)$-module.

Put $\mathfrak{k} = \{ (X, Y) \in \mathfrak{k}_c \mid X = \bar{Y} \}$. Hence $\mathfrak{k}$ is a compact real form of $\mathfrak{g} = \{ (X, \bar{X}) \mid X \in \mathfrak{g} \}$. We denote by K the analytic subgroup of G with respect to $\mathfrak{k}$.

Next, we consider the complexification of parabolic subalgebras. Under the identification: $\mathfrak{g} = \{ (X, \bar{X}) \mid X \in \mathfrak{g} \}$, $\mathfrak{p}_S$ is identified with $\{ (X, \bar{X}) \mid X \in \mathfrak{p}_S \}$. So, the complexification $(\mathfrak{p}_S)_c$ [resp. $(\mathfrak{n}_S)_c$] *of* $\mathfrak{p}_S$ (resp. $\mathfrak{n}_S$) is identified with $\mathfrak{p}_S \times \mathfrak{p}_S$ (resp. $\mathfrak{n}_S \times \mathfrak{n}_S$).

We put $U = U(\mathfrak{g}_c) = U(\mathfrak{g}) \otimes U(\mathfrak{g})$.

Let V be a U-module. If the center $Z(\mathfrak{g}_c)$ of U acts on V by scalar, we say that V has an infinitesimal character. An infinitesimal character is written by the Harish-Chandra homomorphisms. Namely, if we identify $Z(\mathfrak{g}_c)$ with $Z(\mathfrak{g}) \otimes Z(\mathfrak{g})$, the it is written of the form $\chi_\lambda \otimes \chi_\mu$ for some $\lambda, \mu \in \mathfrak{h}^*$. In this case, we say V has an infinitesimal character (λ, μ). We say that V has an integral (resp. a regular) infinitesimal character, if $\lambda, \mu \in P$ (resp. λ and μ are regular). An arbitrary irreducible U-module has an infinitesimal character.

If V is a U-module, put

$$\mathrm{LAnn}(V) = \{ u \in U(\mathfrak{g}) \mid u \otimes 1 \in \mathrm{Ann}_U(V) \},$$

$$\mathrm{RAnn}(V) = \{ u \in U(\mathfrak{g}) \mid 1 \otimes u \in \mathrm{Ann}_U(V) \}.$$

For a U-module V and $\lambda \in P^{++}$, we define as follows.

$$V_{(\mu)} = \{ v \in V \mid U(\mathfrak{k}_c)\, v \text{ is isomorphic to the direct sum of some copies of } E_\mu \}.$$

For a U-module V, we call $v \in V$ is $\mathfrak{k}_c$-finite, if $\dim U(\mathfrak{k}_c)\, v$ is finite. A U-module V is called $(\mathfrak{g}_c, \mathfrak{k}_c)$-module, if all the element of V is $\mathfrak{k}_c$-finite. A $(\mathfrak{g}_c, \mathfrak{k}_c)$-module V is called admissible, if $V_{(\mu)}$ is finite-dimensional (or trivial) for all $\mu \in P^+$. An admissible $(\mathfrak{g}_c, \mathfrak{k}_c)$-module of finite length is called a Harish-Chandra $(\mathfrak{g}_c, \mathfrak{k}_c)$-module. The category of Harish-Chandra $(\mathfrak{g}_c, \mathfrak{k}_c)$-modules is defined as a full subcategory of the category of U-modules.

For $U(\mathfrak{g})$-modules M and N, the dual space of the tensor product $(M \times N)^*$ can be regarded as a U-module in the obvious way. We denote by $L^*(M \otimes N)$ the $\mathfrak{k}_c$-finite part of $(M \otimes N)^*$. Namely,

$$L^*(M \otimes N) = \{ v \in (M \otimes N)^* \mid \dim U(\mathfrak{k}_c)\, v < \infty \}.$$

Hence, $L^*(M \otimes N)$ is a $(\mathfrak{g}_c, \mathfrak{k}_c)$-module.

Next, we construct another $(\mathfrak{g}_c, \mathfrak{k}_c)$-module $L(M, N)$ from $U(\mathfrak{g})$-modules M and N. Hom (M, N) has natural structure of a $U(\mathfrak{g})$-bimodule. We introduce a U-module structure on Hom (M, N) by

$$(u \otimes v)\, \varphi = {}^t\check{u}\, \varphi\, \check{v} \qquad (u, v \in U(\mathfrak{g}), \varphi \in \mathrm{Hom}(M, N)).$$

We denote the $\mathfrak{k}_c$-finite part of Hom (M, N) by $L(M, N)$.

We easily have:

LEMMA 1.4.1 ([Jo2] 4.3). — *Let $\lambda \in \mathfrak{h}^*$ and let* M *be a subquotient of a Verma module. Then, we have an isomorphism of* U-*modules:*

$$L(M, L(\lambda)) \cong L^*(L(\lambda) \otimes M).$$

Next we define the principal series representations. For $\lambda, \mu \in \mathfrak{h}^*$, we define

$$L(\lambda, \mu) = L^*(M(-\lambda) \otimes M(-\mu)).$$

The relation with the usual definition of principal series is found in [D1]. It is known that $L(\lambda, \mu) = 0$ unless $\lambda - \mu \in P$.

Let λ, μ satisfy $\lambda - \mu \in P$. We denote by $V(\lambda, \mu)$ the unique irreducible subquotient of $L(\lambda, \mu)$ containing a $\mathfrak{k}_c$-subrepresentation isomorphic to $E_{\lambda-\mu}$. $V(\lambda, \mu)$ and $L(\lambda, \mu)$ have an infinitesimal character (λ, μ).

The irreducible Harish-Chandra $(\mathfrak{g}_c, \mathfrak{k}_c)$-modules are parametrized as follows.

THEOREM 1.4.2 (Zhelobenko, *see* [D1] 4, [BV4], Proposition 1.8). — (1) *Any irreducible Harish-Chandra Harish-Chandra $(\mathfrak{g}_c, \mathfrak{k}_c)$-module-module is isomorphic to $V(\lambda, \mu)$ for some $\lambda, \mu \in \mathfrak{h}^*$ such that $\lambda - \mu \in P$.*

(2) *Let $\lambda, \mu, \lambda', \mu' \in \mathfrak{h}^*$ and assume $\lambda - \mu, \lambda' - \mu' \in P$. Then, $V(\lambda, \mu) \cong V(\lambda', \mu')$ if and only if there exists some $w \in W$ such that $\lambda = w\lambda'$ and $\mu = w\mu'$.*

The following may be regarded as a special part of the Bernstein-Gelfand-Joseph-Enright category equivalence theorem (*cf.* [BG], [GJ]).

PROPOSITION 1.4.3 (Joseph [Jo2] 4.5). — *Let* $\mu \in \mathfrak{h}^*$ *be dominant and regular. Then for all* $\nu \in \mathfrak{h}^*$ *such that* $\mu - \nu \in P$,

$$L(M(\mu), L(\nu)) \cong V(-\nu, -\mu).$$

Hence, we have another parametrization of irreducible Harish-Chandra $(\mathfrak{g}_c, \mathfrak{k}_c)$-modules with regular infinitesimal characters as follows.

COROLLARY 1.4.4. — *Let* V *be any irreducible Harish-Chandra Harish-Chandra* $(\mathfrak{g}_c, \mathfrak{k}_c)$-*module-module with a regular integral infinitesimal character. Then there exist a unique pair of anti-dominant regular characters* $(\lambda, \mu) \in P^{--} \times P^{--}$ *and a unique* $z \in W$ *such that*

$$V \cong L(M(w_0 \lambda), L(z^{-1} \mu)).$$

We consider the associated variety, the Gelfand-Kirillov dimension, and the multiplicity of a finitely generated $\mathfrak{g}_c$-module V. In this case, Ass (V) is a closed subvariety of $\mathfrak{g}_c^*$ (or $\mathfrak{g}_c = \mathfrak{g} \times \mathfrak{g}$).

1.5. TRANSLATION PRINCIPLE. – Here, we introduce the translation principle. For details, *see* [BG], etc.

For λ, $\mu \in \mathfrak{h}^*$ and for a Harish-Chandra $(\mathfrak{g}_c, \mathfrak{k}_c)$-module V, we say that V has the generalized infinitesimal character (λ, μ), if every irreducible consituent of V has the infinitesimal character (λ, μ). We define a full subcategory $\mathscr{H}$ of the category of U-modules as follows.

$$\mathscr{H} = \{ V \mid V \text{ is a Harish-Chandra } (\mathfrak{g}_c, \mathfrak{k}_c)\text{-module and any irreducible constituent of } V \text{ has an integral infinitesimal character} \}.$$

For λ, $\mu \in P$, we denote by $\mathscr{H}[\lambda, \mu]$ the category of objects of $\mathscr{H}$ with the generalized infinitesimal character (λ, μ).

From the Chinese reminder theorem, we have the following direct sum decomposition of categories.

$$\mathscr{H} = \bigoplus_{\lambda,\ \nu \in P^+} \mathscr{H}[\lambda, \mu].$$

We denote by $P_{\lambda, \mu}$ the projection function of $\mathscr{H}$ onto $\mathscr{H}[\lambda, \mu]$. For $\eta \in P$, we denote by V_η the finite-dimensional irreducible U($\mathfrak{g}$)-module with extreme weight η. For λ, μ, λ', $\mu' \in P$, we define the translation function $T_{\lambda, \mu}^{\lambda', \mu'} : \mathscr{H}[\lambda, \mu] \to \mathscr{H}[\lambda', \mu']$ as follows.

$$T_{\lambda, \mu}^{\lambda', \mu'}(V) = P_{\lambda', \mu'}(V \otimes (V_{\lambda' - \lambda} \otimes V_{\mu' - \mu})) \qquad (V \in \mathscr{H}[\lambda, \mu]).$$

The following is important.

THEOREM 1.5.1. — *The translation functor is exact. If* λ, μ, λ', $\mu' \in P^{++}$, *then* $T_{\lambda, \mu}^{\lambda', \mu'} : \mathscr{H}[\lambda, \mu] \to \mathscr{H}[\lambda', \mu']$ *is an equivalence of categories.*

From the definition, we can easily deduce the following results.

PROPOSITION 1.5.2. — *We assume* λ, μ, λ', $\mu' \in P^{--}$.

(1) *If* $x \in W$, *then*

$$T^{-\mu', -\lambda'}_{-\mu, -\lambda}(L(M(w_0\lambda), L(x^{-1}\mu))) = L(M(w_0\lambda'), L(x^{-1}\mu')).$$

(2) *If* $x, y \in W$ *satisfy* $x\lambda$, $y\mu \in P_S^{++}$, *then*

$$T^{-\mu', -\lambda'}_{-\mu, -\lambda}(L^*(M_S(x\mu) \otimes M_S(y\lambda))) = L^*(M_S(x\mu') \otimes M_S(y\lambda')).$$

(3) *If* $x, y \in W$, *then*

$$T^{\mu', -\lambda'}_{-\mu, -\lambda}(L(L(x\lambda), L(y\mu))) = L(L(x\lambda'), L(y\mu')).$$

Fix λ', $\mu' \in P^{--}$. For λ, $\mu \in P^{-}$, $z \in W$, we define

$$V^{\odot}(z^{-1}; -\mu, -\lambda) = T^{-\mu, -\lambda}_{-\mu', -\lambda'} L(M(w_0\lambda'), L(z^{-1}\mu')).$$

From Proposition 1.5.2, we see the definition of $V^{\odot}(z^{-1}; -\mu, -\lambda)$ does not depend on the choice of λ', $\mu' \in P^{--}$. We also see if λ, $\mu \in P^{--}$, then

$$V^{\odot}(z^{-1}; -\mu, -\lambda) = L(M(w_0\lambda), L(z^{-1}\mu)).$$

For general (possibly singular) λ, $\mu \in P^{-}$, $V^{\odot}(z^{-1}; -\mu, -\lambda)$ is either irreducible or 0. If it is irreducible, then we have

$$V^{\odot}(z^{-1}; -\mu, -\lambda) = L(M(w_0\lambda), L(z^{-1}\mu)).$$

The following generalization of Corollary 1.4.4 is known.

THEOREM 1.5.3. — *Let* V *be any irreducible Harish-Chandra* $(\mathfrak{g}_c, \mathfrak{k}_c)$-*module with an integral infinitesimal character* $(-\mu, -\lambda) \in P^{+} \times P^{+}$. *Then there exists some* $w \in W$ *such that* $V \cong V^{\odot}(w^{-1}; -\mu, -\lambda)$.

1.6. GLOBALIZATIONS OF HARISH-CHANDRA MODULES. — Let H be an admissible continuous representation of G on a Hilbert space on which K acts unitarily and let V be the K-finite part. Harish-Chandra proved V has a natural structure of a Harish-Chandra $(\mathfrak{g}_c, \mathfrak{k}_c)$-module.

Conversely, if we fix a Harish-Chandra module V first, then there exists some admissible Hilbert space G-representation H whose K-finite part is V ([W1]). Here, H is not unique in general.

In their joint work, Casselman and Wallach ([W1], [Ca3]) proved that, if we consider the space H_∞ of C^∞-vectors in H, H_∞ is uniquely determined by V as a Frechet representation. In fact, H_∞ is characterized as a Frechet representation with growth conditions. For details, see the above-mentioned references.

So, we write V_∞ for H_∞. $V \rightsquigarrow V_\infty$ is an exact functor [Ca3].

We denote by V'_∞ the continuous dual space of V.

Let V be a Harish-Chandra $(\mathfrak{g}_c, \mathfrak{k}_c)$-module and let E be a finite dimensional U-module. Then, we can easily see:

LEMMA 1.6.1

$$V'_\infty \otimes E^* = (V \otimes E)'_\infty.$$

2. Additive invariants and an embedding theorem

2.1. COHERENT FAMILIES AND ADDITIVE INVARIANTS. — First, we recall the notion of coherent family and collect some elementary properties.

We consider the category $\mathscr{H}$ (1.5).

A full subcategory $\mathscr{M}$ of $\mathscr{H}$ is called good subcategory if $\mathscr{M}$ satisfies the following condition (G 1) and (G 2).

(G1) For any object V of $\mathscr{M}$, every subquotient of V is an object of $\mathscr{M}$.

(G2) Let E be any finite dimensional representation of $\mathfrak{g}_c$ and let V be an object of $\mathscr{M}$. Then $E \otimes V$ is an object of $\mathscr{M}$.

We remark that $\mathscr{H}$ itself satisfies the above properties (G1) and (G2).

For $d \in \mathscr{N}$, we define a full subcategory $\mathscr{H}_d$ of $\mathscr{H}$ by

$$\mathscr{H}_d = \{ V \in \mathscr{H} \mid \operatorname{Dim}(V) \leqq d \}.$$

$\mathscr{H}_d$ is a good subcategory of $\mathscr{H}$ for each d.

We denote by $\mathscr{M}[\nu, \eta]$ the category of objects of $\mathscr{M}$ with the generalized infinitesimal character (ν, η). Then, the Grothendieck group $K(\mathscr{M}[\nu, \eta])$ is regarded as a subgroup of $K(\mathscr{M})$.

A map $\Theta : P \times P \to K(\mathscr{H})$ is called a coherent family (on $P \times P$) if Θ satisfies the following condition (C1) and (C2).

(C1) $\Theta(\nu, \eta) \in K(\mathscr{H}[\nu, \eta])$ for all $\nu, \eta \in P$.

(C2) For any finite dimensional $\mathfrak{g}_c$-module E and $\lambda, \mu \in P$, we have

$$\Theta(\nu, \eta) \otimes E = \sum_{(\delta_1, \delta_2) \in P \times P} m_{\delta_1, \delta_2} \Theta(v + \delta_1, \eta + \delta_2).$$

Here, m_{δ_1, δ_2} denotes the multiplicity of $\mathfrak{h}_c (= \mathfrak{h} \times \mathfrak{h})$-weight (δ_1, δ_2) in E.

Let Θ_1 and Θ_2 be coherent families. We define the sum $\Theta_1 + \Theta_2$ by $(\Theta_1 + \Theta_2)(\nu, \eta) = \Theta_1(\nu, \eta) + \Theta_2(\nu, \eta)\ (\nu, \eta \in P)$.

A coherent family Θ is called irreducible, if $\Theta(\nu, \eta)$ is the image of an irreducible Harish-Chandra $(\mathfrak{g}_c, \mathfrak{k}_c)$-module into $K(\mathscr{H})$ for every $\nu, \eta \in P^{++}$. Let V be an arbitrary irreducible Harish-Chandra $(\mathfrak{g}_c, \mathfrak{k}_c)$-module with an integral infinitesimal character and let $(-\mu, -\lambda)$ be its ininitesimal character. We can assume $\lambda, \mu \in P^-$. It is known that there exists a unique irreducible coherent family Θ_V such that $\Theta_V(-\mu, -\lambda) = [V]$. If

$w \in W$ and $V = L(M(w_0\lambda), L(w^{-1}(\mu))) \in \mathscr{H}[-\mu, -\lambda]$, we have $\Theta_V(-\mu', -\lambda') = [V^{\odot}(w^{-1}; -\mu', -\lambda')]$ for all $\lambda', \mu' \in P^-$.

Fix λ, $\mu \in P^-$. Then, any $X \in K(\mathscr{H})[-\mu, -\lambda]$ is written by the finite sum $X = \sum_i n_i [V_i]$, where $n_i \in \mathbb{Z}$ and V_i is an irreducible Harish-Chandra $(\mathfrak{g}_c, \mathfrak{k}_c)$-module with the infinitesimal character $(-\mu, -\lambda)$ for all i. Then we define a coherent family Θ_X by $\Theta_X = \sum_i n_i \Theta_{V_i}$. Clearly, we have $\Theta_X(-\mu, -\lambda) = X$ and $X \rightsquigarrow \Theta_X$ defines a homomorphism of abelian groups.

Fix a good subcategory $\mathscr{M}$ of $\mathscr{H}$. We remark that, for $V \in \mathscr{M}$, we have $\Theta_V(\nu, \eta) \in K(\mathscr{M}[\nu, \eta])$ for all $\nu, \eta \in P$. Let λ, $\mu \in P^-$. We introduce a $W \times W$-module structure on $K(\mathscr{M}[-\mu, -\lambda])$ as follows. For $K(\mathscr{M}[-\mu, -\lambda])$ and $w, y \in W$, we define

$$(w, y).X = \Theta_X(-w^{-1}\mu, -y^{-1}\lambda).$$

$W \times W$-representation $K(\mathscr{M}[-\mu, -\lambda])$ (or $K(\mathscr{M}[-\mu, -\lambda]) \otimes_{\mathbb{Z}} \mathbb{C}$) is called a coherent continuation representation.

Next, we introduce the notion of additive invariants (*cf.* [Mat5]). Let $\mathscr{M}$ be a good subcategory of $\mathscr{H}$ and let a be a map of the set of objects in $\mathscr{M}$ to $\mathbb{N}$. a is called an additive invariant on $\mathscr{M}$, if it satisfies the following two conditions (A1) and (A2).

(A1) For all exact sequence in $\mathscr{M}$

$$0 \to M_1 \to M_2 \to M_3 \to 0,$$

we have

$$a(M_2) = a(M_1) + a(M_3).$$

(A2) For any $M \in \mathscr{M}$ and any finite dimensional $U(\mathfrak{g}_c)$-module E, we have

$$a(M \otimes E) = \dim E . a(M).$$

For example, the multiplicity c_d is an additive invariant on $\mathscr{H}_d$ for any $d \in \mathbb{N}$ [Vo1].

The following result is important.

THEOREM 2.1.1 (Vogan [Vo1]). — *Let $\mathscr{M}$ be a good subcategory of $\mathscr{H}$ and let a be an additive invariant on $\mathscr{M}$. Then we have:*

(1) *We take a coherent family Θ on $P \times P$ which takes values in $K(\mathscr{M})$. Then the map*

$$P \times P \ni (\nu, \eta) \rightsquigarrow a(\Theta(\nu, \eta)) \in \mathbb{Z}$$

extends uniquely to a $W \times W$-harmonic polynomial $v[a; \Theta]$ on $\mathfrak{h} \times \mathfrak{h}$.

(2) *Fix λ, $\mu \in P^-$, then the map*

$$\Phi_a:\quad K(\mathscr{M}[-\mu, -\lambda]) \ni X \rightsquigarrow v[a; \Theta_X] \in S(\mathfrak{h} \times \mathfrak{h})$$

is $W \times W$-equivariant.

2.2. Double cells in Weyl groups. — In this section, we review the theory of cells (*cf.* [Jo3, 11], KL1], [BV2, 3, 4], [Lu4, 5, 6, 7], etc.). Especially, we can find most of the following results in [BV4] section 3.

Let $\mu \in P^{--}$. For $w \in W$, we put $J(w\mu) = \mathrm{Ann}_{U(\mathfrak{g})}(L(w\mu))$. For $x, y \in W$, we define preorders $\overset{L}{\leqq}$, $\overset{R}{\leqq}$ as follows. $x \overset{L}{\leqq} y$ iff $J(x\mu) \subseteq J(y\mu)$. $x \overset{R}{\leqq} y$ iff there exists some finite dimensional $U(\mathfrak{g})$-module E such that $L(y\mu)$ is an irreducible constituent of $L(x\mu) \otimes E$. Form the translation principle, it is known that the definitions of $\overset{L}{\leqq}$ and $\overset{R}{\leqq}$ do not depend on the choice of $\mu \in P^{--}$. We define an equivalence relation $\overset{L}{\sim}$ (resp. $\overset{R}{\sim}$) by $x \overset{L}{\sim} y$ iff $x \overset{L}{\leqq} y$ and $y \overset{L}{\leqq} x$ (resp. $x \overset{R}{\sim} y$ iff $x \overset{R}{\leqq} y$ and $y \overset{R}{\leqq} x$).

