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THETA CHARACTERISTICS ON SINGULAR CURVES, SPIN STRUCTURES AND ROHLIN THEOREM

By A. LIBGOBER

A theta characteristic on a (possibly singular) algebraic curve is a line bundle whose square is isomorphic to the dualizing sheaf of the curve (i. e. the cotangent bundle in a non-singular case). A theta characteristic L is called even (respectively odd) if dim $H^0(L)$ is even (respectively odd). J. Harris [H] computed the number of even and odd theta characteristics on a curve with Gorenstein singularities in terms of a local invariant of singularities introduced by him. He raised the question of relating this invariant to more standard invariants of singularities. The purpose of this paper is to show that in the case in which the curve belongs to a non-singular surface, the Harris invariant coincides with the Robertello invariant [R] of the link of singularity. The latter invariant is defined for the proper links, i. e. for the links with the property that the linking number of each component with the union of other components is even. The Robertello invariant also can be described as the Arf invariant of the $\mathbf{Z}_2$-quadratic form $x \to V(x, x) \bmod 2$, where $V(\ , \ )$ is the Seifert form of the link (*cf.* [R]). For a knot (i. e. in the case of a singularity with one branch) the Robertello invariant is just the Legendre symbol $(2/\Delta(-1))$ where $\Delta(t)$ is the Alexander polynomial.

The paper is organized as follows. Section 1 contains a description of the relationship between quadratic forms associated by D. Johnson [J] to a Spin structure on a closed 2-manifold and the theta forms, the relevant linear algebra and reviews the Robertello invariant of links. In the section 2 we show that if an algebraic curve belongs to an algebraic surface where this curve is dual to the second Steifel Whitney class of the surface then this embedding induces the theta characteristic on the curve with the property that the associated theta form coincides with a $\mathbf{Z}_2$-quadratic form topologically defined in these circumstances by Rohlin [Ro]. In the section 3 we consider the behavior of theta characteristics in families of curves on an algebraic surface. In the section 4 we derive the main result described above and point out how one can obtain by analytic means the Rohlin theorem (claiming that the Arf invariant of the $\mathbf{Z}_2$-quadratic form associated with a 2-submanifold X in a 4-manifold M with X dual to $w_2(M)$ is equal to $(\sigma(M)-X^2)/8 \pmod 2)$) in the case when X is an algebraic curve and M is an algebraic surface. Finally in the last section we show how mod 2 Seifert form is related in the case of 1-dimensional unibranched complex singularity to the geometry of the Hodge structure defined by J. Steenbrink on the vanishing cohomology and discuss a possible generalization of some of results of this paper to higher dimensions.

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1. Preliminaries

A. $\mathbf{Z}_2$-QUADRATIC FORMS. — Let H be a finite dimensional vector space over $\mathbf{Z}_2$ and let B be a bilinear form on H. A quadratic form associated with B is a map $q: \mathrm{H} \to \mathbf{Z}_2$ such that

$$q(x+y)=q(x)+q(y)+\mathrm{B}(x, y). \tag{1}$$

Let $\mathrm{R}=\{x \in \mathrm{H} \mid \mathrm{B}(x, v)=0 \text{ for any } v \text{ in } \mathrm{H}\}$ be the radical of the bilinear form B. If $q(x)|_{\mathrm{R}} \neq 0$ then there is only one isomorphism class of quadratic forms associated with B. In this case $q(x)$ takes the values 0 and 1 equal number of times. On the other hand there are two isomorphism classes of the forms such that $q(x)|_{\mathrm{R}}=0$. If $\dim \mathrm{R}=r$ and $\dim \mathrm{H}=2h+r$ then the zero value is taken either $2^{h+r-1}(2^h+1)$ or $2^{h+r-1}(2^h-1)$ times. If the bilinear form on H is written as

$$\mathrm{B}(x_1, \ldots, x_{2h+r}, y_1, \ldots, y_{2h+r})=\sum_{i=1}^{h} x_{2i} y_{2i+1}+x_{2i+1} y_{2i}$$

then these isomorphism classes are represented by the forms

$$q^+(x_1, \ldots, x_{2h}, x_{2h+1}, \ldots, x_{2h+r})=\sum_{i=1}^{h} x_{2i-1} x_{2i}$$

(2) and

$$q^-(x_1, \ldots, x_{2h}, x_{2h+1}, \ldots, x_{2h+r})=\sum_{i=1}^{h} x_{2i-1} x_{2i}+x_1^2+x_2^2$$

respectively. The Arf invariant of a form isomorphic to q^+ (resp. q^-) is defined to be $+1$ (resp. -1) (*cf.* [B], ch. 3, sect. 1).

In B below we will need to show that two families of quadratic forms (on H_1 of a closed surface) parametrized by an affine space (the set of Spin-structures) consist of identical (rather than isomorphic) forms. The rest of this paragraph describes the linear algebra which we will need for this purpose.

Let V be an affine space over a vector space H i.e. a set with a transitive action of $\mathrm{H}: v \to v+h$ $(v \in \mathrm{V},\ h \in \mathrm{H})$ without fixed points. An affine quadratic form associated with a bilinear form B on H is a map $\varphi: \mathrm{V} \to \mathrm{Z}_2$ such that

$$\varphi(v+h_1+h_2)-\varphi(v+h_1)-\varphi(v+h_2)+\varphi(v)=\mathrm{B}(h_1, h_2) \tag{3}$$

for any v in V and h_1, h_2 in H. If φ takes values 0 and 1 unequal number of times we say that the Arf invariant of φ is 0 (resp. 1) provided $\# \{v \in \mathrm{V} \mid \varphi(v)=0\} > \# \{v \in \mathrm{V} \mid \varphi(v)=1\}$ (resp. one has opposite inequality). We shall denote the space of affine quadratic forms on V with a fixed associated bilinear form B by AQ(V).

