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ON THE DEFORMATION OF ARTIN-SCHREIER TO KUMMER

BY T. SEKIGUCHI (*), F. OORT AND N. SUWA

Let k be an algebraically closed field of characteristic $p>0$. We denote by $W(k)$ the Witt vector ring of k. Let C be a (complete, non-singular) curve of genus g over k, and let G be a subgroup of the automorphism group $\mathrm{Aut}_k(C)$ of C. Our aim is to treat the following problem:

(I) *Lift a given pair* (C, G) *to a pair* ($\mathscr{C}$, G) *of a smooth proper curve* $\mathscr{C}$ *and a subgroup* $G \subset \mathrm{Aut}(\mathscr{C})$ *over a suitable discrete valuation ring* A *dominating* W(k).

This problem is equivalent to the following problem:

(II) *Let* C/D *be a Galois covering of curves over k with Galois group* G. *Then lift* C/D *to a Galois covering* $\mathscr{C}/\mathscr{D}$ *over a suitable discrete valuation ring* A *dominating* W(k).

If C/D is unramified, it is well known that C/D has a lifting over W(k) (*cf.* SGA1, Exp. X, Th. 2.1). Moreover, if C/D is tamely ramified, Laudal and Lønsted [8] show that C/D has a lifting over W(k). On the contrary, if C/D is wildly ramified, the answer to our problem is generally negative. For example, if we take as G the full automorphism group $\mathrm{Aut}_k(C)$, then there exists a curve C with $\sharp G > 84(g-1)$ (*cf.* [19], Satz 1, [22], Th. 3.3, 3.3′, or [23], Teil II, Satz 5,6). But in char. 0, the order of the automorphism group of a curve of genus g is at most $84(g-1)$ (as Hurwitz proved).

In this paper, we devote ourselves to the study of the problem in the case of $G = \langle \sigma \rangle$ with ord $\sigma = pm$ and $(p, m) = 1$. Our result is as follows:

Let C *be a (complete non-singular) curve over k, and* σ *be an automorphism of* C *of order pm with* $(p, m)=1$. *Then there exists a lifting* ($\mathscr{C}$, σ) *of* (C, σ) *over* W$(k)[\xi]$ *where* ξ *is a primitive p-th root of unity* (cf. Ch. IV, Th. 2.6).

To attack our problem, we have two methods generally. One is to pile up the infinitesimal arguments (*cf.* Grothendieck, FGA and SGA 1, Exp. III). Unfortunately, there is an obstruction to succeed in solving the lifting problems by this method in the case of characteristic $p>0$. In fact, we have examples of (C, σ) with ord $\sigma = p$ which cannot be lifted over W(k) (*cf.* Oort-Sekiguchi [17], Lemma 2.3, Nakajima [12]). This leads us another method: class field theory (*cf.* Serre [21]), namely to look at our problem from the view point (II). The main tool is the exact sequence

$$0 \to \mathbb{Z}/p \to \mathscr{G}^{(\lambda)} \to \mathscr{G}^{(\lambda^p)} \to 0,$$

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which combines the Artin-Schreier sequence

$$0 \to \mathbb{Z}/p \to \mathbb{G}_a \xrightarrow{\wp} \mathbb{G}_a \to 0,$$

where $\wp x = x^p - x$ for $x \in \mathbb{G}_a$, and the Kummer sequence

$$0 \to \boldsymbol{\mu}_p \to \mathcal{G}_m \xrightarrow{p} \mathbb{G}_m \to 0.$$

(For definition, *see* Ch. I).

Now we explain the organization of this paper.

In Ch. I, we will construct the deformations of $\mathbb{G}_a$ to $\mathbb{G}_m$ and discuss the connection between the theories of Artin-Schreier and of Kummer.

In Ch. II, we calculate the cohomology groups with coefficients in certain group schemes. In the last section, we arrange, following the argument expanded by Breen [3], one of his results restricting ourselves to our case.

In Ch. III, we will construct singular curves over a discrete valuation ring, following Serre's argument, and analyze their generalized Jacobian schemes.

After the preparation mentioned above, in Ch. IV, we give a proof of the main theorem. First we treat tamely ramified Galois coverings of curves by our method, because it would give a visual explanation how to treat the ramification points in our deformation. Next we treat Galois coverings of degree p, which is our main subject.

This article is a souped up version of "On the deformation of Artin-Schreier to Kummer" published as Preprint Nr. 369, the University of Utrecht, 1985.

Tatsuji Kanbayashi and Ryuji Sasaki pointed out the existence of articles [7], [24] and [25], [26] respectively. The authors would like to express their thanks to them. The first author would like to express his hearty thanks to the University of Utrecht for hospitality and excellent working conditions.

Notations

In the first three chapters, A denotes a discrete valuation ring and $\mathfrak{M}$ (resp. K, k) denotes the maximal ideal (resp. the fraction field, resp. residue field) of A, if there is no restrictions. We denote by v the valuation on A. We put $S = \operatorname{Spec} A$, and we denote by η (resp. s) the generic (resp. closed) point of S.

X_{et} (resp. X_{fppf}) denotes the etale site (resp. *fppf*-site) of X.

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I. Deformations $\mathcal{G}^{(\lambda)}$ of $\mathbb{G}^a$ to $\mathbb{G}_m$

1. Definition

Definition 1.1. — For any $\lambda \in \mathfrak{M} \setminus \{0\}$, we define an affine flat commutative group S-scheme $\mathcal{G}^{(\lambda)}$ in the following way:

(1.1.1) $$\mathcal{G}^{(\lambda)} = \mathrm{Spec}(\mathrm{A}[x, 1/(\lambda x + 1)])$$

with

1. *Law of multiplication:*

(1.1.2) $$\begin{cases} m: & \mathcal{G}^{(\lambda)} \times_{\mathrm{S}} \mathcal{G}^{(\lambda)} \to \mathcal{G}^{(\lambda)} \\ & (x, y) \mapsto \lambda xy + x + y \end{cases};$$

i. e.,

(1.1.3) $$\begin{cases} m^*: & \mathrm{A}[x, 1/(\lambda x + 1)] \otimes_{\mathrm{A}} \mathrm{A}[x, 1/(\lambda x + 1)] \leftarrow \mathrm{A}[x, 1/(\lambda x + 1)] \\ & \lambda(x \otimes x) + x \otimes 1 + 1 \otimes x \leftarrow\!\shortmid x \end{cases};$$

2. *Law of inverse:*

(1.1.4) $$\begin{cases} \iota: & \mathcal{G}^{(\lambda)} \to \mathcal{G}^{(\lambda)} \\ x \mapsto & -x/(\lambda x + 1) \end{cases};$$

i. e.,

(1.1.5) $$\begin{cases} \iota^*: & \mathrm{A}[x, 1/(\lambda x + 1)] \leftarrow \mathrm{A}[x, 1/(\lambda x + 1)] \\ & -x/(\lambda x + 1) \leftarrow\!\shortmid x \end{cases};$$

3. *Law of identity:*

(1.1.6) $$\begin{gathered} e: \ \mathrm{S} \to \mathcal{G}^{(\lambda)}: \text{the morphism defined by} \\ e^*: \ \mathrm{A} \leftarrow \mathrm{A}[x, 1/(\lambda x + 1)]. \\ 0 \leftarrow\!\shortmid x \end{gathered}$$

1.2. We define an S-homomorphism $\alpha^{(\lambda)}: \mathcal{G}^{(\lambda)} \to \mathbb{G}_{m,\,A}$ by

$$(1.2.1) \qquad \begin{cases} \alpha^{(\lambda)*}: & A[x,\ 1/(\lambda x+1)] \leftarrow A[u,\ 1/u] \\ & \lambda x + 1 \leftarrow\!\shortmid u \end{cases}.$$

Obviously the generic fibre of $\alpha^{(\lambda)}$:

$$(1.2.2) \qquad \alpha_\eta^{(\lambda)}: \mathcal{G}_\eta^{(\lambda)} = \operatorname{Spec} K[x,\ 1/(\lambda x+1)] \to \mathbb{G}_{m,\,K}$$

is an isomorphism. On the other hand, the special fibre of $\mathcal{G}^{(\lambda)}$ is clearly isomorphic to $\mathbb{G}_{a,\,k}$:

$$(1.2.3) \qquad \mathcal{G}_s^{(\lambda)} = \operatorname{Spec}(A[x,\ 1/(\lambda x+1)] \otimes_A k) \simeq \operatorname{Spec} k[x] \simeq \mathbb{G}_{a,\,k}.$$

That is, the group scheme $\mathcal{G}^{(\lambda)}$ gives a deformation of $\mathbb{G}_a$ to $\mathbb{G}_m$ over $S = \operatorname{Spec} A$.

1.3. Let λ, μ be two elements of $\mathfrak{M}\setminus\{0\}$. Then we get the canonical injection

$$(1.3.1) \qquad h: \ \operatorname{Hom}_{A-gr}(\mathcal{G}^{(\lambda)}, \mathcal{G}^{(\mu)}) \hookrightarrow \operatorname{Hom}_{K-gr}(\mathbb{G}_{m,\,K}, \mathbb{G}_{m,\,K}) \simeq \mathbb{Z}$$

defined by $h(\varphi) = \alpha_\eta^{(\mu)} \circ \varphi_\eta \circ {\alpha_\eta^{(\lambda)}}^{-1}$ for any $\varphi \in \operatorname{Hom}_{A-gr}(\mathcal{G}^{(\lambda)}, \mathcal{G}^{(\mu)})$. The image of h is described in the following way.

PROPOSITION 1.4. — *Let λ, $\mu \in \mathfrak{M}\setminus\{0\}$, and let h be the injection* (1.3.1).

(i) *The equality $h(\varphi) = n$ means that $\varphi(x) = \{(\lambda x+1)^n - 1\}/\mu$.*

(ii) *When we identify* $\operatorname{Hom}_{A-gr}(\mathcal{G}^{(\lambda)}, \mathcal{G}^{(\mu)})$ *with the image of h,*

$$(1.4.1) \qquad \operatorname{Hom}_{A-gr}(\mathcal{G}^{(\lambda)}, \mathcal{G}^{(\mu)}) = \left\{ \pm n \;\middle|\; \begin{matrix} n=0 \text{ or } n \in \mathbb{Z}_{>0} \text{ with} \\ \mu \,\Big|\, \dbinom{n}{i} \lambda^i \text{ for } i=1,\ \ldots,\ n \end{matrix} \right\} \subset \mathbb{Z}.$$

(iii) *If* char. $k=0$, *then*

$$(1.4.2) \qquad \operatorname{Hom}_{A-gr}(\mathcal{G}^{(\lambda)}, \mathcal{G}^{(\mu)}) = \begin{cases} 0 & \text{if } \mu \nmid \lambda \\ \mathbb{Z} & \text{if } \mu \mid \lambda \end{cases}.$$

(iv) *If* char. K = char. $k = p > 0$, *then*

$$(1.4.3) \qquad \operatorname{Hom}_{A-gr}(\mathcal{G}^{(\lambda)}, \mathcal{G}^{(\mu)}) = p^n \mathbb{Z},$$

where n is the smallest non-negative integer such that

$$v(\mu) \leqq p^n v(\lambda).$$

(v) *If* char. K = 0 *and* char. $k = p > 0$, *then*

$$(1.4.4) \qquad \operatorname{Hom}_{A-gr}(\mathcal{G}^{(\lambda)}, \mathcal{G}^{(\mu)}) = p^e \mathbb{Z},$$

where e = the smallest non-negative integer satisfying the inequality

$$v(\mu) \leqq e\,v(p) + v(\lambda)$$

[resp., $v(\mu) \leqq \max(e, e-f+1) v(p) + p^{\min(e, f+1)} . v(\lambda)$], *if*

$$v(p) \leqq (p-1) v(\lambda)$$

[*resp.* $p^f (p-1) v(\lambda) < v(p) \leqq p^{f+1} (p-1) v(\lambda)$ *for some non-negative integer* f].

(vi) *Two group schemes* $\mathscr{G}^{(\lambda)}$ *and* $\mathscr{G}^{(\mu)}$ *are isomorphic if and only if* $v(\lambda) = v(\mu)$.

Proof. — We will check only (v), and the rest remains for the reader.

Let $n \in \mathbb{Z} (\simeq \mathrm{Hom}_{K-gr}(\mathbb{G}_{m,K}, \mathbb{G}_{m,K}))$, and put $n = p^e m$ with $(p, m) = 1$. Then for r with $1 \leqq p^r \leqq n$, we get easily

$$v\left(\binom{n}{ip^r}\right) + ip^r v(\lambda) \leqq v\left(\binom{n}{(i+1)p^r}\right) + (i+1) p^r v(\lambda) \tag{1.4.5}$$

for $i = 1, \ldots, [n/p^r]$ with $p \nmid i+1$. Hence we get the equivalence of

$$v(\mu) \leqq v\left(\binom{n}{i} \lambda^i\right) \quad \text{for} \quad i = 1, \ldots, n \tag{1.4.6}$$

and

$$v(\mu) \leqq v\left(\binom{n}{i} \lambda^i\right) \quad \text{for} \quad 1 \leqq i = p^r \leqq n. \tag{1.4.7}$$

Using this equivalence and the fact that $\mathrm{Hom}_{A-gr}(\mathscr{G}^{(\lambda)}, \mathscr{G}^{(\mu)})$ is a submodule of $\mathbb{Z}$, we can see that (1.4.7) is equivalent to the condition:

$$v(\mu) \leqq v\left(\binom{p^e}{p^i} \lambda^{p^i}\right) \quad \text{for} \quad i = 0, \ldots, e. \tag{1.4.8}$$

On the other hand,

$$v\left(\binom{p^e}{p^i} \lambda^{p^i}\right) - v\left(\binom{p^e}{p^{i+1}} \lambda^{p^{i+1}}\right) = v(p) - p^i (p-1) v(\lambda). \tag{1.4.9}$$

Hence (1.4.8) is equivalent to the inequality

$$v(\mu) \leqq ev(p) + v(\lambda), \tag{1.4.10}$$

if $v(p) \leqq (p-1) v(\lambda)$, and equivalent to the inequality

$$v(\mu) \leqq \max(0, e-f-1) v(p) + p^{\min(e, f+1)} . v(\lambda), \tag{1.4.11}$$

if $p^f (p-1) v(\lambda) < v(p) \leqq p^{f+1} (p-1) v(\lambda)$, respectively. Thus we get our assertion.

