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AUTOMORPHIC REPRESENTATIONS AND LEFSCHETZ NUMBERS

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Introduction

Classically, automorphic functions are holomorphic functions on the upper half plane $X = SO(2) \backslash SL_2(\mathbb{R})$ together with a prescribed transformation rule, *i. e.*, an action of a subgroup Γ of $SL_2(\mathbb{Z})$ on holomorphic functions on X. Given such a space of automorphic functions there is the problem of determining its dimension. This problem can be solved using the Riemann-Roch theorem. The formula for the dimension thus obtained depends on topological invariants of the space X/Γ and on an integer which characterises the transformation rule. A more conceptual explanation of the connection of the dimension with the topology of X/Γ is given by Eichler-Schimura's isomorphism [Sh].

There is a well known generalisation as follows. Let G be a semisimple non compact Lie group and $\Gamma \subset G$ a discrete subgroup of finite covolume, *i. e.* G/Γ has finite volume with respect to some left-invariant measure dg. Let $L^2(G/\Gamma)$ be the space of square integrable functions with respect to dg. If now π is some irreducible unitary representation of G, then π is said to be *automorphic with respect to* Γ, if π occurs discretely with finite multiplicity $m(\pi, \Gamma)$ in $L^2(G/\Gamma)$. Of course here $L^2(G/\Gamma)$ is considered as unitary representation of G where G acts by left translation on functions. The classical situation now can be recognized as follows: If $G = SL_2(\mathbb{R})$, if $\Gamma \subset SL_2(\mathbb{Z})$, and if π is a discrete series representation having a certain lowest $SO(2)$-type which is determined by the transformation rule, then $m(\pi, \Gamma)$ coincides with the dimension of the space of automorphic function with given transformation rule. Back in the general setting, we now assume that π is some given unitary irreducible representation of G. Then the following questions arise:

Is π automorphic with respect to Γ, *i. e.* is $m(\pi, \Gamma) > 0$?

Is $m(\pi, \Gamma)$ related with topological invariants of G/Γ?

What can be said on $m(\pi, \Gamma)$, if Γ shrinks to $\{1\}$?

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There are some unitary irreducible representations of G which are intimately connected with the topology of G/Γ. These are the representations π of G such that $H^{\cdot}(\mathfrak{g}, \mathfrak{k}, \pi\otimes V) \neq \{0\}$ for some V. Here $\mathfrak{g}$ is the Lie algebra of G, $\mathfrak{k}$ is the Lie algebra of a maximal compact subgroup K of G, V is a finite-dimensional irreducible representation of G and $H^{\cdot}(\mathfrak{g}, \mathfrak{k}, \pi\otimes V)$ denotes the relative Lie-algebra cohomology of $\pi\otimes V$. A connection of multiplicities and topology is then provided by Matsushima's formula

$$H^{\cdot}(K\backslash G/\Gamma, V) = \sum_{\pi\in\widehat{G}} H^{\cdot}(\mathfrak{g}, \mathfrak{k}, \pi\otimes V)^{m(\pi,\ \Gamma)}$$

Here we assume for simplicity that G/Γ is compact and that Γ is torsionfree. On the left we have the cohomology of the space $K\backslash G/\Gamma$ in the sheaf of locally constant sections—again denoted by V—of the vector bundle over $K\backslash G/\Gamma$ associated to the representation V. On the right we sum over all classes of irreducible unitary representations of G.

To exploit Matsushima's formula one has to find at first a method which gives some insight on the topological side. In particular methods which yield $H^{\cdot}(K\backslash G/\Gamma, V) \neq \{0\}$ are desirable. One can deduce $H^{\cdot}(K\backslash G/\Gamma, V) \neq \{0\}$from Harder's Gauss-Bonnet-Formula [Ha 2] if and only if rank K = rank G, see [R − S 1, 2] for applications to multiplicities. In this paper we want to establish a method which also works if rank $K \neq$ rank G.

The method we use is inspired by the observation that $H^{\cdot}(\mathfrak{g}, \mathfrak{k}, \pi\otimes V) = \{0\}$ unless π is equivalent to ${}^{\theta}\pi$ and V is equivalent to ${}^{\theta}V$. Here θ denotes the Cartan involution of G corresponding to K and the left upper index θ at a representation indicates the new representation where $g\in G$ acts as $\theta(g)$ on the old representation space. So one can hope that generally $H^{\cdot}(K\backslash G/\Gamma, V) \neq \{0\}$, if θ also acts on the geometrical side. A similar idea occurs first in [H 1] for $SL_2(\mathbb{C})$. To make this precise, let G, K, θ, V be as above. Moreover we assume that G is connected, that Γ is θ-stable arithmetic, torsion-free, and that θ acts linearly in a compatible way on V *i. e.* $\theta(gV) = \theta(g)\theta(V)$ for all $g\in G$, $v\in V$. Then θ acts as θ^i on $H^i(K\backslash G/\Gamma, V)$ and we define a Lefschetz number

$$L(\theta, \Gamma, V) = \sum_{i=0}^{\infty} (-1)^i \operatorname{tr}\theta^i.$$

Here $\operatorname{tr}\theta^i$ is the trace of θ^i. We do not require that G/Γ be compact. Our main result now is as follows:

THEOREM. — *If* Γ *is small enough* (definition 2.8) *and if* V *has a highest weight* λ *satisfying* $\lambda\in P_\theta$ (definition 3.1.1) *then*

$$L(\theta, \Gamma, V) = \chi((K\backslash G/\Gamma)^{\theta}) . \operatorname{tr}(\theta\,|\,V) \neq 0.$$

Here $\chi(K\backslash G/\Gamma)^{\theta})$ *denotes the Euler-Poincaré characteristic of the fixpoint set* $(K\backslash G/\Gamma)^{\theta}$ *of* θ *acting on* $K\backslash G/\Gamma$ *and* $\operatorname{tr}(\theta\,|\,V)$ *is the trace of* θ *on* V.

Applications are given in Proposition 4.8. In particular we prove

PROPOSITION. — *Let* G *be a complex Lie group and* V *a representation having a regular highest* θ*-fixed weight such that* $\operatorname{tr}(\theta \mid V) \neq 0$. *If* Γ *is cocompact and small enough then there is up to equivalence exactly one irreducible unitary principal series representation* π *of* G *which contributes to* $H^{\cdot}(K \backslash G/\Gamma, V) \neq \{0\}$. *The multiplicity* $m(\pi, \Gamma_1)$ *of* π *in* $L^2(G/\Gamma_1)$ *grows at least as* $|(\Gamma/\Gamma_1)^\theta|$ *if* Γ_1 *is* θ*-stable and normal of finite index in* Γ.

Here, of course $|(\Gamma/\Gamma_1)^\theta|$ is the number of fixpoints of θ acting on Γ/Γ_1. We have similar results for $SL_n(\mathbb{R})$, $SL_n(\mathbb{H})$, $SO(n, 1)$, *see* 4.8.

Next, we explain roughly how the main result is proved. Essential is a Lefschetz fixpoint formula

$$L(\theta, \Gamma, V) = \sum_{\gamma \in H^1(\theta, \Gamma)} \chi(F(\gamma)) \operatorname{tr}(\theta_\gamma \mid V),$$

see 1.6. Here $(K \backslash G/\Gamma)^\theta = \bigcup F(\gamma)$ is a finite disjoint union of connected components $F(\gamma)$ parametrized by the classes γ of the non abelian first cohomology $H^1(\theta, \Gamma)$ of θ acting on Γ. We denote by $\chi(F(\gamma))$ the Euler-Poincaré characteristic of $F(\gamma)$ and by $\operatorname{tr}(\theta_\gamma \mid V)$ the trace of θ acting "γ-twisted" on V.

In 1.4 we prove that $F(\gamma)$ is a locally symmetric space of equal rank type, so in particular $\chi(F(\gamma)) \neq 0$ and the sign of $\chi(F(\gamma))$ is determined by the dimension of $F(\gamma)$ mod 4, *see* 1.5.

In 2.8 we introduce the notion "Γ is a small enough". In particular a congruence subgroup is small enough if in the definition of the congruence there occur enough prime divisors. If Γ is small enough we can show, that all $\chi(F(\gamma)) > 0$, *see* 2.10 and that $\operatorname{tr}(\theta \mid V) = \operatorname{tr}(\theta_\gamma \mid V)$ is independent of $\gamma \in H^1(\theta, \Gamma)$. To prove that $\chi(F(\gamma)) > 0$ we associate to $\gamma \in H^1(\theta, \Gamma)$ and all places v of $\mathbb{Q}$ a certain quadratic form $B_v(\gamma)$ over $\mathbb{Q}_v$. Here $\mathbb{Q}_v$ is the completion of $\mathbb{Q}$ with respect to v. These quadratic forms have certain invariants satisfying a product formula due to Weil. If now Γ is small enough this product formula forces a congruence mod 4 on the dimensions of the $F(\gamma)$, $\gamma \in H^1(\theta, \Gamma)$, and therefore $\chi(F(\gamma)) > 0$. At the end of paragraph 2 we give a sharp estimate of the growth of $L(\theta, \Gamma, V)$ if Γ shrinks to $\{1\}$.

In paragraph 3 we compute $\operatorname{tr}(\theta \mid V)$. If θ is inner this is done using Weyl's character formula. If θ is outer we use Kostant's character formula for disconnected groups and reduce the computation of $\operatorname{tr}(\theta \mid V)$ to an application of Weyl's character formula to certain representations of G^{θ_0}. Here $G^{\theta_0} = \{g \in G \mid \theta_0 g = g\}$ where θ_0 is "the diagram automorphism" induced by θ, see 3.1. Our main result is $\operatorname{tr}(\theta \mid V) \neq 0$ if V has an extremal weight which satisfies a mild extra condition, *see* 3.2.5.

In 4.1 we finally can state our main result on the non-vanishing of $L(\theta, \Gamma, V)$. Next we define Lefschetz numbers for θ acting on $H^{\cdot}(\mathfrak{g}, \mathfrak{k}, \pi \otimes V)$, compute these Lefschetz numbers in 4.3, and prove the connection of $L(\theta, \Gamma, V)$ with multiplicities in 4.7. In analogy to Matsushima's formula we obtain for cocompact Γ the equation

$$L(\theta, \Gamma, V) = \sum_{\pi \in \hat{G}} m(\theta, \pi, \Gamma) \dim H^{\cdot}(\mathfrak{g}, \mathfrak{k}, \pi \otimes V).$$

Here $m(\theta, \pi, \Gamma)$ is up to some sign conventions the trace of θ acting on $\mathrm{Hom}_G(\pi, L^2(G/\Gamma))$. We have to sum over all equivalence classes of irreducible $\pi \in \hat{G}$ such that π is equivalent to ${}^\theta\pi$. In 4.8 we give examples of groups G and representations V such that at most one $\pi \in \hat{G}$ contributes to the above sum. In particular complex groups G and representations V with θ-action and a regular extremal weight have this property. Combining this with the main result on Lefschetz numbers 4.1 the Proposition stated above results.

There are many other results on multiplicities of representations in $L^2(G/\Gamma)$. Without any attempt to be complete we mention some of them and some typical methods of proof. Often the connection with topology is exploited using an index theorem in the sense of Atiyah and Singer. Here one applies the theorem of Riemann-Roch (classical: $G=SL_2(\mathbb{R})$), the Gauss-Bonnet theorem [R – S 1], [R – S 2], [Sa] or Dirac operators [B – M] ($\mathrm{rank}_{\mathbb{R}}(G)=1$), [DG – W] ($G/\Gamma$ cocompact). The results obtained in this way are mostly on multiplicities of discrete series representations. These methods give no non trivial information if rank $K \neq$ rank G. If $\mathrm{rank}(K) \neq \mathrm{rank}(G)$ the methods of Lefschetz numbers can be applied. There are results in [H 1], [R 2], [Le – S], [R – Sp 3]. A different approach to multiplicities is provided by the Selberg trace formula. Some typical applications are in [L 1], [L 2], [J – L], [Cl 1], [Cl 2], [La – S]. An application of the twisted trace formula due to Clozel, Delorme and Labesse has been announced in [La].

Notation

0.1 We use the standard notation $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ for natural numbers etc; $\mathbb{H}$ denotes the quaternions over $\mathbb{R}$. If v is a place of $\mathbb{Q}$ then $\mathbb{Q}_v$ is the completion of $\mathbb{Q}$ with respect to v. In particular $\mathbb{R}=\mathbb{Q}_\infty$.

0.2 If M is a set then $|M|$ denotes its cardinality. If a group H acts on M we denote by $M^H=\{m \in M \mid hm=h \text{ for all } h \in H\}$. If $H=\langle h \rangle$ is generated by one element we write $M^h=M^H$. A left action of H on M is denoted by $h(m)$ or hm or ${}^h m$.

0.3 We say that a group H acts on a group G if it acts as a group of automorphisms of G *i. e.* $h(g_1 g_2)={}^h g_1 {}^h g_2$ for all $h \in H$, $g_i \in G$. If H acts on G we denote the first non abelian cohomology set of this action by $H^1(H, G)$, *see* [Se]. If $H=\langle h \rangle$ we write for the cohomology $H^1(h, G)$. A cocycle then is an element $g \in G$ such that $g\,{}^h g=1$ and cocycles g_1, g_2 are equivalent if there is an $a \in G$ such that $g_1=a^{-1} g_2 {}^h a$. By definition $H^1(h, G)$ consists of equivalence classes of cocycles.