Let V_1 (resp. V_2) be an affine space over H_1 (resp. H_2). A map $\varphi: \mathrm{V}_1 \to \mathrm{V}_2$ is called an affine map if the induced map $\mathrm{L}_\varphi: \mathrm{H}_1 \to \mathrm{H}_2$ which takes $h_1 \in \mathrm{H}_1$ into $h_2 \in \mathrm{H}_2$ such that $\varphi(v+h_1)=\varphi(v)+h_2$ is linear. The set of quadratic forms on a vector space H associated with a fixed non degenerate bilinear form B can be given a structure of an affine space by defining the action of H as

$$h(q(x))=q(x+h)-q(h). \tag{4}$$

We will denote this affine space by Q(H). One can also view Q(H) as an affine space over the dual space H* to H with the action $q \to q+\lambda$ $(\lambda \in \mathrm{H}^*)$.

Now let $\mathrm{Aff}_{\mathrm{Id}}(\mathrm{V}, \mathrm{Q(H)})$ be the space of affine maps from V into the affine space Q(H) having the identity as the associated linear map. If $v \to \varphi_v$ is such a map then $v \to \mathrm{Arf}\,\varphi_v$ is an affine quadratic function. Indeed the condition on $v \to \varphi_v$ implies that $\varphi_{v+h}(x)=\varphi_v(x+h)-\varphi_v(h)$. The relation (3) for $\mathrm{Arf}\,\varphi_v$ follows from $\mathrm{Arf}\,\varphi_{v+h}=\mathrm{Arf}\,\varphi_v+\varphi_v(h)$ [the latter is obvious if $\varphi_v(h)=0$ and follows by counting the numbers of times φ_{v+h} and φ_v take value 0 or 1 if $\varphi_v(h)=1$]. Let us consider the map Φ from $\mathrm{Aff}_{\mathrm{Id}}(\mathrm{V}, \mathrm{Q(H)})$ to AQ(H) which takes $(v \to \varphi_v)$ into the quadratic function $v \to \mathrm{Arf}\,\varphi_v$. On the other hand the correspondence $\varphi \to \varphi_v=\varphi(v+x)-\varphi(v)$ defines the map $\Psi: \mathrm{AQ(H)} \to \mathrm{Aff}_{\mathrm{Id}}(\mathrm{V}, \mathrm{Q(H)})$.

PROPOSITION 1.1. — *(a) The following identities take place:*

$$\Phi \circ \Psi(\varphi)(v)=\varphi(v)+\mathrm{Arf}\,\varphi \circ 1_{\mathrm{V}}$$

(1 V *is the function on* V *taking the value* 1 *everywhere) and* $\Psi \circ \Phi=\mathrm{id}$.

(b) The map Φ *is injective and* $\Psi(\varphi_1)=\Psi(\varphi_2)$ *iff* $\varphi_1-\varphi_2$ *is constant.*

Proof. — The formula for $\Phi \circ \Psi$ amounts to the identity

$$\mathrm{Arf}\,\{x \to \varphi(v+x)-\varphi(v)\}+\mathrm{Arf}\,\varphi=\varphi(v) \tag{5}$$

which can be easily verified by counting the number of vectors x on which the form takes values 1 or 0 for various values of $\mathrm{Arf}\,\varphi$ and $\varphi(v)$. This formula implies the second part of (*b*). Now the space AQ(V) of affine quadratic functions contains $2 \circ 2^{\dim \mathrm{V}}$ elements [use for example that the difference of two elements in AQ(V) is an affine linear function on V] while $\mathrm{Aff}_{\mathrm{Id}}(\mathrm{V}, \mathrm{Q(H)})$ has $2^{\dim \mathrm{V}}$ elements because each element in it is determined by a value of a map in a single element of V and Q(H) contains $2^{\dim \mathrm{V}}$ elements. Hence Ψ is onto.

COROLLARY 1.2. — *Let* α *be an affine quadratic function. Let*

$$\varphi_v(x)=\alpha(v+x)-\alpha(v).$$

Then $\alpha(v)=\mathrm{Arf}\,\varphi_{v+x}-\mathrm{Arf}\,\varphi_v+\mathrm{Arf}\,\alpha$.

B. THETA FORMS AND SPIN STRUCTURES. — Let X be an algebraic curve, Pic(X) be the group holomorphic line bundles on X and ω_x (resp. $\mathcal{O}_x$) be the dualizing sheaf (resp. trivial line bundle) of X. Let $S(X)=\{L\in \mathrm{Pic}(X) \mid L^2=\omega_x\}$ be the set of theta characteristics on X. For any $L\in S(X)$ and any bundle ξ of order 2 ($\xi^2=\mathcal{O}_x$) $L\otimes\xi$ is again a theta characteristic. Therefore S(X) is naturally an affine space over the $\mathbf{Z}_2$-space of points of order 2 in Pic(X). If X is non singular and Pic(X) is identified with $H^{0,1}(X)^*/H_1(X, \mathbf{Z})$ (*cf.* for example [ACGH] ch. 1) then the space of points of order 2 in Pic(X) can be identified with $H_1(X, \mathbf{Z}_2)=H_1(X, (1/2)\mathbf{Z})/H_1(X, \mathbf{Z})$. The theta form associated with $L\in S(X)$ is the $\mathbf{Z}_2$-quadratic form on $H_1(X, \mathbf{Z}_2)$ defined by (*cf.* [ACGH] ch. 6 app. B)

$$\varphi_L(\xi)=\dim H^0(X, L\otimes\xi)-\dim H^0(X, L). \tag{6}$$

Next recall (*cf.* [M1]) that the Spin structure on a manifold X of a dimension d is a double cover of the principal $SO(d)$-bundle P associated with the tangent bundle T_X of X such that the restriction of this cover on each fibre is isomorphic to the standard cover $\mathrm{Spin}(d)\to SO(d)$. In the case when X is a non singular algebraic curve the squaring map $L^{-1}\to L^{-1}\otimes L^{-1}=\omega_{X^{-1}}=T_X$ defines the double cover of the principal S^1-bundle associated with T_X i. e. a Spin structure on X. We shall denote it by ξ_L. Vice versa a double cover of the unit tangent bundle is a circle bundle for which the associated $\mathbf{R}^2$-bundle has natural complex structure with which it becomes a theta characteristic. Hence the correspondence $L\to\{\text{unit bundle of } L^*\}$ defines a 1-1 correspondence between S(X) and the set of Spin structures. The action of $H_1(X, \mathbf{Z}_2)$ on S(X) defines the action of $H^1(X, \mathbf{Z}_2)$ on the set of Spin structures which can be described explicitly as follows. Let us consider the Gysin sequence corresponding to principal bundle S^1 bundle P associated with T_X:

$$0\to H^1(X, \mathbf{Z}_2)\to H^1(P, \mathbf{Z}_2)\to H^1(S^1, \mathbf{Z}_2)\to 0. \tag{7}$$

The set of Spin structures can be identified with the non trivial coset of $H^1(P, \mathbf{Z}_2)$ of the subgroup $H^1(X, \mathbf{Z}_2)$ and the natural structure of the affine space on this coset over $H^1(X, \mathbf{Z}_2)$ corresponds to aforementioned structure of affine space on S(X).