Moreover, we have the

THEOREM 1.5 (Watherhouse-Weisfeiler [25], Th. 2.5). — *Let* $\mathscr{G}$ *be a flat* S-*group scheme with* $\mathscr{G}_\eta \simeq \mathbb{G}_{m,K}$ *and* $\mathscr{G}_s \simeq \mathbb{G}_{a,k}$. *Then* $\mathscr{G}$ *is isomorphic to* $\mathscr{G}^{(\lambda)}$ *for some* $\lambda \in \mathfrak{M} \setminus \{0\}$.

DEFINITION 1.6. — By this theorem and Proposition 1.5, (vi), a deformation $\mathscr{G}$ of $\mathbb{G}_a$ to $\mathbb{G}_m$ over S is determined by λ (unique up to a unit) with $\mathscr{G} \simeq \mathscr{G}^{(\lambda)}$. We call λ [or $v(\lambda)$] the *speed* of the deformation $\mathscr{G} \simeq \mathscr{G}^{(\lambda)}$.

Waterhouse and Weisfeiler gave a proof of theorem 1.5 in [25]. Here we give another elementary proof. For our purpose, we prepare the following lemma without proof.

LEMMA 1.7. — *Let* B *be an integral domain, and let* (t) *be a non-trivial principal ideal in* B. *Then* B *is a unique factorization domain if and only if* $B[1/t]$ *is so. Moreover, in this case, if an element* f *of* $B[1/t]$ *is irreducible and* n *is the smallest integer such that* $t^n f \in B$, *then* $t^n f$ *is an irreducible element of* B.

Proof of Theorem 1.7. — Put $B = \Gamma(\mathscr{G}, \mathcal{O}_{\mathscr{G}})$. Then we have an isomorphism $B \otimes_A K \simeq K[u, u^{-1}]$. By the lemma 1.7, B is a unique factorization domain. Since B is smooth over A and A is a discrete valuation ring, B is regular. Let $\mathfrak{a}$ be the augmented ideal of B. Then we have $ht\,\mathfrak{a} + \dim B/\mathfrak{a} = \dim B$. Since we have $\dim B/\mathfrak{a} = \dim A = 1$, we obtain $ht\,\mathfrak{a} = 1$. Since B is a unique factorization domain, $\mathfrak{a}$ is a principal ideal. Put $\mathfrak{a} = tB$. B is isomorphic to $A \oplus \mathfrak{a}$ as A-module. Then $B \otimes_A K$ is isomorphic to $K \oplus (\mathfrak{a} \otimes_A K)$. Consider B as a subring of $K[u, u^{-1}] \simeq B \otimes_A K$. Since $\mathfrak{a} \otimes_A K$ is the augmented ideal, $tB \otimes_A K = (u-1) K[u, u^{-1}] \subset K[u, u^{-1}]$. Then there exists $a \in K$ such that $t = a(u-1)$. Since $\iota^*(\mathfrak{a}) = \mathfrak{a}$, u and u^{-1} are units of B. In fact, $\iota^*(t) = a(u^{-1}-1)$, and therefore, $a(u^{-1}-1)/a(u-1) = (u^{-1}-1)/(u-1) = -u^{-1}$ is a unit of B.

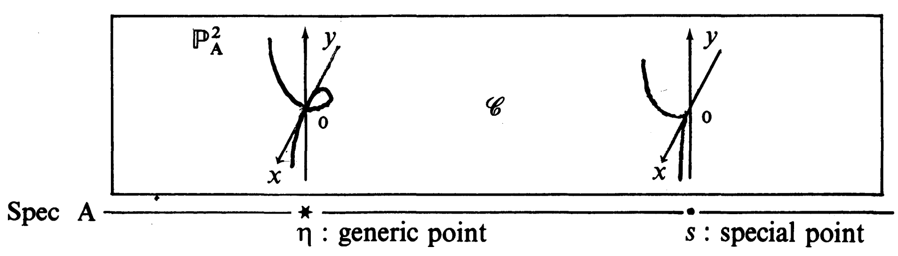
Moreover, since B is faithfully flat over A and $\mathscr{G}_s$ is isomorphic to $\mathbb{G}_a$, $B \otimes_A k$ is isomorphic to the polynomial ring $k[s]$. Since $B \otimes_A k = k \oplus ((tB) \otimes_A k)$, tB is not contained in $\mathfrak{M}B$. Here we see that $a \in K \setminus A$. In fact, if $a \in \mathfrak{M}$, $tB \subset \mathfrak{M}B$. Then $a \in K \setminus \mathfrak{M}$. Moreover, if $a \in A \setminus \mathfrak{M}$, $tB = (u-1)B$. The image of u in $B \otimes_A k$ is a unit of $B \otimes_A k$ and transcendental over k. But $(B \otimes_A k)^\times = k^\times$.

Put $\lambda = a^{-1} \in \mathfrak{M}$. The embedding $A[t, 1/(\lambda t + 1)] \to B$ is a homomorphism of Hopf algebras. Let $\varphi : \mathscr{G} \to \operatorname{Spec} B \to \mathscr{G}^{(\lambda)}$ be the corresponding homomorphism of group S-schemes. Since φ is surjective fiber by fiber, φ is surjective. Since $\mathscr{G}$ is flat over A and $\mathscr{G}^{(\lambda)}$ is smooth over A, φ is flat ([9], Chap. 6, § 2, Lemma 6.12). Then $\operatorname{Ker} \varphi$ is flat over A. Since we have $(\operatorname{Ker} \varphi)_\eta = 0$, we obtain $\operatorname{Ker} \varphi = 0$, that is, the homomorphism $\varphi : \mathscr{G} \to \mathscr{G}^{(\lambda)}$ is an isomorphism and we are done.

Example 1.8. — Let $(A, \mathfrak{M})$ be as above, and $\lambda \in \mathfrak{M} \setminus \{0\}$. We define a plane curve $\mathscr{C}$ by

$$\mathscr{C}: \quad y^2 z - \lambda xyz = x^3 \ (\subset \mathbb{P}^2_A).$$

Then $\mathscr{C}$ is a deformation over A of a cuspidal curve to a nodal curve:



We can see that $\operatorname{Pic}^0_{\mathscr{C}/S}$ is isomorphic to $\mathscr{G}^{(\lambda)}$ (*cf.* Ex. 3.11).

Remark 1.9. — The existence of deformations of $\mathbb{G}_a$ to $\mathbb{G}_m$ was first given by T. Kambayashi and M. Miyanishi [7], and the groups of extensions of such deformation schemes by the additive group scheme $\mathbb{G}_a$ are treated by B. Weisfeiler [24]. Waterhouse and Weisfeiler [25] develop a more general theory concerning the deformations of 1-dimensional algebraic tori.

2. Artin-Schreier to Kummer. — 2.1. Put $A=W(k)[\zeta]$, where $W(k)$ is the Witt vector ring of an algebraically closed field k of characteristic $p(>0)$ and ζ is a primitive p-th root of unity. Put $\lambda=\zeta-1$. We note that A is a complete discrete valuation ring with uniformizing parameter λ, and

(2.1.1) $$\lambda^{p-1}=up,$$

where u is the unit defined by

(2.1.2) $$u=\{(1+\zeta)(1+\zeta+\zeta^2)\dots(1+\zeta+\dots+\zeta^{p-2})\}^{-1}.$$

Now we define an S-homomorphism

(2.1.3) $$\psi:\ \mathscr{G}^{(\lambda)}=\operatorname{Spec}(A[x,\ 1/(\lambda x+1)])\to\mathscr{G}^{(\lambda^p)}=\operatorname{Spec}(A[x,\ 1/(\lambda^p x+1)])$$

by

(2.1.4) $$x\mapsto\{(\lambda x+1)^p-1\}/\lambda^p.$$

By the equality (2.1.1), this morphism is well-defined. Moreover, ψ is faithfully flat and finite. In fact, we have

(2.1.5) $$\operatorname{Ker}\psi=\operatorname{Spec}(A[x,\ 1/(\lambda x+1)]/(((\lambda x+1)^p-1)/\lambda^p)).$$

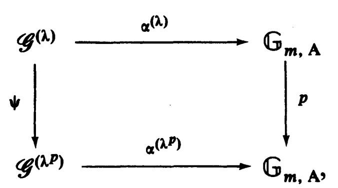
By (2.1.1), the coefficient of the highest term of $\{(\lambda x+1)^p-1\}/\lambda^p$ is a unit, and therefore $\operatorname{Ker}\psi$ is flat over S. On the other hand, we have

(2.1.6) $$x^p+px^{p-1}/\lambda+\dots+px/\lambda^{p-1}=\psi(x),$$

(2.1.7) $$1/(\lambda x+1)^p=\psi(1/(\lambda^p x+1)).$$

This implies that ψ is finite.

2.2. Put $\mathscr{N}=\operatorname{Ker}\psi$. By the definition of ψ, we can see that the following diagram is commutative:



where p denotes the p-th power and that $\mathscr{N}_\eta$ is isomorphic to $\mu_{p,K}$ [cf. Prop. 1.5. (i)].

Moreover, by the definition of ψ and the fact $u \equiv -1 \pmod{\lambda}$ (*cf.* 2.1.2), we see that the special fibre

(2.2.2) $$\psi_s: \quad \mathcal{G}_s^{(\lambda)} \simeq \mathbb{G}_{a,k} \to \mathcal{G}_s^{(\lambda^p)} \simeq \mathbb{G}_{a,k}$$

is nothing but the homomorphism

(2.2.3) $$\wp: \quad \mathbb{G}_{a,k} \to \mathbb{G}_{a,k}$$

defined by $\wp(x) = x^p - x$. Then $\mathcal{N}_s$ is isomorphic to $\mathbb{Z}/p$. Since $\mathcal{N}$ is flat over S, $\mathcal{N}$ is étale over S, and therefore $\mathcal{N}$ is (non-canonically) isomorphic to $\mathbb{Z}/p$. So the exact sequence

(2.2.4) $$0 \to \mathcal{N} \to \mathcal{G}^{(\lambda)} \xrightarrow{\psi} \mathcal{G}^{(\lambda^p)} \to 0$$

gives the connection between the exact sequences of Artin-Schreier and of Kummer. Note that

(2.2.5) $$\mathcal{N}(\mathrm{A}) = \{0, 1, 1+\zeta, \ldots, 1+\zeta+\ldots+\zeta^{p-2}\} \subset \mathcal{G}^{(\lambda)}(\mathrm{A}).$$

We also remark that this $\mathcal{G}^{(\lambda)}$ is the unique deformation of $\mathbb{G}_a$ to $\mathbb{G}_m$ containing the constant group scheme $\mathcal{N} \simeq (\mathbb{Z}/p)_{\mathrm{A}}$.

2.3. Now we embed $\mathcal{G}^{(\lambda)}$ into $\mathbb{P}^1_{\mathrm{A}}$ by $x \mapsto (x:1)$. Then the action of ζ on $\mathcal{G}^{(\lambda)}$ is extended over $\mathbb{P}^1_{\mathrm{A}}$ by $\begin{pmatrix} \zeta & 1 \\ 1 & 1 \end{pmatrix}$. Moreover we get the commutative diagram

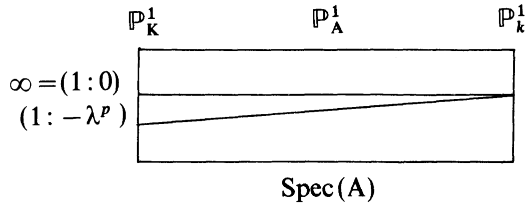
(2.3.1) $$\begin{array}{ccc} \mathcal{G}^{(\lambda)} \xrightarrow{\sim} \mathbb{P}^1_{\mathrm{A}} \setminus \{(1:0), (1:-\lambda)\} & \subset & \mathbb{P}^1_{\mathrm{A}} \\ \psi \downarrow & & \downarrow \Psi \\ \mathcal{G}^{(\lambda^p)} \xrightarrow{\sim} \mathbb{P}^1_{\mathrm{A}} \setminus \{(1:0), (1:-\lambda^p)\} & \subset & \mathbb{P}^1_{\mathrm{A}}, \end{array}$$

where Ψ is defined by $\Psi(x:y) = ((x+y)/\lambda)^p - y^p/\lambda^p : y^p)$. That is, the morphism Ψ gives the quotient morphism $\mathbb{P}^1_{\mathrm{A}} \to \mathbb{P}^1_{\mathrm{A}}/\mathcal{N}$ by the action of $\mathcal{N}$ on $\mathbb{P}^1_{\mathrm{A}}$ passing through the embedding $\mathcal{G}^{(\lambda)} \subsetneq \mathbb{P}^1_{\mathrm{A}}$. Moreover, if we denote by the dotted arrows $\mathbb{P}^1_{\mathrm{A}} \cdots \to \mathcal{G}^{(\lambda)}$ and $\mathbb{P}^1_{\mathrm{A}} \cdots \to \mathcal{G}^{(\lambda^p)}$ the rational maps defined by $(x:y) \mapsto x/y$, then Ψ fits into the cartesian product:

(2.3.2) $$\begin{array}{ccc} \mathbb{P}^1_{\mathrm{A}} & \dashrightarrow & \mathcal{G}^{(\lambda)} \\ \Psi \downarrow & & \downarrow \psi \\ \mathbb{P}^1_{\mathrm{A}} & \dashrightarrow & \mathcal{G}^{(\lambda^p)} \end{array}$$

The ramification locus of Ψ is given by the following figure:

(2.3.3)



This collapse describes the wild ramification of the Artin-Schreier extensions.

Remark 2.4. — It is known by Mumford-Oort [16] that every finite commutative group scheme in char. p can be liftable to one in char. 0. Finally, we note that by using our group scheme $\mathcal{G}^{(\lambda)}$, we can construct also some deformations of finite group schemes.

