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CENTRALIZERS OF ANOSOV DIFFEOMORPHISMS ON TORI

BY J. PALIS AND J. C. YOCCOZ

ABSTRACT. — We prove here that the elements of an open and dense subset of Anosov diffeomorphisms on tori have trivial centralizers: they only commute with their own powers.

1. Introduction

Let M be a smooth connected compact manifold, and $\text{Diff}(M)$ the group of C^∞ diffeomorphisms of M endowed with the C^∞ topology. The diffeomorphisms which satisfy Axiom A and the (strong) transversality condition—every stable manifold intersects transversely every unstable manifold—form an open subset $\mathfrak{A}(M)$ of $\text{Diff}(M)$ and are, by Robbin [4] and a recent result of Mañé [2], exactly the C^1 -structurally stable diffeomorphisms.

We continue here the study, initiated in [3], of centralizers of diffeomorphisms in $\mathfrak{G}(M)$; the concepts we just mentioned are detailed there. We now treat the relevant case where M is the torus \mathbb{T}^n and the diffeomorphisms are such that all of \mathbb{T}^n is a hyperbolic set (Anosov diffeomorphisms). Recall that for $f \in \text{Diff}(M)$, its centralizer $Z(f)$ in $\text{Diff}(M)$ is defined as the set of elements that commute with f . We say that f has trivial centralizer if $Z(f)$ is reduced to the iterates $\{f^n, n \in \mathbb{Z}\}$ of f .

CONJECTURE. — *There is an open and dense subset of $f \in \mathfrak{A}(M)$ which have trivial centralizer.*

N. Kopell [1] proved this conjecture for $M = S^1$. In [3], we investigated the conjecture in higher dimensions. We proved it for $\dim M = 2$, and that (in any dimension) the centralizer is trivial for a residual set of $f \in \mathfrak{A}(M)$. Actually, one can state the question for any connected component of $\mathfrak{A}(M)$, where the topological dynamics stays the same. We then also proved the conjecture when the topological dynamics exhibit one periodic attractor or repeller [3].

The purpose of this paper is to settle the case of Anosov diffeomorphisms on tori.

THEOREM. — *For an open and dense subset of Anosov diffeomorphisms of the n -dimensional torus \mathbb{T}^n , the centralizer is trivial.*

Some basic facts about Anosov diffeomorphisms that we shall use in the sequel are collected in Section 2 and in Section 3 we recall a previous but recent result [3] that is

also needed here. A basic proposition (Proposition 2) is stated in Section 4 and, from it and the result recalled in Section 3, we prove the main theorem. Sections 5 through 7 are dedicated to the proof of this basic proposition. First, in Section 5, we present a statement about the induced linear automorphism (Proposition 3) and show in Section 6 that it implies Proposition 2. Finally, in Section 7, we provide the proof of Proposition 3.

It is worthwhile noting that, while our techniques most probably can be generalized to the case of Anosov diffeomorphisms on infranilmanifolds (which are, conjecturally, the only ones to exist) they do not apply directly to an "abstract" Anosov diffeomorphism: the algebraic structure of the manifold is strongly used. We also point out that there is a version of the above conjecture for flows, which was solved by Sad [5].

2. Anosov diffeomorphisms on tori

We recall some basic material on Anosov diffeomorphisms on tori. For $A \in \text{GL}_n(\mathbb{Z}) = \text{Aut}(H_1(\mathbb{T}^n, \mathbb{Z}))$, we denote by $\text{Homeo}_A(\mathbb{T}^n)$ [resp. $\text{Diff}_A(\mathbb{T}^n)$, resp. $\mathcal{D}_A(\mathbb{T}^n)$] the set of homeomorphisms (resp. diffeomorphisms, resp. Anosov diffeomorphisms) of \mathbb{M} which induce A in homology. Then, $\mathcal{D}_A(\mathbb{T}^n)$ is non empty if and only if A is hyperbolic. We fix from now on some hyperbolic $A \in \text{GL}_n(\mathbb{Z})$. The main result about $\mathcal{D}_A(\mathbb{T}^n)$ is best expressed by considering the finite covering $\tilde{\mathcal{D}}_A(\mathbb{T}^n)$ of $\mathcal{D}_A(\mathbb{T}^n)$ formed by pairs $(f, p) \in \mathcal{D}_A(\mathbb{T}^n) \times \mathbb{T}^n$ such that $f(p) = p$. We recall that any $f \in \mathcal{D}_A(\mathbb{T}^n)$ has at least one fixed point.