0.4 If V is a representation of a group G we write the action of $g \in G$ on $v \in V$ as $v \mapsto gv$. Let θ be an automorphism of G. We say that V is a representation with θ-action if there is given a $C_\theta \in GL(V)$ such that $C_\theta(gv)=\theta(g) C_\theta v$ for all $g \in G$, $v \in V$. Often we also write θ instead of C_θ.

0.5 If G is a Lie group then always $\mathfrak{g}$ denotes its real Lie algebra and $\mathfrak{g}_{\mathbb{C}}=\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ its complexification. The complexification of an automorphism θ of $\mathfrak{g}$ always is again denoted by θ. For roots, weights, Weylgroups etc. we always use standard notation and give some explanations when a symbol appears for the first time.

1. Fixpoints of θ and Lefschetz numbers

We use the notation given in the introduction. In particular G denotes a connected non compact semi-simple Lie group, θ a Cartan involution on G with set of fixpoints K. Let $X = K\backslash G$ be the associated symmetric space and $\Gamma \subset G$ a θ-stable arithmetically defined torsion free subgroup. We do not require that X/Γ be compact. Let V be a finite dimensional complex irreducible representation of G on which θ acts in a compatible way. With the same letter V we denote the associated vector bundle over X/Γ and the sheaf of locally constant sections of this vector bundle. Then θ acts on the sheaf-cohomology

$$\theta^i: \quad H^i(K\backslash G/\Gamma, V) \to H^i(K\backslash G/\Gamma, V)$$

and by definition $L(\theta, \Gamma, V) := \sum_{i=0}^{\infty} (-1)^i \operatorname{tr} \theta^i$ where $\operatorname{tr} \theta^i$ is the trace of θ^i. Since $\dim H^\bullet(K\backslash G/\Gamma, V) < \infty$ this definition makes sense and $L(\theta, \Gamma, V) \in \mathbb{Z}$. It is well known that there is a θ-equivariant isomorphism

$$H^\bullet(K\backslash G/\Gamma, V) \xrightarrow{\sim} H^\bullet(\Gamma, V)$$

where on the right we have abstract group cohomology, [B − W].

In this paragraph we explain how $L(\theta, \Gamma, V)$ depends on the fixpoint set $(X/\Gamma)^\theta$ and give a useful parametrisation of the connected components of $(X/\Gamma)^\theta$ is terms of the non abelian cohomology $H^1(\theta, \Gamma)$. For $G = SL_n(\mathbb{R})$ resp. $G = SO(n, 1)(\mathbb{R})^0$ this has been done in [R 1] resp. [R − S 3].

1.1. *Construction of fixpoints.* — If $\gamma \in \Gamma$ is a cocycle for $H^1(\theta, \Gamma)$ then $\gamma^\theta \gamma = 1$. We have a γ-twisted action on G and Γ given by $\theta_\gamma(g) = \gamma \theta(g) \gamma^{-1}$, $g \in G$, and a γ-twisted action on X given by $x \mapsto {}^\theta x \gamma^{-1}$, $x \in X$. Therefore θ_γ induces on X/Γ the action of θ. Define $X(\gamma) := X^{\theta_\gamma}$ and $\Gamma(\gamma) := \Gamma^{\theta_\gamma}$. Then $X(\gamma)$ is connected and non empty, *see* [He: I. 13.5, 13.3]. We get a natural map

$$X(\gamma)/\Gamma(\gamma) \to (X/\Gamma)^\theta.$$

Since Γ is torsionfree this map is injective. Its image is denoted by $F(\gamma)$ and depends only on the cohomology class determined by γ in $H^1(\theta, \Gamma)$. We note that $F(\gamma)$ is a closed submanifold of X/Γ.

Now we can describe the fixpoint set $(X/\Gamma)^\theta$ as follows

1.2. PROPOSITION. — *We have a decomposition*

$$(X/\Gamma)^\theta = \bigcup_{\gamma \in H^1(\theta, \Gamma)} F(\gamma)$$

into a finite disjoint sum of connected components.

Proof. — The argument given in [R 1] for $G=SL_n(\mathbb{R})$ extends to our situation.

Q.E.D.

We want to understand $X(\gamma)$ for $\gamma\in\Gamma$. Consider γ as an element of G. If $a\in G$ and $\xi=a^{-1}\gamma^{\theta}a$, $X(\xi):=\{x\in X\,|\,{}^{\theta}x\xi^{-1}=x\}$ then $X(\xi)a^{-1}=X(\gamma)$. Therefore $X(\gamma)$ depends up to translation in X only on the image of γ in $H^1(\theta, G)$.

Let T be a maximal torus in K and denote by W_K the Weyl group of T in K. Since θ acts trivially on T we have $H^1(\theta, T)={}_2T:=\{t\in T\,|\,t^2=1\}$. Of course W_K acts on ${}_2T$.

1.3. Proposition. — *The inclusions* $T\subsetneq K\subsetneq G$ *induce bijections*

$$ {}_2T/W_K \xrightarrow{\sim} H^1(\theta, K)\xrightarrow{\sim} H^1(\theta, G).$$

Proof. — The argument given in [R 1 : 1.4] extends verbatim and yields $H^1(\theta, K)\xrightarrow{\sim} H^1(\theta, G)$. It is well known that K consists of conjugates of T and that two elements of T which are conjugate in K are conjugate by an element in the normalizer of T in K. Therefore $H^1(\theta, T)/W_K\xrightarrow{\sim} H^1(\theta, K)$.

Q.E.D.

Let $t\in G$ be a cocycle, *i.e.* $t^{\theta}t=1$. We denote the involution $g\to t\,\theta(g)\,t^{-1}$, $g\in G$, by θ_t. Introduce $X(t):=\{x\in X/{}^{\theta}x\,t^{-1}=x\}$.

1.4. Corollary. — *If* $t\in{}_2T$ *then* θ_t *preserves* K *and* $X(t)\xrightarrow{\sim} K^{\theta_t}\backslash G^{\theta_t}$. *Moreover* G^{θ_t} *contains* T *and* T *is a compact Cartan subgroup of* G^{θ_t}.

Proof. — We have an exact sequence of pointed sets with θ_t-action

$$1\to K\to G\to X\to 1.$$

Hence we get an exact sequence

$$1\to K^{\theta_t}\to G^{\theta_t}\to X(t)\xrightarrow{\partial} H^1(\theta_t, K)\to H^1(\theta_t, G).$$

Using 1.3 and twisting we see that the last arrow is a bijection. Hence the first claim holds. Since θ_t acts trivially on T we get $T\subset G^{\theta_t}$. Now $G^{\theta_t}=\{g\in G/t\,\theta(g)\,t^{-1}=g\}$.

Therefore θ acts on G^{θ_t} as a Cartan involution and this action is given by conjugation with $t\in T\subset G^{\theta_t}$. This means that θ is inner on G^{θ_t}. From [He], IX, 5.7, we deduce that G^{θ_t} has a compact Cartan subgroup. Since $T\subset G^{\theta_t}$ and since T is maximal in K we see that T is a Cartan subgroup of G^{θ_t}.

Q.E.D.

We now can apply Harder's Gauss-Bonnet formula [H 2] and get

1.5. Corollary. — *The Euler-Poincaré characteristic* $\chi(F(\gamma))$ *of* $F(\gamma)$, $\gamma\in H^1(\theta, \Gamma)$, *is not zero. If* $d(\gamma):=\dim F(\gamma)$ *then* $d(\gamma)$ *is even and* $(-1)^{d(\gamma)/2}$ *is the sign of* $\chi(F(\gamma))$.

Proof. — Since $\Gamma(\gamma)$ is an arithmetically defined subgroup of G^{θ_γ} the claim follows directly from [H 2] provided that G^{θ_γ} is semi-simple. In general G^{θ_γ} is reductive. Since

1.4 holds, the center of G^{θ_t} is compact. Therefore we can view $X(\gamma)$ as a symmetric space associated to a semisimple group and [H 2] also applies to this situation.

Q.E.D.

Recall that θ acts on the representation V in a compatible way *i. e.* $\theta(gv) = {}^{\theta}g\,{}^{\theta}v$, $v \in V$, $g \in G$. If $\gamma \in \Gamma$, $\gamma^{\theta}\gamma = 1$ we can define an action of θ_γ on V by $\theta_\gamma(v) = \gamma.\theta(v)$. We write ${}^{\gamma}\Gamma$ for the group Γ with the θ_γ-action given by $\theta_\gamma(y) = \gamma^{\theta} y \gamma^{-1}$, $y \in \Gamma$. Then ${}^{\gamma}\Gamma$ and θ_γ act on V in a compatible way. Observe that the action of θ_γ on V depends up to conjugacy on the class represented by γ in $H^1(\theta, GL(V))$ only. In particular the notion $\operatorname{tr}(\theta_\gamma \mid V)$ for the trace of θ_γ on V where $\gamma \in H^1(\theta, \Gamma)$ makes sense.

We recall the following result which is contained in [R – S 3, R 3].

1.6. Proposition (*Lefschetz fixpoint formula*). — *With the notation introduced above we have*

$$L(\theta, \Gamma, V) = \sum_{\gamma \in H^1(\theta, \Gamma)} \chi(F(\gamma)) \operatorname{tr}(\theta_\gamma \mid V).$$

2. Nonvanishing of Euler-characteristics

In this paragraph we show that the Euler-Poincaré-characteristic of the fixpoint set $(X/\Gamma)^{\theta}$ is positive if $\Gamma \subset G(\mathbb{Z})$ is small enough. For this we write

$$\chi((X/\Gamma)^{\theta}) = \sum_{\gamma \in H^1(\theta, \Gamma)} \chi(F(\gamma))$$

and show that all $\chi(F(\gamma))$ are positive. Using 1.5 we have to prove that $\dim F(\gamma) \equiv 0 \bmod 4$ for all $\gamma \in H^1(\gamma, \Gamma)$. To obtain this we associate to γ a quadratic form $B(\gamma)$. One can do this also locally over $\mathbb{Q}_v$ and one gets invariants satisfying a product formula due to Weil. At the infinite place the invariant we obtain is the signature mod 8 of $B(\gamma)$. If now Γ is a sufficiently small congruence subgroup the product formula forces a fixed signature mod 8 to $B(\gamma)$ from which we can read off our desired congruence for $\dim F(\gamma)$.

To carry out the arithmetical argument, we describe in 2.0 how arithmetic subgroups Γ of connected Lie groups G actually arise. We construct Γ as a subgroup of a semisimple algebraic group $\underline{G}$ defined over $\mathbb{Q}$ such that G is a quotient with compact kernel of $\underline{G}(\mathbb{R})^{\circ}$, the connected component of the real points of $\underline{G}$. In $\underline{G}$ we work with an involution $\underline{\theta}$ defined over $\mathbb{Q}$ which induces our Cartan involution θ on G.

2.0. Preliminaries. — (i) Let $\underline{G}/\mathbb{Q}$ be a semisimple algebraic group defined over $\mathbb{Q}$. Choose a rational embedding $\underline{G} \subsetneq SL_N$ for some $N \in \mathbb{N}$. Then we write $\underline{G}(\mathbb{Z})$ for $\underline{G}(\mathbb{Q}) \cap SL_n(\mathbb{Z})$. All groups commensurable to $\underline{G}(\mathbb{Z})$ are called arithmetic subgroups. It is known, see [B 1] that this notion does not depend on the embedding $\underline{G} \subsetneq SL_N$.

(ii) Let U be an $\mathbb{R}$-linear compact normal subgroup of the connected component $\underline{G}(\mathbb{R})^0$ of $\underline{G}(\mathbb{R})$. Then we have an exact sequence

$$1 \to U \to \underline{G}(\mathbb{R})^0 \xrightarrow{p} G \to 1$$

where G is a $\mathbb{R}$-linear connected Liegroup. If $\Gamma \subset \underline{G}(\mathbb{R})^0$ is an arithmetic subgroup, then $p(\Gamma) \subset G$ also is called arithmetic. Observe that if Γ is torsionfree then $U \cap \Gamma = \{1\}$ and we can identify Γ and $p(\Gamma)$.

(iii) Let G be connected, $\mathbb{R}$-linear and semisimple and θ a Cartan involution on G. We want to establish the existence of θ-stable arithmetic subgroups $\Gamma \subset G$. For this we use a contruction due to Borel [B 3]. To construct $\underline{G}$ as in (ii) it suffices to work on the level of Lie algebras. Let $\mathfrak{g}$ be the Lie algebra of G. Then there exists a θ-stable $\mathbb{Q}$-subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}$ such that $\mathfrak{g}_0 \otimes \mathbb{R} = \mathfrak{g}$, *see* [B 3]. Let $\mathfrak{k}_0 \oplus \mathfrak{p}_0 = \mathfrak{g}_0$ be the Cartan decomposition of θ on $\mathfrak{g}_0$. Let E be a totally real number field with $[E:\mathbb{Q}] \geqq 2$. Choose $u \in E$ that $\sigma(u) < 0$ for all except one embedding σ_0 of E into $\mathbb{R}$. Put $L := E(\sqrt{u})$.

$$\mathfrak{g}_E := \mathfrak{k}_0 \underset{\mathbb{Q}}{\otimes} E \oplus \sqrt{u}\,\mathfrak{p}_0 \underset{\mathbb{Q}}{\otimes} E \subset \mathfrak{g}_0 \otimes L.$$

$\mathfrak{g}_E$ is a Lie algebra over the field E, $\mathfrak{g}_E \underset{E}{\otimes} L = \mathfrak{g}_0 \otimes L$ and $\mathfrak{g}_E \underset{\mathbb{Q}}{\otimes} \mathbb{R} \cong \mathfrak{g} \oplus \sum_{\sigma \neq \sigma_0} \mathfrak{k} \oplus i\,\mathfrak{p}$. On $\mathfrak{g}_E$ we have an E-rational involution $\theta_E : X \oplus \sqrt{u}\, Y \to X \oplus -\sqrt{u}\, Y$, which induces the Cartan involution θ we started with on the first factor of $\mathfrak{g}_E \underset{Q}{\otimes} \mathbb{R}$ and "conjugation" on the other factors.