For example, let ξ be a primitive p^n-root of unity with $n \geqq 2$. We put $A=\mathbb{Z}_p[\xi]$ and $\lambda=\xi-1$. Note that λ is a uniformaizing parameter of A and $(p)=(\lambda^m)$ with $m=p^{n-1}(p-1)$. Now we take the subgroup scheme μ_{p^i} (with $1 \leqq i \leqq n$) in the generic fibre $\mathcal{G}^{(\lambda)}_\eta \simeq \mathbb{G}_{m,K}$, and its flat extension $\mathcal{N}_i$ in $\mathcal{G}^{(\lambda)}$ (*cf.* EGA IV, Prop. 2.8.5). According to the uniqueness of the flat extension, the extension $\mathcal{N}_i$ is also a group scheme. In $\mathcal{G}^{(\lambda)}=\operatorname{Spec}(A[x, 1/(\lambda x+1)])$, the subgroup scheme $\mathcal{N}_i$ is defined by the equation $F_i=0$, where $F_i=\lambda^{-p^i}\{(\lambda x+1)^{p^i}-1\}$. Obviously F_i is a polynomial with coeffitiants in A, and

$$F_i(\operatorname{mod}\lambda)=\begin{cases} x^{p^i} & \text{for } 1 \leqq i \leqq n-1 \\ x^{p^i}-x^{p^{i-1}} & \text{for } i=n. \end{cases} \tag{2.4.1}$$

Therefore, the group scheme N_i gives the following deformation:

$$\mathcal{N}_i=\begin{cases} \text{the generic fibre } (\mathcal{N}_i)_\eta \simeq \mu_{p^i}, \\ \text{the special fibre } (\mathcal{N}_i)_s \simeq \begin{cases} \alpha_{p^i} & \text{for } 1 \leqq i \leqq n-1, \\ \mathbb{Z}/p \times \alpha & \text{for } i=n. \end{cases} \end{cases} \tag{2.4.2}$$

Here, $\mu_{p^i}=\operatorname{Spec}(K[x]/(x^{p^i}-1))$, $\alpha_{p^i}=\operatorname{Spec}(\mathbb{F}_p[x]/(x^{p^i}))$ and $\mathbb{Z}/p=\operatorname{Spec}(\mathbb{F}_p[x]/(x^p-x))$.

Note. — Waterhouse and Weisfeiler [25] discuss finite subgroup schemes of $\mathcal{G}^{(\lambda)}$. Moreover, Waterhouse [26] gives the exact sequence (2.2.4) completely independently.

II. Extensions of an abelian scheme by $\mathcal{G}^{(\lambda)}$

1. The cohomology group $H^1(X, \mathcal{G}^{(\lambda)})$

Lemma 1.1. — *Let* X *be a flat* S-*scheme of finite type. Then the sequence on the étale site* X_{et}

$$0 \to \mathcal{G}^{(\lambda)}_X \xrightarrow{\alpha^{(\lambda)}} \mathbb{G}_{m,X} \to i_* \mathbb{G}_{m,X_\lambda} \to 0 \tag{1.1.1}$$

is exact (*i denotes the closed immersion* $X_\lambda=X \otimes_A (A/\lambda) \hookrightarrow X$).

It is enough to show the exactness for the geometric fibers at each point of X. Then the lemma follows from the

SUBLEMMA 1.2. — *Let* B *be a local ring, flat over* A. *Then the sequence*

$$(1.2.1)\qquad 0 \to \mathcal{G}^{(\lambda)}(\mathrm{B}) \xrightarrow{\alpha^{(\lambda)}} \mathrm{B}^{\times} \to (\mathrm{B}/\lambda\,\mathrm{B})^{\times} \to 0$$

is exact.

Remark 1.3. — $\alpha^{(\lambda)}: \mathcal{G}_{\mathrm{X}}^{(\lambda)} \to \mathbb{G}_{m,\,\mathrm{X}}$ is not injective in the category of S-schemes.

1.4. Let X be a flat proper S-scheme with irreducible fibers. (1.2.1) defines an exact sequence

$$(1.4.1)\qquad 0 \to \Gamma(\mathrm{X},\, \mathcal{G}_{\mathrm{X}}^{(\lambda)}) \to \Gamma(\mathrm{X},\, \mathbb{G}_{m,\,\mathrm{X}}) \to \Gamma(\mathrm{X}_{\lambda},\, \mathbb{G}_{m,\,\mathrm{X}_{\lambda}}) \to \mathrm{H}^1(\mathrm{X},\, \mathcal{G}_{\mathrm{X}}^{(\lambda)}) \to \mathrm{H}^1(\mathrm{X},\, \mathbb{G}_{m,\,\mathrm{X}}) \to \mathrm{H}^1(\mathrm{X}_{\lambda},\, \mathbb{G}_{m,\,\mathrm{X}_{\lambda}}).$$

Since X/S has irreducible fibers, $\Gamma(\mathrm{X},\, \mathbb{G}_{m,\,\mathrm{X}}) = \mathrm{A}^{\times}$ and $\Gamma(\mathrm{X}_{\lambda},\, \mathbb{G}_{m,\,\mathrm{X}_{\lambda}}) = (\mathrm{A}/\lambda)^{\times}$, and therefore $\Gamma(\mathrm{X},\, \mathbb{G}_{m,\,\mathrm{X}}) \to \Gamma(\mathrm{X}_{\lambda},\, \mathbb{G}_{m,\,\mathrm{X}_{\lambda}})$ is surjective. Then we obtain

THEOREM 1.5. — $\mathrm{H}^1(\mathrm{X}_{fppf},\, \mathcal{G}_{\mathrm{X}}^{(\lambda)}) = \mathrm{H}^1(\mathrm{X}_{\mathrm{et}},\, \mathcal{G}_{\mathrm{X}}^{(\lambda)}) = \mathrm{Ker}\,[\mathrm{Pic}\,(\mathrm{X}) \to \mathrm{Pic}\,(\mathrm{X}_{\lambda})]$.

2. THE EXTENSION GROUP $\mathrm{Ext}^1(\mathrm{X},\, \mathcal{G}^{(\lambda)})$. — 2.1. Let X be a commutative group S-scheme and G be an abelian sheaf on $(\mathrm{Sch}/_{\mathrm{S}})_{fppf}$. Then the functorial homomorphism $\mathrm{Hom}_{\mathrm{S-gr}}(\mathrm{X},\, \mathrm{G}) \to \mathrm{Hom}_{\mathrm{S}}(\mathrm{X},\, \mathrm{G})$ induces homomorphisms

$$(2.1.1)\qquad \alpha_{\mathrm{S}}^{j}:\ \mathrm{Ext}_{\mathrm{S}}^{j}(\mathrm{X},\, \mathrm{G}) \to \mathrm{H}^{j}(\mathrm{X},\, \mathrm{G}_{\mathrm{X}}) \qquad \text{for}\quad j \geqq 0.$$

Remark 2.2. — Let $(\mathrm{E}): 0 \to \mathrm{G} \to \mathrm{Y} \to \mathrm{X} \to 0$ be an extension of abelian sheaves on $(\mathrm{Sch}/_{\mathrm{S}})_{fppf}$. $\alpha_{\mathrm{S}}^{1}(\mathrm{E})$ is nothig but the class of the G_{X}-torsor Y of X for the *fppf*-topology.

DEFINITION 2.3. — $a \in \mathrm{H}^{j}(\mathrm{X},\, \mathrm{G}_{\mathrm{X}})$ is said to be primitive if $m^*(a) = p_1^*(a) + p_2^*(a)$, where $m: \mathrm{X} \times_{\mathrm{S}} \mathrm{X} \to \mathrm{X}$ is the multiplication and $p_i: \mathrm{X} \times_{\mathrm{S}} \mathrm{X} \to \mathrm{X}$ is the projection to the i-th factor.

LEMMA 2.4. — *Let* X *be a commutative group* S*-scheme and* G *be an abelian sheaf on* $(\mathrm{Sch}/_{\mathrm{S}})_{fppf}$. *Then* $\alpha_{\mathrm{S}}^{j}: \mathrm{Ext}_{\mathrm{S}}^{j}(\mathrm{X},\, \mathrm{G}) \to \mathrm{H}^{j}(\mathrm{X},\, \mathrm{G}_{\mathrm{X}})$ *has the image in the set of primitive elements.*

Proof. — Let I. be an injective resolution of G. Then we have an exact sequence

$$(2.4.1)\qquad 0 \to \mathrm{Hom}_{\mathrm{S-gr}}(\mathrm{X},\, \mathrm{I.}) \to \mathrm{Hom}_{\mathrm{S}}(\mathrm{X},\, \mathrm{I.}) \xrightarrow{m^* - p_1^* - p_2^*} \mathrm{Hom}_{\mathrm{S}}(\mathrm{X} \times_{\mathrm{S}} \mathrm{X},\, \mathrm{I.}).$$

This implies the lemma.

LEMMA 2.5. — *Suppose that* X *is an abelian scheme over* S *and* G *is a flat affine commutative groupe* S*-scheme. Then* $\mathrm{Ext}_{\mathrm{S}}^{1}(\mathrm{X},\, \mathrm{G}) \to \mathrm{H}^1(\mathrm{X},\, \mathrm{G}_{\mathrm{X}})$ *is injective.*

Proof. — Let $(\mathrm{E}): 0 \to \mathrm{G} \to \mathrm{Y} \to \mathrm{X} \to 0$ be an extension of abelian sheaves on $(\mathrm{Sch}/_{\mathrm{S}})_{fppf}$. Since G is affine over S, Y is representable. Suppose that Y is trivial as G_{X}-torsor of X and choose u a section of Y/X. By a translation in Y, we may assume that $u \circ \varepsilon_X = \varepsilon_Y$ (ε_X, ε_Y denote the section of X or Y respectively). By the rigidity lemma ([9], Cor. 6.4), u is a group homomorphism, that is to say, the extension (E) is trivial.

THEOREM 2.6. — *Suppose that* S *is strictly henselian,* X *is an abelian scheme over* S *and* G *is a smooth affine commutative group* S*-scheme. Then* $\alpha_S^1 : \mathrm{Ext}_S^1(X, G) \to H^1(X, G_X)$ *is injective. Moreover, the image of* α_S^1 *is the set of primitive elements.*

Proof. — It is enough to show that α_S^1 is surjective on the set of primitive elements. Let Y be a G-torsor of X which defines a primitive element of $H^1(X, G_X)$. Consider the cartesian diagram

(2.6.1)
$$\begin{array}{ccc} Y' & \to & Y \\ \downarrow & & \downarrow \\ X \times_S X & \xrightarrow{m_X} & X. \end{array}$$

Then Y′ is a G-torsor of $X \times_S X$. By the hypothesis, Y′ is isomorphic to the G-torsor $(Y \times_S Y) \wedge^{G \times_S G} G$, where $G \times_S G$ acts on G by the multiplication. Then we obtain a commutative diagram

(2.6.2)
$$\begin{array}{ccc} Y \times_S Y & \xrightarrow{m_Y} & Y \\ \downarrow & & \downarrow \\ X \times_S X & \xrightarrow{m_X} & X, \end{array}$$

where m_Y is compatible with the action of G. Since Y is smooth over X and S is strictly henselian, there exists ε_Y a section of Y/S (EGA. IV. Th. 18.5.17). By the translation by G, we may assume $m_Y(\varepsilon_Y, \varepsilon_Y) = \varepsilon_X$. Then m_Y is a group law on Y (*cf.* [21], Chap. VII, § 3, No. 15).

Example 2.7. — $\mathrm{Ext}_S^1(X, \mathbb{G}_{m,S}) = \mathrm{Pic}^0(X) \subset \mathrm{Pic}(X) = H^1(X, \mathbb{G}_{m,X})$.

In fact, $\mathscr{L}$ is primitive $\Leftrightarrow \Lambda(\mathscr{L}) = 0$, where $\Lambda : \mathrm{Pic}(X) \to \mathrm{Hom}_{S-gr}(X, X^t)$ is the homomorphism defined by $\Lambda(\mathscr{L})(x) = T_x^*(\mathscr{L}) \otimes_{\mathscr{O}_X} \mathscr{L}^{-1} \Leftrightarrow \mathscr{L} \in \mathrm{Pic}^0(X)$, by the rigidity lemma ([9], Prop. 6.1 or Cor. 6.2).

Example 2.8. — $\mathrm{Ext}_S^1(X, \mathbb{G}_{a,S}) \simeq H^1(X, \mathbb{G}_{a,X}) \simeq H^1(X, \mathscr{O}_X)$.

In fact, by the Künneth formula, we have a canonical bijection $H^1(H \times_S X, \mathscr{O}_X) \simeq H^1(X, \mathscr{O}_X) \oplus H^1(X, \mathscr{O}_X)$. This implies that any element of $H^1(X, \mathbb{G}_{a,X})$ is primitive.

Combining Theorem 1.5 and Theorem 2.6, we get the following.

Example 2.9:

$$\mathrm{Ext}_S^1(X, \mathscr{G}_S^{(\lambda)}) \simeq H^1(X, \mathscr{G}_X^{(\lambda)}) = \mathrm{Ker}[\mathrm{Pic}(X) \to \mathrm{Pic}(X \otimes_A A/\lambda)]$$
$$= \mathrm{Ker}[\mathrm{Pic}^0(X) \to \mathrm{Pic}^0(X \otimes_A A/\lambda)],$$

where $\lambda \in \mathfrak{M} \setminus \{0\}$.

Example 2.10. — $\mathrm{Ext}_S^1(X, \mathbb{Z}/n) \simeq H^1(X, \mathbb{Z}/n)$.

In fact, by the Künneth formula, we have a canonical bijection $H^1(X \times_S X, \mathbb{Z}/n) \simeq H^1(X, \mathbb{Z}/n) \oplus H^1(X, \mathbb{Z}/n)$. This implies that any element of $H^1(X, \mathbb{Z}/n)$ is primitive.