PROPOSITION 0. — *For $(f, p) \in \tilde{\mathcal{D}}_A(\mathbb{T}^n)$, there is a unique $h = h_f$ in $\text{Homeo}_1(\mathbb{T}^n)$ such that $hf h^{-1} = A$ and $h(p) = 0$. The map $f \mapsto h_f$ is continuous.*

Proof. — We only recall briefly the proof of unicity; more precisely, any homeomorphism h of \mathbb{T}^n which commutes with A and fixes 0 belongs to $\text{GL}_n(\mathbb{Z})$. Indeed, replacing h by hB^{-1} , where B is induced by h in homology and therefore commutes with A , we can assume that $h \in \text{Homeo}_1(\mathbb{T}^n)$. Then, for the lift \tilde{h} of h to \mathbb{R}^n which fixes the origin 0 , we have

$$\text{Sup}_{\mathbb{R}^n} \|\tilde{h}(y) - y\| < +\infty.$$

From this we get, for any $y \in \mathbb{R}^n$,

$$\text{Sup}_{n \in \mathbb{Z}} \|A^n(\tilde{h}(y) - y)\| = \text{Sup}_{n \in \mathbb{Z}} \|\tilde{h}(A^n(y)) - A^n(y)\| < +\infty.$$

As A is hyperbolic, this implies that \tilde{h} is the identity. ■

3. A previous basic result

The proof of the theorem is based on two results, one of which was proved in [3] and that we recall now.

For $f \in \mathfrak{A}(M)$, consider the following properties:

- (i) The spectra of Df at the different fixed points of f are distinct;
- (ii) At each fixed point of f , the eigenvalues of Df are simple, and there is no resonance neither between the stable eigenvalues nor between the unstable eigenvalues.

These properties are clearly satisfied by the elements of an open and dense subset $\mathfrak{A}^*(M)$ of $\mathfrak{A}(M)$. Let \mathcal{U} be an open, connected, simply connected subset of $\mathfrak{A}^*(M)$ and $p: f \mapsto p(f)$ a continuous map from \mathcal{U} to M such that $f(p(f)) = p(f)$. There exist

finite sets I_j , $1 \leq j \leq 4$, and, with $I = \bigcup_{j=1}^4 I_j$, a continuous map $\lambda = (\lambda_i)_{i \in I}$ from $\mathcal{U} \times I$ to \mathbb{C}

such that, for $f \in \mathcal{U}$:

- the $\lambda_i(f)$ with $i \in I_1$ (resp. $i \in I_3$) are the real stable (resp. unstable) eigenvalues of Df at $p = p(f)$;
- the $\lambda_i(f)$ with $i \in I_2$ (resp. $i \in I_4$) are the stable (resp. unstable) eigenvalues with strictly positive imaginary part of Df at p .

Let \mathbb{K}_i be equal to \mathbb{R} for $i \in I_1 \cup I_3$ and to \mathbb{C} for $i \in I_2 \cup I_4$. By Sternberg's linearization theorem (see [3]), there exists, for $f \in \mathcal{U}$, a C^∞ -diffeomorphism $k_s = k_s(f)$ from $\prod_{i \in I_1 \cup I_2} \mathbb{K}_i$

onto $W^s(f, p(f))$ such that $k_s^{-1} f k_s$ is the product of the homotheties of ratio $\lambda_i(f)$ on \mathbb{K}_i , for $i \in I_1 \cup I_2$. Furthermore, k_s is unique up to a product of homotheties in $\prod_{i \in I_1 \cup I_2} \mathbb{K}_i$, and we can choose the map $f \mapsto k_s(f)$ to be continuous. Similarly with $I_3 \cup I_4$,

$W^u(f, p(f))$ and k_u . Then, for $f \in \mathcal{U}$, a diffeomorphism $g \in Z(f)$ must fix p [because of property (i)] and preserve the stable manifold $W^s(f, p(f))$ and the unstable manifold $W^u(f, p(f))$. Moreover $k_s^{-1} g k_s$ and $k_u^{-1} g k_u$ are products of homotheties on their factors \mathbb{K}_i , of respective ratios μ_i . We define the compact part of $Z(f)$ at $p(f)$ by the condition that all ratios $\text{Log} |\mu_i| / \text{Log} |\lambda_i|$, $i \in I$, are equal; this is a subgroup of $Z(f)$ which contains the iterates of f . We then proved in [3]:

PROPOSITION 1. — *There is an open and dense subset \mathcal{U}_1 of \mathcal{U} such that for any $f \in \mathcal{U}_1$ the centralizer $Z(f)$ has trivial compact part at $p(f)$.*

Remark. — A similar result holds if we consider periodic orbits of a fixed period; here, we only need the case of fixed points.