We denote by $\overset{L}{\sim}$ (resp. $\overset{LR}{\sim}$) the relation on W generated by $\overset{L}{\leqq}$ and $\overset{R}{\leqq}$ (resp. $\overset{L}{\sim}$ and $\overset{R}{\sim}$).

Form the definition we can easily see:

Lemma 2.2.1. — *Let $\mu \in P^{--}$ and let w_S be the longest element of W_S. Then, for all $x \in W$, $w_S \overset{R}{\leqq} x$ if and only if $x\mu \in P_S^{++}$.*

We quote:

Theorem 2.2.2 (Joseph [Jo1], also *see* [BV4], 3.10). — *Let λ, $\mu \in P^{--}$ and let $z \in W$. Put $V = L(L(w_0\lambda), L(z^{-1}\mu))$. Then*

$$\mathrm{LAnn}(V) = J(z^{-1}\mu)^{\vee} = J((w_0 z^{-1}(-\mu)),$$
$$\mathrm{RAnn}(V) = J(w_0 z w_0 \lambda)^{\vee} = J(z w_0(-\lambda)).$$

The following results are known.

Theorem 2.2.3 ([KL1], 3.3 Remark). — *The maps $x \rightsquigarrow x w_0$ and $x \rightsquigarrow w_0 x$ reverse each of the preorders $\overset{L}{\leqq}$, $\overset{R}{\leqq}$, $\overset{LR}{\leqq}$ on* W.

Theorem 2.2.4 (*cf.* [BV4] section 3, [Vo2], Theorem 3.2, also *see* Proposition 1.4.3). — *We assume λ, $\mu \in P^{--}$, $x, y \in W$.*

(1) *We have:*

$$x \overset{L}{\leqq} y \textit{ iff } \mathrm{RAnn}(L(M(w_0\lambda), L(x^{-1}\mu))) \subseteq \mathrm{RAnn}(L(M(w_0\lambda), L(y^{-1}\mu))),$$

$$x \overset{R}{\leqq} y \textit{ iff } \mathrm{LAnn}(L(M(w_0\lambda), L(x^{-1}\mu))) \subseteq \mathrm{LAnn}(L(M(w_0\lambda), L(y^{-1}\mu))),$$

$$x \overset{L}{\leqq} y \textit{ iff } x^{-1} \overset{R}{\leqq} y^{-1},$$

$$x \overset{LR}{\leqq} y \textit{ iff } x^{-1} \overset{LR}{\leqq} y^{-1}.$$

(2) $x \overset{L}{\leqq} y$ (resp. $x \overset{R}{\leqq} y$) *if and only if there exists some finite dimensional* $U(\mathfrak{g})$*-module* E *such that* $L(M(w_0\lambda), L(y^{-1}\mu))$ *is an irreducible constituent of* $L(M(w_0\lambda), L(x^{-1}\mu))\otimes(E\otimes 1)$ [*resp.* $L(M(w_0\lambda), L(x^{-1}\mu)]\otimes(1\otimes E)$). *Here,* 1 *means the trivial* $U(\mathfrak{g})$*-module and we regard* $1\otimes E$ *and* $E\otimes 1$ *as* U*-modules.*

(3) $x \overset{LR}{\leqq} y$ *if and only if there exists some finite dimensional* U*-module* E *such that* $L(M(w_0\lambda), L(y^{-1}\mu))$ *is an irreducible constituent of* $L(M(w_0\lambda), L(x^{-1}\mu))\otimes E$.

For $w\in W$, we define full subcategories $\mathscr{H}(w)$ and $\mathscr{H}'(w)$ of $\mathscr{H}$ as follows.

$$\mathscr{H}(w)=\{V\in\mathscr{H}\mid \text{For every irreducible constituent X of V there exist some } \lambda, \mu\in P^- \text{ and } y\in W \text{ such that } w \overset{LR}{\leqq} y \text{ and } X\cong V^{\odot}(y^{-1}; -\mu, -\lambda)\},$$

$$\mathscr{H}'(w)=\{V\in\mathscr{H}\mid \text{For every irreducible constituent X of V there exist some } \lambda, \mu\in P^- \text{ and } y\in W \text{ such that } w \overset{LR}{\leqq} y,\ w \overset{LR}{\not\sim} y, \text{ and } X\cong V^{\odot}(y^{-1}; -\mu, -\lambda)\}.$$

We can easily see $\mathscr{H}(w)$ and $\mathscr{H}'(w)$ are good subcategories of $\mathscr{H}$ form Theorem 2.2.2 (3) and the translation principle. Hence we can define coherent continuation $W\times W$-representations $K(\mathscr{H}(w)[-\mu, -\lambda])$ and $K(\mathscr{H}'(w)[-\mu, -\lambda])$ for all $\lambda, \mu\in P^{--}$. We put

$$\bar{V}^{LR}(w; -\mu, -\lambda)=K(\mathscr{H}(w)[-\mu, -\lambda])\otimes_{\mathbb{Z}}\mathbb{C},$$
$$\tilde{V}^{LR}(w; -\mu, -\lambda)=K(\mathscr{H}'(w)[-\mu, -\lambda])\otimes_{\mathbb{Z}}\mathbb{C},$$
$$V^{LR}(w; -\mu, -\lambda)=\bar{V}^{LR}(w; -\mu, -\lambda)/\tilde{V}^{LR}(w; -\mu, -\lambda).$$

As a representation of $W\times W$, $V^{LR}(w; -\mu, -\lambda)$ [resp. $\bar{V}^{LR}(w; -\mu, -\lambda)$] does not depend on the choice of $\lambda, \mu\in P^{--}$, and it is called a double cell (resp. a double cone) representation (*cf.* [BV4] section 3). So, we often denote the double cell (resp. cone) representation by $V^{LR}(w)$ [resp. $\bar{V}^{LR}(w)$].

From the definition, we see immediately that $x \overset{LR}{\sim} y$ implies $V^{LR}(x; -\mu, -\lambda)=V^{LR}(y; -\mu, -\lambda)$.

Although many deep results are known on double cells, we only remark the following properties (*cf.* [BV4]).

LEMMA 2.2.5. — *Let* $w\in W$ *and* $\lambda, \mu\in P^{--}$.

(1) *The image of* $\{L(M(w_0\lambda), L(y^{-1}\mu))\mid y \overset{LR}{\sim} w\}$ *forms a basis of* $V^{LR}(w; -\mu, -\lambda)$.

(2) *The multiplicities of any irreducible constituent of* $V^{LR}(w; -\mu, -\lambda)$ [*resp.* $\bar{V}^{LR}(w; -\mu, -\lambda)$] *is always one. For any irreducible constituent* V *of* $V^{LR}(w; -\mu, -\lambda)$ [*resp.* $\bar{V}^{LR}(w; -\mu, -\lambda)$], *there exists some irreducible* W*-representation* σ *such that* $V\cong\sigma\otimes\sigma$.

(1) Is clear from the definition. (2) follows from the fact that $V^{LR}(w; -\mu, -\lambda)$ [resp. $\bar{V}^{LR}(w; -\mu, -\lambda)$] is equivalent to some $W\times W$-subquotient (subrepresentation) of the regular representation $\mathbb{C}[W]$ and any irreducible representation W is self-dual.

Lastly, we quote the following:

LEMMA 2.2.6 (*cf.* [Lu1], Lemma 4.1). — *If* $x, y \in W$ *satisfy* $x \overset{L}{\leqq} y$ (resp. $x \overset{R}{\leqq} y$) and $x \overset{LR}{\sim} y$, then $x \overset{L}{\sim} y$ (resp. $x \overset{R}{\sim} y$).

2.3. INDUCED REPRESENTATIONS WITH FINITE-DIMENSIONAL QUOTIENTS. — In this section, we consider some induced representations. The material in this section is more or less known.

First, we fix some terminologies on induced representations. Let $S \subseteqq \Pi$ and let $\mathfrak{p}_S$ (resp. P_S) be the corresponding parabolic subalgebra (group) as above. Let σ be an irreducible finite dimensional (continuous) representation of P_S. (We do not assume σ is holomorphic.) We denote by E_σ the representation space of σ. We define the space of smoothly induced representation by

$$C^\infty (G/P_S; \sigma) = \{ F : G \to E_\sigma \mid F \text{ is of the class } C^\infty \text{ and } F(gp) = \sigma(p^{-1}) F(g) \text{ for all } g \in G \text{ and } p \in P_S \}.$$

G acts on $C^\infty (G/P_S; \sigma)$ by the left translation and there is a natural differential action of U. We denote by $\operatorname{Ind}_{P_S}^G(\sigma)$ the $\mathfrak{k}_c$-finite part of $C^\infty (G/P_S; \sigma)$. Clearly, it is Harish-Chandra $(\mathfrak{g}_c, \mathfrak{k}_c)$-module.

From Sobolev's lemma, we see $\operatorname{Ind}_{P_S}^G(\sigma)_\infty$ coincides with $C^\infty (G/P_S; \sigma)$ (*cf.* [BW] III 7.5).

Nest, we recall the definition of the Casselman-Jacquet module of a Harish-Chandra $(\mathfrak{g}_c, \mathfrak{k}_c)$-module (*cf.* [Ca2], [W1]). For a Harish-Chandra $(\mathfrak{g}_c, \mathfrak{k}_c)$-module V, we define

$$J(V) = \{ v \in V^* \mid \exists k \geqq 1, \forall X_1, \ldots, X_k \in \mathfrak{u}_c, X_1 \ldots X_k v = 0 \}.$$

It is known that J(V) is a U-module of finite length and any irreducible constituent of J(V) is a subquotient of some Verma module of U with respect to the Borel subalgebra $\mathfrak{b}_c = \mathfrak{b} \otimes \mathfrak{b}$.

The following result is a special case of [W1], 5.11.

THEOREM 2.3.1 (Casselman-Wallach). — *For any Harish-Chandra* $(\mathfrak{g}_c, \mathfrak{k}_c)$*-module-module* V, *every element of* J(V) *extend to a continuous linear form on* V_∞ *uuniquely. Namely,* $J(V) \subseteqq V'_\infty$.

For simplicity, we hereafter only consider the integral case. For $\eta, \nu \in P_S^{++} \cap P$, we denote by $E_S(\eta, \nu)$ the finite dimensional irreducible $U((\mathfrak{l}_S)_{\mathbb{C}})$-module $E_S(\eta - \rho) \otimes E_S(\nu - \rho)$ with highest weight $(\eta - \rho, \nu - \rho)$. Then, $E_S(\eta, \nu)$ has a natural L_S-module structure whose differential module structure is the original $U((\mathfrak{l}_S)_{\mathbb{C}})$-module structure. We define that $n \in N_S$ acts on $E_S(\eta, \nu)$ identically. Then, $E_S(\eta, \nu)$ can be regarded as a continuous P_S-representation. We denote by $E_S^*(\eta, \nu)$ the contragredient representation of $E_S(\eta, \nu)$.

From standard arguments, we easily see:

LEMMA 2.3.2. — *Let* $S \subseteq \Pi$ *and* $\eta, \nu \in P_S^{++} \cap P$. *As a Harish-Chandra* $(\mathfrak{g}_c, \mathfrak{k}_c)$*-module* $L^*(M_S(\eta) \otimes M_S(\nu))$ *is isomorphic to* $\mathrm{Ind}_{P_S}^G(E_S^*(\eta, \nu)))$.

Next, we investigate special induced representations. Let $\lambda \in P^{--}$ and let w_S be the longest element of W_S. Then we have $w_S \in P_S^{++}$. So, we can consider the generalized Verma module $M_S(w_S \lambda)$. We can easily see $M_S(w_S \lambda)$ is irreducible. Hence, we have

$$L(M_S(w_S\lambda), M_S(w_S\lambda)) \cong L^*(M_S(w_S\lambda) \otimes M_S(w_S\lambda)) \cong \mathrm{Ind}_{P_S}^G(E_S^*(w_S\lambda, w_S\lambda)),$$

from Lemma 1.4.1 and 2.3.2. In fact, this induced representation has a finite-diemnsional unique irreducible quotient. However, for our purpose, an irreducible subrepresentation is more important. So, we are going to investigate the structure of this induced module.

First, we regard $U(\mathfrak{g})/J(w_S\lambda)$ as a Harish-Chandra $(\mathfrak{g}_c, \mathfrak{k}_c)$-module by $(u \otimes v) X = {}^t\check{u} X \check{v}$ for $u, v \in U(\mathfrak{g})$ and $X \in U(\mathfrak{g})/J(w_S\lambda)$.

Since we can regard $X \in U(\mathfrak{g})/J(w_S\lambda)$ as a linear transformation on $M_S(w_S\lambda) = L(w_S\lambda)$, there exists a natural linear map

$$\Phi: \quad U(\mathfrak{g})/J(w_S\lambda) \to L(M_S(w_S\lambda), M_S(w_S\lambda)).$$

We can immediately see the above Φ is an injective morphism of Harish-Chandra $(\mathfrak{g}_c, \mathfrak{k}_c)$-modules. Here, we quote a deep result.

THEOREM 2.3.3 (Conze-Berline and Duflo [CD] 2.12, 6.13). — *For any* $\lambda \in P^{--}$, $\Phi: U(\mathfrak{g})/J(w_S\lambda) \to L(M_S(w_S\lambda), M_S(w_S\lambda))$ *is isomorphism.*

Remark. – The assumption "$\lambda \in P$" is stronger than enough. Actually, Conze-Berline and Duflo proved the above result under a weaker assumption. Gabber and Joseph showed the assumption can be relaxed further (*cf.* [GJ], 4.4).

An ideal I of $U(\mathfrak{g})$ is called primitive if it is the annihilator of some irreducible $U(\mathfrak{g})$-module. Duflo proved in [D2], any primitive ideal I satisfies $I = {}^tI$. Here ${}^tI = \{ {}^tu \mid u \in I \}$. Since $J(w_S\lambda)$ is primitive, we have:

THEOREM 2.3.4

$$\mathrm{LAnn}(\mathrm{Ind}_{P_S}^G(E_S^*(w_S\lambda, w_S\lambda))) = \mathrm{RAnn}(\mathrm{Ind}_{P_S}^G(E_S^*(w_S\lambda, w_S\lambda))) = J(w_S\lambda)^\vee.$$

An ideal I of $U(\mathfrak{g})$ is called prime, if $J_1 J_2 \subseteq I$ implies $J_1 \subseteq I$ or $J_2 \subseteq I$ for ideals J_1, J_2. We can easily see a primitive ideal is prime. Taking account of the fact that (2-sided) ideals of $U(\mathfrak{g})/J(w_S\lambda)$ correspond to submodules of $\mathrm{Ind}_{P_S}^G(E_S^*(w_S\lambda, w_S\lambda))$, we see the primeness of $J(w_S\lambda)$ can be rephrased as follows.

LEMMA 2.3.5. — *Let* $\lambda \in P^{--}$. (1) $\mathrm{Ind}_{P_S}^G(E_S^*(w_S\lambda, w_S\lambda))$ *has a unique irreducible submodule* [*say* $V_S(\lambda, \lambda)$]. *In other words, the socle of* $\mathrm{Ind}_{P_S}^G(E_S^*(w_S\lambda, w_S\lambda))$ *is irreducible.*

(2) *Let* V *be a subquotient of* $\mathrm{Ind}_{P_S}^G(E_S^*(w_S\lambda, w_S\lambda))/V_S(\lambda, \lambda)$. *Then,* $J(w_S\lambda)^\vee \subsetneqq \mathrm{LAnn}(V)$ *and* $J(w_S\lambda)^\vee \subsetneqq \mathrm{RAnn}(V)$.

(3) $$J(w_S\lambda)^\vee = \mathrm{LAnn}(V_S(\lambda, \lambda)) = \mathrm{RAnn}(V_S(\lambda, \lambda)).$$

Since $\mathrm{Ind}_{P_S}^G(E_S^*(w_S\lambda, w_S\lambda))$ has an infinitesimal character $(-\lambda, -\lambda)$, there exists a unique element v_S of W such that

$$V_S(\lambda, \lambda) = L(M(w_0\lambda), L(v_S^{-1}\lambda)).$$

From Theorem 2.2.2, we have

$$J(w_S\lambda)^\vee = J(v_S^{-1}\lambda)^\vee.$$

Hence, we have:

PROPOSITION 2.3.6

$$w_S \overset{R}{\sim} v_S.$$

Using the translation principle, 2.2.4, and 2.2.6, we finally have:

PROPOSITION 2.3.7. — *Let* $\lambda, \mu \in P^{--}$. *Then we have:*

(1) $\mathrm{Ind}_{P_S}^G(E_S^*(w_S\mu, w_S\lambda))$ *has a unique irreducible submodule* $L(M(w_0\lambda), L(v_{S^{-1}}\mu))$. *Here* $v_S \in W$ *is uniquely determined and independent of the choice of* $\lambda, \mu \in P^{--}$.

(2) $w_S \overset{R}{\sim} v_S$.

(3) *If* $L(M(w_0\lambda), L(x^{-1}\mu))$ *is a subquotient of* $\mathrm{Ind}_{P_S}^G(E_S^*(w_S\mu, w_S\lambda))$, *then* $v_S \overset{R}{\leqq} x$ and $v_S \overset{L}{\leqq} x$. If $L(M(w_0\lambda), L(x^{-1}\mu))$ *appears in* $\mathrm{Ind}_{P_S}^G(E_S^*(w_S\mu, w_S\lambda))/L(M(w_0\lambda), L(v_S - 1\mu))$, *then* $v_S \overset{LR}{\not\sim} x$ and $v_S \overset{LR}{\leqq} x$.

2.4. AN EMBEDDING THEOREM. — In this section, we will prove:

THEOREM 2.4.1. — *Let* $S \subseteqq \Pi$, $\lambda, \mu \in P^{--}$, *and* $z \in W$. *Then the followings are equivalent.*

(1) $w_S \overset{LR}{\leqq} z$.

(2) *There exists some finite dimensional irreducible* L_S-*module* σ *such that* $L(M(w_0\lambda), L(z^{-1}\mu))$ *is a submodule of* $\mathrm{Ind}_{P_S}^G(\sigma)$.

(3) *There exists some finite dimensional irreducible* L_S-*module* σ *such that* $L(M(w_0\lambda), L(z^{-1}\mu))$ *is a subquotient of* $\mathrm{Ind}_{P_S}^G(\sigma)$.

Remark 1. — Even if λ or μ is not integral, we can prove the equivalence of (2) and (3) in the same way as below.

Remark 2. — This result gives an affirmative answer to [Mat4] Working Hypothesis I in a special case. However, if we do not assume the integrality of infinitesimal characters, then there is an easy counterexample of this working hypothesis for *Spin*$(5, \mathbb{C})$.

In the above theorem, (2) $\rightarrow$ (3) is trivial. First, we prove that (3) implies (1).

From Proposition 2.3.7, if $L(M(w_0\lambda), L(z^{-1}\mu))$ is a subquotient of $L(M_S(w_S\lambda), M_S(w_S\mu)) \cong \mathrm{Ind}_{P_S}^G(E_S^*(w_S\mu, w_S\lambda))$, then we have $w_S \overset{LR}{\leqq} z$. Hence, it suffice to prove for any $x, y \in W$ such that $x\lambda, y\mu \in P_S^{++}$ there exists some finite dimensional U-module E such that $\mathrm{Ind}_{P_S}^G(y\mu, x\lambda)$ is a subquotient of $\mathrm{Ind}_{P_S}^G(w_S\mu, w_S\lambda) \otimes E$. But, this statement is easily deduced from the following form of the Mackey tensor product theorem.

LEMMA 2.4.2. — *Let* E (*resp.* V) *be a finite dimensional* L_S-(*resp.* G-) *representation. Then, we have the following functorial isomorphism of Harish-Chandra Harish-Chandra* $(\mathfrak{g}_c, \mathfrak{k}_c)$*-module-modules.*

$$\mathrm{Ind}_{P_S}^G(E) \otimes V \cong \mathrm{Ind}_{P_S}^G(E \otimes V|_{P_S}).$$

In order to prove that (1) implies (2), we should quote several deep results with respect to the cells.

First, let $\gamma_{x,y,z}$ $(x, y, z \in W)$ be a non-negative integer which is defined by Lusztig in [Lu6]. $\gamma_{x,y,z}$ satisfies the following properties.

THEOREM 2.4.3 (Lusztig [Lu6] Theorem 1.8, Corollary 1.9). — (1) *For all* $x, y, z \in W$,

$$\gamma_{x,y,z} = \gamma_{y,z,x} = \gamma_{z,x,y}.$$

(2) *Let* $x, y, z \in W$. *Then* $\gamma_{s,y,z} \neq 0$ *implies*

$$x^{-1} \overset{R}{\sim} y,\ y^{-1} \overset{R}{\sim} z,\ z^{-1} \overset{R}{\sim} x.$$

The following statement is just a rephrasing of [Lu7] 3.1 (*k*) and (1).