In [J] D. Johnson defined the $\mathbf{Z}_2$-quadratic form associated with a Spin structure as follows. Any simple closed curve α on X defines the curve $\tilde\alpha$ in P (the latter we identify with the unit tangent bundle) consisting of unit tangent vectors to α. A different choice of an orientation of α produces the path in P belonging to the same mod 2 homology class in P. On the other hand any homology class x in $H_1(X, \mathbf{Z}_2)$ can be represented by a union of mutually disjoint curves $\alpha_1, \ldots, \alpha_m$ and the homology class in P of the cycle $\bar x=\sum_{i=1}^{m}\tilde\alpha_i+mz$, where z is the class of the fibre, depends only on the homology class of x (*cf.* [J]). Given $\xi\in H^1(P, \mathbf{Z}_2)$ one defines $\varphi_\xi(x)=\langle\xi, \bar x\rangle$. It turns out that $\varphi_\xi(x)$ is a quadratic form having the intersection form as the associated bilinear form [J]:

$$\varphi_\xi(x_1+x_2)=\varphi_\xi(x_1)+\varphi_\xi(x_2)+(x_1, x_2). \tag{8}$$

Vice versa any quadratic form associated with the intersection form corresponds to a unique Spin structure.

PROPOSITION 1.3. — *For any theta characteristic* $L \in S(X)$ *and any* x *in* $H_1(X, \mathbf{Z}_2)$ *one has* $\varphi_L(x) = \varphi_{\xi_L}(x)$.

Proof. — Both functions $\Phi_1 : L \to \varphi_L$ and $\Phi_2 : L \to \varphi_{\xi_L}$ are affine maps $S(X) \to Q(H_1(X, \mathbf{Z}_2))$ of the space of theta characteristics into the affine space of quadratic forms associated with the intersection form. Both Φ_1 and Φ_2 have the identity as the associated linear map. According to 1.1 (*b*) it is enough to check that $\operatorname{Arf} \varphi_L = \operatorname{Arf} \varphi_{\xi_L}$ for any L. $\operatorname{Arf} \varphi_L$ is zero iff $\dim H^0(X, L)$ is even [ACGH] and according to [A], prop. 4.1, the latter occurs iff ξ_L is a Spin boundary. So it is enough to check that $\operatorname{Arf} \varphi_\xi = 0$ iff ξ is a Spin boundary. First note that $\operatorname{Arf} \varphi_\xi$ is a Spin cobordism invariant. Indeed if M is a 3-manifold with a Spin structure and $\partial M = X_1 \cup X_2$ then the exact sequence of the pair $(M, \partial M)$ implies that $rk \operatorname{Ker}(H_1(\partial M) \to H_1(M)) = rk\, H_1(\partial M)/2$ (*cf.* [B], p. 57) and for $\alpha \in H_1(\partial M)$ the path lifts to a connected path in the covering defined by the Spin structure iff the restriction of the Spin structure on α is the trivial element in $H_1(\alpha, \mathbf{Z}_2)$. Combining this with the fact that the Spin structure on α which bounds is the non-trivial element in $H_1(\alpha, \mathbf{Z}_2)$ (*cf.* [M1]) we see that the quadratic form on $H_1(\partial M, \mathbf{Z}_2)$ is trivial on $\operatorname{Ker}(H_1(\partial M) \to H_1(M))$ i.e. on at least half elements of $H_1(\partial M)$. Therefore $0 = \operatorname{Arf} \varphi_\xi |_{H_1(\partial M)} = \operatorname{Arf} \varphi_\xi |_{H_1(X_1)} + \operatorname{Arf} \varphi_\xi |_{H_1(X_2)}$. Secondly the group of Spin cobordisms in dimension 2 is $\mathbf{Z}_2$ (*cf.* [M1]). Hence the vanishing of the Arf invariant for cobordant to zero Spin structures can be verified by checking two examples of Spin non-cobordant structures on torus which is obvious.

Remark. — Another topological interpretation of the theta form is due to Thurston. Let D be a divisor on a curve X corresponding to a theta characteristic L. Let ω be a meromorphic 1-form having 2D as its divisor. Let $\gamma(t)$ $(0 \leqq t \leqq 1)$ be a path on X representing

$$x \in H_1(X, \mathbf{Z}_2) = H_1(X, (1/2)\mathbf{Z})/H_1(X, \mathbf{Z}).$$

Then $\varphi_L(x) = (1/\pi i)\, d\log < \gamma'(t), \gamma(t) > \bmod 2$. Indeed the double cover of the principal bundle $P \to X$ associated with TX defined by the Spin structure corresponding to $\mathcal{O}_X(D)$ coincides, outside of $\pi^{-1}(D)$ with the covering of P branched over $\pi^{-1}(D)$ defined by the cohomology class dual to $\pi^{-1}(D)$ in $H^1(P, \mathbf{Z}_2) = \operatorname{Hom}(H_1(P, \mathbf{Z})/H_1(P, 2\mathbf{Z}), \mathbf{Z}_2)$. This class is represented by the form $1/4\pi i \int_0^1 d\langle v, \omega(\pi(v))\rangle / \langle v, \omega(\pi(v))\rangle$, $v \in P$. (The function $v \to \langle v, \omega(\pi(v))\rangle$ on P has zero or a pole of order 2 along each component of $\pi^{-1}(D)$.) Hence from the proposition 1.3 we can see that the theta form on $H_1(X, \mathbf{Z}_2) = H_1(X, (1/2)\mathbf{Z})/H_1(X, \mathbf{Z})$ is equal to $1/\pi i \int_0^1 d\log \langle \gamma'(t), \gamma(t)\rangle \bmod 2$.