4. A basic proposition and proof of the theorem

The other result which is basic to the proof of the theorem relates, for $(f, p) \in \tilde{\mathcal{D}}_A(\mathbb{T}^n)$ and $f \in \mathfrak{A}^*(\mathbb{T}^n)$, the local linearizing conjugacies k_s , k_u at p , introduced above, and the global conjugacy h_f .

Let \mathcal{D}_1 be the set of $(f, p) \in \tilde{\mathcal{D}}_A(\mathbb{T}^n)$ such that $f \in \mathfrak{A}^*(\mathbb{T}^n)$, and let \mathcal{V} be a connected component of \mathcal{D}_1 . We can then define I_j , $1 \leq j \leq 4$ as above. For $f \in \mathcal{V}$, let J be a non

trivial subset of $I_1 \cup I_2$ (resp. $I_3 \cup I_4$); that is, J is neither the empty set nor the whole

set. We then denote by $W_J(f)$ the image of $\prod_{i \in J} \mathbb{K}_i = \mathbb{K}_J$ under $k_s(f)$ [resp. $k_u(f)$].

PROPOSITION 2. — *There is an open and dense subset \mathcal{V}_1 of \mathcal{V} such that for any $f \in \mathcal{V}_1$ and any non trivial subset J of either $I_1 \cup I_2$ or $I_3 \cup I_4$, the dimension of the linear subspace of \mathbb{R}^n generated by $h_f(W_J(f))$ is strictly greater than the dimension of \mathbb{K}_J .*

Observe that the openness of \mathcal{V}_1 follows from the continuity of the maps $f \rightarrow h_f$, $f \rightarrow k_s(f)$, $f \rightarrow k_u(f)$. Before proving density, we show how to deduce the theorem from Propositions 1 and 2.

Proof of the theorem. — By Proposition 1 there is an open and dense set \mathcal{V}_2 in \mathcal{V}_1 of diffeomorphisms having centralizer with trivial compact part at p . When \mathcal{V} varies among the components of \mathcal{D}_1 , the union of these \mathcal{V}_2 is an open and dense subset of $\tilde{\mathcal{D}}_A(\mathbb{T}^n)$.

Let $f \in \mathcal{V}_2$, $g \in Z(f)$, and assume that g is not an iterate of f . Denote by $(\mu_i)_{i \in 1}$ the ratios of the homotheties associated to g under the linearizing maps k_s, k_u . Then, as g does not belong to the compact part of $Z(f)$ at p , the ratios $\text{Log}|\mu_i|/\text{Log}|\lambda_i(f)|$ are not all equal. Replacing, if necessary, g by $g^k f^l$, with $k, l \in \mathbb{Z}$, we can assume that $Dg(p)$ is hyperbolic, but the stable manifold $W^s(g, p)$ of g at p is distinct from the stable manifold $W^s(f, p)$ and the unstable manifold $W^u(f, p)$ of f at p . Indeed this is true when none of these ratios is zero and they take both signs; and this last condition is satisfied by $g^k f^l$ (for some $k \geq 1, l \in \mathbb{Z}$) if l/k is such that the ratios for g take values on both sides of l/k , none being equal to l/k . Replacing, if necessary, f by f^{-1} , we can assume that $W^s(f, p) \cap W^s(g, p) = W_J(f)$ for some non trivial subset J of $I_1 \cup I_2$.