THEOREM 2.4.4 Lusztig [Lu7]). — *Let* $w, z \in W$. *If* $z \overset{LR}{\sim} w$, *then there exist some* $x, y \in W$ *such that* $x^{-1} \overset{R}{\sim} y \overset{R}{\sim} w$ *and* $\gamma_{x,y,z} \neq 0$.

Next, we recall some results of Joseph in [Jo12].

In [Jo12], A3.3 and A3.6, Joseph defined a map $W \ni y \rightsquigarrow y_* \in W$ which has the following properties.

(*1) $y_{**} = y$ for all $y \in W$. In particular, $y \rightsquigarrow y_*$ is bijection.

(*2) Fix $w \in W$. Then $y \overset{L}{\sim} w$ implies $y_* \overset{L}{\sim} w_0 w$.

(*3) For all $y \in W$, we have $(y^{-1})_* = (y_*)^{-1}$.

Hence, we have $x \overset{L}{\sim} y$ (resp. $x \overset{R}{\sim} y$, $x \overset{LR}{\sim} y$) if and only if $x_* \overset{L}{\sim} y_*$ (resp. $x_* \overset{R}{\sim} y_*$, $x_* \overset{LR}{\sim} y_*$).

The following result is one of the crucial points of our proof.

THEOREM 2.4.5 (Joseph [Jo12] 4.8 Theorem). — *Assume* $\lambda, \mu \in P^{--}$. *Then for all* $x, y \in W$,

$$\mathrm{Soc}\, L(L(x^{-1}w_0\lambda), L(yw_0\mu)) = \bigoplus_{z \in W} L(M(w_0\lambda), L(z^{-1}\mu))^{\gamma_{x*, y*, (w_0 z)*}}.$$

Here, Soc *means the socle.*

Remark 1. — In [Jo12], the above theorem is described in terms of $c_{x, y, z}$ which is defined in [Lu5]. However $\gamma_{x, y, z}$ coincides with the absolute value of $c_{x, y, z}$.

Remark 2. — The statement in [Jo12] is a apparently weaker than the above statement. However, using the translation principle (Theorem 1.5.1 and Proposition 1.5.2), we can deduce the above statement from that of Joseph.

Let λ, $\mu \in P^{--}$. Lemma 2.2.1 implies that, if $w_S \overset{R}{\leqq} x^{-1} w_0$, $w_S \overset{R}{\leqq} y w_0$, $L(yw_0 \mu) \otimes L(x^{-1} w_0 \lambda)$ is a quotient of $M_S(yw_0 \mu) \otimes M_S(x^{-1} w_0 \lambda)$. From Lemma 1.4.1, we see that $w_S \overset{R}{\leqq} x^{-1} w_0$, $w_S \overset{R}{\leqq} y w_0$ implies $L(L(x^{-1} w_0 \lambda), L(yw_0 \mu))$ is a submodule of $L^*(M_S(yw_0 \mu) \otimes M_S(x^{-1} w_0 \lambda)) \cong \mathrm{Ind}_{P_S}^G(E_S^*(yw_0 \mu, x^{-1} w_0 \lambda))$.

Hence, we have only to prove the following lemma.

LEMMA 2.4.6. — *If* $w_S \overset{LR}{\leqq} z$, then there exist some x, y$\in$W *such that* $w_S \overset{R}{\leqq} x^{-1} w_0$, $w_S \overset{R}{\leqq} y w_0$ *and* $\gamma_{x*, y*, (w_0 z)*} \neq 0$.

In order to prove the obove lemma, we need the following deep result.

THEOREM 2.4.7 (Lusztig-Xi Nanhua [LN] Theorem 3.2). — *Let* $x, y \in W$ *be such that* $x \overset{LR}{\leqq} y$, *there exists* $z \in W$ *such that* $x \overset{L}{\leqq} z$ and $z \overset{R}{\sim} y$.

Remark 1. — In [LN], W is an affine Weyl group. However, the same proof is applicable the case of a Weyl group. (The author learned this fact from G. Lusztig.)

Remark 2. — From Theorem 2.2.4 (1), we can interchange L and R in the above statement. Namely, $x \overset{LR}{\leqq} y$ implies the existence of $z \in W$ such that $x \overset{R}{\leqq} z$ and $z \overset{L}{\sim} y$.

Now we prove Lemma We assume $w_S \overset{LR}{\leqq} z$. From the above Remark 2, there exists some $v \in W$ such that $w_S \overset{R}{\leqq} v \overset{L}{\sim} z$. Since $z^{-1} \overset{LR}{\sim} z$ follows from the existence of a Duflo involution σ such that $z \overset{L}{\sim} \sigma$, we have $w_0 z^{-1} \overset{LR}{\sim} w_0 z$ (2.2.3). Hence, using (2.2.3) and (2.2.4 (1)) again, we have

$$v w_0 \overset{L}{\sim} z w_0 \overset{LR}{\sim} w_0 z^{-1} \overset{LR}{\sim} w_0 z.$$

Therefore, we have

$$(v w_0)_* \overset{LR}{\sim} (w_0 z)_*.$$

Theorem 2.4.4 implies that there exist some x', $y' \in W$ such that $(x')^{-1} \overset{R}{\sim} y' \overset{R}{\sim} (v w_0)_*$ and $\gamma_{x', y', (w_0 z)*} \neq 0$. Put $x = (x')_*$ and $y = (y')_*$. Hence, $\gamma_{x*, y*, (w_0 z)*} \neq 0$. Using the above properties of the map $w \rightsquigarrow w_*$, we immediately have $x^{-1} \overset{R}{\sim} y \overset{R}{\sim} v w_0$. From Theorem 2.2.3, we finally have

$$w_S \overset{R}{\leqq} v \overset{R}{\sim} x^{-1} w_0 \overset{R}{\sim} y w_0.$$

Q.E.D.

2.5. Wave fronts sets for complex groups. — Here, we review some notions which represent the singularities of Harish-Chandra $(\mathfrak{g}_c, \mathfrak{k}_c)$-modules. The contents of this section are more or less known. For details, *see* [KV2], [How], [BV1, 2, 3, 4]. For Harish-Chandra $(\mathfrak{g}_c, \mathfrak{k}_c)$-module V, we consider the distribution character Θ_V, which is a distribution on G. (In fact, the classical ressult of Harish-Chandra says that it is a locally integrable function on G.) Let $WF(\Theta_V)$ be the wave front set of Θ_V (*cf.* [Hör], [T]). Usually, wave front sets are defined as closed conic subsets of the cotangent bundle $T^* G$. However, it is more canonical for us to regard $WF(\Theta_V)$ as a subset of $i T^* G$, because there is no reason to define, in order to define wave front sets, the Fourier transform using the character $e^{-i\langle \xi, x\rangle}$ instead of $e^{+i\langle \xi, x\rangle}$. The wave front set WF(V) of V is the fiber of $WF(\Theta_V)$ at the identity element of G. Since the fiber of $T^* G$ at the identity element is canonically identified with the dual of the Lie algebra $\mathfrak{g}^*$, WF(V) is regarded as a closed conic subset of $i\mathfrak{g}^*$. Using the Killing form, we also regard WF(V) as a subset of $i\mathfrak{g}$. If E is a finite-dimensional U-module, then $WF(V \otimes E) = WF(V)$. (Θ_E is a real analytic function on G!) Hence, if $x \overset{LR}{\leqq} y$, then

$$WF(L(M(w_0 \lambda), L(y^{-1}\mu))) \subseteqq WF(L(M(w_0 \lambda), L(x^{-1}\mu))),$$

for all $\lambda, \mu \in P^{--}$. Therefore, wave front sets are uniquely determined for double cells.

Since the distribution character Θ_V is an invariant eigen-distribution (in the terminology of Harish-Chandra), WF(V) is a union of some nilpotent orbits. It is known that the following exact result.

Theorem 2.5.1 (Barbasch-Vogan [BV4] Theorem 3.20). — *for* $w \in W$, *there exists a unique nilpotent orbit* [*say* $\mathcal{O}(w)$] of $\mathfrak{g}$ *such that* $WF(L(M(w_0\lambda), L(z^{-1}\mu))) = i\mathcal{O}(w)$, for all $z \overset{LR}{\sim} w$, $\lambda, \mu \in P^{--}$.

Let V an irreducible Harish-Chandra $(\mathfrak{g}_c, \mathfrak{k}_c)$-module and let $I = \mathrm{Ann}_U(V)$. Then the associated variety Add (U/I) of U/I can be regarded as an algebraic analogue of the wave front set of V (*cf.* [How]). It is known that Ass(U/I) is the closure of some single nilpotent orbit in $\mathfrak{g}_c$ under the adjoint action of $G \times G$. (This statement is studied by several mathematicians, Borho, Brylinski, Joseph, Hotta, etc. and established in full generalities by Joseph.) For the real semisimple case the relation between the wave front set and the associated variety of U/I is rather complicated. For a complex semisimple linear group G, the following exact result is known. Here, we regard $\mathfrak{g}$ as a real form of $\mathfrak{g}_c = \mathfrak{g} \times \mathfrak{g}$ by

$$\mathfrak{g} = \{(X, \bar{X}) \mid X \in \mathfrak{g}\}.$$

as before.

PROPOSITION 2.5.2. — *Let* $w \in W$. *For* λ, $\mu \in P^{--}$ *and* $z \overset{LR}{\sim} w$, *put* $I = \mathrm{Ann}_U(L(M(w_0\lambda), L(z^{-1}\mu)))$, *then we have*

$$\mathrm{Ass}(U/I) = \overline{\mathcal{O}(w)} \times \overline{\mathcal{O}(w)}.$$

So, WF $L(M(w_0\lambda), L(z^{-1}\mu))) = i\mathfrak{g} \cap \mathrm{Ass}(U/I)$.

From a result of Gabber, we have

$$(1) \qquad \dim \mathcal{O}(w) = \frac{1}{2} \mathrm{Dim}(U/I) = \mathrm{Dim}(L(M(w_0\lambda), L(z^{-1}\mu))).$$

We need the following (known) result.

PROPOSITION 2.5.3. — *Let* λ, $\mu \in P^{--}$ *Let* $x, y \in W$ *satisfy* $x \overset{LR}{\leqq} y$ *and* $x \overset{LR}{\not\sim} y$. Then,

$$\mathrm{Dim}(L(M(w_0\lambda), L(y^{-1}\mu))) < \mathrm{Dim}(L(M(w_0\lambda), L(x^{-1}\mu))).$$

For the convenience of readers, we give the proof. For simplicity we put $V_w = L(M(w_0\lambda), L(w^{-1}\mu))$ for $w \in W$ and $I_w = \mathrm{Ann}_U(V_w)$. Since Duflo [D2] proved that $I_w = \mathrm{LAnn}(V_w) \otimes U(\mathfrak{g}) + U(\mathfrak{g}) \otimes \mathrm{RAnn}(V_w)$, from Theorem 2.2.4 (1), we have $I_x \overset{\subset}{\neq} I_y$. From a result of Borho and Kraft ([BoKr], 3.6), we have $\mathrm{Dim}(U/I_y) < \mathrm{Dim}(U/I_x)$. Hence we have the proposition from the above (1).

Now we consider the situation in 2.3. So, we fix $S \subseteqq \Pi$. Let λ, $\mu \in P^{--}$. If we put $I = \mathrm{Ann}_U(L^*(M_S(w_S\mu), M_S(w_S\lambda)))$, then we see $\mathrm{Ass}(U/I) = \overline{\mathcal{O}}_S \otimes \overline{\mathcal{O}}_S$ using irreducibility of Ass (U/I).

Finally, we have

PROPOSITION 2.5.4. — *Let* λ, $\mu \in P^{--}$ *and* $S \subseteqq \Pi$. *Then we have*

(1) *If* $z \overset{LR}{\sim} w_S$, *then*

$$\mathrm{WF}(L(M(w_0\lambda), L(z^{-1}\mu))) = i\overline{\mathcal{O}}_S,$$
$$\mathrm{Dim}(L(M(w_0\lambda), L(z^{-1}\mu))) = \dim \mathcal{O}_S = 2 \dim \mathfrak{n}_S.$$

(2) *If* $w_S \overset{LR}{\leqq} z$ *and* $w_S \overset{LR}{\not\sim} z$, *then* $\mathrm{Dim}(L(M(w_0\lambda), L(z^{-1}\mu))) < \dim \mathcal{O}_S = 2 \dim \mathfrak{n}_S$.

From Proposition 2.3.6, we have:

COROLLARY 2.5.5. — *Let* λ, $\mu \in P^{--}$ *and* $S \subseteqq \Pi$. *Then*

$$\mathrm{Dim}(\mathrm{Ind}_{P_S}^G(E_S^*(w_S\mu, w_S\lambda))/L(M(w_0\lambda), L(v_S^{-1}\mu))) < \mathrm{Dim}(L(M(w_0\lambda), L(v_S^{-1}\mu))) = 2 \dim \mathfrak{n}_S.$$

2.6. SPECIAL REPRESENTATIONS AND GOLDIE RANK POLYNOMIAL REPRESENTATIONS. — In this section, we review some results on special representations and Goldie rank polynomial representations which is defined by Lusztig [Lu2, 3] (also see [Lu4]) and [Jo7, 8] respectively. We also review a result of Barbasch-Vogan [BV2, 3] which relates these two notions.

For an irreducible W-module E (over $\mathbb{C}$), we can attach non-negative integers a_E and b_E. a_E is defined in [Lu4] (4.1.1), using the formal dimension. b_E is the smallest integer $i \geqq 0$ such that the W-module E occurs in the i-th symmetric power of $\mathfrak{h}$ ([Lu4] (4.1.2)).

For an irreducible W-module E and $w \in W$, if $E \otimes E$ occurs in the double cell $V^{LR}(w)$, we write $w \overset{LR}{\sim} E$. The following result is important for our purpose.

THEOREM 2.6.1(Lusztig [Lu4] (4.1.3), 5.27. Corollary). — (1) *For an irreducible* W-*module* E, *we have always* $a_E \leqq b_E$.

(2) *Let* $w \in W$. *Then, there exists a non-negative integer* $a(w)$ *such that* $a(w) = a_E$ *for all irreducible* W-*modules* E *such* that $w \overset{LR}{\sim} E$.

An irreducible W-module E is called special if $a_E = b_E$ holds [*cf.* [Lu4] (4.1.4)].

The following result is also important.

THEOREM 2.6.2 (Barbasch-Vogan [BV2, 3], [BV4] Theorem 3.20). — *Fix* $w \in W$. *Then, there exists just one special representation* E_w *such that* $E_w \otimes E_w$ *occurs in the double cell* $V^{LR}(w)$ *with the multiplicity one. Moreover the correspondence* $E_w \leftrightarrow \mathcal{O}(w)$ (see *Theorem* 2.5.1) *coincides with the Springer correspondence* (*cf.* [Sp1, 2], [KL2], [Hot1, 2, 3], [KT], [BoM1, 2], [BoBr], [BoBrM], [Gi], [HotK], etc.).

The theory of Goldie rank polynomials ([Jo7, 8]) implies:

THEOREM 2.6.3 (*cf.* [Jo8] 3.6, 5,4, also *see* [BV2, 3], [BeyL]). — *For all* $w \in W$, *we denote by* E_w *the special representation which is defined in Theorem* 2.6.2. *Put* $r = \dim \mathfrak{u} = \operatorname{card} \Delta^+$. *Then we have*

(1) $b_{E_w} = r - (1/2) \dim \mathcal{O}(w)$.

(2) *The multiplicity of* E_w *in the* b_{E_w}*-th symmetric power of* $\mathfrak{h}$ *is just one.*

For $w \in W$, we denote by $\sigma(w)$ the unique irreducible W-subrepresentation of the $a(w) (= b_{E_w})$-th symmetry power of $\mathfrak{h}$ which is isomorphic to E_w. We call $\sigma(w)$ the Goldie rank polynomial representation (*cf.*. [Jo7, 8]). Clearly, $\sigma(x) = \sigma(y)$ for all x, $y \in W$ such that $x \overset{LR}{\sim} y$.

Here, we investigate the case $w = w_S$ for some $S \subseteqq \Pi$. Using the Killing form, we identify $\mathfrak{h}$ and $\mathfrak{h}^*$. We define a polynomial p_S on $\mathfrak{h}^*$ by

$$p_S = \prod_{\alpha \in \mathbb{N}S \cap \Delta^+} \alpha.$$

Clearly the degree of p_S is $r-(1/2)\dim \mathcal{O}_S$, where $r=\dim \mathfrak{u}$. Put $d=1/2 \dim \mathcal{O}_S=\dim \mathfrak{n}_S$. Then, $\mathbb{C}[W]p_S \subseteq S^{r-d}(\mathfrak{h})$ coincides with $\sigma(w_S)$ ([Jo7, 8]). The irreducibility of $\mathbb{C}[W]p_S$ is a classical result of MacDonald [Mac].

2.7. MORE ABOUT ADDITIVE INVARIANTS. — Let $w\in W$. From the results in 2.6 (especially Theorem 2.6.2), we see there is a unique surjective $W\times W$-homomorphism

$$\bar{\Phi}_w:\quad \bar{V}^{LR}(w)\to\sigma(w)\otimes\sigma(w)$$

up to some scalar factor. The above homomorphism induces

$$\Phi_w:\quad V^{LR}(w)\to\sigma(w)\otimes\sigma(w).$$

Here, we regard the outer tensor product $\sigma(w)\otimes\sigma(w)$ as a sub-$W\times W$-representation of the $r-d$-th symmetric power of $\mathfrak{h}_c=\mathfrak{h}\times\mathfrak{h}$. Here, $r=\dim \mathfrak{u}=\operatorname{card}\Delta^+$ and $d=(1/2)\dim \mathcal{O}_w$. In this section, we investigate the relation between Φ_w and additive invariants on $\mathscr{H}(w)$.

Let a be an additive invariant on $\mathscr{H}(w)$. Taking account of Theorem 2.1.1, we define the degree deg (a) of a as follows.

$$\deg(a)=\max\{\deg v[a;\Theta_V]\mid V\in\mathscr{H}(w)[-\mu,-\lambda],\ \lambda,\ \mu\in P^{--}\}.$$

Here, $\deg v[a;\Theta_V]$ means the degree as a polynomial and we define $\deg 0=-\infty$.

The following result is convenient to get the degree of additive invariants.

PROPOSITION 2.7.1. — *Let $w\in W$ and let a be an additive invariant on $\mathscr{H}(w)$. We assume there exist some λ, $\mu\in P^{--}$ and $V\in\mathscr{H}(w)$ $[-\mu,-\lambda]$ which satisfy the following condition* (D).

(D) *There exists some $z\in W$ such that $z\overset{LR}{\sim}w$ and $L(M(w_0\lambda), L(z^{-1}\mu))$ is an irreducible constituent of* V.

Then, $\deg(a)=\deg v[a;\Theta_V]$.

In order to prove the above proposition, we prove:

LEMMA 2.7.2. — *Fix $w\in W$, λ, $\mu\in P^{--}$. Let $V\in\mathscr{H}(w)$ $[-\mu,-\lambda]$ and let a be an additive invariant on $\mathscr{H}(w)$. Then, for any irreducible constituent X of V, we have*

$$\deg v[a;\Theta_X]\leqq\deg v[a;\Theta_V].$$

Proof. — Put $\deg v[a;\Theta_V]=n$ and put

$$m=\max\{\deg v[a;\Theta_X]\mid X \text{ is an irreducible constituent of } V\}.$$

We assume $n<m$. We choose an irreducible constituent X of V such that $m=\deg v[a;\Theta_X]$. Let p_X be the m-th homogeneous part of $v[a;\Theta_X]$. Since $P^{++}\times P^{++}$ is Zariski dense in $\mathfrak{h}\times\mathfrak{h}$, there exists some η, $\nu\in P^{++}$ such that $p_X(\eta,\nu)\neq 0$. We, if necessary, exchange X and can assume $p_X(\eta,\nu)<0$. Hence, for sufficiently large $k\in\mathbb{N}$, we have

$$v[a;\Theta_X](k\eta, k\nu)<0.$$

However, this contradicts the positivity of additive invariants. □

Now, we prove Proposition 2.7.1. Put $X=L(M(w_0\lambda), L(z^{-1}\mu))$ and $n=\deg v[a;\Theta_V]$. Since clearly $n\leqq\deg(a)$, we prove $\deg(a)\leqq n$. From the above lemma, we have $\deg v[a;\Theta_X]\leqq n$. From the translation principle and Theorem 2.2.4 (3), for any irreductible object Y in $\mathscr{H}(w)$, there exists some finite dimensional U-module E such that Y is an irreducible constituent of $X\otimes E$. If we use the property (C 2) of coherent families (2.1) and the above lemma again, we can easily see $\deg v[a;\Theta_Y]\leqq n$. □

Using the results in 2.6, we have the following result.