C. ROBERTELLO INVARIANT. — Let N be a link in S^3 and F be a Seifert surface for N i.e. an oriented surface in S^3 with the boundary N. For a fixed side of F (i.e. a unit normal field to F) and a cycle α on F let us denote by $i_*(\alpha)$ the cycle in $S^3 - F$ obtained by pushing α outside of F using chosen normal vector field. The Seifert form is a bilinear form S on $H_1(F, Z)$ defined by $S(x_1, x_2)=lk(\alpha_1, i_*(\alpha_2))$ where α_1 and α_2 are the circles on F representing the homology classes of x_1 and x_2 and $lk(\ ,\ )$ is the linking number of cycles in S^3. One can show that $S(x, y)-S(y, x)=(x, y)$ where $(\ ,\ )$ is the intersection form on F (*cf.* [L]). In particular $S(x, x) \bmod 2$ has the intersection form as the associated bilinear form.

A link is called proper if the linking number of each component with the union of remaining components is even. This is equivalent to the fact that the restriction of the form $x \to S(x, x) \bmod 2$ on the radical of the intersection form is trivial. Indeed the radical fo the intersection form on $H_1(F, \mathbf{Z}_2)$ is generated by the classes of connected components of the boundary N of the Seifert surface F. Let us denote these components $\alpha_1, \ldots, \alpha_e$. Clearly $\alpha_1+\ldots+\alpha_e=0$. Hence

$$S(\alpha_i, \alpha_i)=S(\alpha_i, \sum_{j=1}^{e} \alpha_j-\alpha_i)=lk\left(\alpha_i, \sum_{j=1}^{e} \alpha_j-\alpha_i\right)$$

is zero mod 2 iff the link is proper.

To define the Robertello invariant first let us assume that the link N has only one component, say K. Let M be a 4-manifold and T be a 2-sphere embedded topologically in M in such a way that it has a single singularity with the link K. i. e. for some ball B^4 in M one has $\partial B^4 \cap T=K$. We also assume that T is such that the homology class of T in M is dual to the second Stiefel Whitney class of M (this always can be arranged *cf.* [R]). Then the Robertello invariant of K is defined as $(\sigma(M)-T\circ T)/8 \bmod 2$, where $\sigma(M)$ is the signature of the intersection form $\circ$ on $H_2(M, \mathbf{R})$. If N is an arbitrary proper link in a 3-sphere S^3 and a knot K so that there exist a surface Y of genus zero between these two spheres such that the boundary of Y is $N \cup K$. The Robertello invariant of N is defined as the Robertello invariant of the knot K. A proof of the following proposition is given in [R] in the case of knots.

PROPOSITION 1.4. — The Robertello invariant of a proper link is equal to the Arf invariant of the Seifert form.

Proof. — In notations of the last paragraph one can assume that S^3 containing N belongs to M. Then $T'=F \cup Y \cup (T-T\cap B)$ is homologous to T in M. By the Rohlin theorem [Ro] $(\sigma(M)-T'\circ T')/8 \bmod 2$ is the Arf invariant of the Rohlin form on T^1 (*cf.* the definition in the next section). This implies that the Arf invariant of the Rohlin form on T^1 is the Arf invariant of the restriction of the Rohlin form on the Seifert surface F. To conclude to proof of 1.4 it is enough to show that this restriction coincides with the Seifert form. Let α be a circle in the Seifert surface, $i_*\alpha$ be α "pushed" in S^3, D_2^α be a disk inside B^4 bounded by α. Then D_2^α does not intersect F. Hence the value of the Rohlin form is just the obstruction to extending the vector field normal to α inside the Seifert surface to the vector field normal to D_2^α. If we shall interpret this obstruction as the number of intersectin points of D_2^α with "perturbed" D_2^α with the boundary kept on the Seifert surface and notice that the linking number of α and $i_*\alpha$ can be found as the number intersectin points of "perturbed" D_2^α and the union of the cylinder connecting α and $i_* \alpha$ with D_2^α then we see that this obstruction is just the value of the Seifert form. Q.E.D.

Finally note that for the links with one and two components the Robertello invariant can be found in terms of the Alexander polynomial (*cf.* [L], [Mu1], [Mu2]).

2. Theta characteristics on submanifolds

Let M be a 4-manifold, which for simplicity we shall assume to be simply connected, and let T be a surface in M which is dual to the second Stiefel-Whitney class $w_2(M)$ of M. In this circumstances there is the canonical Spin structure on T which can be defined as follows. Let $q(x)(x \in H_1(T, \mathbf{Z}_2))$ be the Rohlin form. Recall its definition. Let x be represented by an embedded closed curve α and let D_α be an embedded disk in M which has α as the boundary and which is transversal to T. Then $q(x)$ is mod 2 sum of the number of intersections of T and D_α outside α and the obstruction to extending the field of normals to α in T to a non-vanishing normal to D_α in M vector field. The latter obstruction is an element in $\pi_1(SO(2))=\mathbf{Z}$ and can be described as the number of intersections of D_α with a disk D_α obtained from D_α by a perturbation keeping the boundary fixed. $q(x)$ is a $\mathbf{Z}_2$-quadratic form having the intersection form of T as the associated bilinear form. Now the canonical Spin structure on T corresponding to the embedding into M is the element ξ in $H^1(P, \mathbf{Z}_2)$ (P is the principal S^1-bundle associated with the tangent bundle to T) defined by $\langle \xi, \bar{x} \rangle = q(x)$ and $\xi(z)=1$ (as in section 1 B, x is an element from $H_1(T, \mathbf{Z}_2)$, $^-$ denotes the canonical lifting into P and z is the fibre of P; it follows from (7) that $H^1(P, \mathbf{Z}_2)$ is generated by classes x ($x \in H^1(T, \mathbf{Z}_2)$ and z).