The map $h_f g h_f^{-1} = B$ commutes with A and fixes O , hence it belongs to $GL_n(\mathbb{Z})$ (see §2). Moreover, as $Dg(p)$ is hyperbolic, no sufficiently small neighbourhood of p contains a full orbit of g ; the same must be true for B at O , which means that B is hyperbolic. Then $h_f(W_J(f))$ is equal to the intersection of the stable subspaces of A and B , a linear subspace with the same dimension as \mathbb{K}_J . This contradicts the definition of \mathcal{V}_1 and shows that any $f \in \mathcal{V}_2$ has trivial centralizer. ■

5. A statement about the induced linear automorphism

To prove that \mathcal{V}_1 is dense in \mathcal{V} , we consider some $(f, p) \in \mathcal{V}$, and a non trivial invariant subset J of $I_3 \cup I_4$, such that $h_f(W_J(f))$ is equal to a non trivial A -invariant linear subspace of the unstable subspace of A . We will show that there exist arbitrarily small perturbations (in the C^∞ topology) f' of f such that the linear subspace of \mathbb{R}^n generated by $h_{f'}(W_J(f'))$ is equal to the unstable subspace of A .

We choose a minimal non zero A -invariant subspace E of $h_f(W_J(f))$; the dimension of E is 1 or 2, and the restriction of A to E is a similitude of ratio $\lambda \in \mathbb{C}$, with $|\lambda| > 1$. We denote by E^s (resp. E^u) the stable (resp. unstable) subspace of A in \mathbb{R}^n , and also the stable (resp. unstable) manifold $W^s(A, 0)$ in \mathbb{T}^n [resp. $W^u(A, 0)$]. We choose an inner product

$\langle \cdot, \cdot \rangle$ in \mathbb{R}^n , with associated norm $\| \cdot \|$, and a constant $\alpha \in (0, 1)$ such that the following properties hold:

- (1) E^s and E^u are orthogonal for $\langle \cdot, \cdot \rangle$;
- (2) $\|A v\| \leq \alpha \|v\|$, $\|A^{-1} w\| \leq \alpha \|w\|$ for $v \in E^s$, $w \in E^u$;
- (3) $\|A v\| = |\lambda| \|v\|$ for $v \in E$;
- (4) $\|v\| \geq 1$ for $v \in \mathbb{Z}^n - \{0\}$.

Denote by d the distance in \mathbb{T}^n induced by $\| \cdot \|$. The restrictions of $\| \cdot \|$ to E^s , $E^u \subset \mathbb{R}^n$ define distances d_s, d_u on E^s, E^u and we denote by the same letters the images of these distances by the injections of E^s, E^u in \mathbb{T}^n . The closed ball in \mathbb{T}^n (resp. $E^s \subset \mathbb{T}^n$, resp. $E^u \subset \mathbb{T}^n$) with center x and radius r for the distance d (resp. d_s , resp. d_u) will be denoted by $B(x, r)$ [resp. $B_s(x, r)$, resp. $B_u(x, r)$].

A main point in the proof of Proposition 2 is to find a certain number of homoclinic points for A near E in E^u , with “non interacting” orbits; this is made precise in the following statement.

PROPOSITION 3. — *Let K be a neighbourhood of O in E^u , and m be the dimension of E^u . There exist $\delta_0 > 0$, and, for any $\varepsilon_0 > 0$, points z_1, \dots, z_m in E , $y_1, \dots, y_m \in E^u$ such that the following properties hold:*

- (i) $z_i \in K$, for $1 \leq i \leq m$;
- (ii) $d_u(z_i, y_i) < \varepsilon_0$, for $1 \leq i \leq m$;
- (iii) y_1, \dots, y_m is a basis of E^u ;
- (iv) y_i is homoclinic for A in \mathbb{T}^n , for $1 \leq i \leq m$;
- (v) $A^n(y_i) \notin B(z_j, \delta_0) = B_j$ for $n > 0$, $1 \leq i, j \leq m$;
- (vi) $B_i \cap A^n(B_u(z_j, \delta_0)) = \emptyset$ for $n < 0$, $1 \leq i, j \leq m$;
- (vii) $B_i \cap B_j = \emptyset$ for $i \neq j$.

We first assume Proposition 3 to be true and based on it we provide a proof of Proposition 2. We then will be left with proving Proposition 3, where only the hyperbolic linear automorphism A is involved.