2.11. Let $f: X \to S$ be a proper flat morphism. Then there exist functorial isomorphisms

$$(2.11.1) \qquad \Gamma(X, \mathbb{Z}/n) \xrightarrow{\sim} \mathrm{Hom}_{X-gr}(\boldsymbol{\mu}_{n,X}, \mathbb{G}_{m,X}) \xrightarrow{\sim} \mathrm{Hom}_{S-gr}(\boldsymbol{\mu}_{n,S}, f_* \mathbb{G}_{m,X}).$$

Then we have a spectral sequence

$$(2.11.2) \qquad E_2^{qr} = \mathrm{Ext}_S^q(\boldsymbol{\mu}_{n,S}, R^r f_* \mathbb{G}_{m,X}) \Rightarrow H^{q+r}(X, \mathbb{Z}/n)$$

and therefore an exact sequence

$$(2.11.3) \qquad 0 \to \mathrm{Ext}_S^1(\boldsymbol{\mu}_{n,S}, f_* \mathbb{G}_{m,X}) \to H^1(X, \mathbb{Z}/n) \xrightarrow{\beta_S} \mathrm{Hom}_{S-gr}(\boldsymbol{\mu}_{n,S}, \mathrm{Pic}_{X/S}) \\ \to \mathrm{Ext}_S^2(\boldsymbol{\mu}_{n,S}, f_* \mathbb{G}_{m,X}) \to H^2(X, \mathbb{Z}/n).$$

LEMMA 2.12. — *If* S *is strictly henselian, then* β_S *is bijective.*

See [18]. No. 6.

THEOREM 2.13. — *Suppose that* S *is strictly henselian. Let* X *be an abelian scheme over* S *and* X_s *be the fiber of* X *over* s. *In the commutative diagram*

$$(2.13.1) \qquad \begin{array}{ccccc} \mathrm{Ext}_S^1(X, \mathbb{Z}/n) & \xrightarrow{\alpha_S} & H^1(X, \mathbb{Z}/n) & \xrightarrow{\beta_S} & \mathrm{Hom}_{S-gr}(\boldsymbol{\mu}_{n,S}, \mathrm{Pic}_{X/S}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Ext}_s^1(X_s, \mathbb{Z}/n) & \xrightarrow{\alpha_s} & H^1(X_s, \mathbb{Z}/n) & \xrightarrow{\beta_s} & \mathrm{Hom}_{s-gr}(\boldsymbol{\mu}_{n,s}, \mathrm{Pic}_{X_s/s}). \end{array}$$

all the maps are bijective.

Proof. — By the proper base change theorem (SGA4, Exp. XII, Th. 5.1), the middle vertical arrow is bijective. By 2.12 and 2.10, all the horizontal arrows are bijective.

3. THE COHOMOLOGY GROUP $H^1(X, \mathbb{Z}/p)$. — In this section, let k be an algebraically closed field of characteristic $p>0$. Suppose that A is a complete discrete valuation ring dominating $W(k)[\zeta]$ with ζ a primitive p-th root of unity. Put $\lambda = \zeta - 1$ as in Ch. I. Under these notations, we look at the exact sequence (Ch. I, 2.2.4):

$$(3.1.1) \qquad 0 \to \mathbb{Z}/p \to \mathscr{G}^{(\lambda)} \xrightarrow{\psi} \mathscr{G}^{(\lambda^p)} \to 0.$$

Let X be a flat proper S-scheme with irreducible fibers. Then (3.1.1) gives an exact sequence

$$(3.1.2) \qquad 0 \to \Gamma(X, \mathbb{Z}/p) \to \Gamma(X, \mathscr{G}_X^{(\lambda)}) \to \Gamma(X, \mathscr{G}_X^{(\lambda^p)}) \\ \to H^1(X, \mathbb{Z}/p) \to H^1(X, \mathscr{G}_X^{(\lambda)}) \to H^1(X, \mathscr{G}_X^{(\lambda^p)}).$$

Since X/S has irreducible fibers,

$$\Gamma(\mathrm{X}, \mathscr{G}_{\mathrm{X}}^{(\lambda)})=(1+\lambda\mathrm{A})^{\times},\ \Gamma(\mathrm{X}, \mathscr{G}_{\mathrm{X}}^{(\lambda^p)})=(1+\lambda^p\mathrm{A})^{\times}$$

and therefore $\Gamma(\mathrm{X}, \mathscr{G}_{\mathrm{X}}^{(\lambda)}) \to \Gamma(\mathrm{X}, \mathscr{G}_{\mathrm{X}}^{(\lambda^p)})$ is surjective.

On the other hand, consider the commutative diagram

$$\begin{array}{ccc} \mathscr{G}_{\mathrm{X}}^{(\lambda)} & \xrightarrow{\psi_p} & \mathscr{G}_{\mathrm{X}}^{(\lambda^p)} \\ \downarrow{\scriptstyle \alpha^{(\lambda)}} & & \downarrow{\scriptstyle \alpha^{(\lambda^p)}} \\ \mathbb{G}_{m,\mathrm{X}} & \xrightarrow{p} & \mathbb{G}_{m,\mathrm{X}}. \end{array} \tag{3.1.3}$$

We have therefore a commutative diagram

$$\begin{array}{ccc} \mathrm{H}^1(\mathrm{X}, \mathscr{G}_{\mathrm{X}}^{(\lambda)}) & \xrightarrow{\psi_p} & \mathrm{H}^1(\mathrm{X}, \mathscr{G}_{\mathrm{X}}^{(\lambda^p)}) \\ \downarrow & & \downarrow \\ \mathrm{Pic}(\mathrm{X}) & \xrightarrow{p} & \mathrm{Pic}(\mathrm{X}). \end{array} \tag{3.1.4}$$

Then we obtain the

THEOREM 3.2. — $\mathrm{H}^1(\mathrm{X}, \mathbb{Z}/p)=\mathrm{Ker}[{}_p\mathrm{Pic}(\mathrm{X}) \to {}_p\mathrm{Pic}(\mathrm{X}\otimes_{\mathrm{A}}(\mathrm{A}/\lambda))]$.

Remark 3.3. – We have shown that $\mathrm{H}^1(\mathrm{X}, \mathbb{Z}/p) \simeq \mathrm{Hom}_{\mathrm{S-gr}}(\boldsymbol{\mu}_{p,\mathrm{S}}, \mathrm{Pic}_{\mathrm{X/S}})$ (*cf.* Th. 2.13). Then $\mathrm{Hom}_{\mathrm{S-gr}}(\boldsymbol{\mu}_{p,\mathrm{S}}, \mathrm{Pic}_{\mathrm{X/S}})$ is (non canonically) isomorphic to $\mathrm{Ker}[{}_p\mathrm{Pic}(\mathrm{X})] \to {}_p\mathrm{Pic}(\mathrm{X}\otimes_{\mathrm{A}}(\mathrm{A}/\lambda))]$. For example, $g \mapsto g(\mathrm{S})(\zeta)$ gives a bijection of $\mathrm{Hom}_{\mathrm{S-gr}}(\boldsymbol{\mu}_{p,\mathrm{S}}, \mathrm{Pic}_{\mathrm{X/S}})$ to $\mathrm{Ker}[{}_p\mathrm{Pic}(\mathrm{X}) \to {}_p\mathrm{Pic}(\mathrm{X}\otimes_{\mathrm{A}}(\mathrm{A}/\lambda))]$.

COROLLARY 3.4. — *Let* X *be an abelian scheme over* S. *Then* $\mathrm{Ext}^1_{\mathrm{S}}(\mathrm{X}, \mathbb{Z}/p)=\mathrm{H}^1(\mathrm{X}, \mathbb{Z}/p)=\mathrm{Ker}[{}_p\mathrm{X}^t(\mathrm{A}) \to {}_p\mathrm{X}^t(\mathrm{A}/\lambda)]$.

4. A VARIANT OF BREEN'S THEOREM. – In this section, we assume that A is strictly henselian and the residue field is of characteristic $p>0$.

THEOREM 4.1. (*cf.* Breen [3], p. 339, Th.). — *Let* G *be a finite flat group* S-*scheme. Then* $\mathrm{Ext}^2_{\mathrm{S}}(\mathrm{G}, \mathbb{Z}/p^n)=0$.

Proof. – There exists a filtration $0=\mathrm{G}_0 \subset \mathrm{G}_1 \subset \ldots \subset \mathrm{G}_r=\mathrm{G}$ formed by finite flat subgroup S-schemes such that $\mathrm{G}_i/\mathrm{G}_{i-1}$ is finite étale or finite flat connected. Therefore we may assume that G is étale or connected.

Suppose at first that G is étale. Then the theorem is a consequence of vanishing of Ext^i $(i \geqq 2)$ in the category of the abelian groups.

Suppose now that G is connected. In this case, the order of G is a power of p. Let A(G) be the complex of Eilenberg-MacLane algebra of fppf-sheaves over S associated to G (*cf.* [2]). We have two spectral sequences

$$'\mathrm{E}_1^{ij}=\mathrm{Ext}^j_{\mathrm{S}}(\mathrm{A}(\mathrm{G})_i, \mathbb{Z}/p^n) \Rightarrow \mathrm{Ext}^{i+j}_{\mathrm{S}}(\mathrm{A}(\mathrm{G}), \mathbb{Z}/p^n) \tag{a}$$

$$''\mathrm{E}_2^{ij}=\mathrm{Ext}^i_{\mathrm{S}}(\mathrm{H}_j(\mathrm{A}(\mathrm{G})), \mathbb{Z}/p^n) \Rightarrow \mathrm{Ext}^{i+j}_{\mathrm{S}}(\mathrm{A}(\mathrm{G}), \mathbb{Z}/p^n). \tag{b}$$

(*b*) defines an exact sequence

$$(4.1.1)\quad 0 \to \mathrm{Ext}^1_S(\mathrm{H}_0(\mathrm{A}(\mathrm{G})), \mathbb{Z}/p^n) \to \mathrm{Ext}^1_S(\mathrm{A}(\mathrm{G}), \mathbb{Z}/p^n) \to \mathrm{Hom}_{S-gr}(\mathrm{H}_1(\mathrm{A}(\mathrm{G})), \mathbb{Z}/p^n) \to \mathrm{Ext}^2_S(\mathrm{H}_0(\mathrm{A}(\mathrm{G}), \mathbb{Z}/p^n) \to \mathrm{Ext}^2_S(\mathrm{A}(\mathrm{G}), \mathbb{Z}/p^n).$$

Since the order of G is a power of p,

$$(4.1.2)\qquad \mathrm{H}_j(\mathrm{A}(\mathrm{G})) = \begin{cases} \mathrm{G} & \text{if } j=0 \\ 0 & \text{if } 0<j<2p-2. \end{cases}$$

([3], p. 343). Then we get an injection $\mathrm{Ext}^2_S(\mathrm{G}, \mathbb{Z}/p^n) \hookrightarrow \mathrm{Ext}^2_S(\mathrm{A}(\mathrm{G}), \mathbb{Z}/p^n)$.

On the other hand, we have

$$(4.1.3)\qquad \mathrm{Ext}^j_S(\mathrm{A}(\mathrm{G})_i, \mathbb{Z}/p^n) = 0 \quad \text{for } j>0$$

(*loc. cit.* p. 345). Then (*a*) defines isomorphisms

$$(4.1.4)\qquad \mathrm{H}^j(\mathrm{Hom}_{S-gr}(\mathrm{A}(\mathrm{G}), \mathbb{Z}/p^n)) \xrightarrow{\sim} \mathrm{Ext}^j_S(\mathrm{A}(\mathrm{G}), \mathbb{Z}/p^n) \quad \text{for } j \geqq 0.$$

Moreover, we have an isomorphism of complexes

$$(4.1.5)\qquad \mathrm{Hom}_{S-gr}(\mathrm{A}(\mathrm{G}), \mathbb{Z}/p^n) \xrightarrow{\sim} \mathrm{Hom}^{\bullet}_S(\mathrm{X}_., \mathbb{Z}/p^n),$$

where X. is a simplicial pointed S-scheme whose components are cartesian products of the copies of G over S, and $\mathrm{Hom}^{\bullet}_S$ denotes the set of morphisms of pointed S-schemes (*loc. cit.* p. 345). Since X_i is connected and A is strictly henselian, we have $\mathrm{Hom}^{\bullet}_S(\mathrm{X}_., \mathbb{Z}/p^n) = 0$ and therefore $\mathrm{Ext}^2_S(\mathrm{A}(\mathrm{G}), \mathbb{Z}/p^n) = 0$. This implies the theorem.

COROLLARY 4.2. — *Let* G *be a finite flat group* S-*scheme. Then* $\mathrm{Ext}^2_S(\mathrm{G}, \mathbb{Z}/n) = 0$.

COROLLARY 4.3. — $\mathrm{Ext}^2_S(\mathbb{G}_m, \mathbb{Z}/n) = \mathrm{Ext}^3_S(\mathbb{G}_m, \mathbb{Z}/n) = 0$.

Proof. — Consider the exact sequence

$$(4.3.1)\quad \mathrm{Ext}^2_S(\mathbb{G}_m, \mathbb{Z}/n) \xrightarrow{n} \mathrm{Ext}^2_S(\mathbb{G}_m, \mathbb{Z}/n) \to \mathrm{Ext}^2_S(\mu_n, \mathbb{Z}/n) \to \mathrm{Ext}^3_S(\mathbb{G}_m, \mathbb{Z}/n) \xrightarrow{n} \mathrm{Ext}^3_S(\mathbb{G}_m, \mathbb{Z}/n).$$

Since $n: \mathrm{Ext}^i_S(\mathbb{G}_m, \mathbb{Z}/n) \to \mathrm{Ext}^i_S(\mathbb{G}_m, \mathbb{Z}/n)$ is zero, we obtain an exact sequence

$$(4.3.2)\qquad 0 \to \mathrm{Ext}^2_S(\mathbb{G}_m, \mathbb{Z}/n) \to \mathrm{Ext}^2_S(\mu_n, \mathbb{Z}/n) \to \mathrm{Ext}^3_S(\mathbb{G}_m, \mathbb{Z}/n) \to 0.$$

Then 4.2 implies 4.3.

COROLLARY 4.4. — *Let* X *be an abelian scheme over* S. *Then*

$$\mathrm{Ext}^2_S(\mathrm{X}, \mathbb{Z}/n) = \mathrm{Ext}^3_S(\mathrm{X}, \mathbb{Z}/n) = 0.$$

Proof. — Consider the exact sequence

$$(4.4.1)\quad \mathrm{Ext}^2_S(X, \mathbb{Z}/n) \xrightarrow{n} \mathrm{Ext}^2_S(X, \mathbb{Z}/n) \to \mathrm{Ext}^2_S({}_nX, \mathbb{Z}/n) \to \mathrm{Ext}^3_S(X, \mathbb{Z}/n) \xrightarrow{n} \mathrm{Ext}^3_S(X, \mathbb{Z}/n),$$

induced from the exact sequence $0 \to {}_nX \to X \xrightarrow{n} X \to 0$. Since

$$n : \mathrm{Ext}^i_S(X, \mathbb{Z}/n) \to \mathrm{Ext}^i_S(X, \mathbb{Z}/n)$$

is zero, we obtain an exact sequence

$$(4.4.2)\qquad 0 \to \mathrm{Ext}^2_S(X, \mathbb{Z}/n) \to \mathrm{Ext}^2_S({}_nX, \mathbb{Z}/n) \to \mathrm{Ext}^3_S(X, \mathbb{Z}/n) \to 0.$$

So it is sufficient to notice that ${}_nX$ is finite flat over S.