In the case when M is an algebraic surface and T is an algebraic curve one can define the theta characteristic corresponding to this Spin structure as follows. The assumption that T is dual to $w_2(M)$ implies that there exist a divisor D such that $K+T=2D$ where K is the canonical class of M. Therefore $\mathcal{O}_M(2D)|_T = \mathcal{O}_M(K+T)|_T = \Omega_M^2(T) = \Omega_T^1$ i. e. $\mathcal{O}_T(D)$ is a theta characteristic on T. This is the only theta characteristic on T which is a restriction of a bundle on M.

Proposition 2.1. — *The Spin structure on* T *corresponding to* $\mathcal{O}_T(D)$ *coincide with the Spin structure corresponding to the Rohlin form.*

Proof. — We are going to construct for any simple closed curve α on T certain 2-chain in the unit bundle associated to $\Omega_M^2(T)$ such that

(*a*) its boundary is the union of simple closed curves $\alpha', \alpha_1, \ldots, \alpha_N$.

(*b*) In the double cover $\mathcal{O}_M(-D) \otimes \mathcal{O}_M(-D) \to \Omega_M^2(T)^*$ each α_i lifts into a connected curve.

(*c*) α is homologous to $\tilde{\alpha}+z$. Here $\tilde{\alpha}$ and z are considered as elements of $\Omega_M^2(T)|_T^*$ which was identified with the tangent bundle to T.

(*d*) Moreover N is equal to the sum of the number of intersections with T of certain 2-disk D_α in M with boundary α and the obstruction to extending the normal vector field to α in T to a non-vanishing normal field to this 2-disks. This would imply that the value on α of the quadratic form associated to the Spin structure corresponding to $\mathcal{O}_T(D)$ is the same as the value of the Rohlin form because the value of the former quadratic form on α' is the sum of the values on α_i's according to the definition from B of section 1 and its value on α_i is 1 according to (*b*).

Let $\bar{v}$ (resp. v) be a non vanishing tangent vector field (resp. a normal vector field) to D_α. We shall assume that $v|_\alpha$ is tangent to T. Then $(\bar{v}, v)$ defines the section of $\Omega^2_M(T)|^*_{D_\alpha}$ as follows. Let ω^2 be a germ of a section of Ω^2_M which does not need to be holomorphic. Let $s_{\bar{v}, v}(\omega^2)=\omega^2(\bar{v}, v)$. This section vanishes only at the zeroes of v, the intersection points $D_\alpha \cap T$ and also along α. The latter occurs because a form regular near T defines a vanishing along T section of the bundle $\Omega^2_M(T)$. If $U_i (i=1, \ldots, N)$ be small circles in D_α about the zeroes of v and the intersection points of D_α and T then $s_{\bar{v}, v}(D_\alpha - \bigcup U_i)$ can be viewed as a 2-chain in the space of non-zero vectors of the total space of $\Omega^2_M(T)^*$. Each loop $s_{\bar{v}, v}(\partial U_i)$ lifts into a connected path because the covering on each fibre is just the squaring $z \to z^2$ which gives the claim (*b*) above. On the other hand the space of unit vectors in $\Omega^2_M(T)$ restricted to ∂U_i is homeomorphic to $S^1 \times S^1$ where the factors are represented by the class of $s_{\bar{v}, v}(\partial U_i)$ and the class z of the fibre. The tangent vector field in this identification with $S^1 \times S^1$ is in the homology class of the sum of the factors which gives (*c*) and proves the proposition.

Remark. — If one assumes that T is an ample divisor on M then one can give a very simple proof of 2.1 as follows. Both the theta form of the canonical theta characteristic on T coming from the embedding of T in M and the Rohlin form are invariant under the monodromy of a generic pencil in which T moves. The monodromy about a singular member of the pencil is the Picard-Lefshetz transformation which is just the transvection $x \to x-(x, \delta)\delta$ corresponding to the vanishing cycle δ. The invariance of a quadratic form under this transvection implies that the value of this quadratic form on δ is 1. On the other hand the Lefshetz theorem on hyperplane sections implies that $H^1(T, \mathbf{Z}_2)$ is generated by the vanishing cycles. Hence the Rohlin form and the theta form which are the same on the vanishing cycles are equal everywhere. Hence we obtain 2.1.

3. Theta characteristics and the degenerations

Let $\gamma: M \to \mathbf{P}^1$ be a pencil on an algebraic surface M and let M_t be an element of γ corresponding to $t \in \mathbf{P}^1$. Let Sing $\subset \mathbf{P}^1$ be the collection of the points corresponding to the singular fibres of γ. The correspondence relating to any $t \in$ Sing the set of theta characteristics on M_t defines the unbranched covering π of $\mathbf{P}^1-$Sing which is equivalent to the covering which relates to any $t \in \mathbf{P}^1-$Sing the set of order 2 points on the Jacobian of M_t. Let us consider a small perturbation $\tilde{\gamma}: M \to \mathbf{P}^1$ of the pencil γ in which the singular fibre M_{t_0} is replaced by several fibres each of which has only ordinary double point as a singularity. We shall consider the subgroup $G \in \operatorname{Aut}(H_1(M_t, \mathbf{Z}_2))$ ($t \in \mathbf{P}^1$-Sing) generated by the monodromy transformations about the singular fibres in which M_{t_0} splits. The group G acts as well on the set of theta characteristics on M_t. Let α (resp. $\tilde{\alpha}$) be a small loop in the base of the pencil γ (resp. $\tilde{\gamma}$) about t_0 (resp. union of points in which t_0 splits). Let Δ (resp. $\tilde{\Delta}$) be the disk bounded by α (resp. $\tilde{\alpha}$).

PROPOSITION 3.1. — (*a*) *The set of theta characteristics on* M_{t_0} *can be identified with the set of theta characteristics on* M_t ($t \in \tilde{\Delta} - \operatorname{Sing} \tilde{\gamma}$) *invariant under the monodromy group* G *obtained by splitting* M_{t_0} *into fibres with nodes as the only singularities.*

(*b*) *If* M_t *has only unibranched singularities then invariance under* G *can be replaced by the invariance under the monodromy transformation induced by* α.