6. Proof of Proposition 2

6.1. We first define the kind of perturbations of f we will consider.

We define the following subsets of \mathbb{R}^n :

$$\begin{aligned} K_1 &= \{x \in \mathbb{R}^n, |x_i| \leq 1 \text{ for } 1 \leq i \leq n\}, \\ K_2 &= \{x \in \mathbb{R}^n, |x_i| \leq 1 \text{ for } 1 \leq i \leq m \text{ and } x_i = 0 \text{ for } m < i \leq n\}, \\ K_3 &= \left\{x \in \mathbb{R}^n, |x_i| \leq \frac{1}{2} \text{ for } 1 \leq i \leq m \text{ and } x_i = 0 \text{ for } m < i \leq n\right\}, \end{aligned}$$

where $x = (x_1, \dots, x_n)$. Let $|x| = \left(\sum_{i=1}^n x_i^2 \right)^{1/2}$. We fix some C^∞ function $\eta: [0, +\infty) \rightarrow [0, 1]$, satisfying $\eta(t) = 0$ for $t \geq 1$ and $\eta(t) = 1$ for $t \leq 1/2$.

Let $\delta \in (0, 1/2)$, $q_1, \dots, q_l \in K_3$, $q'_1, \dots, q'_l \in K_2$ ($l \geq 1$) be such that:

- (1) $|q_i - q_j| \geq 2\delta$ for $i \neq j$;
- (2) $|q_i - q'_i| \leq \frac{\delta}{2}$ for all $1 \leq i \leq l$.

Then, the C^∞ vector field X in \mathbb{R}^n defined by

$$X(y) = \sum_{j=1}^l \eta(\delta^{-1}|y - q_j|)(q'_j - q_j)$$

has support contained in the interior of K_1 ; it is tangent to K_2 along K_2 , and for a given $y \in K_1$, by (1) at most one term in the sum defining $X(y)$ is non zero. If $(F_t)_{t \in \mathbb{R}}$ is the flow generated by X , one has, for $1 \leq i \leq l$,

$$F_1(q_i) = q'_i.$$

Finally, given a neighbourhood $\tilde{\mathcal{W}}$ of the identity in $\text{Diff}(\mathbb{R}^n)$, there exists $\varepsilon = \varepsilon(\delta, \tilde{\mathcal{W}}) < \delta/2$ such that if we have the stronger condition

- (2') $|q_i - q'_i| \leq \varepsilon$ for $1 \leq i \leq l$,

then F_1 belongs to $\tilde{\mathcal{W}}$.

6.2. We choose a diffeomorphism H of K_1 onto a compact neighborhood of p in \mathbb{T}^n , such that $H(0) = p$ and $H(K_2) \subset W^u(f, p)$. Let $\delta_0 > 0$ be as in Proposition 3 with $K = H(K_3)$. Let $\delta \in (0, 1/2)$ be such that $|y - z| \leq \delta$, for $y, z \in K_1$, implies that $d(h_f H(y), h_f H(z)) < \delta_0$.

Let \mathcal{W} be a neighbourhood of f in the C^∞ topology. We choose $\varepsilon = \varepsilon(\delta, \mathcal{W}) < \delta/2$ such that if the data in (6.1) satisfy (1) and (2'), then $HF_1 H^{-1} f$ belongs to \mathcal{W} . Then, by uniform continuity, there exists $\varepsilon_0 > 0$ such that if $y, z \in E^u$ satisfy $h_f^{-1}(z) \in K$, $d_u(y, z) < \varepsilon_0$, then $h_f^{-1}(y) \in H(K_2)$ and $|H^{-1} h_f^{-1}(y) - H^{-1} h_f^{-1}(z)| < \varepsilon$. For this $\varepsilon_0 > 0$ we apply Proposition 3 and get points $y_1, \dots, y_m \in E^u$, $z_1, \dots, z_m \in E$ satisfying properties (i)-(vii). We now construct as in (6.1) a flow $(F_t)_{t \in \mathbb{R}}$ with data $\delta, l = m$, $q_i = H^{-1} h_f^{-1}(z_i)$ and $q'_i = H^{-1} h_f^{-1}(y_i)$. For $0 \leq t \leq 1$, let $f_t = HF_t H^{-1} f \in \mathcal{W}$; we may assume that $\mathcal{W} \subset \mathcal{D}_A(\mathbb{T}^n)$ and denote by h_t the global conjugacy of f_t and A (so that $h_0 = h_f$).

6.3. We finally prove that f_1 belongs to \mathcal{V}_1 , and more precisely that the linear subspace of \mathbb{R}^n generated by $h_1(W_J(f_1))$ is E^u . As \mathcal{W} is arbitrary and $f_1 \in \mathcal{W}$, this will show that \mathcal{V}_1 is dense in \mathcal{V} .