Let $\underline{H}$ be the simply connected group defined over E corresponding to $\mathfrak{g}_E$. We denote by $\theta_E : \underline{H} \to \underline{H}$ the involution corresponding to θ_E on $\mathfrak{g}_E$ and we put $\underline{G} = \mathrm{Res}_{E \mid \mathbb{Q}} \underline{H}$, $\underline{\theta} = \mathrm{Res}_{E \mid \mathbb{Q}} \theta_E$ where Res denotes Weil restriction. Then $\underline{\theta}$ is a $\mathbb{Q}$-rational involution of $\underline{G}$. We have as in 0.2 an exact sequence

$$1 \to U \to \underline{G}(\mathbb{R})^0 \xrightarrow{p} G \to 1.$$

The involution given by $\underline{\theta}$ on $\underline{G}(\mathbb{R})^0$ factors through p and induces the Cartan involution θ on G.

Since $\underline{\theta}$ is defined over $\mathbb{Q}$ there exist $\underline{\theta}$-stable arithmetic subgroups of $\underline{G}(\mathbb{Q})$, in fact, $\Gamma \cap \underline{\theta}(\Gamma)$ is $\underline{\theta}$ stable arithmetic if $\Gamma \subset \underline{G}(\mathbb{Q})$ is arithmetic. If Γ is $\underline{\theta}$ stable and arithmetic, then Γ contains a $\underline{\theta}$-stable torsion-free congruence subgroup. Minkowski shows [M], that in the setting of (i) a congruence mod 4 suffices.

Borel shows [B 3] that the groups Γ constructed above are cocompact. We will use this in paragraphe 4. The classical groups and their most obvious realisations over $\mathbb{Q}$ usually give rise to non cocompact θ-stable arithmetic groups.

(iv) Let $\underline{G}$ be any semi-simple algebraic group defined over $\mathbb{Q}$ and assume that $\sigma : \underline{G}(\mathbb{R}) \to \underline{G}(\mathbb{R})$ is an involution of the real Lie group $\underline{G}(\mathbb{R})$. Then σ acts isometrically on the space of maximal compact subgroups of $\underline{G}(\mathbb{R})$. Using [He], I 3.5, we see that there is a maximal compact subgroup $\underline{K} \subset \underline{G}(\mathbb{R})^0$ stabilized by σ. Let U be the maximal compact normal subgroup of $\underline{G}(\mathbb{R})$. Then σ preserves U and induces an involution on $G:=G(\mathbb{R})^0/U$. Observe that if θ_0 is the Cartan involution corresponding to $\underline{K}$ then σ and θ_0 commute.

2.0 (v). Definition. — *Let $\underline{\theta}:\underline{G}\to\underline{G}$ be an involution defined over $\mathbb{Q}$. If the involution induced by $\underline{\theta}$ on* G [G *as in* (iv)] *is a Cartan involution we call $\underline{\theta}$ Cartan-like.*

We note that involution constructed in (iii) is Cartan-like. Obviously an involution conjugate in Aut $\underline{G}$ to a Cartan-like involution is Cartan-like. If $\underline{G}$ is $\mathbb{Q}$-simple and if $\underline{G}(\mathbb{R})$ contains a nontrivial compact normal factor, then no Cartan involution of $\underline{G}(\mathbb{R})$ is defined over $\mathbb{Q}$. However, as (iii) shows, it can happen that there exists a Cartan-like involution over $\mathbb{Q}$. A detailed investigation of Cartan-like involutions will appear elsewhere.

(vi) For the rest of paragraph 2 we assume that $\underline{\theta}:\underline{G}\to\underline{G}$ is a Cartan-like involution. Let $\underline{K}\subset\underline{G}(\mathbb{R})$ be a maximal compact subgroup with corresponding Cartan involution θ_0 such that $\underline{\theta}$ and θ_0 commute on $\underline{G}(\mathbb{R})$. Put $X=\underline{K}\diagdown\underline{G}(\mathbb{R})$. We denote the Cartan involution induced by $\underline{\theta}$ and θ_0 on X by θ.

2.1. Next we define global and local invariants for classes in $H^1(\underline{\theta}, \underline{G}(\mathbb{R}))$. For this let F be one of the fields $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{Q}_p$. It $t\in\underline{G}(F)$ represents a class in $H^1(\underline{\theta}, \underline{G}(F))$, *i. e.* $t^\theta t=1$, then we have the t-twisted action $\underline{\theta}_t$ on $\underline{G}(F)$ given for $g\in\underline{G}(F)$ by

$$\underline{\theta}_t(g)=t\,\underline{\theta}(g)\,t^{-1}$$

and this action induces an action denoted by the same symbol on the F-Lie algebra $\mathfrak{g}(F)$ of $\underline{G}(F)$. Here of course $\mathfrak{g}(F)=\mathfrak{g}\otimes_{\mathbb{Q}} F$ where $\mathfrak{g}=\mathfrak{g}(\mathbb{Q})$ is the $\mathbb{Q}$-Lie algebra of $\underline{G}$. Since $\underline{\theta}_t$ acts as an isometry of the Killing form B, the eigen spaces of $\underline{\theta}_t$ in $\mathfrak{g}(F)$ are orthogonal with respect to B. Denote by $\mathfrak{g}(F)(t)$ the set of $\underline{\theta}$-fixed elements in $\mathfrak{g}(F)$. Then $B\,|\,\mathfrak{g}(F)(t)$ is a non degenerate bilinear form. If $t'=a^{-1}\,t\,\underline{\theta}(a)$ represents the same class in $H^1(\underline{\theta}, \underline{G}(F))$ then conjugation with $a\in\underline{G}(F)$ induces an isometry

$$\mathrm{Ad}(a):\quad \mathfrak{g}(F)(t')\to\mathfrak{g}(F)(t).$$

Hence the isometry class of the quadratic space $B_F(t):=(\mathfrak{g}(F)(t),\ B\,|\,\mathfrak{g}(F)(t))$ depends only on the class of t in $H^1(\underline{\theta}, \underline{G}(F))$.

2.2. Lemma. — *The inclusion* $\underline{K}\subset\underline{G}(\mathbb{R})$ *induces a bijection*

$$H^1(\underline{\theta}, \underline{K})\xrightarrow{\sim}H^1(\underline{\theta}, G(\mathbb{R})).$$

Proof. — The argument given in [R 1] holds in our situation.

Q. E. D.

Recall that a quadratic form q on a $\mathbb{Q}$-Vector space V can be diagonalised over $\mathbb{R}$ with r factors 1 and s factors -1 on the diagonal. We write sign $q=r-s$ and call $r-s$ the signature of q. The signature depends only on the isometry class of $q\otimes\mathbb{R}$ in $V\otimes\mathbb{R}$.

2.3. LEMMA. — *Let* $t \in \underline{G}(\mathbb{R})$ *represent a class in* $H^1(\underline{\theta}, \underline{G}(\mathbb{R}))$ *and let* $X(t) := \{x \in X / \underline{\theta}(x) t^{-1} = x\}$ *be the set of fixpoints of* $\underline{\theta}_t$ *on* X. *Then*

$$2 \dim X(t) = \dim \mathfrak{g}(\mathbb{R})(t) + \operatorname{sign}(B_{\mathbb{R}}(t)).$$

Proof. — Using 2.2. we can assume that $t \in \underline{K}$. Then $\underline{\theta}_t$ and θ_0 commute, see 2.0 (iv). Hence we have an eigenspace decomposition

$$\mathfrak{g}(\mathbb{R})(t) = \mathfrak{k}_0 + \mathfrak{p}_0$$

of $\mathfrak{g}(\mathbb{R})(t)$ with respect to the θ_0-action on $\mathfrak{g}(\mathbb{R})(t)$. Since θ_0 is the Cartan involution corresponding to $\underline{K}$ we have that $B|\mathfrak{k}_0$ is negative definite and $B|\mathfrak{p}_0$ is positive definite. The result now follows immediately.

Q. E. D.

Let W(F) be the Grothenendieck-Witt ring of quadratic forms over F, *see* [Sch].

2.4. DEFINITION. — *Denote by* $B_F : H^1(\underline{\theta}, \underline{G}(F)) \to W(F)$ *the map sending a cohomology class t to the class of* $B_F(t) = (\mathfrak{g}(F)(t), B|\mathfrak{g}(F)(t))$. *If* $F = \mathbb{Q}$ *we write* B *instead of* $B_{\mathbb{Q}}$ *and if v is a place of* $\mathbb{Q}$ *we write* B_∞ *instead of* $B_{\mathbb{R}}$ *and* B_p *instead of* $B_{\mathbb{Q}_p}$.

We observe that the inclusions $\mathbb{Q} \subsetneq \mathbb{Q}_v$ induce obvious Hasse maps h in cohomology and of the Witt rings. Therefore we have a commutative diagram

$$\begin{array}{ccc} H^1(\underline{\theta}, \underline{G}(\mathbb{Q})) & \xrightarrow{h} & \prod_v H^1(\underline{\theta}, \underline{G}(\mathbb{Q}_v)) \\ \downarrow B & & \downarrow \prod_v B_v \\ W(\mathbb{Q}) & \xrightarrow{h} & \prod_v W(\mathbb{Q}_v) \end{array}$$

Next, we recall Weil's product formula for invariants of quadratic forms, see [Sch: Chap. V].

2.4. Suppose that (q, V) is a $\mathbb{Q}$-rational quadratic space. Then for every place v of $\mathbb{Q}$ there is defined a Gauβ sum $\gamma_v(q)$ with values in the eight's root of unity. If $v = \infty$ then $\gamma_\infty(q) = \varepsilon^{-\operatorname{sign}(q)}$ where $\varepsilon = (1+i)/\sqrt{2}$ is "the" primitive eight's root of unity. Weil's product formula says

$$\prod_v \gamma_v(q) = 1.$$

Here $\gamma_v(q)$ depends only on the class of $(q \otimes \mathbb{Q}_v, V \otimes \mathbb{Q}_v)$.

2.5. LEMMA. — *Consider the Hasse map*

$$\prod_v h_v = h : H^1(\underline{\theta}, \underline{G}(\mathbb{Q})) \to \prod_v H^1(\underline{\theta}, \underline{G}(\mathbb{Q}_p)).$$

If $t, t' \in \underline{G}(\mathbb{Q})$ *represent classes in* $H^1(\underline{\theta}, \underline{G}(\mathbb{Q}))$ *and if* $h_v(t) = h_v(t')$ *for all* $v \neq \infty$ *then*

$$\dim X(t) \equiv \dim X(t') \bmod 4.$$

Proof. — Using Weil's product formula and the commutative diagram given above we get from our assumption.

$$\gamma_\infty(\mathrm{B}(t))=\gamma_\infty(\mathrm{B}(t')).$$

We now use 2.3. Since $\dim \mathfrak{g}(\mathbb{Q}_p)(t')=\dim \mathfrak{g}(\mathbb{Q}_p)(t)$ for primes, we have $\dim \mathfrak{g}(\mathbb{Q})(t')=\dim \mathfrak{g}(\mathbb{Q})(t)$ and

$$2\dim \mathrm{X}(t)-2\dim \mathrm{X}(t')=\operatorname{sign} \mathrm{B}_{\mathbb{R}}(t)-\operatorname{sign} \mathrm{B}_{\mathbb{R}}(t')\equiv 0 \bmod 8$$

by Weil's formula. Hence our claim holds.

Q. E. D.

Next we produce a $\underline{\theta}$-stable congruence subgroup $\Gamma\subset\underline{\mathrm{G}}(\mathbb{Q})$ such that for every prime p the classes $t\in \mathrm{H}^1(\underline{\theta}, \Gamma)$ all have the same invariants $\gamma_p(\mathrm{B}_p(t))$. For this we need some local cohomological results.

2.6. Recall that we have an embedding $\underline{\mathrm{G}}\subset \mathrm{SL}_\mathrm{N}$ over $\mathbb{Q}$. Denote by $\mathrm{K}_p(j)$ the kernel of the canonical map

$$\mathrm{SL}_\mathrm{N}(\mathbb{Z}_p)\to \mathrm{SL}_\mathrm{N}(\mathbb{Z}_p/p^j\,\mathbb{Z}_p)$$

and by $\mathrm{sl}_\mathrm{N}(\mathbb{Z}_p)$ the set of $\mathrm{N}\times\mathrm{N}$ matrices with coefficients in $\mathbb{Z}_p$ and trace zero. Then the usual exponential series of matrices induces a bijection

$$\exp:\quad p^j\,\mathrm{sl}_\mathrm{N}(\mathbb{Z}_p)\to \mathrm{K}_p(j)$$

if $p>2$ and $j\geqq 1$ or if $p=2$ and $j\geqq 2$.