III. Construction of singular curves over a discret valuation ring

In this chapter, for simplicity and safety we assume that A is complete and k is algebraically closed.

1. CONSTRUCTION. — 1.1. Let $f : \mathscr{C} \to S$ a smooth proper morphism with geometrically irreducible fibers of dimension 1. Note that in this case f becomes automatically projective. Let $\mathfrak{d}$ be an effective divisor of $\mathscr{C}$ flat over S, $\mathscr{F}_{\mathfrak{d}} \subset \mathscr{O}_{\mathfrak{d}}$ the definition ideal of $\mathfrak{d}$ and $\mathscr{A}_{\mathfrak{d}}$ the subsheaf of $f^{-1}(\mathscr{O}_S)$-algebras of $\mathscr{O}_{\mathscr{C}}$ generated by $\mathscr{I}_{\mathfrak{d}}$. Let $\{U_\alpha\}$ be an affine open covering of $\mathscr{C}$. Glueing Spec $\Gamma(U_\alpha, \mathscr{A}_{\mathfrak{d}})$, we obtain a S-scheme $\mathscr{C}_{\mathfrak{d}}$. Let $g : \mathscr{C}_{\mathfrak{d}} \to S$ be the structure morphism and $\psi : \mathscr{C} \to \mathscr{C}_{\mathfrak{d}}$ the morphism corresponding to the homomorphism of rings $\mathscr{A}_{\mathfrak{d}} \to \mathscr{O}_{\mathscr{C}}$.

THEOREM 1.2. — *(i)* $g : \mathscr{C}_{\mathfrak{d}} \to S$ *is flat and proper with geometrically irreducible fibers of dimension* 1.

(ii) $\psi : \mathscr{C} \to \mathscr{C}_{\mathfrak{d}}$ *is finite surjective and induces an isomorphism of* $\mathscr{C} - \mathfrak{d}$ *onto* $\mathscr{C}_{\mathfrak{d}} - \psi(\mathfrak{d})$, *and the closed subscheme* $\psi(\mathfrak{d})$ *defines a section* $\rho : S \to \mathscr{C}_{\mathfrak{d}}$.

Proof. — Since $\mathscr{O}_{\mathscr{C}}/\mathscr{I}_{\mathfrak{d}}$ is of finite type as $f^{-1}(\mathscr{O}_{\mathscr{C}})$-module, $\mathscr{O}_{\mathscr{C}}$ is of finite type as $\mathscr{A}_{\mathfrak{d}}$-module. Then ψ is finite surjective, and therefore, g is proper. Moreover, since $\mathscr{O}_{\mathscr{C}}$ is torsion-free as $f^{-1}(\mathscr{O}_S)$-module, $\mathscr{A}_{\mathfrak{d}}$ is torsion-free as $f^{-1}(\mathscr{O}_S)$-module, that is to say, g is flat.

1.3. Since $g : \mathscr{C}_{\mathfrak{d}} \to S$ is flat and proper with geometrically integral fibers, $\mathrm{Pic}_{\mathscr{C}_{\mathfrak{d}}/S}$ is represented by a separated group S-scheme locally of finite type. Moreover, since g is relatively dimension 1, $\mathrm{Pic}_{\mathscr{C}_{\mathfrak{d}}/S}$ is smooth and we have an exact sequence of commutative groupe S-schemes

$$(1.3.1)\qquad 0 \to \mathrm{Pic}^0_{\mathscr{C}_{\mathfrak{d}}/S} \to \mathrm{Pic}_{\mathscr{C}_{\mathfrak{d}}/S} \xrightarrow{\deg} \mathbb{Z} \to 0.$$

1.4. For any S-scheme X and T, we denote by $\mathrm{Div}(X \times_S T/T)$ the set of relative Cartier divisors of $X \times_S T/T$ and by $\mathrm{Div}_{X/S}$ the contravariant functor on $(\mathrm{Sch}_{/S})$ defined by $T \mapsto \mathrm{Div}(X \times_S T/T)$.

In our case, we have an exact sequence

$$(1.4.1) \quad 0 \to \mathrm{Div.\ Princ.}(\mathscr{C}_{\mathfrak{d}} \times_S T/T) \to \mathrm{Div}(\mathscr{C}_{\mathfrak{d}} \times_S T/T) \to \mathrm{Pic}(\mathscr{C}_{\mathfrak{d}} \times_S T/T) \to 0$$

for any spectrum T of discrete valuation rings $(A', \mathfrak{M}')$ dominating A, where Div. Princ. $(\mathscr{C}_{\mathfrak{d}} \times_S T/T)$ is the subgroup of $\mathrm{Div}(\mathscr{C}_{\mathfrak{d}} \times T/T)$ generated by principal divisors.

In fact, we have only to check the exactness of the last part of (1.4.1). So let $\mathscr{L}$ be an element of Pic $(\mathscr{C}_{\mathfrak{d}} \times_S T/T)$. Since $\mathscr{C}_{\mathfrak{d}} \times_S T$ is an integral scheme, $\mathscr{L}$ is given by a Cartier divisor Z on $\mathscr{C}_{\mathfrak{d}} \times_S T$. Let f be a local equation of Z at the point $\sigma(s')$, where s' is the closed point of T. Here we put $\mathscr{Z} = Z - (f)$. Then obviously $\mathscr{Z}$ is an element of $\mathrm{Div}(\mathscr{C}_{\mathfrak{d}} \times_S T/T)$ lying over $\mathscr{L}$.

PROPOSITION 1.5. (universal property of Albanese type). — *Let* G *be a connected commutative group* S-scheme, and $h : \mathscr{C}_{\mathfrak{d}} - \sigma(\mathfrak{d}) = \mathscr{C} - \mathfrak{d} \to G$ *be an* S*-morphism such that* $h(T)(Z') = 0$ *for any* $T = \mathrm{Spec}\ A'$ *with finite extension* A′ *of* A *and for any principal Cartier divisor* Z′ *of* $\mathscr{C}_{\partial} \times_S T$. *Then there exists a unique* S*-homomorphism*

$$\tilde{h} : \quad \mathrm{Pic}^0_{\mathscr{C}_{\partial}/S} \to G$$

which makes the diagram

$$(1.5.1) \qquad \begin{array}{ccc} \mathrm{Div}^0_{\mathscr{C}_{\delta}/S} & \to & \mathrm{Pic}^0_{\mathscr{C}_{\delta}/S} \\ & {\scriptstyle h} \searrow \quad \swarrow {\scriptstyle \tilde{h}} & \\ & G & \end{array}$$

commutative.

Proof. — By Serre ([21], Chap. V, n° 22, Prop. 13), such a $\tilde{h}$ exists over S_η and S_s, which we denote h_K and h_k, respectively. That is to say, we get a diagram

$$(1.5.2) \qquad \begin{array}{ccc} G_s & \xleftarrow{\ \tilde{h}_k\ } & (\mathrm{Pic}^0_{\mathscr{C}_{\mathfrak{d}}/S})_s \\ \downarrow & & \downarrow \\ G & & \mathrm{Pic}^0_{\mathscr{C}_{\mathfrak{d}}/S} \\ \uparrow & & \uparrow \\ G_\eta & \xleftarrow{\ \tilde{h}_K\ } & (\mathrm{Pic}^0_{\mathscr{C}_{\mathfrak{d}}/S})_\eta. \end{array}$$

Now we take a point $x \in (\mathrm{Pic}^0_{\mathscr{C}_\mathfrak{b}/S})_s$, and we put $y = \tilde{h}_k(x)$, $\mathcal{O}_x = \mathcal{O}_{x, \mathrm{Pic}^0_{\mathscr{C}_\mathfrak{b}/S}}$ and $\mathcal{O}_y = \mathcal{O}_{y,\,G}$. Then the diagram (1.5.2) induces a diagram

$$\begin{array}{ccc} \mathcal{O}_{y,\,G_s} = \mathcal{O}_y \otimes k & \xrightarrow{\quad \tilde{h}_k^* \quad} & \mathcal{O}_x \otimes_A k \\ \uparrow & & \uparrow \\ \mathcal{O}_y & & \mathcal{O}_x \\ \downarrow & & \downarrow \\ \mathcal{O}_y \otimes_A K & \xrightarrow{\quad \tilde{h}_K^* \quad} & \mathcal{O}_x \otimes_A K. \end{array}$$

Suppose that there exists an element $b \in \mathcal{O}_y$ such that $\tilde{h}_K^*(b) \bar{\in} \mathcal{O}_x$. Then $\tilde{h}_K^*(b) = t^{-e} a$ with $e \geqq 1$, $a \in \mathcal{O}_x$ and $t^{-1} a \bar{\in} \mathcal{O}_x$, where t is a uniformizing parameter of A. So the residual class $\bar{a}$ in $\mathcal{O}_x \otimes_A k$ represented by a is not zero. Hence for a suitable DVR $(A', \mathfrak{M}')$ dominating A, there exists a local ring homomorphism $\varphi^* : \mathcal{O}_x \to A'$ such that $\varphi^*(a) \bar{\in} t A'$. The morphism $\varphi : T = \mathrm{Spec}\, A' \xrightarrow{\mathrm{Spec}\, \varphi^*} \mathrm{Spec}\, \mathcal{O}_x \to \mathrm{Pic}^0_{\mathscr{C}_\mathfrak{b}/S}$ defines an element $\mathscr{L} \in \mathrm{Pic}(\mathscr{C}_\mathfrak{b} \times_S T/T)$. On the other hand, by our assumption and the exactness of (1.4.1), there exists a homomorphism $h' : \mathrm{Pic}(\mathscr{C}_\mathfrak{b} \times_S T/T) \to G(T)$ which makes the diagram

$$\begin{array}{ccc} \mathrm{Div}(\mathscr{C}_\mathfrak{b} \times_S T/T) & \to & \mathrm{Pic}(\mathscr{C}_\mathfrak{b} \times_S T/T) \\ & {\scriptstyle h(T)} \searrow \quad \swarrow {\scriptstyle h'} & \\ & G(T) & \end{array}$$

commutative. Let ψ be the T-valued point of G defined by $h'(\mathscr{L})$. Then by our construction we get easily that

$$\psi(\eta') = h'(\varphi(\eta')) = \tilde{h}_{K'}(\eta') \qquad \text{and} \qquad \psi(s') = h'(\varphi(s')) = \tilde{h}_k(s'),$$

where η' and s' are the generic and the closed points of T, respectively, and $K' = f.f.\, A'$. These equalities yield the commutativity of the diagram

$$\begin{array}{ccccc} \mathcal{O}_y \otimes_A k & & \xrightarrow{\qquad \tilde{h}_k^* \qquad} & & \mathcal{O}_x \otimes_A k \\ \uparrow & & & & \uparrow \\ \mathcal{O}_y & \xrightarrow{\ \psi^*\ } & A' & \xleftarrow{\ \varphi^*\ } & \mathcal{O}_x \\ \downarrow & & & & \downarrow \\ \mathcal{O}_y \otimes_A K & & \xrightarrow{\qquad \tilde{h}_K^* \qquad} & & \mathcal{O}_x \otimes_A K. \end{array}$$

Hence $A' \ni \psi^*(b) = \varphi^*(\tilde{h}_K^*(b)) = \varphi^*(t^{-e} a) = t^{-e} \varphi^*(a)$. This contradicts the fact that $\varphi^*(a) \bar{\in} t A'$. So we get that $\tilde{h}_K^*(\mathcal{O}_y) \subset \mathcal{O}_x$ for each point $x \in (\mathrm{Pic}^0_{\mathscr{C}_\mathfrak{d}/S})_s$, and $\tilde{h}_K$ defines a unique homomorphism $\tilde{h} : \mathrm{Pic}^0_{\mathscr{C}_\mathfrak{d}/S} \to G$ which we want.

Q.E.D.

2. GENERALIZED JACOBIAN SCHEMES. — 2.1. The exact sequence of *fppf*-sheaves on $\mathscr{C}_{\mathfrak{d}}$

$$0 \to \mathbb{G}_{m,\,\mathscr{C}_{\mathfrak{d}}} \to \psi_* \mathbb{G}_{m,\,\mathscr{C}} \to \psi_* \mathbb{G}_{m,\,\mathscr{C}}/\mathbb{G}_{m,\,\mathscr{C}_{\mathfrak{d}}} \to 0 \qquad (2.1.1)$$

defines an exact sequence

$$\begin{aligned}(2.1.2)\quad 0 \to g_* \mathbb{G}_{m,\,\mathscr{C}_{\mathfrak{d}}} \to g_* \psi_* \mathbb{G}_{m,\,\mathscr{C}} \to g_* (\psi_* \mathbb{G}_{m,\,\mathscr{C}}/\mathbb{G}_{m,\,\mathscr{C}_{\mathfrak{d}}})\\ \to R^1 g_* \mathbb{G}_{m,\,\mathscr{C}_{\mathfrak{d}}} \to R^1 g_* (\psi_* \mathbb{G}_{m,\,\mathscr{C}}) \to R^1 g_* (\psi_* \mathbb{G}_{m,\,\mathscr{C}}/\mathbb{G}_{m,\,\mathscr{C}_{\mathfrak{d}}}).\end{aligned}$$

Since g (resp. f) is proper with geometrically integral fibers, we get $g_* \mathbb{G}_{m,\,\mathscr{C}_{\mathfrak{d}}} = \mathbb{G}_{m,\,S}$ (resp. $f_* \mathbb{G}_{m,\,\mathscr{C}} = \mathbb{G}_{m,\,S}$), and so $g_* \mathbb{G}_{m,\,C_{\mathfrak{d}}} \to g_* \psi_* \mathbb{G}_{m,\,\mathscr{C}} = f_* \mathbb{G}_{m,\,\mathscr{C}}$ is nothing but id : $\mathbb{G}_{m,\,S} \to \mathbb{G}_{m,\,S}$. Moreover, we have $R^1 g_* (\psi_* \mathbb{G}_{m,\,\mathscr{C}}) = R^1 f_* \mathbb{G}_{m,\,\mathscr{C}}$ by the following

LEMMA 2.2. — *Let* $\psi : X \to Y$ *be a finite morphism of locally noetherian schemes and* G *be a smooth commutative group scheme over* X. *Then* $R^j \psi_* G = 0$ *for* $j > 0$.