(*c*) *For any theta characteristic* L_{t_0} *on* M_{t_0} *there exist a bundle* L *over* $\gamma^{-1}(\Delta)$, *such that for any t,* $L|_{M_t}$ *is a theta characteristic on* M_t *and* $L|_{M_{t_0}} = L_{t_0}$.

(*d*) *The number of theta characteristics on* M_{t_0} *is equal to* $2^{b_1(M_{t_0})}$ *where* $b_1(M_{t_0}) = rk\, H_1(M_{t_0}, \mathbf{Z})$.

Remark. — In [H] slightly different way for finding the total number of theta characteristics is outlined. For example if a curve has one singularity then according to [H], p. 620 this number is equal to $2\tilde{g}+b-1$ where $\tilde{g}$ is the genus of the normalization and b is the number of branches at the singular point. Using for example additivity of Euler characteristic one easily verifies that this is equivalent to (*d*).

Proof of 3.1. — Let $\Omega^1_{M/\mathbf{P}^1}|_{\tilde{\Delta}}$ be the relative dualizing sheaf for $\tilde{\gamma}|_{\tilde{\Delta}}$ and let $\mathscr{L}^2 = \Omega^1_{M/\mathbf{P}^1}$. Then $\mathscr{L}|_{M_t}$ is a G-invariant theta characteristic on M_t. For any space X let $\operatorname{Pic}_{\mathbf{R}}(X)$ be the Picard group of real rank 1 bundle on X (*cf.* [*A*]). The classifying space for such bundles is $\mathbf{RP}^\infty = K(\mathbf{Z}_2, 1)$ and hence $\operatorname{Pic}_{\mathbf{R}}(X) = H^1(X, \mathbf{Z}_2)$. Any real rank 1 bundle satisfies $\xi^2 = \mathrm{id}$. Moreover the bundle $\mathscr{L}|_{M_t} \otimes \xi$ has the natural complex structure because the transition functions on ξ can be chosen to be constant. Hence it is a theta characteristic on M_t. Any theta characteristic on M_t has such form (*cf.* [A]). Clearly $\operatorname{Pic}^G_{\mathbf{R}}(M) = H^1(M_t, \mathbf{Z}_2)^G = H^1(M_{t_0}, \mathbf{Z}_2) = \operatorname{Pic}_{\mathbf{R}}(M_{t_0})$ (where X^G denotes the invariants of X). Therefore $\xi \to \xi \otimes \mathscr{L}|_{M_t}$ can be interpreted as the 1-1 correspondence between the homology classes on M_{t_0} and the G-invariant theta characteristics on M_t. This proves (*a*), (*c*) and (*d*). (*b*) follows from the fact that in unibranched case the monodromy about singular fibre does not have vectors with eigenvalue 1 as one can see from the Wang sequence corresponding to the Milnor fibration (*cf.* [M2] th. 8.5).

4. The main theorem

Now we are in position to prove the following

THEOREM 4.1. — *Let* C *be a reduced curve on a non-singular algebraic surface* V. *Let* $\tilde{g}$ *denotes the genus of the normalization of* C *and k be the rank of the group of line bundles on* C *which pull back on normalization is trivial. If at least one of the links of singularities of* C *is not proper then the number of even and the number of odd theta characteristics on* C *are equal. If the links of all singularities of* C *are proper and if* $R(P_i)$ *is the Robertello invariant of the link of singularity* P_i *of* C ($i = 1, \ldots, N$) *then the number of even (resp. odd) theta characteristics on* C *is equal to* $2^{\tilde{g}+k-1}\left(2^g + \prod_{i=1}^{N} R(P_i)\right)$

$\left[\textit{resp. } 2^{\tilde{g}+k-1}\left(2^g - \prod_{i=1}^{N} R(P_i)\right)\right]$.

Before we shall proceed to the proof let us recall that J. Harris computed the number of even and odd theta characteristics on a curve with Gorenstein singularities as follows. Let P_i be a singular point of C. Let I_{P_i} be the adjoint ideal *i. e.* the annihilator of $\pi_{P_i *} \mathcal{O}_{\tilde{C}_{P_i}}/\mathcal{O}_C$ where $\pi_{P_i}: \tilde{C}_{P_i} \to C$ is the normalization at P_i. Let divisor D_{P_i} be defined by $\pi_{P_i}^* I_{P_i} = \mathcal{O}_{\tilde{C}_{P_i}}(-D_{P_i})$. Then the number of even and the number of odd theta characteristics on C are equal if at least for one singular point P_i the divisor D_{P_i} is not even *i. e.* not all multiplicities in D_{P_i} of the points in $\operatorname{Supp} D_{P_i}$ are even. On the other hand if $D_{P_i} = 2 E_{P_i}$ with $\operatorname{Supp} D_{P_i} = \operatorname{Supp} E_{P_i}$ and if $\varepsilon(P_i) = \dim \Gamma(\mathcal{O}_C/I')$ where I' is the ideal of functions which pullback to $\tilde{C}_{P_i}$ is in $\mathcal{O}_{\tilde{C}_{P_i}}(-E_{P_i})$ then the number of even theta characteristics is $2^{\tilde{g}+k-1}(2^{\tilde{g}} + (-1)^{\Sigma \varepsilon(P_i)})$. We are going to show that the Harris invariant $\varepsilon(P_i)$ and the Robertello invariants $R(P_i)$ in the case of curves on a surface are defined in the same circumstances and that $R(P_i) = (-1)^{\varepsilon(P_i)}$.