We write $\mathfrak{g}(\mathbb{Q}_p)$ for the $\mathbb{Q}_p$-Lie algebra of $\underline{\mathrm{G}}(\mathbb{Q}_p)$ considered as a subset of the $\mathrm{N}\times\mathrm{N}$-matrices with coefficients in $\mathbb{Q}_p$. Then we get an induced bijection

$$\exp:\quad p^j\,\mathrm{sl}_\mathrm{N}(\mathbb{Z}_p)\cap \mathfrak{g}(\mathbb{Q}_p)\to \Gamma_p(j),$$

where $\Gamma_p(j)=\mathrm{K}_p(j)\cap\underline{\mathrm{G}}(\mathbb{Q}_p)$.

2.7. Lemma. — *Let $\mathrm{U}_p\subset\underline{\mathrm{G}}(\mathbb{Z}_p)$ be an open $\underline{\theta}$-stable subgroup. Then there exists an open $\underline{\theta}$-stable normal subgroup V_p of U_p such that the map*

$$\mathrm{H}^1(\underline{\theta}, \mathrm{V}_p)\to \mathrm{H}^1(\underline{\theta}, \mathrm{U}_p)$$

induced by the inclusion $\mathrm{V}_p \subsetneq \mathrm{U}_p$ is trivial.

Proof. — Since U_p is open there is a j such that $\Gamma_p(j)\subset \mathrm{U}$. Choose $j\geqq 1$ if $p\neq 2$ and $j\geqq 2$ if $p=2$. Define $\mathrm{V}'_p=\Gamma_p(j+1)\cap\underline{\theta}\,\Gamma_p(j+1)$ and $\mathrm{V}_p:=\bigcap_{u\in \mathrm{U}_p} u\,\mathrm{V}'_p u^{-1}$. Then V_p is open normal in U_p and $\underline{\theta}$-stable.

If $v\in \mathrm{V}_p$ represents a class in $\mathrm{H}^1(\underline{\theta}, \mathrm{V}_p)$ then $v\,\underline{\theta}(v)=1$, *i. e.* $\underline{\theta}(v)=v^{-1}$. Using 2.6 we find an $\mathrm{X}\in p^{j+1}\,\mathrm{sl}_\mathrm{N}(\mathbb{Z}_p)\cap\mathfrak{g}(\mathbb{Q}_p)$, $\exp \mathrm{X}=v$ and $\exp(-\mathrm{X})=\underline{\theta}(v)$. Put $c=\exp(-\mathrm{X}/2)$. Then $c\in\Gamma_p(j)\cap\underline{\theta}\,\Gamma_p(j)$ and $\underline{\theta}(c)=\exp(-\underline{\theta}\,\mathrm{X}/2)=\exp(\mathrm{X}/2)$ since $\underline{\theta}\,\mathrm{X}=-\mathrm{X}$. Therefore $c^{-1}v\,\underline{\theta}(c)=1$ *i. e.* v is a coboundary in U_p.

Q. E. D.

Our embedding $\underline{G} \subsetneq SL_N$ has been choosen without care with respect to the $\underline{\theta}$ action. Therefore $\underline{\theta}$ will not preserve all $\underline{G}(\mathbb{Z}_p) := \underline{G}(\mathbb{Q}_p) \cap SL_N(\mathbb{Z}_p)$. But of course there exists a finite set S_0 of primes such that $\underline{G}$, $\underline{\theta}$ and the embedding are defined over $\mathbb{Z}_{S_0}$. Enlarging S_0 we assume that $2 \in S_0$. Then $\underline{\theta}$ preserves $\underline{G}(\mathbb{Z}_p)$ for all $p \notin S_0$. If S is a finite set of primes containing S_0 we define

$$\Gamma(S) = \underline{G}(\mathbb{Q}) \cap \prod_{p \in S} V_p \times \prod_{p \notin S} \underline{G}(\mathbb{Z}_p).$$

Then $\Gamma(S)$ is a $\underline{\theta}$-stable congruence subgroup of $\underline{G}(\mathbb{Z})$.

2.8. Definition. — *We call a $\underline{\theta}$-stable arithmetic subgroup $\Gamma \subset \underline{G}(\mathbb{Q})$ small enough if the image under the natural map*

$$\prod_p h_p : \quad H^1(\underline{\theta}, \Gamma) \to \prod_p H^1(\underline{\theta}, \underline{G}(\mathbb{Q}_p))$$

is trivial.

2.9. Lemma. — *If* S *is big enough, then* $\Gamma(S)$ (*as defined above*) *is small enough and torsion free.*

Proof. — We start with $\Gamma(S_0)$. According to [B-S] the set $H^1(\underline{\theta}, \Gamma(S_0))$ is finite. If $\gamma \in H^1(\underline{\theta}, \Gamma(S_0))$ and if there is a prime p such that $h(\gamma) \neq 1$ consider $S = S_0 \cup \{p\}$ and $\Gamma(S)$. Then $\Gamma(S)$ is a subgroup of $\Gamma(S_0)$ and for all $y \in H^1(\underline{\theta}, \Gamma(S))$ we have $h_p(y) = 1$. Therefore γ is not in the image of $H^1(\underline{\theta}, \Gamma(S))$ in $H^1(\underline{\theta}, \Gamma(S_0))$. After finitely many of such steps we arive at a $\Gamma(S)$ such that all classes of $H^1(\theta, \Gamma(S))$ are trivial at all finite primes. By construction $\Gamma(S)$ is a subgroup of the full congruence subgroup mod 4 of $SL_N(\mathbb{Z})$. This congruence subgroup is known to be torsionfree *see* [M]. Hence $\Gamma(S)$ is torsion free.

Q. E. D.

2.10. *Proposition.* — *If* Γ *is small enough and torsion free, then* $\chi((X/\Gamma)^{\theta}) > 0$.

Proof. — Using 2.5 we see that all fixpoint components have the same dimension mod 4. Therefore our claim follows from 1.5 since $\underline{\theta}$ induces the Cartan involution θ on X.

Q. E. D.

Remark. — (i) The examples given in [R 1] and [R-S3] show that sometimes very low congruences suffice to produce a small enough Γ.

(ii) Even if Γ is small enough the image of $H^1(\underline{\theta}, \Gamma)$ in $H^1(\underline{\theta}, \underline{G}(\mathbb{R}))$ is non trivial in general, *see* for example [R 1]. Therefore we have to expect that $(X/\Gamma)^{\theta}$ contains components of dimension bigger than zero.

(iii) If Γ is as in 2.10 then for $\Gamma^0 := \underline{G}(\mathbb{R})^0 \cap \Gamma$ the conclusion of 2.10 holds as well. In the setting of 2.0 (ii) we can consider Γ^0 as a subgroup of G. In paragraph 4 we will use such Γ's.

If we evaluate the Lefschetz number $L(\underline{\theta}, \Gamma, V)$ as explained in 1.6, then there are factors $\operatorname{tr}(\underline{\theta}_\gamma | V)$. We now show that under quite general circumstances $\operatorname{tr}(\underline{\theta}_\gamma | V)$ does not depend on $\gamma \in H^1(\underline{\theta}, \Gamma)$.

2.11. LEMMA. — *If there is a prime p such that the natural map*

$$h_p: \quad H^1(\underline{\theta}, \Gamma) \to H^1(\underline{\theta}, \underline{G}(\mathbb{Q}_p))$$

is trivial, then $\operatorname{tr}(\underline{\theta}_\gamma | V) = \operatorname{tr}(\underline{\theta} | V)$ *for all* $\gamma \in H^1(\underline{\theta}, \Gamma)$.

Proof. — The representation $\mu: \underline{G}(\mathbb{R})^0 \to G \to GL(V)$ extends to a representation $\mu_{\mathbb{C}}$ of $\underline{G}(\mathbb{C})$ on $GL(V)$. If L, $[L:\mathbb{Q}] < \infty$, $L \subset \mathbb{C}$ is a splitting field of $\underline{G}$, there is a L-rational representation μ_0 of $\underline{G} \times_{\mathbb{Q}} L$ on a L-Vectorspace V_0 with an action of $\underline{\theta}$ such that $\mu_{\mathbb{C}}$ is obtained by extension of scalars from L to $\mathbb{C}$. Then $\underline{\theta}_\gamma$ acts on V_0 and $\operatorname{tr}(\underline{\theta}_\gamma | V) = \operatorname{tr}(\underline{\theta}_\gamma | V_0)$. Let w be a place of L extending p. Then $\underline{\theta}_\gamma$ acts on $V_0 \otimes_L L_w$ and $\operatorname{tr}(\underline{\theta}_\gamma | V) = \operatorname{tr}(\underline{\theta}_\gamma | V_0 \otimes_L L_w)$. By assumption the composition of canonical maps

$$H^1(\underline{\theta}, \Gamma) \xrightarrow{h_p} H^1(\underline{\theta}, \underline{G}(\mathbb{Q}_p)) \to H^1(\underline{\theta}, \underline{G}(L_w)) \to H^1(\underline{\theta}, GL(V_0 \otimes_L L_w))$$

is trivial. We have observed, see 1.6, that $\operatorname{tr}(\underline{\theta}_\gamma | V_0 \otimes_L L_w)$ only depends on the cohomology class determined by γ in $H^1(\underline{\theta}, GL(V_0 \otimes_L L_w))$. Since this class is trivial $\operatorname{tr}(\underline{\theta}_\gamma | V_0 \otimes_L L_w) = \operatorname{tr}(\underline{\theta} | V_0 \otimes_L L_w) = \operatorname{tr}(\underline{\theta} | V_0)$ and our claim holds.

Q. E. D.

2.12. PROPOSITION. — *Suppose that* $\Gamma \subset \underline{G}(\mathbb{Q})$ *is small enough and torsionfree. If* Γ_1 *is* θ*-stable arithmetic and normal in* Γ *then*

$$|L(\theta, \Gamma_1, V)| \geqq |(\Gamma/\Gamma_1)^\theta| \, |\operatorname{tr}(\theta | V)|.$$

Proof. — We have $L(\theta, \Gamma_1, V) = \sum_{\gamma \in H^1(\theta, \Gamma_1)} \chi(F(\gamma)) \operatorname{tr}(\theta | V)$. Here we use that Γ_1 is small enough, we use 1.7 and 2.11. Using 2.10 we have $\chi(F(\gamma)) > 0$ for all $\gamma \in H^1(\theta, \Gamma_1)$. Now Γ/Γ_1 acts on X/Γ_1 and $(\Gamma/\Gamma_1)^\theta$ acts as a permutation of the components $F(\gamma)$ of $(X/\Gamma_1)^\theta$. Therefore $\Sigma \chi(F(\gamma))$ is divisible by $|(\Gamma/\Gamma_1)^\theta|$ and our claim holds.

Q. E. D.

Remark. — If Γ is chosen with more care, which means that Γ has to be smaller in a certain way than "small enough", then one can show

$$|L(\theta, \Gamma_1, V)| = |(\Gamma/\Gamma_1)^\theta| \, |L(\theta, \Gamma, V)|.$$

Therefore the crude estimate in 3.12 gives the right order of growth for $L(\theta, \Gamma_1, V)$ if Γ_1 shrinks to $\{1\}$.

3. The trace of a Cartan involution on a finite dimensional representation

Assume that G is a real linear semi-simple Lie group without compact factors and θ a Cartan involution acting on an irreducible complex finite dimensional representation V of G in a compatible way, *i.e.* $\theta(gv)=\theta(g)\,\theta(v)$ for all $g\in G$, $v\in V$. In this paragraph we compute $\operatorname{tr}(\theta\,|\,V)$, the trace of θ on V, and show that $\operatorname{tr}(\theta\,|\,V)\neq 0$ if the highest weight of V lies in a certain big sublattice of the lattice of weights.

3.1. FORMULATION OF THE MAIN RESULT. — Before we can do so, we have to introduce some notation and we have to collect some easy observations.

3.1.0. *Preliminaries.* — Assume that G is as above with Cartan involution θ.

(i) If V is an irreducible representation of G admitting an action of θ in two ways given by $C_\theta^i\in GL(V)$, $i=1,2$ then for all $g\in G$, $v\in V$ we get $C_\theta^1\,C_\theta^2(gv)=g\,C_\theta^1\,C_\theta^2\,v$. Since V is irreducible and $(C_\theta^i)^2=1$ we get $C_\theta^1=\pm C_\theta^2$. That is: if V admits an action of θ this action is unique up to sign.

(ii) If V is an irreducible representation of G we have a new action of $g\in G$ on $v\in V$ given by $g\,.\,v=\theta(g)\,v$. We denote this new representation by ${}^\theta V$ and observe that if V admits a θ-action given by $C_\theta\in GL(V)$ then $C_\theta: V\to {}^\theta V$ is an equivalence of representations. Conversely, if $C: V\to {}^\theta V$ is an equivalence of representations then $C(gv)=\theta(g)\,C\,v$ and $C^2=\alpha\,\mathrm{Id}$, since V is irreducible. Therefore $C_\theta=\pm\sqrt{\alpha}\,C$ defines a θ-action on V.