Proof. — By the definition, $R^j \psi_* G$ is the *fppf*-sheaf associated to the presheaf $T \mapsto H^j(T_{fppf}, G_T)$. Since G is smooth over X, $H^j(T_{fppf}, G_T) = H^j(T_{et}, G_T)$ (GB, Th. 11.7). Since ψ is finite, we have $R^j \psi_{et*} G = 0$ for $j > 0$ (SGA4, Exp. VIII, Cor. 5.6), and therefore, $R^j \psi G_* = 0$ for $j > 0$.

2.3. By 2.1 and 2.2, we get an exact sequence

$$\begin{aligned}(2.3.1)\quad 0 \to g_* (\psi_* \mathbb{G}_{m,\,\mathscr{C}}/\mathscr{C}_{m,\,\mathscr{C}_{\mathfrak{d}}})\\ \to R^1 g_* \mathbb{G}_{m,\,\mathscr{C}_{\mathfrak{d}}} \to R^1 g_* (\psi_* \mathbb{G}_{m,\,\mathscr{C}}) \to R^1 g_* (\psi_* \mathbb{G}_{m,\,\mathscr{C}}/\mathbb{G}_{m,\,\mathscr{C}_{\mathfrak{d}}}).\end{aligned}$$

The sheaf $\psi_* \mathbb{G}_{m,\,\mathscr{C}}/\mathbb{G}_{m,\,\mathscr{C}_{\mathfrak{d}}}$ has the support in $\sigma(S)$ and isomorphic to $\sigma_* (\pi_* \mathbb{G}_{m,\,\mathfrak{d}}/\mathbb{G}_{m,\,S})$, where $\pi = f|_{\mathfrak{d}} : \to S$. So $g_* (\psi_* \mathbb{G}_{m,\,\mathscr{C}_{\mathfrak{d}}})$ is isomorphic to $\pi_* \mathbb{G}_{m,\,\mathfrak{d}}/\mathbb{G}_{m,\,S}$ and $R^1 g_* (\psi_* \mathbb{G}_{m,\,\mathscr{C}}/\mathscr{C}/\mathbb{G}_{m,\,\mathscr{C}_{\mathfrak{d}}})0$. $\pi_* \mathbb{G}_{m,\,\mathfrak{d}}$ is represented by a smooth affine S-scheme (Weil restrictiction of $\mathbb{G}_{m,\,\mathfrak{d}}$ with respect to $\pi : \mathfrak{d} \to S$) (*cf.* [4], Chap. I, § 1.6.6). Then we obtain exact sequences of group S-schemes

$$0 \to \pi_* \mathbb{G}_{m,\,\mathfrak{d}}/\mathbb{G}_{m,\,S} \to \mathrm{Pic}_{\mathscr{C}_{\mathfrak{d}}/S} \to \mathrm{Pic}_{\mathscr{C}/S} \to 0, \qquad (2.3.2)$$

and

$$0 \to \pi_* \mathbb{G}_{m,\,\mathfrak{d}}/\mathbb{G}_{m,\,S} \to \mathrm{Pic}^0_{\mathscr{C}_{\mathfrak{d}}/S} \to \mathrm{Pic}^0_{\mathscr{C}/S} \to 0. \qquad (2.3.3)$$

Hereafter we put $\mathscr{J}_{\mathfrak{d}} = \mathrm{Pic}^0_{\mathscr{C}_{\mathfrak{d}}/S}$, $\mathscr{J} = \mathrm{Pic}^0_{\mathscr{C}/S}$ and $\mathscr{L}_{\mathfrak{d}} = \pi_* \mathbb{G}_{m,\,\mathfrak{d}}/\mathbb{G}_{m,\,S}$ for a fixed $\mathscr{C}$.

Example 2.4. — When $\mathfrak{d}$ is isomorphic to $\mathrm{Spec}(A[T]/(T-a)^2)$, $\mathscr{L}_{\mathfrak{d}} = \mathbb{G}_{a,\,S}$.

Example 2.5. — When $\mathfrak{d}$ is isomorphic to $\mathrm{Spec}(A[T]/(T-a)(T-b))$ $(a \neq b)$, $\mathscr{L}_{\mathfrak{d}} = \mathscr{G}^{(\lambda)}$ if $\lambda = a - b \in \mathfrak{M} \setminus \{0\}$ and $\mathscr{L}_{\mathfrak{d}} = \mathbb{G}_{m,\,S}$ if $a - b$ is a unit of A.

2.6. Next, we will study the difference between $\mathscr{J}_{\mathfrak{d}'} \subset \mathrm{O} \rightarrow \mathscr{J}_{\mathfrak{d}}$, when the flat effective divisors $\mathfrak{d}'$ and $\mathfrak{d}$ of $\mathscr{C}/\mathrm{S}$ satisfy $\mathfrak{d}' > \mathfrak{d}$. By virtue of the construction of $\mathscr{C}_{\mathfrak{d}}$ and $\mathscr{C}_{\mathfrak{d}'}$, there exist canonical commutative diagrams:

$$\begin{array}{ccc} \mathscr{C} & \rightarrow & \mathscr{C}_{\mathfrak{d}'} \\ & \searrow \swarrow & \\ & \mathscr{C}_{\mathfrak{d}} & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathscr{J}_{\mathfrak{d}'} & \rightarrow & \mathscr{J}_{\mathfrak{d}} \\ & \nwarrow \nearrow & \\ & \mathscr{J} & \end{array}$$

Now we put $\mathscr{H}_{\mathfrak{d}'/\mathfrak{d}} = \ker(\mathscr{J}_{\mathfrak{d}'} \rightarrow \mathscr{J}_{\mathfrak{d}})$. In particular, $\mathscr{H}_{\mathfrak{d}'/\mathfrak{d}} = \mathscr{L}_{\mathfrak{d}}$. The we get exact sequences

$$(2.6.1) \qquad \begin{array}{l} 0 \rightarrow \mathscr{H}_{\mathfrak{d}'/\mathfrak{d}} \rightarrow \mathscr{J}_{\mathfrak{d}'} \rightarrow \mathscr{J}_{\mathfrak{d}} \rightarrow 0 \\ \text{and} \\ 0 \rightarrow \mathscr{H}_{\mathfrak{d}'/\mathfrak{d}} \rightarrow \mathscr{L}_{\mathfrak{d}'} \rightarrow \mathscr{L}_{\mathfrak{d}} \rightarrow 0. \end{array}$$

PROPOSITION 2.7. — *If* $\mathfrak{d}$ *is an effective divisor of* $\mathscr{C}/\mathrm{S}$, *then* $\mathscr{L}_{\mathfrak{d}} = \pi_* \mathbb{G}_{m,\,\mathfrak{d}}/\mathbb{G}_{m,\,\mathrm{S}}$ *is a successive extension of* $\mathbb{G}_{a,\,\mathrm{S}}$*'s,* $\mathbb{G}_{m,\,\mathrm{S}}$*'s and* $\mathscr{G}^{(\lambda_i)}$*'s.*

Proof. — The proposition follows from 2.5 and 2.6 by induction on r.

2.8. For later use, we investigate more explicit structure of $\mathscr{L}_{\mathfrak{d}} = \pi_* \mathbb{G}_{m,\,\mathfrak{d}}/\mathbb{G}_{m,\,\mathrm{S}}$ for a divisor $\mathfrak{d}$ of type as in the above proposition.

Let $\mathfrak{s}_1^{(i)}, \ldots, \mathfrak{s}_{n_i}^{(i)} : \mathrm{S} \rightarrow \mathscr{C}$ be given n_i sections with $\mathfrak{s}_1^{(i)}(s) = \mathfrak{s}_2^{(i)}(s) = \ldots = \mathfrak{s}_{n_i}^{(i)}(s)$; say $\mathrm{P}_i \in \mathscr{C}_0 \subset \mathscr{C}$ for each $i = 1, \ldots, r$. We denote by $\mathfrak{d}^{(i)}$ $(i = 1, \ldots, r)$ and $\mathfrak{d}$ the formal sums

$$(2.8.1) \qquad \mathfrak{d}^{(i)} = \mathfrak{s}_1^{(i)} + \ldots + \mathfrak{s}_{n_i}^{(1)} \qquad \text{and} \qquad \mathfrak{d} = \sum_{i=1}^{r} \mathfrak{d}^{(i)}.$$

Moreover, we put $\mathfrak{s}_j^{(i)}(\eta) = \mathrm{Q}_j^{(i)} \in \mathscr{C}_\eta \subset \mathscr{C}$ for each (i, j). Here we assume that

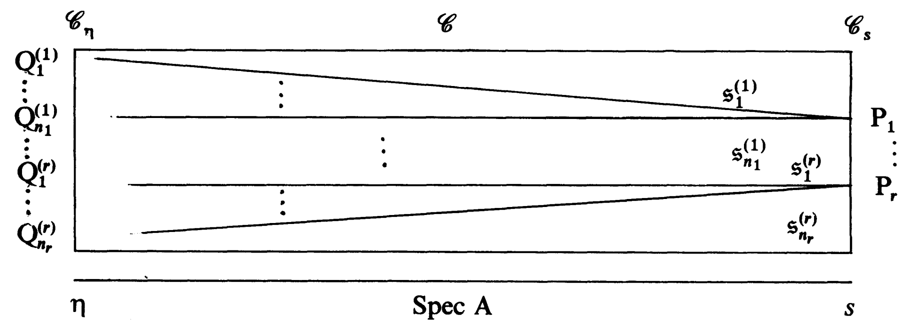
$$\mathrm{P}_k \neq \mathrm{P}_l \qquad \text{if} \quad k \neq l$$

and

$$\mathrm{Q}_j^{(i)} \neq \mathrm{Q}_{j'}^{(i')} \qquad \text{if} \quad (i, j) \neq (i', j').$$

We can draw the following figure:

(2.8.2)



If we take an affine open subset $\mathrm{U} = \mathrm{Spec}(\mathrm{B})$ of $\mathscr{C}$ containing the points $\mathrm{P}_1, \ldots, \mathrm{P}_r$, then since $\mathfrak{s}_j^{(i)}$'s are closed subschemes of $\mathscr{C}$, these sections $\mathfrak{s}_j^{(i)}$'s are contained in U. So, the section $\mathfrak{s}_j^{(i)}$ is defined by a prime ideal $\mathfrak{p}_j^{(i)}$ of B for each (i, j). Our assumption implies that

$$\mathfrak{p}_1^{(i)} \otimes_A k = \ldots = \mathfrak{p}_{n_i}^{(i)} \otimes_A k \subset B \otimes_A k \quad \text{for} \quad i=1, \ldots, r. \tag{2.8.3}$$

We denote $\mathfrak{p}_1^{(i)} \otimes k = \overline{\mathfrak{M}(p_i)}$ for each i. The equations (2.8.3) imply that

$$\mathfrak{p}_k^{(i)} + t\,B = \mathfrak{p}_l^{(i)} + t\,B \quad \text{for any } k, l; \tag{2.8.4}$$

which we denote by $\mathfrak{M}(P_i)$. If we take U to be suitably small, then we can choose an element $\bar{x} \in B \otimes_A k$ such that $\bar{x} - (\mathfrak{s}_j^{(i)*} \otimes k)(\bar{x})$ is a local parameter of $\overline{\mathfrak{M}(P_i)}$ for each i. Let x be an element of B lying over $\bar{x}$ by the canonical surjection $B \to B \otimes_A k$. We put $x_j^{(i)} = x - \mathfrak{s}_j^{(i)*}(x) \in \mathfrak{p}_j^{(i)}$ for each (i, j). Then we can see that

$$\mathfrak{p}_j^{(i)} = (x_j^{(i)}). \tag{2.8.5}$$

Obviously by the definition, we get

$$B_{\mathfrak{d}} = \Gamma(U, \mathscr{A}_{\mathfrak{d}}) = A + \prod_{i,\, j} \mathfrak{p}_j^{(i)}. \tag{2.8.6}$$

Moreover we get the isomorphy

$$\Gamma(S, \pi_* \mathbb{G}_{m,\,\mathfrak{d}}/\mathbb{G}_{m,\,S}) = \Gamma(U, \mathscr{I}_{\mathfrak{d}}) \tag{2.8.7}$$
$$= \left(\bigcap_{i=1}^{r} \mathcal{O}_{\mathscr{C},\, P_i} \right)^{\times} \Big/ (A + \prod_{i,\, j} (x_j^{(i)}))^{\times} / A^{\times} \cong \prod_{i=1}^{r} (\mathcal{O}_{\mathscr{C},\, P_i} / \prod_{i,\, j} (x_j^{(i)}))^{\times} / A^{\times}.$$

Moreover, in our case,

$$B \supset (t, x_1^{(i)}) = (t, x_2^{(i)}) = \ldots = (t, x_{n_i}^{(i)}) = \mathfrak{M}(P_i) \tag{2.8.8}$$

and

$$\mathcal{O}_{\mathscr{C},\, P_i} = B_{\mathfrak{M}(P_i)} \text{: regular local ring.} \tag{2.8.9}$$

Hence, by the structure theorem of complete local rings, we get

$$\hat{\mathcal{O}}_{\mathscr{C},\, P_i} = \hat{B}_{\mathfrak{M}(P_i)} \cong A[[x_j^{(i)}]] \qquad (j=1, \ldots, n_i). \tag{2.8.10}$$

Therefore, we can rewrite (2.8.7) in the following form:

$$\Gamma(S, \mathscr{L}_{\mathfrak{d}}) \cong \prod_{i=1}^{r} (A[[x_j^{(i)}]] / (\prod_{i,\, j} x_j^{(i)}))^{\times} / A^{\times}. \tag{2.8.11}$$

Now we consider the A-algebra

$$A[[x_j^{(i)}]] / (\prod_{i,\, j} x_j^{(i)}),$$

and its spectrum

$$\pi_T : T=\operatorname{Spec}(A\,[[x_j^{(i)}]]/(\prod_{i,\,j} x_j^{(i)}) \to S=\operatorname{Spec} A.$$

Then obviously we get

$$\Gamma(S, (\pi_T)_* \mathbb{G}_{m,\,T})=(A\,[[x_j^{(i)}]]/(\prod_{i,\,j} x_j^{(i)}))^\times \supset A^\times,$$

and the canonical injection $\mathbb{G}_{m,\,S} \to (\pi_T)_* (\mathbb{G}_{m,\,T})$.
Here we put

$$\mathscr{L}^{(x_1^{(i)},\,\dots,\,x_{n_i}^{(i)})}=(\pi_T)_* (\mathbb{G}_{m,\,T})/\mathbb{G}_{m,\,S}. \tag{2.8.12}$$

Then we get the isomorphism

$$\mathscr{L}_\mathfrak{d} \cong (\mathbb{G}_{m,\,S})^{r-1} \times \prod_{i=1}^{r} \mathscr{L}^{(s_1^{(i)},\,\dots,\,x_{n_i}^{(i)})}. \tag{2.8.13}$$

IV. The liftability of *p*-cyclic coverings

Throughout this chapter, k denotes an algebraically closed field of characteristic $p>0$.