THEOREM 2.7.3 [(2) is due to Vogan and Joseph]. — *Let $w\in W$ and let a be a non-trivial additive invariant on $\mathscr{H}(w)$. Put $r=\operatorname{card}\Delta^+=\dim\mathfrak{u}$ and $d=\dim\mathcal{O}_w$. Then,*

(1) *We always have $2r-d\leqq\deg(a)$.*

(2) *The multiplicity c_d is an additive invariant on $\mathscr{H}(w)$ and $\deg(c_d)=2r-d$.*

(3) *If $2r-d=\deg(a)$, then there exists some positive constant k such that $a=kc_d$. Moreover, Φ_a (cf. Theorem 2.1.1) coincides with $\bar{\Phi}_w$ up to scalar factor.*

Proof. — If $w\overset{LR}{\leqq}y$, then $\dim\mathcal{O}_w\geqq\dim\mathcal{O}_y$. So, $a(w)\leqq a(y)$. Hence, Theorem 2.6.1, 2.6.2, and 2.6.3 implies $\sigma(w)\otimes\sigma(w)$ is the only irreducible sub-$W\times W$-module of $\bigoplus_{i\leqq 2r-d} S^i(\mathfrak{h}\times\mathfrak{h})$ which occurs in $\bar{V}^{LR}(w)$. Hence, we have (1) and the latter part of (3). (2) is deduced from [Jo8] 5.7. Theorem. The positivity of the above k follows from the positivities of additive invariants.

Q.E.D.

3. Whittaker vectors and Whittaker polynomials

3.1. WHITTAKER DATA AND $C^{-\infty}$-WHITTAKER VECTORS. — Next we define the space of C^{∞}-continuous Whittaker vectors. Fix $S\subseteqq\Pi$. Here, we consider $\mathfrak{n}_S$ as a real Lie algebra. Let $\psi:\bar{\mathfrak{n}}_S\to\mathbb{C}$ be a character, namely one dimensional representation. Put $\Psi=(\bar{\mathfrak{n}}_S,\psi)$. We call the above pair Ψ a Whittaker datum. If the image of ψ is contained in $i\mathbb{R}$, Ψ is called unitary. Namely, ψ is a differential of a unitary character of $\bar{N}_S$ in this case.

For a Whittaker datum $\Psi=(\bar{\mathfrak{n}}_S,\psi)$, we denote by the same letter the complexification $\psi:(\bar{\mathfrak{n}}_S)_{\mathbb{C}}\to\mathbb{C}$ of ψ. Since the complexification $(\bar{\mathfrak{n}}_S)_c$ of $\mathfrak{n}_S$ is identified with $\bar{\mathfrak{n}}_S\times\bar{\mathfrak{n}}_S$, we regard ψ as an element (ψ_L,ψ_R) of $\bar{\mathfrak{n}}_S^*\times\bar{\mathfrak{n}}_S^*$. If ψ is unitary, we have $\psi=\psi_L=-\psi_R$.

We call $\Psi=(\bar{\mathfrak{n}}_S,\psi)$ admissible if ψ_L and ψ_R are both contained in the Richardson orbit $\mathcal{O}_S$.

We consider a generalized flag variety $X=G/P_S$. An admissible Whittaker datum Ψ is called strongly admissible, it the moment map $\mu: T^*X \to \bar{\mathcal{O}}_S$ (for example see [BoBr], [BoBrM], etc.) is birational. In [BoM2], the degree of the moment map is given in terms of the Springer representations.

If an admissible (resp. strongly admissible) Whittaker datum exists, then we call $\mathfrak{p}_S$ admissible (resp. strongly admissible). Admissible parabolic subgroups are classified by Lynch [Ly] except E_6, E_7, and E_8. It is known that, if $G=SL(n, \mathbb{C})$, any nilpotent orbit is the Richardson orbit of some strongly admissible parabolic subalgebra (*cf.* [Y1], [OW]). Parabolic subalgebras associated by a $\mathfrak{sl}_2$-triples which contain even nilpotent elements are strongly admissible. If $\mathfrak{g}$ consists of only factors of type A, admissibility implies strong admissibility.

Taking account of [Y1], we say that a Whittaker datum $\Psi=(\bar{\mathfrak{n}}_S, \psi)$ is permissible, if the restriction of ψ to $\bar{\mathfrak{n}}_S \cap \mathrm{Ad}(l\tilde{w})\mathfrak{p}_S$ is non-trivial for all $l \in L$ and $w \in W$ such that $\tilde{w} \notin L$. Here, $\tilde{w}$ is a representative of $w \in W$ in G.

The following result is known:

THEOREM 3.1.1 (Yamashita [Y1] Lemma 3.3, Proposition 3.4). — *A strongly admissible Whittaker datum is permissible.*

I do not know an example of an admissible character which is not permissible.

Let V be a Harish-Chandra $(\mathfrak{g}_c, \mathfrak{k}_c)$-module. We define

$$\mathrm{Wh}_\Psi^\infty(V)=\{v \in V'_\infty \mid X.v=\psi(X)v\,(X \in \mathfrak{n}_S)\}.$$

We call an element of $\mathrm{Wh}_\Psi^\infty(V)$ a $C^{-\infty}$-Ψ-Whittaker vector for V. (For simplicity, we call it a $C^{-\infty}$-Whittaker vector.)

3.2. EXACTNESS OF Wh_Ψ^∞. — Hereafter, we fix a permissible unitary Whittaker datum $\Psi=(\bar{\mathfrak{n}}_S, \psi)$ and denote by $(\bar{\mathfrak{n}}_S)_c$ the complexification of $\bar{\mathfrak{n}}_S$. The following is one of crucial point of this paper.

PROPOSITION 3.2.1. — *Let* $S \subseteq \Pi$. *We assume* $\Psi=(\bar{\mathfrak{n}}_S, \psi)$ *is a permissible unitary Wittaker datum. Then* $\mathcal{H}(w_S) \ni V \rightsquigarrow \mathrm{Wh}_\Psi^\infty(V)$ *is an exact functor from* $\mathcal{H}(w_S)$ *to the category of complex vector spaces.*

Remark. — W. Casselman proved the exactness of Wh_Ψ^∞ when P_S is a minimal parabolic subgroup. The above proposition does not contain his result, since Casselman proved the exactness without the assumptions "G is complex" and "V has an integral infinitesimal character". He described a sketch of proof in [Ca1].

We consider the (twisted) $(\bar{\mathfrak{n}}_S)_c$-cohomology (*cf.* [Ko], [Ly], [W3], also *see* [Mat5] 2.2, 2.3). For a $U((\bar{\mathfrak{n}}_S)_c)$-module M and $i \in \mathbb{N}$, the 0-th $(\bar{\mathfrak{n}}_S)_c$-cohomology group $H^0((\bar{\mathfrak{n}}_S)_c, M)$ is defined by

$$H^0((\bar{\mathfrak{n}}_S)_c, M)=\{v \in M \mid Xv=0\,(X \in ((\bar{\mathfrak{n}}_S)_c,)\}.$$

$M \rightsquigarrow H^0((\bar{\mathfrak{n}}_S)_c, M)$ is a left exact functor from the category of $U((\bar{\mathfrak{n}}_S)_c)$-module to the category of vector spaces. The i-th $(\bar{\mathfrak{n}}_S)_c$-cohomology group $H^i((\bar{\mathfrak{n}}_S)_c, M)$ is defined as an i-th right derived functor of a functor $M \rightsquigarrow H^0((\bar{\mathfrak{n}}_S)_c, M)$.

For a Whittaker datum $\Psi=(\bar{\mathfrak{n}}_S, \psi)$, we define a one-dimensional $U((\bar{\mathfrak{n}}_S)_c)$-module $\mathbb{C}_{-\psi}$ by $Xv=-\psi(X)v$ $(X\in(\bar{\mathfrak{n}}_S)_c, v\in\mathbb{C}_{-\psi})$.

Since Wh_Ψ^∞ is a left exact contravairant functor from $\mathscr{H}$ to the category of complex vector spaces, taking account of the exactness of $V \rightsquigarrow V'_\infty$ $(V\in\mathscr{H})$, we immediately see we have only to prove $H^1((\bar{\mathfrak{n}}_S)_c, V'_\infty\otimes\mathbb{C}_{-\psi})=0$ for all $V\in\mathscr{H}(w_S)$. From a standard argument using a long exact sequence, we easily *see* Proposition 3.2.1 is deduced to the following lemma (In fact, we only need the vanishing of the 1st cohomology.)

LEMMA 3.2.2. — *Let $\Psi=(\bar{\mathfrak{n}}_S, \psi)$ be a permissible unitary Whittaker datum and let* V *be an irreducible Harish-Chandra $(\mathfrak{g}_c, \mathfrak{k}_c)$-module with contained in $\mathscr{H}(w_S)$. Then we have*

$$H^i((\bar{\mathfrak{n}}_S)_c, V'_\infty\otimes\mathbb{C}_{-\psi})=0,$$

for all $i>0$.

The following result will be proved in § 4 using Casselman's idea.

PROPOSITION 3.2.3. — *We assume $\Psi=(\bar{\mathfrak{n}}_S, \psi)$ is a permissible unitary Whittaker datum. For all finite dimensional irreducible representation σ of L_S, we have*

$$H^p((\bar{\mathfrak{n}}_S)_c, C^\infty(G/P_S;\sigma)'\otimes\mathbb{C}_{-\psi})=0,$$

for all $p>0$.

Using this proposition, we prove Lemma 2.2.2. First, we show that we can assume $V\in\mathscr{H}(w_S)[\rho, \rho]$.

Let $V\in\mathscr{H}$ and let E be a finite dimensional $U((\bar{\mathfrak{n}}_S)_c)$-module. E always bas a $U((\bar{\mathfrak{n}}_S)_c)$-stable finite filtration whose grading module is a direct sum of the copies of the trivial $U((\bar{\mathfrak{n}}_S)_c)$-module $\mathbb{C}$. From a standard argument using a long exact sequence (or a spectral sequence), we see:

LEMMA 3.2.4. — *Let $V\in\mathscr{H}$ and let* E *be a finite dimensional* U*-module such that* $E\neq 0$. *Then,* $H^p((\bar{\mathfrak{n}}_S)_c, ((V\otimes E)'_\infty\otimes\mathbb{C}_{-\psi})=0\ (p>0)$ *if* $H^p((\bar{\mathfrak{n}}_S)_c, (V'_\infty)\otimes\mathbb{C}_{-\psi})=0\ (p>0)$.

Let $\lambda, \mu\in P^-$ and let $V\in\mathscr{H}(w)[-\mu, -\lambda]$ be irreducible. Then there exists some irreducible Harish-Chandra $(\mathfrak{g}_c, \mathfrak{k}_c)$-module $V_0\in\mathscr{H}(w)[\rho, \rho]$ such that $V=T_{\rho,\ \rho}^{-\mu,\ -\lambda}(V_0)$. Since V is a direct summand of $V_0\otimes V_{-\mu-\rho,\ -\lambda-\rho}$, we see that we can assume $V\in\mathscr{H}(w_S)[\rho, \rho]$ in order to prove Lemma 3.2.2. So, hereafter we assume $V\in\mathscr{H}(w_S)[\rho, \rho]$).

Put $d=\dim(\bar{\mathfrak{n}}_S)_c$. Since $(\bar{\mathfrak{n}}_S)_c$-cohomologies are computed by a Koszul complex, we always have

$$H^p((\bar{\mathfrak{n}}_S)_c, V'_\infty\otimes\mathbb{C}_{-\psi})=0$$

for all $p>d$.

So, we prove

$$(*)_k \qquad H^p((\bar{\mathfrak{n}}_S)_c, V'_\infty\otimes\mathbb{C}_{-\psi})=0 \quad \text{for all } p>k.$$

by the descending induction on k. We assume $(*)_k$ holds. Using long exact sequences, we see that $H^{k+1}((\bar{\mathfrak{n}}_S)_c, M'_\infty \otimes \mathbb{C}_{-\psi})=0$ for all $M \in \mathscr{H}(w_S)[\rho, \rho]$.

From Theorem 2.4.1, there exists some irreducible finite dimensional L_S-representation σ such that V is a submodule of $\mathrm{Ind}_{P_S}^G(\sigma)$. Put $M=\mathrm{Ind}_{P_S}^G(\sigma)/V$. Theorem 2.4.1 also implies $M \in \mathscr{H}(w_S)[\rho, \rho]$. Hence, Lemma 2.2.2 follows from Proposition 2.2.3 and the following long exact sequence.

$$\ldots \to H^k((\bar{\mathfrak{n}}_S)_c, C^\infty(G/P_S, \sigma)' \otimes \mathbb{C}_{-\psi}) \to H^k((\bar{\mathfrak{n}}_S)_c, V'_\infty \otimes \mathbb{C}_{-\psi}) \to H^{k+1}((\bar{\mathfrak{n}}_S)_c, M'_\infty \otimes \mathbb{C}_{-\psi}) \to \ldots$$

Q.E.D.

Remark. — The above argument is suggested in [Ca1] and Casselman uses his subrepresentation theorem in stead of Theorem 2.4.1.

3.3. WHITTAKER POLYNOMIALS. — We fix $S \subseteq \Pi$ and a permissible unitary Whittaker datum $\Psi=((\bar{\mathfrak{n}}_S)_c, .$ First, we have:

LEMMA 3.3.1. — *Let* $S \subseteq \Pi$ *and let* $\Psi=(\bar{\mathfrak{n}}_S, \psi)$ *be a permissible Whittaker datum. Let* V *be a Harish-Chandra* $(\mathfrak{g}_c, \mathfrak{k}_c)$*-module and let* E *be a finite dimensional* U*-module. If* $\dim \mathrm{Wh}_\Psi^\infty(V) < \infty$, *then*

$$\dim \mathrm{Wh}_\Psi^\infty(V \otimes E) = \dim E \dim \mathrm{Wh}_\Psi^\infty(V).$$

Proof. — Since we have $(V \otimes E)'_\infty = V'_\infty \otimes E^*$. Hence, we have the lemma from [Ly] Theorem 4.2 (also see [Ko2]). □

We need:

THEOREM3.3.2 (Yamashita [Y1] Theorem 3.7). — *Let* $S \subseteq \Pi$ *and let* $\Psi=(\bar{\mathfrak{n}}_S, \psi)$ *be a permissible Whittaker datum. Let* (σ, E_σ) *be an irreducible finite-dimensional continuous representation of* L_S.

Then we have

$$\dim \mathrm{Wh}_\Psi^\infty(\mathrm{Ind}_{P_S}^G(\sigma)) \leqq \dim E_\sigma.$$

Remark 1. — Yamashita proved the above result for real semisimple (reductive) Lie groups.

Remark 2. — In [Y1], the above result is stated under a stronger assumption on Ψ, namely Ψ should be of GGGR type (*cf.* [Kaw1, 2, 3]). However, in his proof, Yamashita only uses the permissibility of a character.

Remark 3. — In [W3], independently, Wallach proved a related result.

Hereafter, we put $w_\Psi(V)=\dim \mathrm{Wh}_\Psi^\infty(V)$ for all $V \in \mathscr{H}(w_S)$. Using the above results, we have:

PROPOSITION 3.3.3. — *Let* $S \subseteq \Pi$ *and let* $\Psi=(\bar{\mathfrak{n}}_S, \psi)$ *be a permissible Whittaker datum. For an arbitrary* $V \in \mathscr{H}(w_S)$, *we have* $w_\Psi(V) < \infty$.

Proof. – Let $V \in \mathscr{H}(w_S)$. From Proposition 3.2.1, we can assume V is irreducible. Hence, there exists some $\lambda, \mu \in P^{--}$ and $z \in W$ such that $V \cong V^{\odot}(z^{-1}; -\mu, -\lambda)$ and $z \overset{LR}{\leqq} w_S$. Since $V = T_{\rho, \rho}^{-\mu, -\lambda}(L(M(\rho), L(-z^{-1}\rho)))$, V is a direct summand of $V_{-\mu-\rho, -\lambda-\rho} \otimes L(M(\rho), L(-z^{-1}\rho))$. From Lemma 3.3.1, we can assume V has infinitesimal character (ρ, ρ). The proposition follows from Theorem 3.3.2, Theorem 2.4.1, and Proposition 3.2.1. □

If Ψ is unitary and permissible Wh_{Ψ}^{∞} is non-trivial, namely we have:

LEMMA 3.3.4. — *Put* $\rho_S = 1/2(\rho + w_S \rho)$. *Then, sufficiently large* $k \in \mathbb{N}$, *we have*

$$w_{\Psi}(\mathrm{Ind}_{P_S}^{G}(E_S^{*}(-w_S \rho - 2k\rho_S, -w_S \rho - 2k\rho_S)) = 1.$$

Proof. – First, we remark that the dimension of $\sigma_k = E_S^{*}(-w_S \rho - 2k\rho_S, -w_S \rho - 2k\rho_S)$ is one. Hence $f \in C^{\infty}(G/P_S; \sigma_k)$ can be regard as a function on G. For $f \in C^{\infty}(G/P_S; \sigma_k)$, we define the Whittaker integral (*cf.* [Jc], [Sc]) by

$$j^{\Psi}(f) = \int_{\bar{N}_S} \psi(\bar{n})^{-1} f(\bar{n})\, d\bar{n}.$$

Here, $d\bar{n}$ is a Haar measure on $\bar{N}_S$. The absolute convergence of the above integral for sufficiently large k is proved in just the same way as the case of intertwining integral (*cf.* [Kn], Theorem 7.22). Clearly Whittaker integral defines non-zero element of $Wh_{\Psi}^{\infty}(\mathrm{Ind}_{P_S}^{G}(\sigma_k))$ for sufficient large $k \in \mathbb{N}$. Hence, $1 \leqq w_{\Psi}(\mathrm{Ind}_{P_S}^{G}(\sigma_k))$. The other inequality is just Theorem 3.3.2. □

Proposition 3.2.1, Lemma 3.3.1, Proposition 3.3.3, and Proposition 2.7.1 imply:

COROLLARY 3.3.5. — *Let* $S \subseteqq \Pi$ *and let* $\Psi = (\bar{\mathfrak{n}}_S, \psi)$ *be a permissible unitary Whittaker datum. Then,* w_{Ψ} *is an additive invariant on* $\mathscr{H}(w_S)$.

For $V \in \mathscr{H}(w_S)$ with an (integral) infinitesimal character, we define the Whittaker polynomial of V with respect to the admissible unitary Whittaker datum Ψ by

$$p_{\Psi}[V] = v[w_{\Psi}; \Theta_V].$$

Here, the right hand side is a polynomial defined in Theorem 2.1.1. If $\lambda, \mu \in P^{--}$ and $V \in \mathscr{H}(w_S)[-\mu, -\lambda]$ is irreducible, then

$$p_{\Psi}[V] = p_{\Psi}[T_{-\mu, -\lambda}^{-\mu', -\lambda'}(V)]$$

for all $\lambda', \mu' \in P^{-}$ such that $T_{-\mu, -\lambda}^{-\mu', -\lambda'}(V) \neq 0$.

Remark. – Whittaker polynomials are first introduced by Lynch [Ly] for the algebraic analogue of Wh_{Ψ}^{∞}.

Hereafter, we put $r = \mathrm{card}\, \Delta^{+}$ and $d = \dim \mathcal{O}_S = \dim (\mathfrak{n}_S)_c = 2 \dim \mathfrak{n}_S$. Theorem 3.3.2, Proposition 2.3.7, Proposition 2.7.1, and Weyl's dimension formula imply

$\deg(w_\Psi) \leqq 2r-d$. From Lemma 3.3.4 and Theorem 2.7.3, we have the main result of this paper.

THEOREM 3.3.6. — *Let* $S \subseteq \Pi$ *and let* Ψ *be an permissible unitary Whittaker datum. Then we have*

(1) $w_\Psi (= \dim \mathrm{Wh}_\Psi^\infty)$ *is an additive invariant on* $\mathscr{H}(w_S)$ *and* $\deg(w_\Psi) = 2r-d$.

(2) $\Phi_{w_\Psi}: \bar{V}^{LR}(w_S) \to S(\mathfrak{h} \times \mathfrak{h})$ (cf. *Theorem* 2.1.1) *induces a surjective* $W \times W$-*homomorphism*

$$\varphi_{w_\Psi}: \quad V^{LR}(w_S) \to \sigma(w_S) \otimes \sigma(w_S).$$

Here, $\sigma(w_S) \subseteq S^{2r-d}(\mathfrak{h})$ *is generated by* p_S *as a* W-*module.*

(3) *There exists some positive constant k such that* $w_\Psi = kc_d$. *Hence, for all* $V \in \mathscr{H}(w_S)$, *we have* $p_\Psi[V] = kv[c_d; \Theta_V]$.

(4) *For* $\lambda, \mu \in P^{--}$ *and* $w_S \overset{LR}{\leqq} z$, $p_\Psi[L(M(w_0\lambda), L(z^{-1}\mu)))] \neq 0$ *if and only if* $z \overset{LR}{\sim} w_S$.

Remark. — If Ψ is permissible but non-unitary, then w_Ψ is trivial.

From Theorem 2.6.2, the above φ_{w_Ψ} is determined up to a scalar factor. The following result determines the scalar factor.

PROPOSITION 3.3.7. — *Let* $S \subseteq \Pi$ *nd let* Ψ *be a permissible unitary Whittaker datum. Let* $\lambda, \mu \in P^{--}$. *Then*

$$p_\Psi[L(M(w_0\lambda), L(v_S^{-1}\mu))](\eta, \nu) = \frac{p_S(\eta)p_S(\nu)}{p_S(\rho)^2}.$$

Proof. — We can assume $\lambda = \mu = -\rho$. From Proposition 2.3.7 and Theorem 3.3.6, we have

$$p_\Psi[L(M(\rho), L(-z^{-1}\rho))] = p_\Psi[\mathrm{Ind}_{P_S}^G(E_S^*(-w_S, \rho, -w_S\rho))].$$

[Jo4], 6.7 Remark and Weyl's dimension formula implies $p_\Psi[L(M(\rho), L(-z^{-1}\rho))]$ is proportional to $p_S \otimes p_S$. So, the proposition follows from Lemma 3.3.4. □

Remark. — Since the actions of $W \times W$ on the double cells are, in principle, computable, we can, in principle, compute Whittaker polynomials using the above proposition. However, in actual computations, this fact is not so useful (for example, see a discussion in the introduction of [BV3]).