(iii) Let $\mathfrak{k}\oplus p$ be the Cartan decomposition of the real Lie algebra $\mathfrak{g}$ of G with respect to θ. We choose a fundamental Cartan subalgebra $\mathfrak{h}=\mathfrak{t}\oplus\mathfrak{a}$ in $\mathfrak{g}$, i.e. $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{k}$, denote by $\mathfrak{h}_\mathbb{C}$ it's complexification in $\mathfrak{g}_\mathbb{C}=\mathfrak{g}\otimes_\mathbb{R}\mathbb{C}$ and choose a θ-stable Borel subalgebra $\mathfrak{b}_\mathbb{C}\subset\mathfrak{g}_\mathbb{C}$ containing $\mathfrak{h}_\mathbb{C}$. Now an irreducible representation V of G is uniquely determined by its highest integral dominant weight $\lambda\in P$ in the weight lattice P determined by $\mathfrak{h}_\mathbb{C}$. The Cartan involution θ acts on P and using (ii) we see that V admits a θ-action if and only if $\theta(\lambda)=\lambda$. We now fix the action of θ on V with highest weight $\lambda=\theta\lambda$ by the requirement that θ acts identically on highest weight vectors.

(iv) Let $\mathfrak{b}_\mathbb{C}$ be θ-stable as in (iii). We have the rootspace decomposition $\mathfrak{g}_\mathbb{C}=\mathfrak{h}_\mathbb{C}\oplus\sum_{\alpha\in\Delta}\mathfrak{g}_\alpha$ with system $\Delta^+\subset\Delta$ of positive roots determined by $\mathfrak{b}_\mathbb{C}$. Let $\alpha_1,\ldots,\alpha_l$ be a basis of Δ^+. Then there exists exactly one automorphism θ_0 of $\mathfrak{g}_\mathbb{C}$ with the following properties

(*a*) $\theta_0\,|\,\mathfrak{h}_\mathbb{C}=\theta\,|\,\mathfrak{h}_\mathbb{C}$

(*b*) If $\theta(\alpha_i)=\alpha_i$ then $\theta_0\,|\,\mathfrak{g}_{\alpha_i}=\mathrm{Id}\,|\,\mathfrak{g}_{\alpha_i}$

If $\theta(\alpha_i)\neq\alpha_i$ then $\theta_0\,|\,\mathfrak{g}_{\alpha_i}=\theta\,|\,\mathfrak{g}_{\alpha_i}$.

The existence and uniqueness of θ_0 follows from [Bou], Ch. VIII, § 4, Thm. 2 (i). The automorphism θ_0 has the properties $\theta_0\,\theta=\theta\theta_0$, $\bar{\theta}_0=\theta_0$ *i.e.* θ_0 is defined over $\mathbb{R}$, and θ_0 preserves $\mathfrak{b}_\mathbb{C}$. This essentially follows also from Thm. 2 *loc cit.* One has to use that θ is defined over $\mathbb{R}$. We call θ_0 the diagramm automorphism determined by θ.

3.1.1. DEFINITION. — *We use the notation established above.*

(i) *A root* $\alpha \in \Delta$ *is called* non compact *if*

$$0 \neq (\mathfrak{g}_\alpha + \mathfrak{g}_{\theta_0(\alpha)})^{\theta_0} \subset p_{\mathbb{C}}$$

(ii) *Let be* $P_\theta := \{\lambda \in P^\theta / (\lambda, \alpha)/(\alpha, \alpha) \in \mathbb{Z}$ for all non-compact roots $\alpha \in \Delta\}$.

Remark. — (i) If $\theta = \theta_0$ is the outer automorphism determined by θ, then $P_\theta = P^\theta$ since there are no non compact roots. If θ is inner then $\theta_0 = \mathrm{Id}$ and our definition of a non-compact root is just the usual one. We have $2P^\theta \subset P_\theta \subset P^\theta$ since $2(\lambda, \alpha)/(\alpha, \alpha) \in \mathbb{Z}$ for all $\lambda \in P$.

(ii) Since θ acts on $\mathfrak{h}_{\mathbb{C}}$ it acts on $W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ and this action is trivial if θ is inner on $\mathfrak{g}_{\mathbb{C}}$. Denote by W^θ the fixpoints of the θ-action on W. Then W^θ acts on P_θ.

(iii) If $w_0 \in W$ is the element of W mapping Δ^+ to $-\Delta^+$ then $w_0 \in W^\theta$. Hence the highest weight λ of irreducible representation V is in P_θ if and only if the lowest weight $w_0 \lambda$ is in P_θ. We will use this in paragraph 4.

(iv) The geometric meaning of $\lambda \in P_\theta$ is as follows. Assume that $\lambda \in P^\theta$ is integral dominant with respect to $\mathfrak{b}_{\mathbb{C}}$ and let V_λ be the corresponding irreducible representation with highest weight λ. If $w \in W^\theta$ then $w\lambda$ is integral dominant with respect to ${}^w\mathfrak{b}_{\mathbb{C}}$, the w-conjugents of $\mathfrak{b}_{\mathbb{C}}$. Let $V_{w\lambda}$ be the corresponding irreductible representation. Put a θ-action on V_λ resp. on $V_{w\lambda}$ as explained in 3.1.0 (iii). These actions depend on $\mathfrak{b}_{\mathbb{C}}$ resp. on ${}^w\mathfrak{b}_{\mathbb{C}}$. Then $\lambda \in P_\theta$ if and only if the natural equivalence of representations $V_\lambda \xrightarrow{\sim} V_{w\lambda}$ is θ-equivariant. This statement will become clear on the following pages. Since we will not need this observation we don't give a formal proof.

We will prove the following statement in 3.2 and 3.3.

3.1.2. PROPOSITION. — *Assume that* $\lambda \in P_\theta$ *is the highest weight of* V. *Then* $\mathrm{tr}(\theta \mid V) > 0$.

Remark. — We believe that the sufficient condition $\lambda \in P_\theta$ is also necessary for $\mathrm{tr}(\theta \mid V) \neq 0$.

3.1.3. *A reduction.* We decompose $\mathfrak{g} = \mathfrak{g}_1 \times \ldots \times \mathfrak{g}_s$ where $\mathfrak{g}_i$ are simple non-compact $\mathbb{R}$-Lie algebras. Then $\theta = \theta_1 \times \ldots \times \theta_s$ where θ_i is a Cartan involution of $\mathfrak{g}_i$. An irreductible representation V of $\mathfrak{g}$ then is a product $V_1 \otimes \ldots \otimes V_s$, where the V_i's are irreductible and V admits a θ action if and only all V_i's do. Since $\mathfrak{h}_{\mathbb{C}}$ and $\mathfrak{b}_{\mathbb{C}}$ are also decomposed we fix the θ_i-action on V_i according to 3.1 (iii). We have $\mathrm{tr}(\theta \mid V) = \prod_{i=1}^{s} \mathrm{tr}(\theta_i \mid V_i)$. Therefore we may assume that $\mathfrak{g}$ is simple over $\mathbb{R}$.

3.2. THE TRACE OF AN INNER CARTAN INVOLUTION. — We assume that θ is an inner automorphism of $\mathfrak{g}_{\mathbb{C}}$.

3.2.1. *Preliminaries.* — (i) Since θ is inner $\mathfrak{g}$ contains a compact Cartan subalgebra $\mathfrak{h} = \mathfrak{t}$ in the notation of 3.1 (iii). Since θ stabilizes $\mathfrak{b}_{\mathbb{C}}$ and $\mathfrak{h}_{\mathbb{C}}$ there is a $\theta_0 \in \mathfrak{t}$ such that $\theta = \exp 2\pi i \,\mathrm{ad}\, \theta_0$.

(ii) If V_λ is irreducible with a highest weight vector $v \neq 0$, if $0 \neq X_\alpha \in \mathfrak{g}_\alpha$ are root vectors, $\alpha \in \Delta$, if Δ^+ denotes the system of positive roots given by $\mathfrak{b}_\mathbb{C}$ then θ acts as

$$\theta(X_{-\alpha_{i_1}} \ldots X_{-\alpha_{i_t}} v) = \exp 2\pi i \langle -\lambda, \theta_0 \rangle \theta(X_{-\alpha_{i_1}}) \ldots \theta(X_{-\alpha_{i_t}}) v, \qquad \alpha_{i_j} \in \Delta^+,$$

see 3.1 (iii).

(iii) Denote by Δ_K^+ the set of roots $\alpha \in \Delta^+$ such that $\mathfrak{g}_\alpha \subset \mathfrak{k}_\mathbb{C}$ and denote by W_K the subgroup of W generated by refections at these roots. Denote $W^K = \{ u \in W / \Delta_K^+ \subset u\Delta^+ \}$. Then one has a well known bijection

$$W_K \times W^K \xrightarrow{\sim} W \text{ given by } (s, u) \mapsto su$$

(iv) Let λ be an integral dominant weight with respect to $\mathfrak{h}_\mathbb{C} = \mathfrak{t}_\mathbb{C}$ and $\mathfrak{b}_\mathbb{C}$. We write $\rho = (1/2) \sum_{\alpha \in \Delta^+} \alpha$ and $\rho_K := (1/2) \sum_{\alpha \in \Delta_K^+} \alpha$. If $u \in W^K$ then $u(\lambda + \rho) - \rho_K$ is integral dominant with respect to $\mathfrak{t}_\mathbb{C}$ and $\mathfrak{k}_\mathbb{C} \cap \mathfrak{b}_\mathbb{C}$. We denote by $V_{u(\lambda+\rho)-\rho_K}$ the irreducible $\mathfrak{k}_\mathbb{C}$-modules of highest weight $u(\lambda + \rho) - \rho_K$.

3.2.3. Proposition. — *We use the notation established in* 3.2.1.

If V *is irreducible of highest weight* λ *then*

$$\operatorname{tr}(\theta \mid V) = 2^{-l} \sum_{u \in W^K} \exp 2\pi i \langle u\lambda - \lambda, \theta_0 \rangle \dim V_{u(\lambda+\rho)-\rho_K}$$

where $l = |\Delta^+ \setminus \Delta_K^+|$ *is the number of non-compact positive roots.*

Proof. — We use Weyl's character formula and get

$$\begin{aligned} \operatorname{tr}(\theta \mid V) &= \exp 2\pi i \langle -\lambda, \theta_0 \rangle \operatorname{tr}(2\pi i \theta_0 \mid V) \\ &= \exp 2\pi i \langle -\lambda, \theta_0 \rangle \lim_{t \to 0} \frac{\sum_{w \in W} \det w \exp 2\pi i \langle w(\lambda+\rho) - \rho, \theta_0 + t \rangle}{\sum_{w \in W} \det w \exp 2\pi i \langle w(\rho) - \rho, \theta_0 + t \rangle} \end{aligned}$$

where $t + \theta_0 \in \mathfrak{h}_\mathbb{C}$ is regular.

We write $w = su$ with $s \in W_K$, $u \in W^K$ and use that

$$\exp 2\pi i \langle su(\lambda+\rho), \theta_0 \rangle = \exp 2\pi i \langle u(\lambda+\rho), \theta_0 \rangle$$

which holds since $\langle \alpha, \theta_0 \rangle \in \mathbb{Z}$ for $\alpha \in \Delta_K^+$. Moreover we have $\exp 2\pi i \langle u(\rho) - \rho, \theta_0 \rangle = \det u$. To see this we recall that if $t := |u\Delta^+ \cap -\Delta^+|$ then $(-1)^t = \det u$. But $u(\rho) - \rho = -\Sigma\beta$, $\beta \in u\Delta^+ \cap -\Delta^+$. Since $\Delta_K^+ \subset u\Delta^+$ all the β's are in $\Delta^+ \setminus \Delta_K$. Since $\langle \beta, \theta_0 \rangle \in 1/2 + \mathbb{Z}$ the claim holds. Therefore we have

$$\begin{aligned} &\sum_{w \in W} \det w \exp 2\pi i \langle w(\lambda+\rho) - \rho, \theta_0 + t \rangle \\ &= \sum_{u \in W^K} \exp 2\pi i \langle u\lambda, \theta_0 \rangle \sum_{s \in W_K} \det s \exp 2\pi i \langle su(\lambda+\rho) - \rho, t \rangle. \end{aligned}$$

For the denominator of Weyl's character formula we get

$$\sum_{w \in W} \det w \exp 2\pi i \langle w(\rho)-\rho, \theta_0+t \rangle = \prod_{\alpha \in \Delta^+} (1-\exp 2\pi i \langle -\alpha, \theta_0+t \rangle).$$

Since $\langle -\alpha, \theta_0 \rangle \in \mathbb{Z}$ if $\alpha \in \Delta_K^+$ and $\langle -\alpha, \theta_0 \rangle \in 1/2+\mathbb{Z}$ if $\alpha \in \Delta_K^+ - \Delta^+$, the last expression is equal to

$$\prod_{\alpha \in \Delta_K^+} (1-\exp 2\pi i \langle -\alpha, t \rangle) \prod_{\alpha \in \Delta^+ \setminus \Delta_K^+} (1+\exp 2\pi i \langle -\alpha, t \rangle)$$

and by the denominator formula of the Weyl character formula for $\mathfrak{k}_{\mathbb{C}}$ the first part of the last expression is

$$\exp 2\pi i < -\rho_K, t \rangle \sum_{s \in W_K} \det s \exp 2\pi i \langle s\rho_K, t \rangle.$$

If we now substitute our formulas in the equation for $\operatorname{tr}(\theta \mid V)$ and compute the limit for $t \to 0$ using the Weyl dimension formula over $\mathfrak{k}_{\mathbb{C}}$ for the representations $V_{u(\lambda+\rho)-\rho_K}$ the result we claimed follows.

Q.E.D.