With the notation of Proposition 3, the image under h_f of the support of $HF_t H^{-1}$ ($0 \leq t \leq 1$) is contained in $\bigcup_{i=1}^l B_i$ (by the definition of δ); by property (vi), p does not

belong to the support of $\text{HF}_t H^{-1}$ and the manifolds $W_J(f)$, $W_J(f_1)$ coincide in a neighbourhood of p in $W^u(f, p)$. By property (v), we also have, for $1 \leq i \leq m$, $0 \leq t \leq 1$ and $n \geq 0$:

$$f_t^n(h_f^{-1}(y_i)) = f^n(h_f^{-1}(y_i)) = h_f^{-1}(A^n(y_i)),$$

hence $h_f^{-1}(y_i)$ belongs to $W^s(f, p)$. On the other hand, by relation (vi), we have, for $1 \leq i \leq m$, $0 \leq t \leq 1$, and $n < 0$:

$$(3) \quad f_t^n(h_f^{-1}(y_i)) = f^n \text{HF}_t^{-1} H^{-1} h_f^{-1}(y_i).$$

But $H^{-1} h_f^{-1}(y_i)$ belongs to K_2 , hence $\text{HF}_t^{-1} H^{-1} h_f^{-1}(y_i)$ belongs to $W^u(f, p)$, and we conclude that $h_f^{-1}(y_i)$ belongs to $W^u(f, p)$, and indeed it is homoclinic for f_t .

As the set of homoclinic points of A is totally discontinuous, and h_t depends continuously on t , the map $t \rightarrow h_t(h_f^{-1}(y_i))$ is constant and equal to $h_0 h_f^{-1}(y_i) = y_i$. By construction of F_1 , we have $\text{HF}_1 H^{-1}(h_f^{-1}(z_i)) = h_f^{-1}(y_i)$ for $1 \leq i \leq m$, hence by relation (3) we have that $f_1^n(h_f^{-1}(y_i)) = f^n(h_f^{-1}(z_i))$ for $1 \leq i \leq m$, $n < 0$. But $h_f^{-1}(z_i)$ belongs to $W_J(f) \supset h_f^{-1}(E)$, and $W_J(f)$ and $W_J(f_1)$ coincide near p in $W^u(f, p)$, hence $h_f^{-1}(y_i)$ belongs to $W_J(f_1)$ for $1 \leq i \leq m$. As $h_1 h_f^{-1}(y_i) = y_i$, we get the conclusion we were after by property (iii) of Proposition 3. This concludes the proof of Proposition 2. ■

It remains to show Proposition 3, which will be done in the next section.

7. Proof of Proposition 3

7.1. The lemma we now state will be used at a crucial step in the proof of Proposition 3. Let $\theta > 1$, $N_0 \in \mathbb{N}^* = \mathbb{N} - \{0\}$ be such that $\theta^{N_0} > 6(N_0 + 1)$. Define $l_0 = 2(1/(\theta - 1) + N_0(\theta - 1)/3\theta)$, and denote by T the homothety of ratio θ in \mathbb{R} . Let $k \in \mathbb{N}^*$, and suppose we are given, for each $1 \leq i \leq k N_0$, a family $(L_{i,j})$ of intervals of length 1.

LEMMA. — Let $l \geq l_0$; assume that for each $1 \leq i \leq k N_0$ the centers of the intervals $L_{i,j}$ are mutually distant apart by at least $3\theta l/(\theta - 1)$. Then, the set:

$$L = T^{k N_0}([0, l]) - \bigcup_{i=1}^{k N_0} \bigcup_j T^{k N_0 - i}(L_{i,j})$$

contains an interval of length l .

Proof. — We prove the assertion for $k = 1$; the assertion for $k \geq 2$ follows by iterating k -times the process. The number of intervals $L_{i,j}$ (with fixed i , $1 \leq i \leq N_0$) contained in the interior of $[0, \theta^i l]$ is at most $[\theta^i(\theta - 1)/3\theta] + 1$, hence the number of components of L is less than:

$$1 + \sum_{i=1}^{N_0} \left(\left[\frac{\theta - 1}{3\theta} \theta^i \right] + 1 \right) \leq 1 + N_0 + \frac{1}{3} \theta^{N_0} < \frac{1}{2} \theta^{N_0}.$$

On the other hand, the Lebesgue measure of the intersection $[0, \theta^i l] \cap (\bigcup_j L_{i,j})$ is less than $[\theta^i(\theta-1)/3\theta] + 1$, hence the measure of $[0, \theta^{N_0} l] - L$ is less than:

$$\sum_{i=1}^{N_0} \theta^{N_0-i} \left(\left[\frac{\theta-1}{3\theta} \theta^i \right] + 1 \right) \leq \theta^{N_0} \left(\frac{1}{\theta-1} + N_0 \frac{\theta-1}{3\theta} \right) = \frac{1}{2} l_0 \theta^{N_0},$$

For $l \geq l_0$, the measure of L is therefore bigger than or equal to $(1/2) l \theta^{N_0}$ and the result follows.