Using the lemmas 10.6 and 10.7 from [M2] we can assume that singularity P_i is equivalent to a singularity P such that P is a single singular point of a plane curve C_0 of degree d which is congruent to $\pm 1 \bmod 8$. Let us fix a pencil containing C_0 such that the generic curve C_t in it is non-singular. Any element in this pencil represents a homology class dual to $w_2(\mathbf{CP}^2)$ because the degree of C_t is odd. Then we have the following

Proposition 4.2. — *In the situation as above let q_{C_0} and q_{C_t} be the theta forms corresponding to the canonical theta characteristic on the curves C_0 and C_t respectively. Let l_P be the Seifert form of the link of the singularity P of C_0. Then $q_{C_t} = q_{C_0} + l_P$.*

Proof. — Let us consider the blow up V of $\mathbf{CP}^2$ at the base points of the chosen pencil. The pullback of $\mathcal{O}_{\mathbf{CP}^2}((d-3))/2$ to V restricted to C_t ($|t|$ small) is the canonical theta characteristic L. According proposition 3.1 for any order 2 bundle ξ over C_0 there is a bundle L_ξ over the union of the curves C_t ($|t|$ as above) such that $L_\xi|_{C_0} = \xi \otimes L|_{C_0}$. Theorem 1.10 from [H] implies that the theta form q_{C_t} restricted to the invariant part of $H_1(C_t, \mathbf{Z}_2)$ is isomorphic to the form q_{C_0}. On the other hand $\operatorname{Ker}(H_1(C_t, \mathbf{Z}_2) \to H_1(C_0, \mathbf{Z}_2))$ is the subgroup of the vanishing cycles which in turn is isomorphic to H_1 of the Seifert surface of the link of singularity of C_0 obtained by pushing of the part of C_t inside a small ball D^4 about the singularity of C_0 into the boundary of this ball (this always can be done according to [M2] sect. 5). Now according to proposition 2.1 the Rohlin form on $H_1(C_t, \mathbf{Z}_2)$ can be identified with the theta form of the theta characteristic $(\mathcal{O}(d-3)/2)|_{C_t}$. On the other hand the Rohlin form restricted to the first homology group of the Seifert surface obtained from $C_t \cap D^4$ as above coincide with the Seifert form. Indeed one can make calculation of the Rohlin form on $\tilde{C}_t$ which is obtained from C_t by replacing $C_t \cap D^4$ by the Seifert surface. The Rohlin forms on $\tilde{C}_t$ and C_t are the same because the surfaces are isotopic and the result follows from proposition 1.4.

To conclude the proof notice that theorem 1.11 and lemma 5.6 from [H] combined with the fact that the intersection number of two algebraic curves is the linking number of links of these curves at the intersection point implies that the Harris invariant for a curve on the surface is defined if and only if the link of the singularity is proper. Next the Harris invariant of the singularity of C_0 is the Arf invariant of q_{C_0} as one can see for example by considering the number of times q_{C_0} takes value zero or one and using the fact that the canonical theta characteristic on C_0 is even ($d = \pm 1 \mod 8$) (*e.g.* if the Harris invariant is 0 then the number of even theta characteristics on C_0 is bigger then the number of odd ones, *i.e.* $q_{C_0} = \dim H^0(\xi \otimes L) - \dim H^0(L) = \dim H^0(L \otimes \xi) \mod 2$ (here L is the canonical theta characteristic which is even) takes more even values *i.e.* $\operatorname{Arf} q_{C_0}$ is zero). On the other hand the Arf invariant of q_{C_0} is the Arf invariant of the Seifert form *i.e.* the Robertello invariant of the link and the theorem follows.

Example. — Let P be an ordinary singularity *i.e.* locally given by $x^n = y^n$. Corresponding link consists of n circles such that the linking number of any two circles is equal to 1 (it can be also described as the union of n fibres of the Hopf fibration $S^3 \to S^2$). Hence the Robertello invariant is defined if and only if n is odd. Let $n = 2m+1$. In $\mathbf{CP}^2$ the curve given by $x^n = y^n$ has the self intersection $(2m+1)^2$. Hence using definition from sect. 1 the Robertello invariant is determined by $(1-n^2)/8 \mod 2$ and is equal to $(-1)^{m(m+1)/2}$ (*cf.* [H]).

One can easily work out the Robertello invariant in unibranched case using the formula with Legendre symbol mentioned in the introduction via the value of the Alexander polynomial at -1. For example for singularity $x^2 = y^n$ (n odd)

$$\Delta(t) = t^{n-1} - t^{n-2} + \ldots + 1.$$

Therefore the Robertello invariant is $(2/\Delta(-1)) = (-1)^{(n^2-1)/8}$. In the case of two branches one can use the Murasugi formula [Mu2].

Remark. — According to proposition 2.1 the Arf invariant of the Rohlin form for a non-singular algebraic curve S on an algebraic surface V such that $S + K = 2D$ (K is the canonical class of V) is equal to $\dim H^0(S, \mathcal{O}_S(D)) \mod 2$. Hence the Rohlin theorem claims that

$$\dim H^0(S, \mathcal{O}_S(D)) = (\mathcal{O}(V) - S^2)/8 \mod 2. \tag{9}$$

One may wounder about algebro-geometric proof of this statement. Indeed the Riemann Roch theorem for the bundle $\mathcal{O}_V(D)$ on V gives that the right hand side of (9) is equal to $\chi(\mathcal{O}_V(D))$. Hence the Rohlin theorem would follow (in analytic case) from the congruence $\dim H^0(S, \mathcal{O}_S(D)) = \chi(\mathcal{O}_V(D))$. This is a special case of the theorem of Atiyah and Rees ([AR], th. 7.4).

5. Steenbrink Jacobian and the Seifert form

Let $f(x, y)$ be a germ of a reduced curve having singularity at the origin which we shall assume in this section has a single branch. J. Steenbrink [S] put the mixed Hodge structure on the cohomology of the Milnor fibre B_f of f which with our assumptions is in fact pure of weight 1 because the monodromy operator is semisimple in this case. Let $F^1 \subset H^1(B_f, \mathbf{C})$ be the corresponding element of the Hodge filtration. Then the torus $J(B_f) = (F^1)^*/H_1(B_f, \mathbf{Z})$ will be called the Steenbrink Jacobian of the singularity B_f. Moreover in the unibranched case the intersection form is unimodular, the Riemann relations are satisfied and therefore $J(B_f)$ is in fact a principally polarized abelian variety. We shall denote by $\Theta(B_f)$ the theta divisor of $J(B_f)$.