3.2.4. LEMMA. — *We have* $\lambda \in P_\theta$ *if and only if* $\exp 2\pi i \langle w\lambda-\lambda, \theta_0 \rangle = 1$ *for all* $w \in W$.

Proof. — Assume that $\lambda \in P_\theta$. If $s_\alpha \in W$ is the reflection determined by $\alpha \in \Delta^+$, then $\langle s_\alpha \lambda - \lambda, \theta_0 \rangle = \langle 2(\lambda, \alpha)/(\alpha, \alpha)\alpha, \theta_0 \rangle$. We have $\langle \alpha, \theta_0 \rangle \in \mathbb{Z}$ for all $\alpha \in \Delta_K^+$. If α is a non-compact root and if $\lambda \in P_\theta$ then $\langle 2(\lambda, \alpha)/(\alpha, \alpha), \theta_0 \rangle \in \mathbb{Z}$. Therefore $\exp 2\pi i \langle s_\alpha \lambda - \lambda, \theta_0 \rangle = 1$ for all $\alpha \in \Delta^+$. Suppose now that $w = s_\alpha u$ where w, $u \in W$. Then $u(\lambda) \in P_\theta$ and $\langle s_\alpha u\lambda - \lambda, \theta_0 \rangle = \langle s_\alpha u\lambda - u\lambda + u\lambda - \lambda, \theta_0 \rangle$. Hence by induction on the length of w we get $\exp 2\pi i \langle w\lambda - \lambda, \theta_0 \rangle = 1$ for all $w \in W$. If conversely $\exp 2\pi i \langle w\lambda - \lambda, \theta_0 \rangle = 1$ for all $w \in W$, this holds in particular for s_α. If α is a non-compact root this exactly means that $(\lambda, \alpha)/(\alpha, \alpha) \in \mathbb{Z}$ since $\langle \alpha, \theta_0 \rangle \equiv 1/2 \bmod \mathbb{Z}$. Therefore $\lambda \in P_\theta$.

Q.E.D.

3.2.5. COROLLARY. — *If* $\lambda \in P_\theta$ *then* $\operatorname{tr}(\theta \mid V) > 0$.

3.3. THE TRACE OF AN OUTER CARTAN INVOLUTION. — We assume throughout that $\mathfrak{g}$ is an $\mathbb{R}$-simple Lie algebra and that θ is outer on $\mathfrak{g}_{\mathbb{C}}$. In case that $\mathfrak{g}$ is a complex simple Lie algebra considered as a real Lie algebra the trace of a Cartan involution on a representation can be computed in a few lines. However the method we use to handle the absolutely simple case applies as well to a complex $\mathfrak{g}$, so that we don't separate these cases.

3.3.1. *Preliminaries.* — (i) Recall that $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$, $\mathfrak{t} = \mathfrak{h} \cap \mathfrak{k}$, $\mathfrak{a} \subset \mathfrak{p}$ and that $\mathfrak{h}$ is a fundamental θ-stable Cartan subalgebra. We have $\mathfrak{h}_{\mathbb{C}} \subset \mathfrak{b}_{\mathbb{C}}$ where $\mathfrak{b}_{\mathbb{C}}$ is a θ-stable Borel subalgebra and $\dim \mathfrak{a} \geqq 1$ since θ is assumed to be outer on $\mathfrak{g}_{\mathbb{C}}$.

(ii) If $Q \subset \mathfrak{h}_{\mathbb{C}}^*$ denotes the root lattice of $\mathfrak{g}_{\mathbb{C}}$, and $\bar{T} := \mathrm{Hom}_{\mathbb{Z}}(Q, \mathbb{C}^*)$ the corresponding complex maximal torus in the adjoint group for $\mathfrak{g}_{\mathbb{C}}$, we have a sequence

$$1 \to \bar{T} \to \mathrm{Aut}(\mathfrak{g}_{\mathbb{C}}, \mathfrak{b}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) \xrightarrow{\varepsilon} \mathrm{Aut}(D) \to 1,$$

see [Bou], V, III, § 5, n° 2. Here D denotes the Dynkin diagram for $\mathfrak{g}_{\mathbb{C}}$, Aut(D) its group of automorphisms and $\mathrm{Aut}(\mathfrak{g}_{\mathbb{C}}, \mathfrak{b}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ the group of automorphisms of the Lie algebra $\mathfrak{g}_{\mathbb{C}}$ leaving $\mathfrak{b}_{\mathbb{C}}$ and $\mathfrak{h}_{\mathbb{C}}$ stable.

Let θ_0 be the diagram automorphism associated to θ in 3.0 (iv). Since $\varepsilon(\theta_0) = \varepsilon\theta$ there is a $t_0 \in \bar{T}$ such that $\theta = \mathrm{int}(t_0) \circ \theta_0$. Since $\mathrm{int}({}^{\theta_0}t_0) = \theta_0\,\theta = \theta\theta_0 = \mathrm{int}(t_0)$ we have $t_0 \in \bar{T}^{\theta_0}$.

Finally we observe that θ acts on $\mathfrak{g}_{\mathbb{C}}^{\theta_0}$ and induces here the inner automorphism given by $\mathrm{int}(t_0)$.

(iii) If $\mathfrak{g}$ is a simple complex Lie algebra with Dynkin diagram D^0 then $D = D^0 \cup D^0$ (disjoint union), $\mathfrak{g}_{\mathbb{C}} \xrightarrow{\sim} \mathfrak{g} \times \mathfrak{g}$ and θ_0 acts on $\mathfrak{g}_{\mathbb{C}}$ by switching the copies of $\mathfrak{g}$.

If $\mathfrak{g}$ is absolutely simple then $\mathfrak{g}_{\mathbb{C}}$ is of type A_l, $l \geqq 2$, D_l, $l \geqq 4$ or E_6 with the obvious involution θ_0 on the associated diagram.

Now we can explain the idea of our proof. We consider V as a representation of the disconnected group $G \times \{1, \theta\}$. We use Kostant's character formula and express $\mathrm{tr}(\theta \mid V)$ as a quotient of two Weyl character formulas for certain representations of $\mathfrak{g}_{\mathbb{C}}^{\theta_0}$ evaluated at t_0, *see* 3.3.1 (ii). For this we need some results on $\mathfrak{g}^{\theta_0}$ which are essentially well known, *see* [C], Chap. 13 and [Bou], Ch. VIII, § 7. Exercises (13). Therefore we give only a sketch of a proof.

3.3.2. Proposition. — *Let* $\mathfrak{g}_{\mathbb{C}}^0 := \mathfrak{g}_{\mathbb{C}}^{\theta_0}$ *and* $\mathfrak{h}_{\mathbb{C}}$ *resp.* $\mathfrak{t}_{\mathbb{C}}$ *be as in* 3.3.1 (i). *Then*

(i) *The Lie algebra* $\mathfrak{g}_{\mathbb{C}}^0$ *is semisimple with Cartan subalgebra* $\mathfrak{t}_{\mathbb{C}} \cong \mathfrak{h}_{\mathbb{C}}^{\theta_0}$.

(ii) *The restriction* $\mathrm{res} : \mathfrak{h}_{\mathbb{C}}^* \to \mathfrak{t}_{\mathbb{C}}^* \cong (\mathfrak{h}^*)^{\theta_0}$ *induces an isorphism*

$$W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})^{\theta_0} \xrightarrow{\sim} W(\mathfrak{g}_{\mathbb{C}}^0, \mathfrak{t}_{\mathbb{C}}).$$

(iii) *If* $\bar{T}$ *denotes the complex torus corresponding to* $\mathfrak{h}_{\mathbb{C}}$ *in the adjoint group of* $\mathfrak{g}_{\mathbb{C}}$, *then* $\bar{T}^{\theta_0}$ *is connected.*

(iv) *If* $\lambda \in P^{\theta_0}$ *is a* θ_0*-invariant weight of* $\mathfrak{h}_{\mathbb{C}}$, *then* $\mathrm{res}\,\lambda$ *is a weight of* $\mathfrak{t}_{\mathbb{C}}$.

Proof. — Statement (i) is proved in Bourbaki, *loc. cit.* and statement (ii) appears up to some identifications in Carter *loc. cit.* Let Q be the root lattice of $\mathfrak{g}_{\mathbb{C}}$. Then θ_0 acts on Q and the root lattice of $\mathfrak{g}_{\mathbb{C}}^0$ is $Q/(1-\theta_0)Q$, *see* Bourbaki's exercise. Since $\bar{T}^{\theta_0} = \mathrm{Hom}_{\mathbb{Z}}(Q, \mathbb{C}^*)^{\theta_0} \cong \mathrm{Hom}_{\mathbb{Z}}(Q/(1-\theta_0)Q, \mathbb{C}^*)$ canonically, and since θ_0 acts as a permutation of the simple roots corresponding to $\mathfrak{b}_{\mathbb{C}}$ the $\mathbb{Z}$- module $Q/(1-\theta_0)Q$ is free. This means that $\bar{T}^{\theta_0}$ is a connected torus and (iii) holds. Bourbaki, *loc. cit.*, gives formulas which allow to express the coroots for $\mathfrak{g}_{\mathbb{C}}^0$ and $\mathfrak{t}_{\mathbb{C}}$ in terms of coroots for $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{h}_{\mathbb{C}}$. From this (iv) follows easily.

Q.E.D.

Remark. — Assume that $\mathfrak{g}$ is absolutely simple. Then Bourbaki's exercise gives the type of $\mathfrak{g}_{\mathbb{C}}^0$ as follows. If $\mathfrak{g}_{\mathbb{C}}$ is at type A_{2k}, A_{2k-1}, D_{k+1}, E_6 then $\mathfrak{g}_{\mathbb{C}}^0$ is of type D_k, C_k, B_k, F_4.

3.3.3. Lemma. — *Let ρ be half the sum of positive roots in $\mathfrak{b}_{\mathbb{C}}$ and denote by ρ^0 half the sums of positive root of $\mathfrak{t}_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}^0 \cap \mathfrak{b}_{\mathbb{C}}$. Then $\rho_N := \rho - \rho^0$ is an integral dominant weight of $\mathfrak{t}_{\mathbb{C}}$.*

Proof. — Our claim is obvious if $\mathfrak{g}$ is a complex simple Lie algebra. If $\mathfrak{g}$ is absolutely simple our claim is easily checked using the explicit description of coroots given in Bourbaki's exercise.

Q.E.D.

3.3.4. Proposition. — *We use the notation introduced above and denote the irreducible representation of $\mathfrak{g}_{\mathbb{C}}^0$ with extremal weight ρ_N resp. $\lambda+\rho_N$ by U_{ρ_N} resp. $U_{\lambda+\rho_N}$. Then θ acts on U_{ρ_N} and $U_{\lambda+\rho_N}$, we have $\operatorname{tr}(\theta \mid U_{\rho_N}) \neq 0$ and*

$$\operatorname{tr}(\theta \mid V) = \frac{\operatorname{tr}(\theta \mid U_{\lambda+\rho_N})}{\operatorname{tr}(\theta \mid U_{\rho_N})}.$$

Proof. — We recall that $\theta = \operatorname{int}(t_0)\theta_0$ acts on $\mathfrak{g}_{\mathbb{C}}^0 = \mathfrak{g}_{\mathbb{C}}^{\theta_0}$ as $\operatorname{int}(t_0)$. Since $\bar{T}_{\theta_0}$ is connected see 3.3.2 (iii) we can find a $\tau_0 \in \mathfrak{A}_{\mathbb{C}}$ such that $\exp 2\pi i \tau_0 = t_0 \in \bar{T}^{\theta_0}$. To prove that $\operatorname{tr}(\theta \mid U_{\rho_N}) \neq 0$ we apply 3.2.4 for $\lambda = \operatorname{res} \rho - \rho^0 = \rho_N$. We have to consider $\langle w\lambda - \lambda, \tau_0 \rangle$ for $w \in W^{\theta_0}$, $W = W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$.

Here we use 3.3.2 (ii). Now

$$w\lambda - \lambda = w \operatorname{res} \rho - w\rho^0 - \operatorname{res} \rho + \rho^0 = -\sum_{\substack{\alpha \in \Delta^+ \\ w(\alpha) < 0}} \operatorname{res} \alpha + \sum_{\substack{\bar{\alpha} \in \bar{\Delta}^+ \\ w(\bar{\alpha}) < 0}} \bar{\alpha}.$$

Here Δ^+ resp. $\bar{\Delta}^+$ is the system of positive roots determined by $\mathfrak{b}_{\mathbb{C}}$ resp. $\mathfrak{b}_{\mathbb{C}} \cap \mathfrak{g}_{\mathbb{C}}^0$. Since $\mathfrak{g}_{\mathbb{C}}^0 = \mathfrak{t}_{\mathbb{C}} + \sum_{\alpha \in \Delta} (\mathfrak{g}_\alpha + \mathfrak{g}_{\theta\alpha}) \cap \mathfrak{g}_{\mathbb{C}}^0$ all roots in $\bar{\Delta}^+$ occur as restrictions of roots in Δ^+. Therefore the above sum is orthogonal to τ_0 and using 3.2.4 we have $\operatorname{tr}(\theta \mid U_{\rho_N}) \neq 0$.

We choose $\mathfrak{h} \in \mathfrak{t}_{\mathbb{C}}$ such that $\tau_0 + h$ is regular in $\mathfrak{h}_{\mathbb{C}}$. Then $\theta_h := \exp(2\pi i \operatorname{ad} \tau_0 + h)$ acts without fixpoints in the unipotent radical of $\mathfrak{b}_{\mathbb{C}}$, *i.e.* θ_h is regular in the sense of Kostant [K], p. 377.