Remark. — Casselman [Ca1] and Wallach ([W3], Theorem 7.2) got a similar result for real semisimple groups (also *see* [Mat5], Corollary 7.3.8). It is possible to prove Proposition 3.3.7 without using Joseph's result.

3.4. $C^{-\infty}$-WHITTAKER VECTORS AND WAVE FRONT SET. — In this section we prove:

THEOREM 3.4.1. — *Let* $S \subseteq \Pi$ *and let* Ψ *be a strongly admissible unitary Whittaker datum. For an irreducible Harish-Chandra* $(\mathfrak{g}_c, \mathfrak{k}_c)$-*module* V *with an integral infinitesimal character the followings are equivalent.*

(1) $\mathrm{Wh}_\Psi^\infty(V) \neq 0$ *and* $\dim \mathrm{Wh}_\Psi^\infty(V) < \infty$.

(2) $\mathrm{WF}(V) = i\bar{\mathcal{O}}_S$.

Proof. — From Theorem 2.5.1 and Proposition 2.5.4, we can easily see (2) implies the following (3).

(3) There exist some λ, $\mu \in P^-$ and $z \in W$ such that $V \cong L(M(w_0\lambda), L(z^{-1}\mu)) \neq 0$ and $z \overset{LR}{\sim} w_S$.

From Theorem 3.3.6, we see that (3) implies (1).

Next, we remark that (2) is equivalent to the following (4) (*cf.* 2.5).

(4) $\mathrm{Ass}(U/I) = \bar{\mathcal{O}}_S \times \bar{\mathcal{O}}_S$, where $I = \mathrm{Ann}_U(V)$.

We quote:

LEMMA 3.4.2 ([Mat2] Theorem 2, [Mat5], Theorem 2.9.4). — *Let* $S \subseteq \Pi$ *and let* Ψ *be an admissible unitary Whittaker datum. For an irreducible Harish-Chandra* $(\mathfrak{g}_c, \mathfrak{k}_c)$-*module* V, *put* $I = \mathrm{Ann}_U(V)$.

Then,

(1) $\mathrm{Wh}_\Psi^\infty(V) \neq 0$ *implies* $\bar{\mathcal{O}}_S \times \bar{\mathcal{O}}_S \subseteq \mathrm{Ass}(U/I)$.

(2) *If* $\mathrm{Dim}(V) > \dim \mathcal{O}_S$, *their either* $\mathrm{Wh}_\Psi^\infty(V) = 0$ *or* $\dim \mathrm{Wh}_\Psi^\infty(V) = \infty$.

(1) $\to$ (4) *is clear from the above lemma.*

Q.E.D.

4. Vanishing of twisted $\mathfrak{n}_S$-cohomologies of an induced representation

4.1. A VANISHING THEOREM. — In this 4, to complete the proof of our main theorem, we prove a vanishing theorem of twisted $\mathfrak{n}_S$-cohomology groups of some induced representations (Proposition 3.2.3). We would like to stress how much the contents in this 4 owes to Casselman.

In order to state the result in a general form, we abandon the notations of 1.4, and use the following notations. (We retain the notations in 1.1-1.3.)

Let $\mathfrak{g}_0$ be a real form of $\mathfrak{g}$ and let G be a connected semisimple linear Lie group whose Lie algebra is $\mathfrak{g}_0$. We fix a minimal parabolic subgroup P_m of G whose complexified Lie algebra $\mathfrak{p}_m$ contains $\mathfrak{b}$. Let S_m be the subset of Π corresponding to $\mathfrak{p}_m$. Hereafter, we fix a parabolic subgroup P of G such that $P_m \subseteq P$. Let S be the subset of Π corresponding to $\mathfrak{p}$ and we write $\mathfrak{l}$, $\mathfrak{n}$, $\bar{\mathfrak{n}}$, $\mathfrak{a}$, and $\mathfrak{m}$ for $\mathfrak{l}_S$, $\mathfrak{n}_S$, $\bar{\mathfrak{n}}_S$, and $\mathfrak{m}_S$ respectively. Let $P = M_\# A_\# N$ be a Langlands decomposition of P and we denote by $\mathfrak{m}_\#$ and $\mathfrak{a}_\#$ the complexified Lie algebras of $M_\#$ and $A_\#$ respectively. We assume $\mathfrak{a}_\# + \mathfrak{m}_\# = \mathfrak{l}$, $\mathfrak{a}_\# \subseteq \mathfrak{a}$, and $\mathfrak{m} \subseteq \mathfrak{m}_\#$. We put $L = M_\# A_\#$. We denote by $\bar{N}$ the opposite nilpotent subgroup to N. Let $P_m = M_m A_m U_m$ be the Langlands decomposition which has the same properties as the above Langlands decomposition of P. Let $\bar{U}_m$ be the opposite subgroup to U_m and let $G = KA_m U_m$ be the Iwasawa decomposition which is compatible with the above definitions. We denote the complexified Lie algebras of A_m, K, M_m, $\bar{U}_m$, and U_m by $\mathfrak{a}_m$, $\mathfrak{k}$, $\mathfrak{m}_m$, $\bar{\mathfrak{u}}_m$, and $\mathfrak{u}_m$ respectively. We denote by Σ the restricted root system with respect to $(\mathfrak{g}, \mathfrak{a}_m)$ and denote by Σ^+ the positive system corresponding to $\mathfrak{u}_m$. We denote by Π_m the simple root system of Σ^+. Then P corresponding to a subset S_m of Π_m, namely S_m is a simple root system of $(\mathfrak{l}, \mathfrak{a}_m)$. Let W_m *be the little Weyl group*.

Let (σ, E_σ) be a finite dimensional continuous L-representation. We define that N acts on E_σ trivially and regard σ as a P-representation. We define

$$C^\infty(G/P; \sigma) = \{F : G \to E_\sigma \mid F \text{ is of the class } C^\infty \text{ and } F(gp) = \sigma(p^{-1}) F(g) \text{ for all } g \in G \text{ and } p \in P\}.$$

We regard the above space as a Frechet G-module in the standard manner. Then there is a differential action of $U(\mathfrak{g})$ on $C^\infty(G/P; \sigma)$.

Let $\psi : \bar{\mathfrak{n}} \to \mathbb{C}$ be a character and we define a 1-dimensional $\bar{\mathfrak{n}}$-module $\mathbb{C}_{-\psi}$ by $Xz = -\psi(X) z$ for $z \in \mathbb{C}_{-\psi}$ and $X \in \bar{\mathfrak{n}}$. Using the Killing form, we can regard ψ as an element of $\mathfrak{g}$. We call ψ permissible, if the restriction of ψ to $\bar{\mathfrak{n}} \cap \mathrm{Ad}(l\tilde{w})\mathfrak{p}$ is non-trivial for all $l \in L$ and $w \in W_m$ such that $\tilde{w} \notin L$. Here, $\tilde{w}$ means a representative of w in K.

The purpose of § 4 is to prove:

THEOREM 4.1.1. — *For a permissible unitary character* ψ *of* $\bar{\mathfrak{n}}$ *and an irreducible finite dimensional continuous* L*-representation* σ, *we have*

$$H^i(\bar{\mathfrak{n}}, C^\infty(G/P; \sigma)' \otimes \mathbb{C}_{-\psi}) = 0,$$

for all $i > 0$. *Here,* $C^\infty(G/P; \sigma)'$ *is the continuous dual space of* $C^\infty(G/P; \sigma)$.

Proposition 3.2.3 is clearly a special case of the above theorem.

The above theorem is proved by Casselman (*cf.* [Ca1]) when P is a minimal parabolic subgroup. Our proof of the above theorem is essentially the same as that of Casselman which is sketched in [Ca1]. Since L is not stable under the conjugations of W_m and since $\bar{N}$ does not act transitively on the Schubert cells in G/P for a general P, our proof is technically more complicated. In particular, we reduce the theorem to the class one case and consider the Bruhat filtration on G/P in stead of that of G/N.

4.2. THE FIRST REDUCTION. — Here, we reduce Theorem 4.1.1 to the following lemma using an idea in [LeW] and the technique of wall crossing.

LEMMA 4.2.1. — *Let* τ *be a one-dimensional continuous* L*-representation such that* $\tau|_{M_\sharp} \equiv \mathrm{id}_{M_\sharp}$. *Let* ψ *be a permissible unitary character on* $\bar{\mathfrak{n}}$. *Then we have*

$$H^i(\bar{\mathfrak{n}}, C^\infty(G/P; \tau)' \otimes \mathbb{C}_{-\psi}) = 0,$$

for all $i > 0$.

In 4.2, we assume Lemma 4.2.1 and deduce Theorem 4.1.1.

First, we remark that we can assume G has a simply-connected complexification. Let $\tilde{G}$ be the covering group of G whose complexification is simply-connected. Let $\tilde{P}$ and $\tilde{L}$ be the corresponding subgroups to P and L respectively. For a continuous finite dimensional L-representation σ, we denote by $\tilde{\sigma}$ the lifting of σ to $\tilde{L}$. Then clearly we have $C^\infty(G/P;\sigma)' \cong C^\infty(\tilde{G}/\tilde{P};\tilde{\sigma})'$ as $U(\mathfrak{g})$-modules. Hence, hereafter, we assume the complexification $G_{\mathbb{C}}$ of G is simply-connected.

Next, we investigate the precise structure of L and its finite dimensional irreducible representations.

Let $L_{\mathbb{C}}$ be the analytic subgroup of $G_{\mathbb{C}}$ with respect to $\mathfrak{l}$. Then, we have $L \subseteq L_{\mathbb{C}}$. For $\lambda \in P_S^{++} \cap P$, we denote by $E_{\lambda-\rho}$ the finite-dimensional irreducible $U(\mathfrak{l})$-module with the highest weight $\lambda-\rho$. It is not difficult see (for example, *see* [Mat5], Lemma 7.1.3) $E_{\lambda-\rho}$ can be lifted to an $L_{\mathbb{C}}$-representation. We denote the above lifting of $E_{\lambda-\rho}$ to an $L_{\mathbb{C}}$-representation by the same letter $E_{\lambda-\rho}$. We also denote the restriction of $E_{\lambda-\rho}$ to an L-representation by $E_{\lambda-\rho}$. Let $\chi: L \to \mathbb{C}^\times$ be a linear character, namely one dimensional representation. We denote by $\mathbb{C}_\chi$ the representation space of χ. We denote by $\hat{X}$ the complexified differential representation of χ and denote the restriction of $\hat{\chi}$ to $\mathfrak{h}$ by the same letter. We denote by $L^\vee$ the set of linear character on L. For $\chi \in L^\vee$, we put $\Lambda_\chi = \hat{\chi} + P \subseteq \mathfrak{h}^*$. We denote by $E_{\lambda-\rho}(\chi)$ an L-representation $E_{\lambda-\rho} \otimes \mathbb{C}_\chi$ for all $\lambda \in P_S^{++} \cap P$ and $\chi \in L^\vee$.

For $\alpha \in \Pi_m$, we put $e_\alpha = \exp(\pi i \check{H}_\alpha)$, where $\check{H}_\alpha \in \mathfrak{a}_m$ is defined by $\lambda(\check{H}_\alpha) = 2\langle \lambda, \alpha \rangle / \langle \alpha, \alpha \rangle (\lambda \in \mathfrak{a}_m^*)$. We denote by Z_S the group generated by $e_\alpha \alpha \in \Pi_m$. Since G is linear and connected, we have $L = L_0 Z_S$. Here, L_0 is the identity component of L. Using this fact, it is not difficult to prove:

LEMMA 4.2.2. — *For any irreducible finite dimensional* L-*representation* V, *there exist some* $\lambda \in P_S^{++} \cap P$ *and* $\chi \in L^\vee$ *such that* $\chi|_{M_\sharp}$ *and* $V \cong E_{\lambda-\rho}(\chi)$.

(I learned this lemma from D. A. Vogan.) The details of the proof of this lemma are left to readers. However, if G is complex, we have $L = L_0$. So, in this case, the above lemma is trivial.

Put $\rho_S = 1/2(w_S\rho + \rho)$. For $\chi \in L^\vee$ and $\lambda \in \Lambda_\chi \cap P_S^{++}$, we define

$$C^\infty(G/P;\lambda,\chi) = C^\infty(G/P; E_{\lambda-\hat{\chi}+2\rho_S-\rho}(\chi)).$$

We can easily see $C^\infty(G/P;\lambda,\chi)$ has an infinitesimal character λ.

Let Δ_χ, Δ_χ^+, and W_χ be the integral root system, the integral positive root system, and the integral Weyl group with respect to Λ_χ. We denote by Π_χ the set of simple roots of Δ_χ^+. We remark $S \subseteq \Pi_\chi$.

The following result is similar to Lemma 3.2.4.

LEMMA 4.2.3. — *Let* V *be an arbitrary* $U(\mathfrak{g})$-*module and let* E *be a finite dimensional* $U(\mathfrak{g})$-*module. We assume that* $H^p(\bar{\mathfrak{n}}, V \otimes \mathbb{C}_{-\psi}) = 0$ *for all* $p > 0$. *Then we have* $H^p(\bar{\mathfrak{n}}, (V \otimes E) \otimes \mathbb{C}_{-\psi}) = 0$ *for all* $p > 0$.

Next, we consider translation functors. Fix $\chi \in L^\vee$. For $\lambda \in \Lambda_\chi$ and a $U(\mathfrak{g})$-module V such that $Z(\mathfrak{g})$ acts on V locally finitely, we define $P_\lambda(V)$ in the same way as 1.5. Namely,

$$P_\lambda(V) = \{ v \in V \mid \exists n \in \mathbb{N}, \forall u \in Z(\mathfrak{g})\, (\chi_\lambda(u) - u)^n v = 0 \}.$$

Clearly $P_\lambda(V)$ is a direct summand of V. For $\eta \in P$, we denote by V_η the irreducible finite dimensional $U(\mathfrak{g})$-module with an extreme weight η. We also denote by $P(V_\eta)$ the set of $\mathfrak{h}$-weights of V_η.

Let λ, $\mu \in \Lambda_\chi$ and let V be a $U(\mathfrak{g})$-module with an infinitesimal character μ. Then, we define the translation functor by

$$T_\mu^\lambda(V) = P_\lambda(V \otimes V_{\lambda-\mu}).$$

Using Lemma 4.2.3, we immediately have:

LEMMA 4.2.4. — *Let $\chi \in L^\vee$. Let λ, $\mu \in \Lambda_\chi$ and let V be a $U(\mathfrak{g})$-module with an infinitesimal character μ. We assume $H^p(\bar{\mathfrak{n}}, V \otimes \mathbb{C}_{-\psi}) = 0$ for all $p > 0$. Then, we have $H^p(\bar{\mathfrak{n}}, T_\mu^\lambda(V) \otimes \mathbb{C}_{-\psi}) = 0$ for all $p > 0$.*

For $\lambda \in \Lambda_\chi$, we define

$$A(\lambda) = \{\alpha \in \Delta_\chi^+ \mid \langle \lambda, \check{\alpha} \rangle \geqq 0\},$$
$$W_\lambda = \{w \in W \mid w\lambda = \lambda\}.$$

The following is well-known:

LEMMA 4.2.5 (*cf.* [Vo3]). — (1) *Fix $\chi \in L^\vee$ and let λ, $\mu \in \Lambda_\chi$. We assume λ is regular and $A(\lambda) = A(\mu)$. Then, the equation $\lambda + \eta = w\mu$ for $\eta \in P(V_{\mu-\lambda})$ and $w \in W$ holds if and only if $w \in W_\mu$ and $\eta = \mu - \lambda$. The equation $\mu + \eta = w\lambda$ for $\eta \in P(V_{\lambda-\mu})$ and $w \in W$ holds if and only if $w \in W_\mu$ and $\lambda = w\mu$.*

(2) *Let λ, $\mu \in P_S^{++}$ and let V be a finite dimensional completely reducible L-representation. We assume that $E_{\lambda-\rho}$ is an irreducible constituent of $E_{\mu-\rho} \otimes V$ as a L-module. Then there exists some $w \in W_S$ and a weight $\eta \in P(V)$ of V such that $\lambda + \eta = w\mu$.*

Here, we recall that the following Mackey tensor product theorem.

LEMMA 4.2.6. — *For a finite dimensional continuous L-representation E and a finite dimensional continuous G-representation V, we have*

$$C^\infty(G/P; E) \otimes V \cong C^\infty(G/P; E \otimes V|_P).$$

From Lemma 4.2.5, Lemma 4.2.6, it is not difficult to see:

LEMMA 4.2.7. — *Fix $\chi \in L^\vee$ and let λ, $\mu \in \Lambda_\chi \cap P_S^{++}$. We assume λ is regular and $A(\lambda) = A(\mu)$. Then*

$$T_\lambda^\mu(C^\infty(G/P; \lambda, \chi)) = C^\infty(G/P; \mu, \chi).$$

From Lemma 4.2.2, we immediately have:

LEMMA 4.2.8. — *Let $\chi \in L^\vee$. Then there exists some regular $\lambda \in \Lambda_\chi \cap P_S^{++}$ such that $A(\lambda) = \mathbb{N} S \cap \Delta^+$ and $E_{\lambda - \hat{\chi} + 2\rho_S - \rho}(\chi)$ is one dimensional continuous L-representation such that $\tau|_{M_\#} = \mathrm{id}_{M_\#}$.*

From Lemma 4.2.4, Lemma 4.2.7, and Lemma 4.2.8, we immediately see that Lemma 4.2.1 implies:

LEMMA 4.2.9. — *Let* $\chi \in L^\vee$. *We assume that* $\lambda \in \Lambda\chi \cap P_S^{++}$ *satisfies that* $A(\lambda) = \mathbb{N} S \cap \Delta^+$. *Then,* $H^p(\bar{\mathfrak{n}}, C^\infty(G/P; \lambda, \chi)' \otimes \mathbb{C}_{-\psi}) = 0$ *for all* $p > 0$.

For regular $\lambda \in P_S^{++} \cap \Lambda_\chi$, we define a non-negative integer $n(\lambda)$ as follows.

$$n(\lambda) = \operatorname{card} A(\lambda) - \operatorname{card} \mathbb{N} S \cap \Delta^+.$$

In order to deduce Theorem 4.1.1 from Lemma 4.2.9, we have only to prove for all $k \geqq 0$ the following statement.

$(A)_k$: For all regular $\lambda \in \Lambda_\chi \cap P_S^{++}$ such that $n(\lambda) \geqq k$, we have

$$H^p(\bar{\mathfrak{n}}, C^\infty(G/P; \lambda, \chi)' \otimes \mathbb{C}_{-\psi}) = 0$$

for all $p > 0$.

(Here, from Lemma 4.2.4 and Lemma 4.2.7, we see we can assume λ is regular.)

We prove A_k by the induction on k. A_0 is just Lemma 4.2.9. We assume $(A)_k$ holds and $\lambda \in \Lambda_\chi \cap P_S^{++}$ is a regular weight such that $n(\lambda) = k+1$.

We denote by $\Delta_\chi^+(\lambda)$ the positive root system of Δ_χ such that λ is antidominant with respect to $\Delta_\chi^+(\lambda)$. Let w_λ be the unique element of W_χ such that $w_\lambda \lambda \in P^{--}$, namely $w_\lambda \lambda$ is antidominant with respect to Δ_χ^+. Since $n(\lambda) > 0$, we have $l(w_\lambda) > l(w_S)$, where l means the length function on W_χ with respect to Π_χ.

We need:

CLAIM 1. — *There exists some simple root* α_λ *of* $\Delta_\chi^+(\lambda)$ *such that* $s_{\alpha_\lambda} \lambda \in P_S^{++} \cap \Lambda_\chi$ *and* $n(s_{\alpha_\lambda} \lambda) = k$.