1. Tamely ramified case. — 1.1. Let C_0 be a complete non-singular curve of genus g over k. G_0 be a finite abelian subgroup of the automorphism group $\operatorname{Aut}(C_0)$ of C_0. We denote by $\pi : C_0 \to D_0 : =C_0/G_0$ the canonical morphism, by $\mathfrak{d}_0$ the conductor of C_0/D_0, and by $J_{\mathfrak{d}_0}$ the generalized Jacobien variety of the singular curve $D_{0_{\mathfrak{d}_0}}$. Then we get the canonical exact sequence

$$(E_0) \quad 0 \to L_{\mathfrak{d}_0} \to J_{\mathfrak{d}_0} \to J=J(D_0) \to 0. \tag{1.1.1}$$

Moreover, by Lang's class field theory, there exists an extension

$$0 \to G_0 \to J'_{\mathfrak{d}_0} \to J_{\mathfrak{d}_0} \to 0, \tag{1.1.2}$$

which gives the covering C_0/D_0:

$$\begin{array}{ccc} C_0 & \dashrightarrow & J'_\mathfrak{d} \\ \downarrow & \square & \downarrow \\ D_0 & \dashrightarrow & J_{\mathfrak{d}_0}, \end{array} \tag{1.1.3}$$

where the dotted arrow of the bottom is the natural rational map. The sequence (1.1.2) can be decomposed into the following commutative diagram:

$$
(1.1.4)\qquad
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & N_0 & \to & G_0 & \to & H_0 & \to & 0 \\
 & & \downarrow & \square & \downarrow & & \downarrow & & \\
0 & \to & L'_{\mathfrak{d}} & \to & J'_{\mathfrak{d}_0} & \to & J' & \to & 0 \\
 & & {\scriptstyle f_0}\downarrow & \square & \downarrow & & \downarrow{\scriptstyle g_0} & & \\
0 & \to & L_{\mathfrak{d}_0} & \to & J_{\mathfrak{d}_0} & \to & J & \to & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

We denote by E'_0 and E_0 the second and the third horizontal lines in (1.1.4), respectively. Moreover, we choose arbitrarily a lifting $\mathscr{D}$ over $W(k)$ of D_0 (*cf.* SGA 1, Exp. III, Cor. 7.4.).

THEOREM 1.2. — *Suppose* G_0 *has order* n *with* $(n, p)=1$. *In this case, the conductor* $\mathfrak{d}_0$ *is given by* $\mathfrak{d}_0=\sum_{i=1}^{r} P_{0i}$, *where* P_{0i}*'s are mutually distinct ramification points* (*cf.* [21], Chap. VI, § 2, n° 12, Ex. 1°). *We choose arbitrary sections* $\mathfrak{s}_1, \ldots, \mathfrak{s}_r$ *of the structure morphism* $\mathscr{D} \to \operatorname{Spec}(W(k))$ *such that* $\mathfrak{s}_i(s)=P_{0i}$ *for each* $i=1, \ldots, r$ *where s is the special point of* $\operatorname{Spec}(W(k))$. *We put* $\mathfrak{d}=\sum_{i=1}^{r} \mathfrak{s}_i$. *Then there exists a lifting* $\mathscr{C}/\mathscr{D}$ *of* C_0/D_0 *over* $W(k)$ *with ramification locus* $\mathfrak{d}$.

Proof. — By our assumption, the exact sequence

$$(1.2.1)\qquad (E)\quad 0 \to \mathscr{L}_{\mathfrak{d}} \to \mathscr{J}_{\mathfrak{d}} \to \mathscr{J}=\mathscr{J}(\mathscr{D}) \to 0$$

is a deformation of the exact sequence E_0 over $W(k)$ (*cf.* Ch. II,2). Now we take the decompositions into cyclic groups:

$$(1.2.2)\qquad N_0=\prod_{i=1}^{r_1} \mu_{a_i, k},\ H_0=\prod_{j=1}^{r_2} \mu_{b_j, k}, \quad \text{and} \quad G_0=\prod_{l=1}^{r_3} \mu_{c_l, k};$$

and we put

$$(1.2.3)\qquad \mathscr{N}=\prod_{i=1}^{r_1} \mu_{a_i}, W(k), \qquad \mathscr{H}=\prod_{j-1}^{r_2} \mu_{b_j}, W(k),$$

$$\mathscr{G}=\prod_{l=1}^{r_8} \mu_{c_l}, W(k),$$

where a_i, b_j, c_l's are the invariants of N_0, H_0 and G_0, respectively.

Since the reduction maps

$$(1.2.4)\qquad \begin{array}{ccc} \mathrm{Ext}^1(\mathscr{J}, \mathscr{H}) & \to & \mathrm{Ext}^1(\mathrm{J}, \mathscr{H}_s) \\ \wr\downarrow & & \wr\downarrow \\ \prod_i \mathscr{J}^t[a_i](\mathrm{W}(k)) & \to & \prod_i \mathrm{J}^t[a_i](k) \end{array}$$

are bijective, there exists a unique lifting

$$(1.2.5)\qquad 0 \to \mathscr{H} \to \mathscr{J}' \xrightarrow{g} \mathscr{J} \to 0$$

of the third vertical line of (1.1.4), over $\mathrm{W}(k)$. In our case,

$$\mathscr{L}_{\mathfrak{d}} \simeq (\mathbb{G}_{m,\,\mathrm{A}})^{r-1} \leftarrow \mathrm{L}_{\mathfrak{d}_0} \simeq (\mathbb{G}_{m,\,k})^{r-1}$$

and the reduction map

$$(1.2.6)\qquad \begin{array}{ccc} \mathrm{Ext}^1(\mathscr{L}_{\mathfrak{d}}, \mathscr{N}) & \to & \mathrm{Ext}^1(\mathrm{L}_{\mathfrak{d}_0}, \mathrm{N}_0) \\ \wr\downarrow & & \wr\downarrow \\ (\prod_i \mathbb{Z}/a_i)^{r-1} & \to & (\prod_i \mathbb{Z}/a_i)^{r-1} \end{array}$$

is an isomorphism. Therefore, there exists also a unique lifting

$$(1.2.7)\qquad 0 \to \mathscr{N} \to \mathscr{L}'_{\mathfrak{d}} \to \mathscr{L}_{\mathfrak{d}} \to 0$$

of the first vertical line of (1.1.4) over $\mathrm{W}(k)$. Moreover, since $\mathrm{Ext}^2(\mathscr{J}', \mathscr{N})=0$ (*cf.* Ch. II, 4), we get the exact sequence

$$(1.2.8)\qquad 0 \to \mathrm{Ext}^1(\mathscr{J}', \mathscr{N}) \to \mathrm{Ext}^1(\mathscr{J}', \mathscr{L}'_{\mathfrak{d}}) \to \mathrm{Ext}^1(\mathscr{J}', \mathscr{L}_{\mathfrak{d}}) \to 0.$$

Hence, there exists an extension

$$(1.2.9)\qquad (\mathrm{E}')\quad 0 \to \mathscr{L}'_{\mathfrak{d}} \to \mathscr{J}'_{\mathfrak{d}} \to \mathscr{J}' \to 0$$

such that $f_*(\mathrm{E}')=g^*(\mathrm{E})$. Now we take the special fiber E'_s of the extension E'. Since the reduction map

$$(1.2.10)\qquad \mathrm{Ext}^1(\mathscr{J}', \mathscr{N}) \to \mathrm{Ext}^1(\mathrm{J}', \mathrm{N}_0)$$

is surjective (*cf.* Ch. II. 2), and noting that

$$(1.2.11)\qquad \mathrm{E}'_0 - \mathrm{E}'_s \in \mathrm{Ker}((f_0)_*) = \mathrm{Ext}^1(\mathrm{J}', \mathrm{N}_0),$$

there exists an extension $E'' \in \mathrm{Ext}^1(\mathscr{J}', \mathscr{L}_{\mathfrak{d}})$ such that $E''_s = E'_0 - E'_s$. Replacing E' by $E' + E'' \in \mathrm{Ext}^1(\mathscr{J}', \mathscr{L}'_{\mathfrak{d}})$, which we denote also by E', we get the commutative diagram

$$
(1.2.12) \qquad
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \mathscr{N} & \rightarrow & \mathscr{G} & \rightarrow & \mathscr{H} & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \mathscr{L}'_{\mathfrak{d}} & \rightarrow & \mathscr{J}'_{\mathfrak{d}} & \rightarrow & \mathscr{J}' & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \mathscr{L}_{\mathfrak{d}} & \rightarrow & \mathscr{J}_{\mathfrak{d}} & \rightarrow & \mathscr{J} & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

whose special fiber is just equal to (1.1.4). Now we take the fiber product $\mathscr{C} = \mathscr{D} \times_{\mathscr{J}_{\mathfrak{d}}} \mathscr{J}'_{\mathfrak{d}}$:

$$
(1.2.13) \qquad
\begin{array}{ccc}
\mathscr{C} & \dashrightarrow & \mathscr{J}'_{\mathfrak{d}} \\
\downarrow & & \downarrow \\
\mathscr{D} & \dashrightarrow & \mathscr{J}_{\mathfrak{d}}
\end{array}
$$

where the dotted arrow of the bottom is the natural rational map. Then obviously this covering $\mathscr{C}/\mathscr{D}$ is the required one.

Q.E.D.

2. Proof of the main theorem. — 2.1. We assume now $G_0 = \langle \sigma_0 \rangle$ be a cyclic subgroup of order p of $\mathrm{Aut}(C_0)$. We put $A = W(k)[\xi]$, where ζ is a primitive p-th root of unity. Then we get our main theorem.

Theorem 2.2. — *There exists a lifting $\mathscr{C}/\mathscr{D}$ of the given p-cyclic covering C_0/D_0, over* A.

We will prove the main theorem in two steps: (1) $D_0 = \mathbb{P}^1_k$; (2) D_0 general. We start with the following general lemma.

Lemma 2.3. — *Let* $(A, \mathfrak{M})$ *be a discrete valuation ring. We put* $S = \mathrm{Spec}\, A$. *We denote by* η *and s the generic and special points of* S, *respectively. Let* $\pi : \mathscr{C} \to S$ *be a projective flat morphism with geometrically integral curves as fibres. Assume that* $g(\mathscr{C}_\eta) = g(\mathscr{C}_s)$, *where g means the geometric genus. Then the normalization* $\tilde{\pi} : \tilde{\mathscr{C}} \to S$ *of* $\mathscr{C}$ *has the non-singular models of* $\mathscr{C}_\eta$ *and* $\mathscr{C}_s$ *as fibres.*

Proof. — By the compatibility of the normalization and the localization, we get $\tilde{\mathscr{C}}_\eta = (\mathscr{C}_\eta)^{\sim}$, where the symbol $\sim$ means the normalization. Hence the generic fibre $\tilde{\mathscr{C}}_\eta$ is a non-singular model of $\mathscr{C}_\eta$, and $p_a(\tilde{\mathscr{C}}_\eta) = g(\mathscr{C}_\eta)$, where p_a means the arithmetic genus. Since the structure morphism $\tilde{\pi}$ is flat, we get $p_a(\tilde{\mathscr{C}}_\eta) = p_a(\tilde{\mathscr{C}}_s)$. Hence we get the equality $g(\tilde{\mathscr{C}}_s) = p_a(\tilde{\mathscr{C}}_s)$. On the other hand, there exists the inequalities $p_a(\tilde{\mathscr{C}}_s) \geqq g(\tilde{\mathscr{C}}_s) \geqq g(\mathscr{C}_s)$. Hence we get the equalities $p_a(\tilde{\mathscr{C}}_s) = g(\tilde{\mathscr{C}}_s) = g(\mathscr{C}_s)$ which imply that $\tilde{\mathscr{C}}_s$ is non-singular.

Q.E.D.

2.4. Let (C_0, σ_0) be a pair of complete non-singular curve C_0 over k and an automorphism σ_0 of C_0 of order p. We assume that $C_0/\langle\sigma_0\rangle = \mathbb{P}^1_k$. Let $m_0 = \sum_{i=1}^{r} n_i P_{0\,i}$ [with $P_{0\,i} = (\alpha_{0\,i} : 1) \in \mathbb{P}^1_k$; $\alpha_{0\,i} \in k$; $\alpha_{0\,i} \neq \alpha_{0\,j}\,(i \neq j)$] be the conductor of $C_0/\mathbb{P}^1_k$, where $(x:y)$ are the homogeneous coordinates of $\mathbb{P}^1_k$. It is well-known that $n_i \geqq 2$ and $(n_1 - 1, p) = 1$ for each i (*cf.*, e. g., [21], Chap. VI, n° 12, Ex. 2°). Moreover, by the theory of Artin-Schreier, there exists a function

$$(2.4.1) \qquad f(x) = g(x) \Big/ \prod_{i=1}^{r} (x - \alpha_{0\,i})^{n_i - 1},$$

with $(x - \alpha_{0\,i}) \nmid g(x)$ and $\deg(g) = \sum_{i=1}^{r} (n_i - 1)$, such that C_0 can be given by the following cartesian products:

$$(2.4.2) \qquad \begin{array}{ccccc} C_0 & \longrightarrow & \mathbb{P}^1_k & \dashrightarrow & \mathbb{G}_{a,k} \\ \downarrow & \square & \downarrow \Phi & \square & \downarrow \; \wp : x \longmapsto x^p - x. \\ \mathbb{P}^1_k & \xrightarrow{\;f\;} & \mathbb{P}^1_k & \dashrightarrow & \mathbb{G}_{a,k} \end{array}$$

In this case, the genus of C_0 is given by

$$(2.4.3) \qquad g(C_0) = (p-1)(-2 + \sum n_i)/2$$

(*cf.* Hasse [5], p. 43). Under these notations, we get the following.