Then by Kostant's character formula

$$\operatorname{tr}(\theta \mid V) = \lim_{h \to 0} \operatorname{tr}(\theta_h \mid V) = \lim_{h \to 0} \exp 2\pi i \langle -\lambda; \tau_0 \rangle \frac{\sum_{w \in W^0} \operatorname{set} w \exp 2\pi i \langle w(\lambda+\rho), \tau_0 + h \rangle}{\sum_{w \in W^0} \det w \exp 2\pi i \langle w\rho, \tau_0 + h \rangle}.$$

Here we abbreviate $W^0 = W^{\theta_0}$.

We now use Weyl's character formula for the occuring ~~nominator and denominator~~, apply 3.2.3 and our claim follows if we take in account our normalization of θ-actions on representations, see 3.1.0 (iii).

Q.E.D.

3.3.5. COROLLARY. — *If the highest weight* λ *of* V *is in* P_θ, *then* $\operatorname{tr}(\theta \mid V) > 0$.

Proof. — By definition a non-compact root α of Δ is exactly a root with non-trivial restriction res α to $t_{\mathbb{C}}$ such that res α is non compact with respect to θ in g^{θ_0}. Hence we can apply 3.2.5. We get $\operatorname{tr}(\theta \mid U_{\lambda+\rho_N}) \neq 0$ and our claim follow from 3.3.4.

Q.E.D

4. Main result and applications

This paragraph contains our main result: Theorem 4.1 which says that $L(\theta, \Gamma, V)$ does not vanish in general. In 4.7 we compute $L(\theta, \Gamma, V)$ in terms of multiplicites and $(\mathfrak{g}, \mathfrak{k})$-cohomology of irreducible representations. Here Γ is assumed to be cocompact. We apply these results in 4.9 and obtain non-vanishing for multiplicities of unitary representations of a group G with $\operatorname{rank}(K) \neq \operatorname{rank}(G)$. These results are almost as satisfactory as the corresponding ones for discrete series representations if $\operatorname{rank}(K) = \operatorname{rank}(G)$. In particular here the case of complex groups is covered.

We combine 1.6, 2.10 and 2.11 and get as a main result:

4.1. THEOREM. — *Let* G *be a connected semi-simple Lie group with Cartan involution* θ. *Let* V *be an irreducible finite dimensional representation with* θ*-action having a highest weight in* P_θ (*definition* 3.1.1). *If* Γ *is small enough* (*definition* 2.8) *then*

$$L(\theta, \Gamma, V) = \chi((K \backslash G/\Gamma)^\theta) \operatorname{tr}(\theta \mid V) > 0.$$

Next we give an analog of Matsushima's formula, *i. e.* we express $L(\theta, \Gamma, V)$ in terms of multiplicities and $(\mathfrak{g}, \mathfrak{k})$-cohomology of unitary irreducible representations.

4.2 (i). Let V be as above with θ-action given by a linear map $C_V: V \to V$. The letter π always denotes an unitary irreducible representation of G. Let

$$H^*(\mathfrak{g}, \mathfrak{k}, \pi \otimes V) \text{ be the relative } (\mathfrak{g}, \mathfrak{k})\text{-cohomology}$$

of the representation $\pi \otimes V$ of G. Here and in the following we do not distinguish in our notation a representation, the space on which the representation acts, the representation on the underlying set of C^∞-vectors or the induced infinitesimal representation on the underlying $(\mathfrak{g}, \mathfrak{k})$-module.

(ii) If π is unitary and irreducible we denote by ${}^\theta\pi$ the representation space of π together with a new action, where $g \in G$ acts as $\theta(g)$. We write $\pi \sim \sigma$ for unitary equivalence of representations. Then $\pi \sim {}^\theta\pi$ if and only if π admits a θ-action, *see* 3.1.

We write $C_\pi: \pi \to {}^\theta\pi$ for the equivalence. We observe that C_π is unique up to sign since π is irreducible. If convenient we identify π with its unitary equivalence class $\pi \in \hat{G}$. In particular we then write $\pi = {}^\theta\pi$ instead of $\pi \sim {}^\theta\pi$.

(iii) We recall the definition of a θ-action on $H^{\bullet}(\mathfrak{g}, \mathfrak{k}, \pi\otimes V)$, *see* [R-S 3: 2.2], assuming that $\pi={}^{\theta}\pi$. Let $C_{\pi}=\pi\to{}^{\theta}\pi$ be as in (ii) and $C_V: V\to{}^{\theta}V$ as in (i). Then $C:=C_{\pi}\otimes C_V$ is an equivalence $\pi\otimes V\to{}^{\theta}(\pi\otimes V)$. Now $H^{\bullet}(\mathfrak{g}, \mathfrak{k}, \pi\otimes V)$ is computed from a complex $\mathrm{Hom}_{\mathfrak{k}}(\Lambda^{*}\mathfrak{p}, \pi\otimes V)$, see [B-W:], and we define a homomorphism $\theta^{\bullet}$ of this complex by

$$(\theta^{i} w)(X_{j_1}\wedge \quad \wedge X_{j_1}):=C^{-1}(w(\theta(X_{j_1})\wedge \quad \wedge \theta(X_{j_i})))$$

where $X_{j_1}\wedge \quad \wedge X_{j_i}\in\Lambda^{i}\mathfrak{p}$, θ acts a usual on $\mathfrak{p}$, *i. e.* $\theta(X_{j_s})=-X_{j_s}$ and

$$w\in\mathrm{Hom}_{\mathfrak{k}}(\Lambda^{i}\mathfrak{p}, \pi\otimes V).$$

The map induced by θ^{i} in $(\mathfrak{g}, \mathfrak{k})$-cohomology again is denoted by θ^{i} and the trace of θ^{i} on cohomology is denoted by $\operatorname{tr}\theta^{i}$.

We define

$$L(\theta, \pi\otimes V)=\begin{cases}\sum_{i=0}^{\infty}(-1)^{i}\operatorname{tr}\theta^{i} & \text{if } \pi={}^{\theta}\pi\\ 0 & \text{if } \pi\neq{}^{\theta}\pi.\end{cases}$$

Since our equivalences C_{π} and C_V are unique up to sign $|L(\theta, \pi\otimes V)|\in\mathbb{N}$ is well defined.

4.3. PROPOSITION. — *For all $\pi\in\hat{G}$ we have*

$$|L(\theta, \pi\otimes V)| = \sum_{i=0}^{\infty}\dim H^{i}(\mathfrak{g}, \mathfrak{k}, \pi\otimes V)=:\dim H^{\bullet}(\mathfrak{g}, \mathfrak{k}, \pi\otimes V).$$

Proof. — There is nothing to prove if $H^{\bullet}(\mathfrak{g}, \mathfrak{k}, \pi\otimes V)=0$. So let us assume that $H^{\bullet}(\mathfrak{g}, \mathfrak{k}, \pi\otimes V)\neq 0$. Thanks to Vogan and Zuckerman [V-Z], 5.3, then $\pi\sim A_{\mathfrak{q}}(\lambda)$, $\pi\sim{}^{\theta}\pi$, where $\mathfrak{q}\subset\mathfrak{g}_{\mathbb{R}}$ is a certain θ-stable parabolic subalgebra containing a θ-stable Cartan algebra $\mathfrak{h}_{\mathbb{C}}$ and where λ is a certain character of the Levi component of $\mathfrak{q}$ with $\lambda\,|\,\mathfrak{h}_{\mathbb{C}}=-\gamma$. Here γ is the lowest weight of V. Since $H^{\bullet}(\mathfrak{g}, \mathfrak{k}, \pi\otimes V)\neq 0$ we have

$$H^{\bullet}(\mathfrak{g}, \mathfrak{k}, A_{\mathfrak{q}}(\lambda)\otimes V)=\mathrm{Hom}_{\mathfrak{k}}(\Lambda^{\bullet}\mathfrak{p}, A_{\mathfrak{q}}(\lambda)\otimes V),$$

see [B-W], I.3.1.

Now in $A_{\mathfrak{q}}(\lambda)$ there is a lowest $\mathfrak{k}$-type $\mu(\mathfrak{q}, \lambda)$ which occurs exactly once, *see* [V-Z], 6.1, and as in [V-Z], 3.7, there is an isomorphism induced by the inclusion $\mu(\mathfrak{q}, \lambda)\subset A_{\mathfrak{q}}(\lambda)$

$$\mathrm{Hom}_{\mathfrak{k}}(\Lambda^{\bullet}\mathfrak{p}, A_{\mathfrak{q}}(\lambda)\otimes V)\xrightarrow{\sim}\mathrm{Hom}_{\mathfrak{k}}(\Lambda^{\bullet}\mathfrak{p}, \mu(\mathfrak{q}, \lambda)\otimes\mathbb{C})$$

Here $\otimes\,\mathbb{C}$ picks up the lowest weight space of V.

But our $C_{\pi}: A_{\mathfrak{q}}(\lambda)\to{}^{\theta}A_{\mathfrak{q}}(\lambda)$ has to preserve $\mu(\mathfrak{q}, \lambda)$ and acts up to sign as the identity on $\mu(\mathfrak{q}, \lambda)$. The map C_V acts identically on the lowest weight space of V. Hence our claim holds.

Q. E. D.

Remark. — If $\theta = x\theta' x^{-1}$, $x \in K$ then $|L(\theta, \pi \otimes V)| = |L(\theta', \pi \otimes V)|$. In particular if $\operatorname{rank} K = \operatorname{rank} G$ then θ is inner and $L(\theta, \pi \otimes V) = \chi(\pi \otimes V)$ is the Euler-Poincaré characteristic of $H^{\cdot}(\mathfrak{g}, \mathfrak{k}, \pi \otimes V)$. Of course here we choose $C_\pi = \mathrm{Id}$ and $C_V = \mathrm{Id}$.

4.4. (i) Let $L^2_{\text{cusp}}(G/\Gamma)$ be the space of cuspidal square integrable functions on G/Γ, *see* [H-Ch]. We consider $L^2_{\text{cusp}}(G/\Gamma)$ as a left G-module and unitary representation of G. If $\pi \in \hat{G}$, then it is well known that the multiplicity

$$m(\pi, \Gamma) = \dim \operatorname{Hom}_G(\pi, L^2_{\text{cusp}}(G/\Gamma))$$

is finite and

$$L^2_{\text{cusp}}(G/\Gamma) = \hat{\bigoplus_{\pi \in \hat{G}}} \pi^{m(\pi, \Gamma)}$$

as unitary representation. Here $\hat{\oplus}$ means: completed direct Hilbert sum.

(ii) If $\pi = {}^\theta\pi$ we define an action of θ on $T \in \operatorname{Hom}_G(\pi, L^2_{\text{cusp}}(G/\Gamma))$ by $\theta(T)(a)(g) = T(C_\pi(a))(\theta(g))$, $a \in \pi$, $g \in G/\Gamma$. Here $C_\pi : \pi \to {}^\theta\pi$ is as in 4.1 (iii). The trace of θ acting on $\operatorname{Hom}_G(\pi, L^2_{\text{cusp}}(G/\Gamma))$ is denoted by $m(\theta, \pi, \Gamma)$. We put $m(\theta, \pi, \Gamma) = 0$ if $\pi \neq {}^\theta\pi$.

(iii) We have a natural injective map

$$\tau : \quad \pi \otimes V \otimes \operatorname{Hom}_G(\pi, L^2_{\text{cusp}}(G/\Gamma)) \to L^2_{\text{cusp}}(G/\Gamma) \otimes V$$

sending $a \otimes v \otimes T$ to $T(a) \otimes v$. The image of this map can be identified with the isotypical component of π in $L^2_{\text{cusp}}(G/\Gamma)$ tensored with V. Since θ acts on G/Γ and in a fixed way on V we have an action of θ on $L^2_{\text{cusp}}(G/\Gamma) \otimes V$.

We observe that τ is equivariant if the left sides carries the θ-action which is the product of the actions described above, *see* [R-S 3], 2.3.1.

(iv) We define $H^{\cdot}_{\text{cusp}}(\Gamma, V) := H^{\cdot}(\mathfrak{g}, \mathfrak{k}, L^2_{\text{cusp}}(G/\Gamma) \otimes V)$.

Since θ acts on $L^2_{\text{cusp}}(G/\Gamma) \otimes V)$ there is again a θ-action on $H^{\cdot}_{\text{cusp}}(\Gamma, V)$. The Lefschetz number of this action is denoted by $L_{\text{cusp}}(\theta, \Gamma, V)$. Using (ii) and the argument given in [B-W], Chap. VII, we easily arrive at

4.5. Proposition. — *We use the notation explained above. Then*

$$L_{\text{cusp}}(\theta, \Gamma, V) = \sum_{\pi \in \hat{G}} m(\theta, \pi, \Gamma)\, L(\theta, \pi \otimes V).$$

Observe that there are only finitely many non zero contributions and that only π's with $\pi = {}^\theta\pi$ contribute.