Proof. — Put $\tau = w_\lambda^{-1}$ and $\mu = w_\lambda \lambda \in P^{--}$. For all $\sigma \in W$, put $\Phi_\sigma = \sigma \Delta_\chi^+ \cap -\Delta_\chi^+$. We remark that $l(\sigma) = \operatorname{card} \Phi_\sigma$. We assume $\tau \Pi_\chi \cap -\Delta_\chi^+ \subseteqq \Phi_{w_S}$. So, $\tau \Pi_\chi \subseteqq \Phi_{w_S} \cup \Delta_\chi^+ = (\mathbb{Z} S \cap \Delta_\chi) \cup \Delta_\chi^+$. If $\beta, \gamma \in (\mathbb{Z} S \cap \Delta_\chi) \cup \Delta_\chi^+$ and $\beta + \gamma \in \Delta_\chi$, then $\beta + \gamma \in (\mathbb{Z} S \cap \Delta_\chi) \cup \Delta_\chi^+$. Hence, $\tau \Delta_\chi^+ \subseteqq \Phi_{w_S} \cup \Delta_\chi^+$. So, we have $\Phi_\tau = \tau \Delta_\chi^+ \cap -\Delta_\chi^+ \subseteqq \Phi_{w_S}$. This contradicts $l(w_S) < l(\tau)$. So, there exists some $\alpha \in \tau \Pi_\chi \cap -\Delta_\chi^+$ such that $\alpha \notin \Phi_{w_S}$. Hence we have $s_\alpha \tau \Delta_\chi^+ = \tau \Delta_\chi^+ \cup \{-\alpha\} - \{\alpha\}$. So, we have $\Phi_{s_\alpha \tau} = \Phi_\tau - \{\alpha\} \supseteqq \Phi_{w_S}$. So, we have $s_\alpha \tau \mu \in P_S^{++}$. Since easily we see $\Delta_\chi^+(\lambda) = \tau \Delta_\chi^+$, we have the claim. □

We quote:

LEMMA 4.2.10 (*cf.* [Vo3], Lemma 7.3.6). — *Let* $\chi \in L^\vee$ *and let* $\lambda \in \Lambda_\chi \cap P_S^{++}$ *satisfy* $n(\lambda) = k+1 > 0$. *Then there exists some* $\lambda' \in P_S^{++} \cap \Lambda_\chi$ *such that* $\langle \alpha_\lambda, \lambda' \rangle = 0$ *and* $\langle \beta, \lambda' \rangle \neq 0$ *for* $\beta \in \Delta_\chi^+(\lambda) - \{\alpha_\lambda\}$.

Choose λ' as the above lemma. We choose a regular weight $\omega \in P_S^{++} \cap \Lambda_\chi$ such that $A(\lambda) = A(\omega)$, $\omega - \lambda' \in P_S^{++}$, and $s_{\alpha_\lambda} \omega - \lambda' \in P_S^{++}$. The existence of such an ω can be proved easily. For example, put $\kappa = -w_\lambda^{-1} \rho$. Then $\omega = \lambda + m\kappa$ satisfies the above condition for sufficiently large positive integer m.

We prove:

CLAIM 2. — *There is an exact sequence*

$$0 \to C^\infty(G/P;\omega,\chi) \to T_{\lambda'}^{\omega}(C^\infty(G/P;\lambda',\chi)) \to C^\infty(G/P;s_{\alpha_\lambda}\omega,\chi) \to 0.$$

Proof. — Put $\eta = \omega - \lambda' \in P_S^{++} \cap P$ and $\xi = s_{\alpha_\lambda}\omega - \lambda' \in P_S^{++} \cap P$. Since V_η has η and ξ as extreme weights, there are embeddings of $L_{\mathbb{C}}$-modules $E_\eta \subsetneq V_\eta$ and $E_\xi \subsetneq V_\eta$. Put $E_1 = E_{\omega - \hat{\chi} + 2\rho_S - \rho}(\chi)$ and $E_2 = E_{s_{\alpha_\lambda}\omega - \hat{\chi} + 2\rho_S - \rho}(\chi)$. Then there exist embeddings of L-modules

$$E_1 \subsetneq E_{\lambda' - \hat{\chi} + 2\rho_S - \rho}(\chi) \otimes V_\eta,$$
$$E_2 \subsetneq E_{\lambda' - \hat{\chi} + 2\rho_S - \rho}(X) \otimes V_\eta.$$

Let F be the P-subrepresentation of $E_{\lambda' - \hat{\chi} + 2\rho_S - \rho}(\chi) \otimes V_\eta$ which generated by E_1. Since E_1 is closed under L, we have $F = U(\mathfrak{n}) E_1$. We assume E_2 is an irreducible constituent of F as an L-module. From Lemma 4.2.5, there exists some $\eta \in P(U(\mathfrak{n}))$ and $w \in W_S$ such that $ws_{\alpha_\lambda}\omega = \omega + \eta$. Put $n = -\langle \lambda, \check{\alpha}_\lambda \rangle$. Then n is a positive integer, since $\alpha_\lambda \in \Delta_\chi^+(\lambda)$. Hence we have $n\alpha \in \eta + \mathbb{Z}S$. Since $\alpha_\lambda \in -\Delta_\chi^+$ and $\alpha_\lambda \notin \Phi_{wS}$ (*cf.* the proof of claim 1), we have the root space of α appears in $\bar{\mathfrak{n}}$. Hence, we have a contradiction. So, we have $E_2 \cap F = 0$. We have the claim form Lemma 4.2.5 and Lemma 4.2.6. □

From Claim 2, we have the following exact sequence.

$$0 \to C^\infty(G/P;s_{\alpha_\lambda}\omega,\chi)' \otimes \mathbb{C}_{-\psi} \to T_{\lambda'}^{\omega}(C^\infty(G/P;\lambda',\chi))' \otimes \mathbb{C}_{-\psi}$$
$$\to C^\infty(G/P;\omega,\chi)' \otimes \mathbb{C}_{-\psi} \to 0.$$

From the assumption of the induction, we have

$$H^p(\bar{\mathfrak{n}}, C^\infty(G/P;s_{\alpha_\lambda}\omega,\chi)' \otimes \mathbb{C}_{-\psi}) = 0$$

for all $p>0$. From Lemma 4.2.7 and Lemma 4.2.4, we have

$$H^p(\bar{\mathfrak{n}}, T_{\lambda'}^{\omega}(C^\infty(G/P;\lambda',\chi))' \otimes \mathbb{C}_{-\psi}) = 0$$

for all $p>0$. Hence, using the long exact sequence associated to the above short exact sequence, we have

$$H^p(\bar{\mathfrak{n}}, C^\infty(G/P;\omega,\chi)' \otimes \mathbb{C}_{-\psi}) = 0$$

for all $p>0$. We use Lemma 4.2.7 and Lemma 4.2.4 again and have

$$H^p(\bar{\mathfrak{n}}, C^\infty(G/P;\lambda,\chi)' \otimes \mathbb{C}_{-\psi}) = 0$$

for all $p>0$.

Q.E.D.

4.3. THE SECOND REDUCTION (Casselman's Bruhat filtration). — Now we can use the Casselman's ingenious idea in [Ca1]. He constructs the Bruhat filtration using his theory of functions of Schwartz class on real algebraic varieties, which he refers in [Ca1] to. It seems that Casselman's Schwartz class is the dual of the tempered distributions in the sense of Kashiwara [Kas] (also see [Lo], [Mar]). I understand that Casselman and Kashiwara developed their theory independently. W. Casselman told me that around 1975 he had got his basic results, the first the result that the Schwartz space of a real algebraic variety was invariantly definable, the second the filtration of the Schwartz space associated to a stratification.

However, unfortunately, Casselman's theory has not been published at this time. So, for the convenience of readers, the following construction of Bruhat filtration, which is also ascribed to Casselman, will be depends on Kashiwara's results on tempered distributions.

First, we recall the original notion of tempered distributions by Schwartz [Sch]. A distribution u on $\mathbb{R}^n$ is called tempered, if there exists some positive C and non-negative integers m, r such that u satisfies the following condition

$$(2) \qquad \left|\int u(x)\,\varphi(x)\,dx\right| \leqq C \sum_{|\alpha|\leqq m} \sup \left|(1+|x|^2)^r \frac{\partial^{|\alpha|}}{\partial x^\alpha}\varphi(x)\right| \quad \text{for any } \varphi\in C_0^\infty(\mathbb{R}^n).$$

Here, $C_0^\infty(\mathbb{R}^n)$ denotes the space of C^∞-functions on $\mathbb{R}^n$ with compact support, $\alpha=(\alpha_1, \ldots, \alpha_n)$ is a multi-index, $|x|^2=x_1^2+\ldots+x_n^2$, $dx=dx_1 \wedge \ldots \wedge dx_n$, and $|\alpha|=\alpha_1+\ldots+\alpha_n$. We denote by $\mathscr{S}\mathbb{R}^n)'$ the space of the tempered distributions on $\mathbb{R}^n$ in the meaning of Schwartz.

We regard $\mathbb{R}^n \cup \{\infty\}$ *as a* n-dimension sphere $\mathbb{S}^n$ naturally. Then, the following holds.

THEOREM 4.3.1 (Schwartz [Sch] VII § 4 Théorème V). — *A distribution u on $\mathbb{R}^n$ is tempered if and only if there exists some distribution v on $\mathbb{S}^n$ such that $v|_{\mathbb{R}^n}$.*

The following result follows from, for example, the above theorem and a standard argument.

LEMMA 4.3.2 (*cf.* [Ca1]). — *Let $0<m\leqq n$ and we consider an Euclidean global coordinate $(x_1, \ldots, x_n)$. Let $\delta(x_1, \ldots, x_m)$ be the delta function on $\mathbb{R}^m$. Namely we have*

$$\int \delta(x_1, \ldots, x_m)\,\varphi(x_1, \ldots, x_m)\,dx=\varphi(0, \ldots, 0)$$

for $\varphi\in C_0^\infty(\mathbb{R}^m)$. We assume u is a tempered distribution on $\mathbb{R}^n$ whose support is contained in $\mathbb{R}^m=\{x_1=\ldots=x_m=0\}$. Then, u is uniquely written as follows:

$$u=\sum_{|\alpha|\leqq h} \frac{\partial^{|\alpha|}}{\partial x^\alpha}\delta(x_1, \ldots, x_m)\otimes v_\alpha.$$

Here, h is a finite non-negative integer, α is the multi-index $(\alpha_1, \ldots, \alpha_m)$, and v_α is a tempered distribution on $\mathbb{R}^{n-m}=\{(x_{m+1}, \ldots, x_n)\}$ for each α.

We review the theory of tempered distributions in the sense of [Kas], § 3. Let X be an n-dimensional real analytic paracompact manifold. We, for simplicity, assume X is orientable and fix a real analytic volume form $d\xi$ on X. For an open set U of X, we define the space $\mathscr{D}b(U)$ of distributions on U by $C_0^\infty(U)' \otimes (d\xi)^{-1}$. If we write the pairing as follows, then it well behave under local coordinate changes.

$$\int_X u(\xi)\,\varphi(\xi)\,d\xi$$

for all $\varphi \in C_0^\infty(X)$.

A distribution u defined on an open subset U of X is called tempered at a point p of X if there exist a neighborhood V of p and a distribution v defined on V such that $u|_{V\cap U} = v|_{V\cap U}$. If u is tempered at any point, then we say that u is tempered. We denote by $\mathscr{T}_X(U)$ the space of the tempered distributions on U. It is clear that the definition of temperedness dose not depend on the choice of $d\xi$.

Here, we describe some of the elementary properties of tempered distributions. First, the following result justify the name "tempered distribution" in the viewpoint of Theorem 4.3.1.

LEMMA 4.3.3 ([Kas] Lemma 3.2). — *Let u be a distribution defined on an open subset* U *of* X. *Then the following conditions are equivalent.*

(1) *u is tempered.*

(2) *u is tempered at any point of $\partial U = \bar{U} - U$.*

(3) *There exists a distribution w defind on* X *such that $u = w|_U$.*

For a subset A of $\mathbb{R}^n$ and a point x of $\mathbb{R}^n$, we denote

$$d(x, A) = \inf\{\,|y-x|\;|\,y \in A\,\}.$$

LEMMA 4.3.4 ([Kas] Lemma 3.3). — *Let u be a distribution defined on a relatively compact open subset* U *of $\mathbb{R}^n$. Then the following conditions are equivalent:*

(1) *u is tempered at any point of $\mathbb{R}^n$.*

(2) *There exist a positive constant* C *and a positive integer m such that*

$$\left|\int u(x)\,\varphi(x)\,dx\right| \leqq C \sum_{|\alpha| \leqq m} \sup \left|\frac{\partial^{|\alpha|}}{\partial x^\alpha}\varphi\right|$$

for any $\varphi \in C_0^\infty(U)$.

(3) *There exist a positive constant* C *and a positive integer m and r such that*

$$\left|\int u(x)\,\varphi(x)\,dx\right| \leqq C \sum_{|\alpha| \leqq m} \sup \left(d(x, \partial U)^{-r} \left|\frac{\partial^{|\alpha|}}{\partial x^\alpha}\varphi\right|\right)$$

for any $\varphi \in C_0^\infty(U)$.

From Lojasiewicz's inequality ([L], also see [Kas], Lemma 3.7), we immediately have:

LEMMA 4.3.5. — *Let* Z *be a closed subset of* $\mathbb{R}^n$ *and* $f \in \mathbb{C}(x_1, \ldots, x_n)$ *be a rational function which is defined (and has a finite value) at any point in* $\mathbb{R}^n - Z$. *Then, for any relatively compact open subset* U *of* $\mathbb{R}^n$, *there exist a positive constant* C *and* $r \in \mathbb{N}$ *such that*

$$|f(x)| \leqq C(d(x, Z))^{-r},$$

for all $x \in U$.

We call a C^∞-function f on $\mathbb{R}^n$ is tame, if for all constant coefficient differential operator P there exist some positive $C > 0$ and $r \in \mathbb{N}$ such that $|Pf(x)| \leqq C(1 + |x|^2)^r$ for all $x \in \mathbb{R}^n$. For example, a rational function $f \in \mathbb{C}(x_1, \ldots, x_n)$ which is defined on the entire $\mathbb{R}^n$ is tame.

Here, we assume on X and $d\xi$ the following conditions.

$(x1)$ X is compact.

$(x2)$ X is covered by a finite number of real analytic local coordinate system [say $(x_1^{(i)}, \ldots, x_n^{(i)}; U_i)$ $1 \leqq i \leqq k$] such that $(x_1^{(i)}, \ldots, x_n^{(i)})$ gives the real analytic surjective diffeomorphism $\varphi_i : U_i \cong \mathbb{R}^n$.

$(x3)$ On each U_i, if we write $dv|_{U_i} = f_i \, dx^{(i)}$, then the both f_i and f_i^{-1} are tame real analytic functions. Here, we put $dx^{(i)} = dx_1^{(i)} \wedge \ldots \wedge dx_n^{(i)}$.

$(x4)$ For all distinct $1 \leqq i, j \leqq k$,

$$(\varphi_i)|_{U_i \cap U_j} \circ (\varphi_j)^{-1}|_{\varphi_j(U_i \cap U_j)} : \varphi_j(U_i \cap U_j) \tilde{\to} \varphi_i(U_i \cap U_j)$$

extend to a tuple of rational functions $(z_1^{(i,j)}, \ldots, z_n^{(i,j)})$, $(z^{(i,j)} \in \mathbb{C}(x_1^{(j)}, \ldots, x_n^{(j)}))$.

Remark. — In $(x4)$, we automatically have $z^{(i,j)} \in \mathbb{R}(x_1^{(j)}, \ldots, x_n^{(j)})$. Hence, under the above conditions, X is a rational non-singular algebraic variety over $\mathbb{R}$.

We have:

LEMMA 4.3.6. — *Let* X *be a real analytic manifold which satisfies the above conditions* $(x1)$-$(x4)$. *Then, under the identification* $\varphi_i : U_i \tilde{\to} \mathbb{R}^n$, $\mathscr{T}_X(U_i)$ coincides $\mathscr{S}(\mathbb{R}^n)'$ *for all i.*

Proof. — We can assume $i = 1$. We identify U_i and $\mathbb{R}^n$ by φ_1. We assume $u \in \mathscr{T}_X(U_1)$. Since X is compact, there exists a finite open covering $\{V_1, \ldots, V_m\}$ such that for each $1 \leqq j \leqq m$ there exists some $1 \leqq s_j \leqq k$ such that V_j is a relatively compact subset of U_{s_j}. Put $W_j = U_1 \cap V_j$ for $1 \leqq j \leqq m$. Let $\{\zeta_j \mid 1 \leqq j \leqq m\}$ be a C^∞-partition of unity subordinate to $\{V_1, \ldots, V_m\}$. Put $\eta_j = \zeta_j|_{U_1}$ and $u_j = \eta_j u$. Hence we have $u_j \in \mathscr{T}_X(U_1)$. Thus, we have $u_j \in \mathscr{S}(\mathbb{R}^n)'$ from Lemma4.3.3. Hence $u = u_1 + \ldots + u_m \in \mathscr{S}(\mathbb{R}^n)'$.

Conversely, we assume $u \in \mathscr{S}(\mathbb{R}^n)'$. Since we can easily see each η_j is a tame C^∞-function on U_1, we have $u_j \in \mathscr{S}(\mathbb{R}^n)'$. From Lemma 4.3.3 and Lemma 4.3.4, we see that $u_j \in \mathscr{T}_{U_{s_j}}(W_j)$. Since W_j is relatively compact in U_{s_j}, we have $u_j \in \mathscr{T}_X(U_1)$. $\square$

Now we consider the case $X = G/P$. Clearly X satisfies $(x\,1)$.

For $w \in W_m$, we define $\Phi_w^m = w^{-1}\Sigma^+ \cap -\Sigma^+$ and put

$$W_m^S = \{ w \in W_m \mid \Phi_w^m \cap \mathbb{Z}S = \varnothing \}.$$

We denote by W_S^m the Weyl group of $(\mathfrak{I}, \mathfrak{a}_m)$. For $w \in W_m$, let $l_m(w)$ be the length of w with respect to Π_m.

We quote:

LEMMA 4.3.7 (Kostant [Ko1]). — *Each* $w \in W_m$ *is uniquely written as* $w = xy$, $x \in W_m^S$, $y \in W_S^m$.

For each $w \in W_m$, we fix a representative $\tilde{w} \in K$. For $w \in W_m^S$, we put $\bar{U}_w = \bar{U}_m \cap \tilde{w}\bar{N}\tilde{w}^{-1}$.

Put $Y_w = \bar{U}\tilde{w}P/P \subseteq X$. Clearly, $Y_w = \bar{U}_w\tilde{w}P/P$.

LEMMA 4.3.8 (Bruhat, Harish-Chandra, *cf.* [War], Theorem 1.2.3.1)

$$X = \bigcup_{w \in W_m^S} Y_w \quad (\textit{disjoint union}).$$

We put $V_w = \tilde{w}\bar{N}P/P \subseteq X$ for $w \in W_m^S$. Then each V_w is an open subset of X and $Y_w \subseteq V_w$. Hence $X = \bigcup_{w \in W_m^S} V_w$. Let $\bar{\mathfrak{n}}_0$ be the real form of $\bar{\mathfrak{n}}$ corresponding to $\bar{N}$. Then $\iota_w : \bar{\mathfrak{n}}_0 \ni X \to \tilde{w}\exp(X)P/P$ defines a real analytic coordinate system $\varphi_w : V_w \xrightarrow{\sim} \bar{\mathfrak{n}}_0$.

Hence, X satisfies $(x\,2)$. If we consider the complexification $X_C = G_C/P_C$ of X, then it is a projective non-singular rational algebraic variety. We can easily see the complexification of V_w and φ_w are a Zariski open subset of X_C and an isomorphism in the category of complex algebraic varieties respectively. Hence, we have $(x\,4)$ holds for X. Let $d\xi$ be a K-invariant volume form on X, which is unique up to positive scalar factors. Then, $(x\,3)$ follows from, for example, [Kn] (5.25) and Proposition 7.17 (also see [He]).

Hence, we have:

PROPOSITION 4.3.9. — *Let* $X = G/P$ *and let* $d\xi$ *be a* K-*invariant volume form on* X, *which is unique up to positive scalar factors. Then* $\mathscr{T}_X(V_w)$ *coincides the space of tempered distributions in the sense of Schwartz for all* $w \in W_m^S$ *under the identification by* $\varphi : V_w \cong \bar{\mathfrak{n}}_0 \cong \mathbb{R}^n$. *Here, we put* $n = \dim_{\mathbb{R}} \bar{N} = \dim_{\mathbb{R}} X$.

In order to quote some results in [Kas], we introduce the notion of semianalytic sets (*cf.* [GorM], p. 43). A semianalytic subset A of a real analytic manifold X is a subset which can be covered by open sets $U \subseteq X$ such that each $U \cap A$ is a union of connected components of sets of the form $g^{-1}(0) - f^{-1}(0)$, where g and f belong to some finite collection of real valued analytic functions in U. Semianalytic subsets are always subanalytic [Hi1, 2], also *see* [GorM]). We can easily see any subset Z of $X = G/P$ of the following form is semianalytic.

$$Z = \bigcup_{x \in A} Y_x \cup \bigcup_{y \in B} V_y - \bigcup_{z \in C} Y_z \qquad (A, B, C \subseteq W_m^S). \tag{3}$$

In particular the closure $\bar{Y}_w$ of Y_w is semianalytic.

Let A be a locally closed subset of X and let $\iota : A \hookrightarrow X$ be the embedding map. For any sheaf F on X, we put $F|_A = \iota^{-1} F$ and $F_A = \iota_! \iota^{-1} F$. Hence, we have $(F|_A)_x = (F_A)_x = F_x$ for all $x \in A$ and $(F_A)_y = 0$ for all $y \in X - A$.

Next, we introduce the notion of $\mathbb{R}$-constructible sheaves ([Kas], Definition 2.6, 2.7). Let F be a sheaf of $\mathbb{C}$-vector spaces on a real analytic manifold X. We say that F is $\mathbb{R}$-constructible if there exists a locally finite family $\{X_j \mid j \in J\}$ of subanalytic subsets of X such that, for all $j \in J$, $F|_{X_j}$ is a locally constant sheaf on X_j whose stalk is finite dimensional and that $X = \bigcup X_j$. Hence, for all Z of the form (3) above, $\mathbb{C}_Z \, (= (\mathbb{C}_X)_Z)$ is a $\mathbb{R}$-constructible sheaf on $X = G/P$.