PROPOSITION 2.5. — *There exists a pair* $(\mathscr{C}, \sigma)$ *of a proper smooth curve* $\mathscr{C}$ *over* A *and an automorphism* σ *of* $\mathscr{C}$ *such that* $(\mathscr{C}, \sigma) \otimes_A k \simeq (C_0, \sigma_0)$.

Proof. — We choose elements $\alpha_1, \ldots, \alpha_r \in A$ so that $\alpha_i \,(\text{mod}\ \mathfrak{M}) = \alpha_{0\,i}$ for each i. Now we put

$$(2.5.1) \qquad G_0(x, y) = y^N g(x/y),$$

where $N = \sum_{i=1}^{r} (n_i - 1)$. Then we can choose a homogeneous polynomial $G(x, y) \in A[x, y]$ of degree N satisfying the following conditions:

$$(2.5.2) \qquad G(x, y) \,(\text{mod}\ \mathfrak{M}) = G_0(x, y),$$

and the equation

$$(2.5.3) \qquad F(x, 1) = -1/\lambda^p$$

has simple roots, where

$$(2.5.4) \qquad F(x, y) = G(x, y) \Big/ \prod_{i=1}^{r} (x - \alpha_i y)^{n_i - 1}.$$

In fact, first we choose arbitrary a homogeneous polynomial G_1 satisfying (2.5.2). Then the equation (2.5.3) can be rewritten in the form:

$$\lambda^p G_1(x, 1)+\prod_i (x-\alpha_i)^{n_i-1}=0. \tag{2.5.5}$$

Let

$$\lambda^p G_1(x, 1)+\prod_i (x-\alpha_i)^{n_i-1}=(\lambda^p-1)\prod_{j=1}^{s} H_j(x)^{e_j} \tag{2.5.6}$$

be the decomposition of the left hand side of (2.5.5) into monic irreducible factors $H_j(x)$'s in $A[x]$. Now we choose elements a_{jl} ($j=1, \ldots, s$; $l=1, \ldots, e_j$) of A so that

$$H_j(x-a_{jl}\lambda^p)\neq H_{j'}(x-a_{j'l'}\lambda^p) \quad \text{if} \quad (j, l)\neq(j', l'). \tag{2.5.7}$$

Here we put

$$G(x, 1)=\frac{\lambda^p-1}{\lambda^p}\left\{\prod_{j=1}^{s}\prod_{l=1}^{e_j} H_j(x-a_{jl}\lambda^p)-\prod_{j=1}^{s} H_j(x)^{e_j}\right\}+G_1(x, 1). \tag{2.5.8}$$

Then obviously this satisfies our conditions.

Now we define $\mathscr{C}'$ by the cartesian product:

$$\begin{array}{ccc} \mathscr{C}' & \longrightarrow & \mathbb{P}^1_A \\ \downarrow & \square & \downarrow \psi \\ \mathbb{P}^1_A & \longrightarrow & \mathbb{P}^1_A, \end{array} \tag{2.5.9}$$

where ψ is the morphism defined by Ch. I.1. Obviously, the fibres $F^{-1}(1, -\lambda^p)$ and $F^{-1}(1, 0)$ consist of N distinct points and r distinct points, respectively, and these points are all the ramification points of $\mathscr{C}'_\eta/\mathbb{P}^1_k$ [note $(n_i-1, p)=1$]. Hence, by Hurwitz' theorem, we get

$$g(\mathscr{C}'_\eta)=(p-1)\left(-2+\sum_{i=1}^{r} n_i\right)\Big/2. \tag{2.5.10}$$

So, by (2.4.3), we get the equality

$$g(\mathscr{C}'_\eta)=g(\mathscr{C}'_0)=g(C_0)=(p-1)(-2+\textstyle\sum n_i)/2. \tag{2.5.11}$$

Therefore, by Lemma 2.3, the normalization $\mathscr{C}=\tilde{\mathscr{C}}'$ gives a lifting of the Galois covering $C_0\to\mathbb{P}^1$ [*i.e.*, of (C_0, σ_0)] over A.

2.6. We pass to the proof in the general case. Since the unramified case is already known, we may assume that C_0/D_0 has ramifications, and so the conductor $\mathfrak{d}_0$ of C_0/D_0 is non-trivial. In our case, the conductor $\mathfrak{d}_0$ is of the type

$$\mathfrak{d}_0=\sum_{i=1}^{r} n_i P_{0\,i}, \tag{2.6.1}$$

with $n_i \geqq 2$ and $(n_i - 1, p) = 1$(*cf.* [21], Chap. VI, n° 12, Ex. 2), and in the diagram (4.1.3), $N_0 = G_0$ and $H_0 = \{0\}$. Now we choose arbitrary distinct elements $\alpha_{0\,1}, \ldots, \alpha_{0\,r} \in k$ and we put

$$(2.6.2) \qquad P'_{0\,i} = (\alpha_{0\,i} : 1) \in \mathbb{P}^1_k \qquad (i = 1, \ldots, r).$$

Moreover we put

$$(2.6.3) \qquad \mathfrak{d}'_0 = \sum_{i=1}^{r} \cdot n_i P'_{0\,i} \qquad \text{[compare with (4.2.1)]},$$

and take the singular curve $(\mathbb{P}^1_k)_{\mathfrak{d}'_0}$. Then obviously we get a natural isomorphism

$$(2.6.4) \qquad J((\mathbb{P}^1_k)_{\mathfrak{d}'_0}) \simeq L_{\mathfrak{d}'_0}.$$

Hence, by (*loc. cit.*, Chap. VI, n° 11, Prop. 9.10), the first vertical line of (1.1.3) determines a cyclic covering $C'_0/\mathbb{P}^1_k$ by the cartesian product:

$$(2.6.5) \qquad \begin{array}{ccc} & & 0 \\ & & \downarrow \\ & & N_0 \\ & & \downarrow \\ C'_0 & \dashrightarrow & L'_{\mathfrak{d}_0} \\ \downarrow & \square & \downarrow f_0 \\ \mathbb{P}^1 & \dashrightarrow & J((\mathbb{P}^1_k)_{\mathfrak{d}'_0}) \simeq L_{\mathfrak{d}_0}. \end{array}$$

Obviously the conductor of this covering is just equal to $\mathfrak{d}'_0$. By virtue of Proposition 2.5, we can extend the diagram (2.6.5) over A:

$$(2.6.6) \qquad \begin{array}{ccc} & & 0 \\ & & \downarrow \\ & & N \\ & & \downarrow \\ \mathscr{C}' & \dashrightarrow & \mathscr{L}' \\ \downarrow & \square & \downarrow f \\ \mathbb{P}^1_k & \dashrightarrow & \mathscr{L}, \end{array}$$

where $\mathscr{N}=(\mathbb{Z}/p)_A$. By the proof of Proposition 2.5, there exist elements $\beta_j^{(i)} \in A$ $(i=1, \ldots, r; j=1, \ldots, n_i)$ such that when we put $P_j^{(i)}=(\beta_j^{(i)}: 1) \in \mathbb{P}_A^1$,

$$\mathscr{L} \simeq (\mathbb{G}_{m,\,A})^{r-1} \times_S \prod_{i=1}^{r} \mathscr{L}^{(x_1^{(i)},\, \ldots,\, x_{n_i}^{(i)})}, \tag{2.6.7}$$

where $S=\operatorname{Spec}(A)$, $x_j^{(i)}=x-\beta_j^{(i)}$ and x is the affine coordinate of $\mathbb{P}_A^1$ (*cf.* Ch. III, 2.8.12, 2.8.13). By the isomorphy (2.8.13) of Ch. III, we can choose N sections $\mathfrak{s}_1, \ldots, \mathfrak{s}_N$ of $\mathscr{D}/S$ such that if we put $\mathfrak{d}=\sum_{i=1}^{N} \mathfrak{s}_i$, then the special fibre $\mathfrak{d}_s=\mathfrak{d}_0$ (*cf.* 2.6.1), and $\mathscr{L}_{\mathfrak{d}}=\mathscr{L}$. Hence we get the diagram:

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ & & & & \mathscr{N} & & & & \\ & & & & \downarrow & & & & \\ & & & & \mathscr{L}' & & & & \\ & & & & {\scriptstyle f}\downarrow & & & & \\ 0 & \to & \mathscr{L} & \simeq & \mathscr{L}_{\mathfrak{d}} & \to & \mathscr{J}_{\mathfrak{d}} & \to & \mathscr{J} \to 0, \end{array} \tag{2.6.8}$$

where $\mathscr{J}_{\mathfrak{d}}=\mathscr{J}(\mathscr{D}_{\mathfrak{d}})$ and $\mathscr{J}=\mathscr{J}(\mathscr{D})$. By Ch. II, Cor. 4.4, we get the exact sequence

$$0 \to \operatorname{Ext}^1(\mathscr{J}, \mathscr{N}) \to \operatorname{Ext}^1(\mathscr{J}, \mathscr{L}') \xrightarrow{f_*} \operatorname{Ext}^1(\mathscr{J}, \mathscr{L}) \to 0. \tag{2.6.9}$$

By the surjectivity of f_*, we can complete the diagram (2.6.8) in the following way:

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & \mathscr{N} & = & \mathscr{G} & & \\ & & \downarrow & & \downarrow & & \\ 0 & \to & \mathscr{L}' & \longrightarrow & \mathscr{J}' & \to & \mathscr{J} \to 0 \\ & & {\scriptstyle f}\downarrow & & \downarrow & & \\ & & \mathscr{L} \simeq \mathscr{L}_{\mathfrak{d}} & \to & \mathscr{J}_{\mathfrak{d}} & \to & \mathscr{J} \to 0. \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array} \tag{2.6.10}$$

Let $E \in \operatorname{Ext}^1(\mathscr{J}, \mathscr{L}')$ be the element corresponding to the extension

$$0 \to \mathscr{L}' \to \mathscr{J}' \to \mathscr{J} \to 0. \tag{2.6.11}$$

Since $(f_0)_*(E_s)=(f_0)_*(E_0)$, $E_0-E_s\in \mathrm{Ext}^1(J, \mathbb{Z}/p)$. Then, by virtue of Ch. II, Th. 2.13, there exists an element $E'\in \mathrm{Ext}^1(\mathscr{J}, (\mathbb{Z}/p)_A)$ such that $E'_s=E_0-E_s$. So if we replace (2.6.11) by the extension corresponding to the element $E+E'\in \mathrm{Ext}^1(\mathscr{J}, \mathscr{L}')$ (NB. 2.6.11), then we get such a commutative diagram like (2.6.10), whose special fibre is just equal to (1.1.4). We denote again this modified diagram by (2.6.10). Now we take the fibre product

$$\begin{array}{ccc} \mathscr{C}=\mathscr{D}\times_{\mathscr{J}_{\mathfrak{d}}}\mathscr{J}' : \mathscr{C} & \dashrightarrow & \mathscr{J}' \\ \downarrow & & \downarrow \\ \mathscr{D} & \dashrightarrow & \mathscr{J}_{\mathfrak{d}}. \end{array}$$

By the choice of f, the conductor of the generic fibre $\mathscr{C}_\eta/\mathscr{D}_\eta$ is just equal to $\Sigma_i(\mathfrak{s}_i)_\eta$. Hence, by Hurwitz' theorem,

$$g(\mathscr{C}_\eta)=p.g(D_0)+(p-1)(-2+\Sigma n_i)/2. \tag{2.6.13}$$

This is nothing but the genus of $\mathscr{C}_s=C_0$. Therefore, by Lemma 2.2, the normalization $\tilde{\mathscr{C}}$ of $\mathscr{C}$ gives the required one.

Q.E.D.

Combining Theorem 1.2 and Theorem 2.2, we get the following.

COROLLARY 2.7. — *Under the notation in* 1.1, *suppose that the Galois group* $G_0=\langle \sigma_0 \rangle$ *is a cyclic group of order* pm *with* $(p, m)=1$. *Then there exists a lifting* $(\mathscr{C}, \sigma)$ *of* (C_0, σ_0), *over* A.

Proof. — We put $\sigma_{01}=\sigma_0^p$ and $\sigma_{02}=\sigma_0^m$. Moreover, we put $D_{01}=C_0/\langle \sigma_{01} \rangle$, $D_{02}=C_0/\langle \sigma_{02} \rangle$. Then C_0 is nothing but the normalization of the cartesian product C'_0:

$$\begin{array}{ccccc} & \swarrow & C'_0 & \searrow & \\ D_{01} & & & & D_{02}. \\ & \searrow & D_0 & \swarrow & \end{array} \tag{5.7.1}$$

By the theorem 2.6, there exists a lifting $\mathscr{D}_1\to\mathscr{D}$ over A of $D_{01}\to D_0$. On the other hand, by the theorem 1.2, we can choose a lifting $\mathscr{D}_2\to\mathscr{D}$ over A of $D_{02}/\mathscr{D}$ so that if a point $P\in D_0$ is ramefied in both extensions D_{01}/D_0 and D_{01}/D_0, the ramification A-section of $\mathscr{D}_2/\mathscr{D}$ passing through P is one of $\mathscr{D}_2/\mathscr{D}$. Now we take the fibre product $\mathscr{C}'=\mathscr{D}_1\times_{\mathscr{D}}\mathscr{D}_2$. Then obviously the geometric genera of the generic and the special fibres of $\mathscr{C}'$ coincide. Hence by Lemma 2.3, the normalization $\tilde{\mathscr{C}}$ of $\mathscr{C}$ has the non-singular models of the fibres of $\mathscr{C}$, and it is the one we wanted.

Q.E.D.

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