7.2. We choose a unit vector w in E and denote by D the orthogonal subspace (with respect to $\langle \cdot, \cdot \rangle$) of $\mathbb{R}w$ in E^u , and, for $\varepsilon \geq 0$, by P_ε the endomorphism of E^u equal to the identity on $\mathbb{R}w$ and to the multiplication by ε on D . We may assume that $K \cap \mathbb{R}w = \{tw, |t| \leq a\}$ for some $a > 0$. We pick m distinct real numbers u_1, \dots, u_m in $((1/2)a|\lambda|^{-1}, (1/2)a)$ (recall that A/E is a similitude of ratio $\lambda \in \mathbb{C}$) and choose a basis t_1, \dots, t_m of E^u such that $P_0(t_i) = u_i w$ for $1 \leq i \leq m$.

For $\delta_1 > 0$, $1 \leq i \leq m$, let J_i (resp. J'_i) the line segment $\{u_i w + tw, 0 \leq t \leq \delta_1\}$ (resp. $\{t_i + tw, 0 \leq t \leq \delta_1\}$). Denote by \bar{V}_i (resp. $V_{i,u}$) the δ_1 -neighbourhood of J_i in \mathbb{T}^n (resp. E^u) for the distance d (resp. d_u). We choose $\delta_1 \in (0, (1/4)a)$ sufficiently small so that the following properties hold:

- (4) $J_i \subset K$, for $1 \leq i \leq m$;
- (5) $\bar{V}_i \cap \bar{V}_j = \emptyset$, $0 \notin \bar{V}_i$ for $1 \leq i \leq m$;
- (6) $\bar{V}_i \cap A^n(\bar{V}_{j,u}) = \emptyset$ for all $1 \leq i, j \leq m$, $n < 0$;
- (7) if $y_i \in J'_i$ for $1 \leq i \leq m$, then y_1, \dots, y_m is a basis of E^u .

Next we choose an open neighbourhood \mathcal{O} of O in E^s such that $A(\mathcal{O}) \subset \mathcal{O}$ and $\mathcal{O} \cap \bar{V}_i = \emptyset$ for $1 \leq i \leq m$; let $\beta > 0$ be such that for any $x \in E^u$, the ball $B_u(x, \beta)$ intersects \mathcal{O} . Define $r = 2a + 4\beta$.

7.3. In the context of the Lemma in (7.1), let $\theta = |\lambda|$, and $N_0 \geq 1$ satisfying $\theta^{N_0} > 6(N_0 + 1)$. This defines l_0 ; let N_1 be an integer such that $\theta^{N_1} > l_0 r \delta_1^{-1}$. For $\delta_0 > 0$, consider the union $\tilde{B}(\delta_0)$ of the balls $B_u(q, a)$ for $q \in B_s(0, \delta_0) \cap E^u$. We choose $\delta_0 \in (0, \delta_1/2)$ small enough to have:

- (8) $A^n V'_i \cap V'_j = \emptyset$ for $0 < n \leq N_1$, $1 \leq i, j \leq m$, where V'_i is the δ_0 -neighbourhood of J'_i in \mathbb{T}^n ;
- (9) $d_u(q, q') \geq (3\theta/(\theta-1)l_0 + 1)r$ for distinct q, q' in $B_s(0, \delta_0) \cap E^u$.

7.4. Let $\varepsilon_0 \in (0, \delta_0)$ be given (see Proposition 3). We choose $\varepsilon_1 > 0$ small enough so that we have:

$$(10) \quad \|P_{\varepsilon_1}(y) - P_0(y)\| < \frac{1}{2} \varepsilon_0 \quad \text{for } y \in \bigcup_{i=1}^m J'_i.$$

As P_{ε_1} is a linear automorphism of E^u , any family y_1, \dots, y_m with $y_i \in P_{\varepsilon_1}(J'_i)$ is a basis of E^u [see property (7) above]. Therefore we can choose $0 < \varepsilon_2 < (1/2)\varepsilon_0$ such that:

(11) any family y_1, \dots, y_m , with $d_u(y_i, P_{\varepsilon_1}(J'_i)) < \varepsilon_2$ for $1 \leq i \leq m$, is a basis of E^u .