Let $\mathscr{D}_X$ be the sheaf of real analytic differential operators on X. Following [Kas], Definition 3.1.3, we introduce a contravariant functor TH from the category of $\mathbb{R}$-constructible sheaves to the category of $\mathscr{D}_X$-modules. For $\mathbb{R}$-constructible sheaf F on X, TH (F) is the subsheaf of $\mathscr{H}om_{\mathbb{C}}(F, \mathscr{D}b_X)$ defined as follows: for any open subset U of X

$$\Gamma(U, TH(F)) = \{ \varphi \in \Gamma(U, \mathscr{H}om_{\mathbb{C}}(F, \mathscr{D}b_X)) \mid \varphi \text{ satisfies the following condition (T)} \}$$

(T) For any relatively compact open subanalytic subset V of U and $s \in F(V)$, $\varphi(s)$ is a tempered distribution.

We quote some results about TH.

LEMMA 4.3.10 (Kashiwara [Kas], Corollary 3.16). — *For any open subanalytic subset* U *and an open subset* Ω *of* X, *we have*

$$\Gamma(\Omega, TH(\mathbb{C}_U)) = \{ u \in \Gamma(U \cap \Omega, \mathscr{D}b_X) \mid u \textit{ is tempered at any point of } \Omega \}.$$

LEMMA 4.3.11 (Kashiwara [Kas] Proposition 3.14). — *For any* $\mathbb{R}$*-constructible sheaf* F, TH (F) *is a soft sheaf.*

LEMMA 4.3.12 (Kashiwara [Kas] Proposition 3.22). — *If* Z *is a closed subanalytic subset of* X *and if* F *is* $\mathbb{R}$*-constructible sheaf on* X, *then we have*

$$\Gamma_Z(TH(F)) = TH(F_Z).$$

Especially, the following is crucial.

THEOREM 4.3.13 (Kashiwara [Kas] Theorem 3.18). — *TH is an exact functor.*

Now we consider the case $X = G/P$.

For $w \in W_m^S$, we denote $l_m(w) = \dim_{\mathbb{R}} Y_w = \dim_{\mathbb{R}} \bar{U}_w$. For $i \in \mathbb{N}$, we put

$$Z_i = \bigcup_{l_m(w) \leq i, \; w \in W_m^S} Y_w.$$

Then Z_i is a closed subanalytic subset of X for each $0 \leqq i \leqq n = \dim_{\mathbb{R}} X$ and $X = Z_n$. From Lemma 4.3.11 and Lemma 4.3.9, we have

$$\begin{aligned}\Gamma(X, TH(\mathbb{C}_{Z_i})) &= \Gamma(X, \Gamma_{Z_i} TH(\mathbb{C}_X)) \\ &= \Gamma(X, \Gamma_{Z_i} \mathscr{D}b_X) \\ &= \{ u \in \mathscr{D}b_X(X) \mid \operatorname{supp}(u) \subseteqq Z_i \}.\end{aligned} \tag{4}$$

For all $w \in W_m^S$, we remark that $\bar{Y}_w \cap V_w = Y_w$. We also have from Lemma 4.3.11 and Lemma 4.3.9,

$$\begin{aligned}\Gamma(X, TH(\mathbb{C}_{Y_w})) &= \Gamma(X, \Gamma_{\bar{Y}_w} TH(\mathbb{C}_{V_w})) \\ &= \{ u \in \mathscr{T}_X(V_w) \mid \operatorname{supp}(u) \subseteqq Y_w \}.\end{aligned} \tag{5}$$

If $0 \leqq i < n$, we have

$$0 \to \mathbb{C}_{Z_i} \to \mathbb{C}_{Z_{i+1}} \to \bigoplus_{i+1 = l_m(w)} \mathbb{C}_{Y_w} \to 0. \tag{6}$$

From the above (3)-(4), Theorem 4.3.12 and Lemma 4.3.10, we have

LEMMA 4.3.14. — *For all $0 \leqq i < n$, the cokernel of the natural inclusion*

$$\Gamma_{Z_i} \mathscr{D}b_X(X) \hookrightarrow \Gamma_{Z_{i+1}} \mathscr{D}b_X(X)$$

is just

$$\bigoplus_{i+1 = l_m(w)} \{ u \in \mathscr{T}_X(V_w) \mid \operatorname{supp}(u) \subseteqq Y_w \}.$$

We call the filtration $\{ \Gamma_{Z_i} \mathscr{D}b_X(X) \}$ of $\mathscr{D}b_X(X)$ the Bruhat filtration (*cf.* [Ca1]).

Remark. — In [Ca1], Casselman considered not G/P_m but G/U_m and proved Theorem 4.1.1 for a minimal parabolic subgroups without the first reduction in 4.2.

Now we consider $C^\infty(G/P; \tau)'$, where τ satisfies the condition in Lemma 4.2.1. In this case, $C^\infty(G/P; \tau)$ can be identified with $C^\infty(G/P)$, since the line bundle associated to τ is trivial as a real analytic line bundle. Here, $\mathfrak{g}$ acts on $C^\infty(G/P)$ not as vector field but as first order differential operators. Under the above identification, we put

$$C_w^\infty(G/P; \tau) = \{ u \in \mathscr{T}_X(V_w) \mid \operatorname{supp}(u) \subseteqq Y_w \},$$

for $w \in W_m^S$. Hence, Lemma 4.2.1 (and Theorem 4.1.1) is reduced to the following lemma.

LEMMA 4.3.15. — *Let τ be a one-dimensional continuous* L*-representation such that $\tau|_{M_\sharp} \equiv \mathrm{id}_M$. Let ψ be a unitary admissible character on $\bar{\mathfrak{n}}$. Then we have*

$$H^p(\bar{\mathfrak{n}}, C_e^\infty(G/P; \tau)' \otimes \mathbb{C}_{-\psi}) = 0$$

for all $p>0$ *and*

$$H^p(\bar{\mathfrak{n}}, C_w^\infty(G/P;\tau)'\otimes\mathbb{C}_{-\psi})=0$$

for all $p\in\mathbb{N}$ *and* $w\in W_S^m-\{e\}$. *Here, e is the identity element of* W_m.

Next, using again Casselman's idea, we further reduce the above lemma.

First, we consider the case $w=e$. This case, clearly $C_e^\infty(G/P;\tau)'\cong\mathscr{S}(\bar{N})'$. Here, we identify $\bar{N}$ and $\bar{\mathfrak{n}}_0\cong\mathbb{R}^n$ by the exponential map. The action of $\bar{\mathfrak{n}}$ is induced from the left regular action of $\bar{N}$ and $\bar{\mathfrak{n}}_0\cong\mathbb{R}^n$ by the exponential map. The action of $\bar{\mathfrak{n}}$ is induced from the left regular action of $\bar{N}$. So, this case is reduced to:

LEMMA 4.3.16. — *Let* ψ *be a unitary character on* $\bar{\mathfrak{n}}$. *Then, we have*

$$H^p(\bar{\mathfrak{n}}, \mathscr{S}(\bar{N})'\otimes\mathbb{C}_{-\psi})=0$$

for all $p>0$.

Next we consider the case $w\in W_m^S-\{e\}$. Put $k=l_m(w)=\dim_{\mathbb{R}}Y_w$. We denote by $\bar{\mathfrak{u}}_w$ (resp. $\bar{\mathfrak{n}}_w$) the real Lie algebra of $\bar{U}_w$ (resp. $\tilde{w}\bar{N}\tilde{w}^{-1}$). Since $\bar{\mathfrak{u}}_w$ is a subspace of $\bar{\mathfrak{n}}_w$, we can choose a basis $\{X_1, \ldots, X_n\}$ of $\bar{\mathfrak{n}}_w$ such that $X_1, \ldots, X_k\in\bar{\mathfrak{u}}_w$. We introduce on V_w an Euclidean coordinate $(x_1, \ldots, x_n)$ by $\bar{\mathfrak{n}}_w\ni X\rightsquigarrow\exp(X)\,\tilde{w}P/P$, where $X=\sum x_iX_i$. Hence, we see that $Y_w=\{x_{k+1}=\ldots=x_n=0\}$.

Let $\bar{\mathfrak{p}}_m$ be the complexified Lie algebra of $\bar{P}_m=M_mA_m\bar{U}_m$. The action of $\mathfrak{g}$ on $\mathscr{S}(V_w)'$ is the dual action to the left regular action on the following space.

$$C_0^\infty(V_w)=\{f\in C^\infty(G/P;\tau)\mid \operatorname{supp} f\subseteq V_w\}.$$

For $X\in\mathfrak{g}$, we denote the first order differential operator by which X acts on $C_0^\infty(V_w)$ by

$$P_X=\sum a_i^X(x_1, \ldots, x_n)\frac{\partial}{\partial x_i}+F_X(x_1, \ldots, x_n).$$

Since Y_w is a $\bar{P}_m$-orbit, for all $X\in\bar{\mathfrak{p}}_m$ we have

$$a_i^X(x_1, \ldots, x_k, 0, \ldots, 0)\equiv 0 \quad \text{for all } k+1\leqq i\leqq n.$$

Hence $U(\bar{\mathfrak{p}}_m)$ preserves $S_w=\mathscr{S}(Y_w)'\otimes\delta(x_{k+1}, \ldots, x_n)\subseteq\mathscr{S}(V_w)'$. Moreover, we assume $X\in\bar{\mathfrak{n}}$. Since we see $\bar{N}\bar{U}_w\subseteq\bar{U}_w\tilde{w}M_\sharp N\tilde{w}^{-1}$, F_X vanishes on Y_w. Hence, as a $U(\bar{\mathfrak{n}})$-module, S_w is isomorphic to $\mathscr{S}(\bar{U}_m/\bar{U}_m^w)'$. Here, $\bar{U}_m^w=\bar{U}_m\cap\tilde{w}P\tilde{w}^{-1}$ and $U(\bar{\mathfrak{n}})$-action on $\mathscr{S}(\bar{U}_m/\bar{U}_m^w)'$ is induced from the left regular action of $\bar{N}$.

From Lemma 4.3.2, there is a surjective $U(\mathfrak{g})$-homomorphism

$$T:\quad U(\mathfrak{g})\otimes_{U(\bar{\mathfrak{p}}_m)}S_w\to C_w^\infty(G/P;\tau)=\{u\in\mathscr{S}(V_w)\mid \operatorname{supp} u\subseteq Y_w\}.$$

From the Poincaré-Birkhoff-Witt Theorem, we have

$$U(\mathfrak{g})\otimes_{U(\bar{\mathfrak{p}}_m)}S_w=U(\mathfrak{u}_m)\otimes_{\mathbf{C}}S_w.$$

Let $U_p(u_m)$ the space of the elements of $U(\bar{u}_m)$ which is spanned by at most p products of u_m. If we put $E_p = U_p(u_m) \otimes S_w$, then clearly E_p is a sub $U(\bar{n})$-module of $U(\mathfrak{g}) \otimes_{U(\bar{\mathfrak{p}}_m)} S_w$. Since cohomology commutes with a direct limit, we have only to prove

$$H^i(\bar{n}, T(E_p) \otimes \mathbf{C}_{-\psi}) = 0$$

for all i and p. Since it is not difficult to see there is a finite filtration $1 \otimes S_w = L_q \subseteq L_{q-1} \subseteq \ldots \subseteq L_0 = E_p$ such that $\bar{n} L_j \subseteq L_{j+1}$. From, for example, Corollary 4.4.4 in the next 4.4, we have $H^i(\bar{n}, T(L_j)/T(L_{j+1}) \otimes \mathbf{C}_{-\psi}) = 0$ for $j < q$. Hence, we have only to prove

$$H^i(\bar{n}, S_w \otimes \mathbf{C}_{-\psi}) = 0$$

for all i. Hence, we see that Theorem 4.1.1 is reduced to Lemma 4.3.16 and the following lemma.

LEMMA 4.3.17. — *Let ψ be a permissible character on $\bar{n}$. Then, we have*

$$H^p(\bar{n}, \mathscr{S}(\bar{U}_m/\bar{U}_m^w)' \otimes \mathbb{C}_{-\psi}) = 0$$

for all $p \in \mathbb{N}$ and $w \in W_m^S - \{e\}$.

4.4. THE FINAL STEP. — In this section, we prove Lemma 4.3.16 and Lemma 4.3.17. In the last part, we use, following Yamashita [Y1], Hilbert's Nullstellensatz. First, we collect some results of elementary homological algebra which we need.

For a ring Λ, we denote the category of left Λ-modules by Λ-Mod. Let A be a unital (non-commutative) $\mathbb{C}$-algebra and let J be a 2-sided ideal of A. We consider the following functor $\Gamma_{A/J}$ from A-Mod to A/J-Mod.

$$\Gamma_{A/J}(V) = \{ v \in V \mid \forall a \in J \ av = 0 \} \qquad (V \in A\text{-Mod}).$$

Clearly, $\Gamma_{A/J}$ is a left exact additive covariant functor. So, we can define the right derived functor $\mathbf{R}^p \Gamma_{A/J}$. For example, if A is the universal enveloping algebra of a complex Lie algebra and J is the augmentation ideal, then $\mathbf{R}^p \Gamma_{A/J}$ is the Lie algebra cohomology.

On the other hand, we have a functorial isomorphism $\Gamma_{A/J}(V) \cong \mathrm{Hom}_A(A/J, V)$ of abelian groups. So, as an abelian group, we have $\mathbf{R}^p \Gamma_{A/J}(V) \cong \mathrm{Ext}_A^p(A/J, V)$ for all $V \in A$-Mod and $p \geqq 0$. We consider how A/J acts on $\mathrm{Ext}_A^p(A/J, V)$. First we investigate the simplext case. Namely, we assume $x \in A$ is contained in the center of A and is not a zero divisor. Put $I = xA = Ax$. Then we have the following free resolution of A/I.

$$0 \leftarrow A/I \leftarrow A \overset{x}{\leftarrow} A \leftarrow 0.$$

Here, « x » means the map $a \rightsquigarrow ax$. Since x is contained in the center, the arrows in the above exact sequence are all not only left A-module homomorphisms but also right A-module homomorphism. Since $\mathrm{Ext}_A^p(A/I, V)$ is the p-th cohomology of the following complex, we can define a left A/I-module structure on $\mathrm{Ext}_A^p(A/I, V)$.

$$0 \to \mathrm{Hom}_A(A, V) \xrightarrow{x^*} \mathrm{Hom}_A(A, V) \to 0.$$

If we construct a suitable double complex, we immediately see this A-module structure on $\mathrm{Ext}_A^p(A/I, V)$ coincides with that on $\mathbf{R}^p\Gamma_{A/I}(V)$. Hence we have:

LEMMA 4.4.1. — *We assume $x \in A$ is contained in the center of* A *and is not a zero divisor. Put* $I = xA = Ax$. *Then, we have*

$$\mathbf{R}^1\Gamma_{A/I}(V) = V/xV,$$
$$\mathbf{R}^p\Gamma_{A/I}(V) = 0 \ (p \geqq 2).$$

In particular,

$$\mathrm{Ann}_A(V) + I \subseteqq \mathrm{Ann}_{A/I}(\mathbf{R}^p\Gamma_{A/I}(V)),$$

for all $p \in \mathbb{N}$.

It is not difficult to check:

LEMMA 4.4.2. — (1) *Let* J *be a 2-sided ideal of* A. *Then* $\Gamma_{A/J}$ *preserves injective objects.*

(2) *Let* R *be a unital* $\mathbb{C}$*-subalgebra of* An *such that* A *is flat over* R. *Then, the forgetful functor* Fgt_A^R *preserves the injective objects.*

Then we have:

LEMMA 4.4.3. — *Let* J *be a 2-sided ideal which satisfies the following conditions.*

(F 1) *There exists a positive integer m and a sequence of* $\mathbb{C}$*-albegra* $A_0, \ldots, A_m$ *such that* $A_0 = A$, $A_m = A/J$, *and* A_{i+1} *is a quotient of* A_i.

(F 2) *We denote by* I_i *the kernel of the projection* $A_i \to A_{i+1}$. *Then,* I_i *is generated by an element* x_i *of* A_i *such that* x_i *is contained in the center of* A_i *and* x_i *is not a zero-divisor.*

Then, there exists some positive integer l such that

$$\mathrm{Ann}_A(V)^l + J \subseteqq \mathrm{Ann}_A(\mathbf{R}^q\Gamma_{A/J}(V))$$

for all $V \in A\text{-Mod}$ *and* $q \geqq 0$.

Proof. — We use the induction on m. Let I be the kernel of the projection $A \to A_{m-1}$. From the assumption of the induction, we have $\mathrm{Ann}_A(V)^{l'} + I \subseteqq \mathrm{Ann}_A(\Gamma_{A/I}(V))$ for some l'. From Lemma 4.4.2, we have a Grothendieck special sequence (*cf.* [HS], VIII, Theorem 9.3)

$$E_1^{p,q} = \mathbf{R}^p\Gamma_{A_{m-1}/I_{m-1}}\mathbf{R}^{q-p}\Gamma_{A/I}(V) \quad \Rightarrow \quad \mathbf{R}^q\Gamma_{A/J}(V).$$

From Lemma 4.4.1, we have $E_1^{p,q} = 0$ for $p \neq 0,1$. Thus, $\mathbf{R}^q\Gamma_{A/J}(V)$ has a A-submodule V_q such that V_q [resp. $\mathbf{R}^q\Gamma_{A/J}(V)/V_q$] is a subquotient of $E_1^{0,q}$ (resp. $E_1^{1,q-1}$). Hence Lemma 4.4.1 implies the lemma. □

If we consider an upper central series, we easily deduce:

COROLLARY 4.4.4. — *Let* $\mathfrak{q}$ *be a nilpotent Lie algebra and let* $\psi : \mathfrak{q} \to \mathbb{C}$ *be a non-trivial character. Let* V *be a* $U(\mathfrak{q})$*-module such that* $Xv=0$ *for all* $v \in V$ *and* $X \in \mathfrak{q}$. *Then* $H^p(\mathfrak{q}, V \otimes \mathbb{C}_{-\psi}) = 0$ *for all* $p \in \mathbb{N}$.

From Lemma 4.2.2, we also have:

LEMMA 4.4.5. — *Let* R *be a unital* $\mathbb{C}$*-subalgebra of* A *such that* A *is flat over* R. *Let* I (*resp.* J) *be a 2-sided ideal in* R (*resp.* A) *such that* $J \cap R = I$ *and* $J = AI = IA$. *Then we have the following isomorphism of functors for all* $p \in \mathbb{N}$.

$$\mathbf{R}^p \Gamma_{R/I} \circ \mathrm{Fgt}_A^R \cong \mathrm{Fgt}_{A/J}^{R/I} \circ \mathbf{R}^p \Gamma_{A/J}$$

The proof of the following lemma is similar to that of Lemma 4.4.3.

LEMMA 4.4.6. — *Let* A *be a unital* $\mathbb{C}$*-algebra and let* $A = A_1, A_2, \ldots, A_k$ *be a sequence of unital* $\mathbb{C}$*-algebras which satisfy the following conditions.*

(E 1) *For each* $2 \leqq i \leqq k$, *there exists a flat algebra extention* $\tilde{A}_i \supseteqq A_i$ *such that* $\tilde{A}_i$ *is a quotient algebra of* A_{i-1}.

(E 2) *We denote by* I_i *the kernel of the projection* $p_i : A_i \to \tilde{A}_{i+1}$ *for* $1 \leqq i < k$. *Then,* I_i *is generated by an element* x_i *of* A_i *such that* x_i *is contained in the center of* A_i *and* x_i *is not a zero-divisor.*

Fix $V \in A$-Mod. *We define inductively a 2-sided ideal* J_i *of* A_i *for each* $0 \leqq i \leqq k$ *as follows.*

$$J_0 = \mathrm{Ann}_A(V).$$
$$J_{i+1} = A_{i+1} \cap (p_i(J_i^2 + I_i)) \qquad (0 \leqq i < k).$$

Put

$$\Phi = \mathrm{Fgt}_{A_k}^{\tilde{A}_k} \circ \Gamma_{\tilde{A}_{k-1}/I_{k-1}} \circ \ldots \circ \mathrm{Fgt}_{A_2}^{\tilde{A}_2} \circ \Gamma_{\tilde{A}_1/I_1} : A-\mathrm{Mod} \to A_k-\mathrm{Mod}.$$

Then we have

$$J_k \subseteqq \mathrm{Ann}_{A_k}(\mathbf{R}^p \Phi(V)).$$

Now, we prove Lemma 4.3.16. We fix a basis $\{X_1, \ldots, X_n\}$ of $\bar{\mathfrak{n}}_0$ and define a local coordinate system on $\bar{N}$ by $\exp(x_1 X_1 + \ldots + x_n X_n)$. Put $Q_i = \sum_{j=1}^{i} \mathbb{C}_{x_j} \subseteqq \bar{\mathfrak{n}}$. We assume that

$$0 = Q_0 \subseteqq Q_1 \subseteqq \ldots \subseteqq Q_n = \bar{\mathfrak{n}}$$

is a refinement of the upper central series of $\bar{\mathfrak{n}}$. Hence, Q_{i+1}/Q_i is contained in the center of $\bar{\mathfrak{n}}/Q_i$ for each i.