4.6. We want to compare $L_{\text{cusp}}(\theta, \Gamma, V)$ with the topologically defined Lefschetz number $L(\theta, \Gamma, V)$ of paragraph 1. At first we recall that there is a natural θ-equivariant isomorphism

$$H^{\cdot}(\mathfrak{g}, \mathfrak{k}, C^\infty(G/\Gamma) \otimes V) \xrightarrow{\sim} H^{\cdot}(K \backslash G/\Gamma, V) = H^{\cdot}(\Gamma, V)$$

see [B-W], VII, § 2, and [R-S 3], 2.1.1. Let ${}^{\infty}L^2_{\text{cusp}}(G/\Gamma)=L^2_{\text{cusp}}(G/\Gamma)\cap C^{\infty}(G/\Gamma)$. Then the inclusion ${}^{\infty}L^2_{\text{cusp}}(G/\Gamma)\subsetneq C^{\infty}(G/\Gamma)$ induces a θ-equivariant map $H^{\cdot}_{\text{cusp}}(\Gamma, V)\to H^{\cdot}(\Gamma, V)$. Borel [B 3] shows that the last map is injective. If we define $L_{\text{Eis}}(\theta, \Gamma, V)$ in the obvious way on a θ-stable complement of $H^{\cdot}_{\text{cusp}}(\Gamma, V)$ in $H^{\cdot}(\Gamma, V)$, we get a formula

$$L(\theta, \Gamma, V)=L_{\text{cusp}}(\theta, \Gamma, V)+L_{\text{Eis}}(\theta, \Gamma, V).$$

Little is known about $L_{\text{Eis}}(\theta, \Gamma, V)$, *see* however [R-S 3], 2.4.1, [R 2] and [Le-S]. We will pursue this further elsewhere. If G/Γ is compact then $L^2_{\text{cusp}}(G/\Gamma)=L^2(G/\Gamma)$ and $L(\theta, \Gamma, V)=L_{\text{cusp}}(\theta, \Gamma, V)$. We will give applications of this identity in 4.8.

We now choose C_π, *see* 4.2 (iii), such that $L(\theta, \pi\otimes V)=\dim H^{\cdot}(\mathfrak{g}, \mathfrak{k}, \pi\otimes V)$. Then combining 4.1, 4.3, 4.4, 4.5 and 4.6 we obtain.

4.7. Proposition. — *In the setting of* 4.1 *let* G/Γ *be compact. Then*

$$L(\theta, \Gamma, V)=\sum_{\pi\in\hat{G}} m(\theta, \pi, \Gamma)\dim H^{\cdot}(\mathfrak{g}, \mathfrak{k}, \pi\otimes V)>0.$$

We have an obvious consequence:

4.8. Corollary. — *In the setting of* 4.7 *there is at least one* $\pi\in\hat{G}$ *with* $\pi={}^{\theta}\pi$ *such that* $m(\pi, \Gamma)>0$.

Of course we now can conclude that π occurs in $L^2(G/\Gamma)$, if there is at most one π such that $L(\theta, \pi\otimes V)\neq\{0\}$. The following result shows that this argument often applies.

4.9. Proposition. — *Assume that* V *is an irreducible representation with a regular highest* θ*-invariant weight.*

(*i*) *If* $G=SL_{2n+1}(\mathbb{R})$, $SO(2n+1, 1)(\mathbb{R})^0$ $n\geqq 1$; $SL_n(\mathbb{H})$, $n\geqq 2$, *or if* G *is a complex Lie group there is up to equivalence exactly one irreducible unitary representation* π *so that* $H^{\cdot}(\mathfrak{g}, \mathfrak{k}, \pi\otimes V)\neq\{0\}$.

(ii) *If* $G=SL_{2n}(\mathbb{R})$, $n\geqq 1$, *there are exactly two representations which have non trivial Lie algebra cohomology with* V*-twisted coefficients.*

Proof. — Our claims on $SL_n(\mathbb{R})$, $n\geqq 2$, are proved in [Sp]. For $G=SO(2n+1, 1)(\mathbb{R})^0$, *see* [R-S 3], 1.3.

Assume now that $\mathfrak{g}$ is a complex Lie-algebra. Choose $\mathfrak{k}\subset\mathfrak{g}$ a compact real form. Then $\mathfrak{g}=\mathbb{C}\otimes_{\mathbb{R}}\mathfrak{k}$ and conjugation with respect to $\mathfrak{k}$ is a Cartan involution θ on $\mathfrak{g}$. We have a natural isomorphism

$$\mathfrak{g}_{\mathbb{C}}=\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}=\mathbb{C}\otimes_{\mathbb{R}}\mathfrak{k}\otimes_{\mathbb{R}}\mathbb{C}\xrightarrow{\sim}\mathfrak{g}\times\mathfrak{g}$$

mapping $z_1\otimes X\otimes z_2$ to $(z_1 z_2\times X;\ \bar{z}_1 z_2\otimes X)$, $z_i\in\mathbb{C}$, $X\in\mathfrak{k}$. Then $\mathfrak{g}\cong\mathfrak{g}\otimes 1$ is identified with $\{(X, \theta X)/X\in\mathfrak{g}\}$ and the complex extension of θ acts on $\mathfrak{g}\oplus\mathfrak{g}$ mapping (x_1, x_2) to (x_2, x_1). We choose $\mathfrak{t}$, a Cartan subalgebra in $\mathfrak{k}$. Then $\mathfrak{h}=\mathfrak{t}\otimes\mathbb{C}=\mathfrak{t}+i\,\mathfrak{t}\subset\mathfrak{g}$ is a Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{h} \xrightarrow{\sim} \{(X, \theta X)/X \in \mathfrak{h}\}$ as subalgebra of $\mathfrak{g} \times \mathfrak{g}$. We put $\mathfrak{a} = i\mathfrak{t}$ and identify $\mathfrak{a}$ with $\{(x, -x)/x \in \mathfrak{a}\}$. Since $\theta\mathfrak{h} = \mathfrak{h}$ we can identify $\mathfrak{h}_{\mathbb{C}}$ with $\mathfrak{h} \times \mathfrak{h}$ in $\mathfrak{g} \times \mathfrak{g}$. Choose a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ containing $\mathfrak{h}$. Then $\mathfrak{b} \times \mathfrak{b}$ is a Borel subalgebra in $\mathfrak{g} \times \mathfrak{g}$ with corresponding system of positive roots $\Delta^+ \cup \Delta^+$ if Δ^+ is the positive system determined by $\mathfrak{b}$ in $\mathfrak{g}$. We have $W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) \xrightarrow{\sim} W(\mathfrak{g}, \mathfrak{h}) \times W(\mathfrak{g}, \mathfrak{h})$ and

$$W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})^{\theta} \xrightarrow{\sim} W(\mathfrak{g}, \mathfrak{h}).$$

If now V is an irreducible representation of $\mathfrak{g}_{\mathbb{C}}$ with highest weight $(\lambda, \mu) \in \mathfrak{h}_{\mathbb{C}}^*$ then $V = V_\lambda \otimes V_\mu$ where V_μ resp. V_λ are irreducible of $\mathfrak{g}$ with highest weight μ resp. λ. Assume that θ acts on V. Then $\mu = \lambda$. We denote also by (λ, λ) the character induced on $\mathfrak{h}$ by (λ, λ). Then (λ, λ) is trivial on $i\mathfrak{t}$. We denote the unitarily induced representation determined by $\mathfrak{b}$ and (λ, λ) on G by $X(\lambda, \lambda)$.

We observe that $X(\lambda, \lambda)$ has the same infinitesimal character as V and that

$$H^{\cdot}(\mathfrak{g}, \mathfrak{k}, X(-\lambda - \lambda) \otimes V) \cong \Lambda^{\cdot}\mathfrak{t} \otimes \mathbb{C}$$

see [B-W], III, § 3.

By a result of Zelobenko [Z] all other unitarily induced principal series representation with the same infinitesimal character are unitarily equivalent to $X(-\lambda, -\lambda)$. But according to [V-Z], 6.2, 6.3, since (λ, λ) is assumed to be regular, all irreducible π's with $H^{\cdot}(\mathfrak{g}, \mathfrak{k}, \pi \otimes W) \neq \{0\}$ are unitarily induced principal series representations. Hence our claim holds.

Now suppose $G = SL_n(\mathbb{H})$ where $\mathbb{H}$ means quaternions. For $g \in G$ put $\theta g = {}^t\bar{g}^{-1}$. Here the bar denotes the canonical antiinvolution on $\mathbb{H}$. We have a maximal θ-stable torus

$$H = \left\{ (z_1, \overline{z_1}, z_2, \overline{z_2}, \ldots, z_n, \overline{z_n})/z_i \in \mathbb{C}^*, \prod_{i=1}^{n} |z_i| = 1 \right\}$$

diagonally in G. The natural inclusions $S^1 = \{z \in \mathbb{C}^*/|z| = 1\} \subset \mathbb{C}^*$ and $\mathbb{R}^* \subset \mathbb{C}^*$ give a decomposition $H = TA$ where $T = (S^1)^n$ and $A = (\mathbb{R}_+^*)^n$. The upper triangular matrices in G make up a parablic subgroup P with Levipart $SU(2)^n$ sitting blockwise on the diagonal.

Let $\mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition corresponding to θ. Then a θ-invariant weight λ in $\mathfrak{h}_{\mathbb{C}}$ gives by restriction a weight on $\mathfrak{t}_{\mathbb{C}}$ which we can write $\lambda = (\lambda_1, \ldots, \lambda_n)$ in correspondence to the decomposition $T = (S^1)^n$. The Weyl group $W(A) = N_K(A)/Z_K(A)$ of A, $K = Sp_n$, acts as the group S_n of permutations of the factors of T. This is easily seen by inspection. Denote by $\pi_{\lambda i}$ the irreducible representation of SU(2) with highest weight λ_i. Then according to [E] the unitarily induced representation

$$I(\lambda_1, \ldots, \lambda_n) = \mathrm{Ind}_P^G(\pi_{\lambda_1} \otimes \ldots \otimes \pi_{\lambda_n} \otimes 1)$$

is irreducible and if V′ is the contragredient representation of the irreducible representation V of G with highest weight λ then $H^{\cdot}(\mathfrak{g}, \mathfrak{k}, I(\lambda_1, \ldots, \lambda_n) \otimes V') \cong \Lambda^{\cdot} \mathfrak{a} \otimes \mathbb{C} \neq \{0\}$.

Now G has up to conjugacy just one θ-stable Cartan subgroup. Since λ is regular every unitary π with $H^{\cdot}(\mathfrak{g}, \mathfrak{k}, \pi \otimes V'_\lambda) \neq \{0\}$ has to be of the form $I(\mu_1, \ldots, \mu_n)$. Here we use [V-Z], 6.2, 6.3. Since then $I(\mu_1, \ldots, \mu_n)$ and V′ have to have the same infinitesimal character, $(\mu_1, \ldots, \mu_n)$ is a permutation of $(\lambda_1, \ldots, \lambda_n)$. But since this permutation can be realized by an element in W(A) we get from [K-Z], Thm. 14.2 that $I(\mu_1, \ldots, \mu_n)$ and $I(\lambda_1, \ldots, \lambda_n)$ are unitarily equivalent. Hence our claim holds.

Q.E.D.

Remark. — The other classical groups which are not of equal rank and have not been dealt with in 5.8 are of the form $SO(p, q)$ where p and q are odd. It is known that for $SO(p, q)$, $p \geqq q > 1$, there is more than one class of representations with non trivial cohomology.

Observe that the two representations which occur for $SL_{2n}(\mathbb{R})$ in 4.9 are constituents of one irreducible representation of $SL_{2n}(\mathbb{R})_+ = \{A \in GL_n(\mathbb{R}) / \det A = \pm 1\}$; *see* [Sp]. Since

$$SO(2n) \diagdown SL_{2n(\mathbb{R})} \xrightarrow{\sim} O(2n) \diagdown SL_{2n}(\mathbb{R})_+$$

we get that if one of these representations contributes to $H^{\cdot}(\mathfrak{g}, \mathfrak{k}, L^2(G/\Gamma) \otimes V)$ then both do and with the same multiplicity. Here we assume that the diagonal element $(1, \ldots, 1, -1)$ of $SL_n(\mathbb{R})_+$ normalizes Γ.

4.10. Proposition. — *Assume that* G, V *and* Γ *are as in* 4.7 *and denote by* π *one of the representations described in* 4.8. *Then*

(i) $m(\pi, \Gamma) > 0$.

(ii) *If* Γ_1 *is* θ*-stable, normal in* Γ *and arithmetic then there is a constant* $a(\Gamma) > 0$ *depending on* Γ *such that* $m(\pi, \Gamma_1) \geqq |(\Gamma/\Gamma_1)^\theta| \, a(\Gamma)$.

Proof. — The first claim follows from 4.7 and 4.8. To prove the second claim observe first that the Cartan involution for G as described in 4.8 actually is "the" outer automorphism. Hence 3.1.2 applies and $\mathrm{tr}(\theta | V) \neq 0$. Then the second claim follows using 2.12.

Q.E.D.

Remark. — (i) If Γ is chosen with more care, we even have $m(\pi, \Gamma_1) \geqq |(\Gamma/\Gamma_1)^\theta| \, m(\pi, \Gamma)$.

(ii) We do not know whether $|(\Gamma/\Gamma_1)^\theta|$ gives the right order of growth for $m(\pi, \Gamma_1)$, *see* 2.12: remark.

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Added in proof: The proof of Prop. 4.3.2 in [R-S 3] contains an error concerning orientations. The proof can be corrected as follows: delete from the last five sentences of the proof the first four. After the last sentence add: this equation implies that $E(\varphi, 0)=0$ iff $\varphi=0$ (in the notation of 4.3.1). Hence the second claim follows as in 4.3.1.