We choose $k \in \mathbb{N}^*$ and define $N_2 = N_1 + kN_0$ such that $\alpha^{N_2}\beta < \varepsilon_2$ [where α is defined in property (2) of Section 5].

7.5. Fix $1 \leq i \leq m$; let D_1 be the linear subspace of E^u orthogonal to $A^{N_2}w$ and let $B = B_{D_1}(0, \beta)$ be the closed ball in D_1 with center O and radius β .

For $N_1 < j \leq N_2$, let L_j be the set of points x in $A^j(P_{\varepsilon_1}(J'_i))$ such that $x + A^{j-N_2}(B)$ intersects $\tilde{B}(\delta_0)$. The diameter of $A^{-N_2+j}(B)$ in E^u is at most 2β . If two points $x, x' \in L_j$ are such that $x + A^{j-N_2}(B)$ and $x' + A^{j-N_2}(B)$ intersect the same component of $\tilde{B}(\delta_0)$, we have:

$$(12) \quad d_u(x, x') \leq 2a + 4\beta = r.$$

Otherwise, by (9), we have:

$$(13) \quad d_u(x, x') \geq \frac{3\theta}{\theta-1} l_0 r.$$

By the definition of N_1 , the length of $A^{N_1}(P_{\varepsilon_1}(J'_i))$ is at least $l_0 r$; relations (12) and (13) show that we can apply the Lemma in (7.1) and conclude that there exists J'_i of length $\overset{N_2}{\text{bigger than or equal to } l_0 r}$ contained in $A^{N_2}(P_{\varepsilon_1}(J'_i)) - \bigcup_{j=N_1+1}^{N_2} A^{N_2-j}(L_j)$. Observe that

$l_0 \geq 1$, hence $l_0 r \geq 2\beta$. By definition of β , there exists some $y'_i \in J'_i$ such that $y'_i + B$ intersects \mathcal{O} in some point y''_i . Let $y_i = A^{-N_2}(y'_i)$, $y''_i = A^{-N_2}(y'_i)$ and $z_i = P_0(y''_i)$.

7.6. We finally check for y_i, z_i ($1 \leq i \leq m$) the conclusions of Proposition 3. Clearly $y_i \in E^u$ and $z_i \in E$. Recall that $\mathcal{O} \subset E^s$.

PROPERTY (i). — We have $y''_i \in P_{\varepsilon_1}(J'_i)$, $P_0(P_{\varepsilon_1}(J'_i)) = P_0(J'_i) = J_i$, hence (i) is a consequence of (4).

PROPERTY (ii). — We have that

$$d_u(y_i, y''_i) < \alpha^{N_2}\beta < \varepsilon_2 < \frac{1}{2}\varepsilon_0,$$

$$d_u(y''_i, z_i) < \frac{\varepsilon_0}{2} \text{ [see property (10) above].}$$

PROPERTY (iii). — This results from $d_u(y_i, y''_i) < \varepsilon_2$ and (11).

PROPERTY (iv). — Indeed $y'_i \in E^u \cap \mathcal{O}$, and $y_i = A^{-N_2}(y'_i)$.

PROPERTY (v). — One has that $y_i \in B_i \subset V_i$ for $1 \leq i \leq m$.

Let $1 \leq i, j \leq m$; for $1 \leq n \leq N_1$, $A^n(y_i)$ does not belong to V_j' according to (8) above. The same holds for $n \geq N_2$ because then $A^n(y_i) = A^{n-N_2}(y_i'') \in \mathcal{O}$ and $\mathcal{O} \cap V_j' = \emptyset$. For $N_1 < n < N_2$, $A^n(y_i)$ does not belong to $\tilde{B}(\delta_0)$ by the construction of J_i'' . As $\tilde{B}(\delta_0)$ contains $B_j \cap E^u$, we have again $A^n(y_i) \notin B_j$.

PROPERTIES (vi), (vii). — These follow immediately from (5), (6) and $\delta_0 < \delta_1$. \square

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