Since an element of the enveloping algebra $U(\bar{\mathfrak{n}})$ acts on $\mathscr{S}(\bar{N})'$ as a differential operator with polynomial coefficients, we can see $U(\bar{\mathfrak{n}})$ is injectively embedded into the Weyl algebra $\mathscr{A}_n = \mathbb{C}[x_1, \ldots, x_n; \partial_1, \ldots, \partial_n]$. Here, we put $\partial_i = \partial/\partial x_i$. From the Campbell-Hausdorff hormula, it is not difficult to see X_i are written as follows.

$$X_i = \partial_i + \sum_{i<j} f_j(x_i, \ldots, x_{j-1})\, \partial_j. \tag{1}$$

Hence, the above injection induces the isomorphism

$$\mathbb{C}[x_1, \ldots, x_n] \,\#\, U(\bar{\mathfrak{n}}) \xrightarrow{\sim} \mathscr{A}_n.$$

Here, # is the smash product. Namely, as a $\mathbb{C}[x_1, \ldots, x_n]$-$U(\bar{\mathfrak{n}})$-bimodule, we have

$$\mathbb{C}[x_1, \ldots, x_n] \,\#\, U(\bar{\mathfrak{n}}) \cong \mathbb{C}[x_1, \ldots, x_n] \otimes U(\bar{\mathfrak{n}}).$$

The product on the smash product is generated by

$$(f \otimes X)(g \otimes u) = fg \otimes Xu + fX(g) \otimes u \quad (f, g \in \mathbb{C}[x_1, \ldots, x_n],\ X \in \bar{\mathfrak{n}},\ u \in U(\bar{\mathfrak{n}})).$$

Hence, $U(\bar{\mathfrak{n}})\mathscr{A}_n$ is a flat extension. Let I be the kernel of $\psi : U(\bar{\mathfrak{n}}) \to \mathbb{C}$. Clearly, we have $I = \sum_{i=1}^{n} U(\bar{\mathfrak{n}})(X_i - \psi(X_i))$. We put $J = \mathscr{A}_n I = \sum_{i=1}^{n} \mathscr{A}_n (X_i - \psi(X_i))$. Then, we have

$$\begin{aligned} H^0(\bar{\mathfrak{n}}, \mathscr{S}(\bar{N})' \otimes \mathbb{C}_{-\psi}) &\cong \Gamma_{U(\bar{\mathfrak{n}})/I}(\mathscr{S}(\mathbb{R}^n)') \\ &\cong \mathrm{Hom}_{U(\bar{\mathfrak{n}})}(U(\bar{\mathfrak{n}})/I, \mathscr{S}(\mathbb{R}^n)') \\ &\cong \mathrm{Hom}_{\mathscr{A}_n}(\mathscr{A}_n/J, \mathscr{S}(\mathbb{R}^n)'). \end{aligned}$$

Since $U(\bar{\mathfrak{n}}) \subseteq \mathscr{A}_n$ is a flat extension, we have the following isomorphism of abelian groups.

$$H^p(\bar{\mathfrak{n}}, \mathscr{S}(\bar{\mathfrak{n}})' \otimes \mathscr{C}_{-\psi}) \cong \mathrm{Ext}^p_{\mathscr{A}_n}(\mathscr{A}_n/J, \mathscr{S}(\mathbb{R}^n)').$$

(Note : $V \rightsquigarrow V \otimes \mathbb{C}_{-\psi}$ preserves injectivity.)

Hence we have only to prove $\mathrm{Ext}^p_{\mathscr{A}_n}(\mathscr{A}_n/J, \mathscr{S}(\mathbb{R}^n)') = 0$ for $p > 0$. Since ψ vanishes on $[\bar{\mathfrak{n}}, \bar{\mathfrak{n}}]$, from (1) we see

$$J = \sum_{i=1}^{n} \mathscr{A}_n (\partial_i - \psi(X_i)).$$

Put $F = F(x_1, \ldots, x_n) = \exp\left(\sum \psi(X_i)\, x_i\right)$. Since ψ is unitary, the both F and F^{-1} are tame. Then $u \rightsquigarrow Fu$ gives an isomorphism of $\mathscr{S}(\mathbb{R}^n)'$. If we twist the action of $\mathscr{A}_n$ under this map we have

$$J = \sum_{i=1}^{n} \mathscr{A}_n \partial_i.$$

Put $\mathscr{P}_n = \mathbb{C}[\partial_1, \ldots, \partial_n]$ and

$$J_0 = \sum_{i=1}^{n} \mathscr{P}_n \partial_i.$$

Since $\mathscr{A}_n$ is flat over $\mathscr{P}_n$, we have only to show

$$\operatorname{Ext}^p_{\mathscr{P}_n}(\mathscr{P}_n/J_0, \mathscr{S}(\mathbb{R}^n)') = 0$$

for $p > 0$.

From Lemma 4.4.1 and Grothendieck's spectral sequence, we can easily prove this statement using the induction on n and the following claim.

CLAIM 1 : *For a positive integer m, ∂_m defines a surjective map of $\mathscr{S}(\mathbb{R}^m)'$ to $\mathscr{S}(\mathbb{R}^m)'$.*

The above claim is well-known but I do not know the reference. So, for the convenience of readers, we give a proof here. Twisting by the Fourier transform, the claim is reduced to the surjectivity of a multiplication operator x_m. Let $\mathscr{S}(\mathbb{R}^m)$ be the space of rapidly decreasing functions. $\mathscr{S}(\mathbb{R}^m)'$ is the topological dual space of $\mathscr{S}(\mathbb{R}^m)$. We can easily see the image of $\mathscr{S}(\mathbb{R}^m)$ under x_m coincides with:

$$V = \{ f \in \mathscr{S}(\mathbb{R}^m) \mid f(x_1, \ldots, x_{m-1}, 0) \equiv 0 \}.$$

Clearly V is a closed subspace of a Frechet space $\mathscr{S}(\mathbb{R}^m)$. From the open mapping theorem, there is a continuous inverse $F : V \to \mathscr{S}(\mathbb{R}^m)$ of x_m. For $T \in \mathscr{S}(\mathbb{R}^m)'$, we define a continuous functional T' on V by $T' \circ F$. From Hahn-Banach's Theorem there is an extension $\bar{T}$ of T' to $\mathscr{S}(\mathbb{R}^m)$. Them, clearly we have $x_m \bar{T} = T$. □

Lastly, we prove Lemma 4.3.17. We put

$$\bar{N}_w = \bar{N} \cap \tilde{w} \bar{N} \tilde{w}^{-1},$$

$$\bar{N}^w = \bar{N} \cap \tilde{w} P \tilde{w}^{-1},$$

$$\bar{U}^w_m = \bar{U}_m \cap \tilde{w} P \tilde{w}^{-1}.$$

$$\bar{U}_S = \bar{U}_m \cap L,$$

$$\bar{U}(w) = U \bar{U}_S \cap \tilde{w} \bar{N} \tilde{w}^{-1},$$

$$\bar{U}^w = \bar{U}_S \cap \tilde{w} P \tilde{w}^{-1}.$$

We denote by $\bar{\mathfrak{n}}_w$, $\bar{\mathfrak{n}}^w$, $\bar{\mathfrak{u}}_S$, $\bar{\mathfrak{u}}(w)$, and $\bar{\mathfrak{u}}^w$ the complexified Lie algebras of $\bar{N}_w$, $\bar{N}^w$, $\bar{U}_S$, $\bar{U}(w)$ and $\bar{U}^w$, respectively. Then we have $\bar{\mathfrak{u}} = \bar{\mathfrak{n}} \oplus \bar{\mathfrak{u}}_S$ and $\bar{\mathfrak{n}} = \bar{\mathfrak{n}}_w \oplus \bar{\mathfrak{n}}^w$. Let $\bar{\mathfrak{u}}_0$ be the real form of $\bar{\mathfrak{u}}_S$ corresponding to $\bar{U}_S$. Put $k = \dot{\mathfrak{u}}_w$ and $l = \dim \bar{\mathfrak{n}}_S$. Then, $n - k = \dim \bar{\mathfrak{n}}^w = \dim \bar{\mathfrak{u}}(w)$ and $l - n + k = \dim \bar{\mathfrak{u}}^w$. We choose a basis $\{X_1, \ldots, X_k\}$

(resp. $\{Y_1, \ldots, Y_{n-k}\}$) of $\bar{\mathfrak{n}}_0$ (resp. $\bar{\mathfrak{u}}_0$) which is compatible with the root space decomposition with respect to the restricted root system Σ. Put

$$\bar{\mathfrak{n}}_i = \sum_{j=1}^{i} \mathbb{C} X_j \qquad (1 \leqq i \leqq k),$$

$$\bar{\mathfrak{u}}_i = \bar{\mathfrak{n}} + \sum_{j=1}^{i} \mathbb{C} Y_j \qquad (1 \leqq i \leqq n-k).$$

We can assume (and do) that $0 \subseteqq \bar{\mathfrak{n}}_1 \subseteqq \ldots \bar{\mathfrak{n}}_k \subseteqq \bar{\mathfrak{u}}_1 \subseteqq \bar{\mathfrak{u}}_{n-k} = \bar{\mathfrak{u}}$ is a refinement of the upper central series do $\bar{\mathfrak{u}}$.

Let $1 \leqq \xi_1 < \ldots < \xi_k \leqq n$ (resp. $1 \leqq \eta_1 < \ldots < \eta_{n-k} \leqq n$) be such that $\bar{\mathfrak{n}}_w = \sum_{i=1}^{k} \mathbb{C} X_{\xi_i}$ (resp. $\bar{\mathfrak{n}}^w = \sum_{i=1}^{n-k} \mathbb{C} X_{\eta_i}$). So, $\{\xi_1, \ldots, \xi k\} \cup \{\eta_1, \ldots, \eta_{n-k}\} \triangleq 1, \ldots, n\}$. Similarly, we define $1 \leqq \zeta_1 < \ldots < \zeta_{n-k} \leqq l$ (resp. $1 \leqq \rho_1 < \ldots < \rho_{l-n+k} \leqq l$) be suth that $\bar{\mathfrak{u}}(w) = \sum_{i=1}^{n-k} \mathbb{C} Y_{\zeta_i}$ (resp. $\bar{\mathfrak{u}}^w = \sum_{i=1}^{l-n+k} \mathbb{C} Y_{\rho_i}$). Put $y = (y_1, \ldots, y_{n-k})$, $x = (x_1, \ldots, x_k)$, and

$$e(y, x) = \exp(y_1 Y_{\zeta_1} + \ldots + y_{n-k} Y_{\zeta_{n-k}} + x_1 X_{\xi_1} + \ldots + x_k X_{\xi_k}).$$

We define an Euclidean coordinate on $\bar{U}/\bar{U}_m^w$ by $e(y, x)\bar{U}_m^w/\bar{U}_m^w$. For $f \in \mathscr{S}(\bar{U}_m/\bar{U}_m^w)$ and $X \in \bar{\mathfrak{n}}$, we define:

$$(L_X f)(e(y, x)\bar{U}_m^w/\bar{U}_m^w) = \frac{d}{dt} f(\exp(-tX)\, e(y, x)\bar{U}_m^w/\bar{U}_m^w)\big|_{t=0},$$

$$(R_X f)(e(y, x)\bar{U}_m^w/\bar{U}_m^w) = \frac{d}{dt} f(e(y, x)\exp(-tX)\bar{U}_m^w/\bar{U}_m^w)\big|_{t=0}.$$

$X \in \bar{\mathfrak{n}}$ acts on $\mathscr{S}(\bar{U}_m/\bar{U}_m^w)$ by $-L_X$. We consider a smash product

$$B = \mathbb{C}[y, x] \,\#\, U(\bar{\mathfrak{n}}).$$

Then, $\mathscr{S}(\bar{U}_m/\bar{U}_m^w)$ has naturally a B-module structure.

We consider the relation between L_X and R_X. At the point (y, x), R_X is $L_{\mathrm{Ad}(e(y,x))X}$. Let $X \in \bar{\mathfrak{n}}^w$. Then, we have $R_X = 0$. This means an element $\mathrm{Ad}(e(y, x))X$ of B acts on $\mathscr{S}(\bar{U}_m/\bar{U}_m^w)'$ by 0. Since ψ is permissible, there exists some η_i such that $\psi(X_{\eta_i}) \neq 0$. We fix such η_i. From the assumptions on the basis, we see

$$\mathrm{Ad}(e(y, x))X_{\eta_i} = X_{\eta_i} + \sum_{\eta_i > j} F_j(y, x) X_j \in B.$$

Clearly $F_j(0, 0) = 0$ for all $\eta_i > j$. For $k \geqq i > j \geqq 1$, we write $i \sim j$ if $X_j \subseteqq [\bar{\mathfrak{u}}(w), \ldots, [\bar{\mathfrak{u}}(w), X_i] \ldots]$ (n-times) for some n. Clearly $F_j(y, x)$ only depend on y

for $\eta_i \sim j$. Since ψ vanishes on $[\bar{\mathfrak{n}}, \bar{\mathfrak{n}}]$, we have $\psi(X_j)=0$ for all $j<\eta_i$ such that $\eta_i \not\sim j$ and $F_j \neq 0$. Again, from the assumption on $X_1, \ldots, X_k, Y_1, \ldots, Y_{n-k}$, we can see F_j only depends on $(y, x_{j+1}, \ldots, x_k)$. Put $A_0 = \mathbb{C}[y, x_2, \ldots, x_k] \# U(\bar{\mathfrak{n}})$. We have proved:

CLAIM 2. — *There exists some* η_i *such that* $\psi(X_{\eta_i}) \neq 0$ *and*

$$X_{\eta_i} - \psi(X_{\eta_i}) + \sum_{\eta_i \sim j} F_j(y)(X_j - \psi(X_j)) + \sum_{\eta_i < j,\ \eta_i \not\sim j} F_j(y, x_{j+1}, \ldots, x_k) X_j \in \mathrm{Ann}_{A_0}(\mathscr{S}(\bar{U}_m/\bar{U}_m^w)').$$

For $2 \leqq i \leqq k$, we put

$$\tilde{A}_i = \mathbb{C}[y, x_i, \ldots, x_k] \# U(\bar{\mathfrak{n}}/\bar{\mathfrak{n}}_{i-1}),$$
$$A_i = \mathbb{C}[y, x_{i+1}, \ldots, x_k] \# U(\bar{\mathfrak{n}}/\bar{\mathfrak{n}}_{i-1}).$$

We define $A_k = \mathbb{C}[y]$. Then, X_i is contained in the center of A_i. We put $I_i = A_i X_i$. Then clearly $\tilde{A}_{i+1}$ is isomorphic to A_i/I_i. Put $C = \mathbb{C}[y] \otimes U(\bar{\mathfrak{n}})$. (We easily see that y_i commutes with $X \in \bar{\mathfrak{n}}$.)

If we define a functor Φ as in Lemma 4.4.6, we have

$$\Gamma_{C/C\bar{\mathfrak{n}}} \mathrm{Fgt}^{C}_{A_1} = \Phi.$$

Hence, Lemma 4.4.6 and claim 2 avove imply: for some positive integer l,

$$(\psi(X_{\eta_i}) + \sum_{\eta_i \sim j} F_j(y)\, \psi(X_j))^l \in \mathrm{Ann}_{\mathbb{C}[y]}(\mathbf{R}^p \Gamma_{C/C\bar{\mathfrak{n}}}(\mathscr{S}(\bar{U}_m/\bar{U}_m^w)')) \quad (p \in \mathbb{N}).$$

Since C is flat over $U(\bar{\mathfrak{n}})$, from Lemma 4.4.5, we can regard $H^p(\bar{\mathfrak{n}}, \mathscr{S}(\bar{U}_m/\bar{U}_m^w)' \otimes \mathbb{C}_{-\psi})$ as a $\mathbb{C}[y]$-module and we have:

$$H(y) = (\psi(X_{\eta_i}) + \sum_{\eta_i \sim j} F_j(y)\, \psi(X_j))^l \in \mathrm{Ann}_{\mathbb{C}[y]}(H^p(\bar{\mathfrak{n}}, \mathscr{S}(\bar{U}_m/\bar{U}_m^w)' \otimes \mathbb{C}_{-\psi})) \quad (p \in \mathbb{N}).$$

Here, $H(0) = \psi(X_{\eta_i})^l \neq 0$. Fix some value of y (say y_0). We consider $\psi' = \mathrm{Ad}(e(y_0, 0)^{-1}\psi$ in stead of ψ. Since $e(y_0, 0) \in \bar{U}(w) \subseteqq L$, ψ' is again a permissible character on $\bar{\mathfrak{n}}$. We choose X_{η_i} such that $\psi'(X_\eta) \neq 0$. Then the above argument implies that there exists some $H \in \mathbb{C}[y]$ such that $H(y_0) \neq 0$ and $H \in \mathrm{Ann}_{\mathbb{C}[y]}(H^p(\bar{\mathfrak{n}}, \mathscr{S}(\bar{U}_m/\bar{U}_m^w) \otimes \mathbb{C}_{-\psi}) = 0$ for all $p \in \mathbb{N}$. From Hilbert's Nullstellensatz, we have Lemma 4.3.17. Hence, we complete the proof of Theorem 4.1.1.

Q.E.D.

5 The real semisimple case — discussion

In this section we discuss about the possibility of extending our results to the general real semisimple (reductive) linear Lie groups.

There are several difficulties to get the such extensions.

First, in order to show $\dim \mathrm{Wh}_{\Psi}^{\infty}$ defines an additive invariant, we should extend Proposition 3.2.1. Let P be a parabolic subalgebra of a real semisimple (reductive) linear Lie group and σ be an irreducible representation of P. Let $\bar{\mathrm{N}}$ be the nilradical of the opposite parabolic subgroup of P and we assume there is a permissible unitary character on $\bar{\mathrm{N}}$. We consider the following problem:

Let σ *be a finite-dimensional irreducible representation of* P and let V *be an arbitrary irreducible subquotient of* $\mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}(\sigma)$. *Is there exists any finite dimensional irreducible* P-*representation* τ *such that* V *is a subrepresentation of* $\mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}(\tau)$?

If we can give affirmative answer to the above question for a parabolic subgroup P which also satisfies the above assumption, then, using Theorem 4.1.1, we can deduce the higher cohomology vanishing of a Harish-Chandra module whose irreducible subquotients are subquotients of induced representations from finite dimensional irreducible P-representations.

Of course, Casselman's subrepresentation theorem gives the affirmative answer for a minimal parabolic subgroup P, and Casselman's result [Ca1] assures the exactness of $\mathrm{Wh}_{\Psi}^{\infty}$ in this case. Theorem 2.4.1 also gives affirmative answer in some special cases.

However, in general case, the above question seems quite difficult. W. Casselman told me existence of a relation between the above question and his work on p-adic groups (*cf.* [BW]).

The second problem is that only a little, other than the results stated in [Vo5], is known about coherent continuation representations for Harish-Chamdra modules for general real semisimple (reductive) Lie groups. However, Barbasch-Vogan determined the cell structure for $\mathrm{U}(p, q)$ ([BV5], Theorem 4.2). For example, if we construct cell representations for $\mathrm{U}(p, q)$ in the same way as the double cell representations for complex semisimple groups, they are all irreducible.

The third problem is that the intersection of a complex nilpotent orbit and a real form of a complex Lie algebra need not be a single real nilpotent orbit any more.

For example, we assume $\mathrm{G}=\mathrm{U}(p, q)$ and $\bar{\mathrm{N}}$ is the nilradical of a minimal parabolic subgroup $\bar{\mathrm{P}}$ of G. We also assume the integrality of the infinitesimal characters of Harish-Chandra modules. If $p>q$, then the above results of Casselman and Barbasch-Vogan assure that we can deduce the corresponding results to Theorem 3.4.1. Details are left to readers. (For the dimension of the space of C^{∞}-continuous Whittaker vectors, *see* [Mat5] § 8.) However, if $p=q$, then the third problem occurs and we cannot deduce the corresponding statement in the same way as the case $p>q$.

Remark. — Although I do not know whether $\dim \mathrm{Wh}_{\Psi}^{\infty}$ defines an additive invariant for general real semisimple case, we can define Whittaker polynomaials. Namely, the following result can be easily deduced from some results in [Mat5], namely Corollary 2.9.1, Theorem 2.9.3, and Corollary 5.1.4.

PROPOSITION 5.1.7. — (*We use the notation in c* 4. *For simplicity, we also assume* G *is simply-connected.*) *Let* $\Psi=(\bar{\mathfrak{n}}, \psi)$ *be an admissible Whittaker datum and let* V *be an* *irreducible Harish-Chandra* $(\mathfrak{g}, K)$*-module with a dominant infinitesimal character* λ. *We assume* $\dim \mathrm{Wh}_{\Psi}^{\infty}(V) < \infty$.

Then, there exists some W*-harmonic polynomial* $p_{\Psi}[V]$ *on* $\mathfrak{h}^*$ *such that:*

(1) *For all dominant* $\mu \in \lambda + P$, *we have*

$$\dim \mathrm{Wh}_{\Psi}^{\infty}(T_{\lambda}^{\mu}(V)) = p_{\Psi}[V](\mu).$$

(2) *If* $\mathrm{Dim}(V) < \dim \bar{\mathfrak{n}}$, *then* $p_{\Psi}[V] = 0$.

It seems likely to $p_{\Psi}[V]$ above is proportional to a Goldie rank polynomial. However, I do not even know whether $p_{\Psi}[V]$ is a homogeneous polynomial of degree card $\Delta^{+} - \dim \bar{\mathfrak{n}}$, in general.

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