Next we are going to construct a certain base element in $J(B_f)$ which we shall call the canonical theta characteristic. To do this recall that the mixed Hodge structure on $H^1(B_f)$ is constructed in such a way that the exact sequence

$$(10) \qquad 0 \to H^1(C_0) \to H^1(C_\infty) \to H^1(B_f) \to 0$$

is a sequence of the mixed Hodge structures in which C_∞ is the fibre product $\tilde{C} \times_{\Delta^*} \Delta$ where $\tilde{C}$ is the restriction of the pencil C_t considered in the last section on Δ^*, $\Delta^* = \{t,\ 0 < |t| < \varepsilon\} = \Delta - 0$ and $\Delta \to \Delta^*$ is the exponential map $\alpha \to \exp(\alpha)$. The Hodge filtration on $H^1(C_\infty)$ is the limit when t tends to 0 of the Hodge filtration on $H^1(C_{\log t})$ (the filtration is independent of the value of $\log t$ because the action of semisimple monodromy preserves the Hodge filtration and the limit exist by the Schmid nilpotent orbit theorem). Because $H_1(C_\infty, \mathbf{Z}) = H_1(B_f, \mathbf{Z}) \oplus H_1(C_0, \mathbf{Z})$ the sequence of Jacobians corresponding to (10) splitts and hence $J(B_f) \oplus J(C_0) = J(C_\infty)$. The canonical theta characteristic on each C_t in limit produces the element $k(\infty)$ in $J(C_\infty)$ projection of which on $J(B_f)$ [resp. on $J(C_0)$] we denote by $k(B_f)$ [resp. by $k(C_0)$].

THEOREM 5.1. — *Let Θ be the theta divisor of* $J(B_f)$ *and* $K(B_f)$ *be its canonical theta characteristic. Then for* $\alpha \in H^1(B_f, Z_2)$ *the multiplicity of* Θ *at* $K(B_f) + \alpha$ *is equal* mod 2 *to* $l(\alpha, \alpha)$ *where l is the Seifert form of* B_f *considered as a surface in* S^3.

Proof. — According to the last section the theta form of canonical theta characteristic is the Rohlin form and the former can be identified with multiplicity mod 2 of the theta divisor. Hence in the limit one obtains the same relationship in $J(C_\infty)$ because the mult mod 2 of the theta divisor in a point which is a theta characteristic is invariant in holomorphic deformations as is aparent from interpretation the theta divisor as the zero of the theta function with corresponding theta characteristic (*cf.* [ACGH] p. 292). Obviously

$$\operatorname{mult}_{k(\infty)+\alpha+\beta}(C_\infty) = \operatorname{mult}_{k(0)+\alpha}(C_0) + \operatorname{mult}_{k(B_f)+\beta}(B_f).$$

Hence taking $\alpha = 0$ we obtain that the function relating to a homology class the multiplicity of the theta divisor mod 2 is the restriction of the Rohlin form which by 1.4 is the Seifert form.

Q.E.D.

Remark. — S. Oshanine [O] introduced a Kervaire invariant of Spin cobordism of $(8k+2)$ dimensional Spin manifold S by embedding it into a $(8k+4)$-dimensional Spin^C manifold M where S is dual to $w_2(M)$ and by letting $k(S) = (\sigma(M) - (S \circ S))/8 \bmod 2$ (σ denotes the signature and $S \circ S$ is a transversal self-intersection inside M). On the other hand one has the invariant $a(S)$ introduced by Atiyah [A] which is the mod 2 dimension of the space of harmonic spinors on S which is independent of a choice of a metric and which depends only on the Spin cobordism class of S. The results of this paper show that $a(S) = k(S)$ for Spin manifolds of dimension 2. It turns out this relation fails in higher dimensions as show the calculations with hypersurfaces in $\mathbf{CP}^n$. We would like to know if there is a natural homomorphism $\Omega^{\text{spin}}_{8k+2} \rightarrow \mathbf{Z}_2[q]$ such that the Atiyah and Kervaire invariants are the specializations of thsi homomorphism in two values of q in a way as elliptic genus of Landweber-Stong-Oshanine-Witten for oriented cobordisms specializes into L- and A- genera.

REFERENCES

[A] M. Atiyah, *Riemann Surfaces and Spin Structures* (*Ann. Scient. Ec. Norm. Sup.*, 4 série, T. 4, 1971).

[AR] M. Atiyah and E. Rees, *Vector Bundles on Projective 3-Space* (*Inv. Math.*, Vol. 35, 1976).

[ACGH] Arbarello, Cornalba, Griffiths and Harris, *Geometry of Algebraic Curves*, Vol. 1, Springer Verlag, 1985.

[B] W. Browder, *Surgery on Simply Connected Manifolds*, Springer Verlag, 1972.

[J] D. Johnson, *Spin Structures and Quadratic Forms on Surfaces* (*Journ. of London Math. Soc.*, Vol. 2, (22), 1980, pp. 365-373).

[H] J. Harris, *Theta Characteristics on Algebraic Curves* (*Trans. Amer. Math. Soc.*, Vol. 271, 1982, pp. 611-638).

[L] J. Levine, *Polynomial Invariants of Knots of Codimention Two* (*Ann. of Math*, Vol. 84, 1966).

[M1] J. Milnor, *Spin Structures on Manifolds* (*L'enseignement Math.*, Vol. 9, 1963, pp. 198-203).

[M2] J. Milnor, *Singular Points of Complex Hypersurfaces* (*Ann. of Math. Studies*, Princeton, 1968).

[Mu1] K. Murasugi, *Arf Invariant of Knot Types* (*Proc. A.M.S.*, vol. 21, 1969, p. 69-72).

[Mu2] K. Murasugi, *On the Arf Invariant of Links* (*Math. Proc. Camb. Phil. Soc.*, Vol. 95, 1984, pp. 61-69).

[O] S. Oshanine, *Signature et invariant de Kervaire généralisés*, *C.R. Acad. Sci. Paris*, 285, série A, 1977, pp. 211-213.

[R] R. Robertello, *An Invariant of Knot Cobordisms* (*Comm. Pure Appl. Math.*, Vol. 18, 1965, pp. 543-555).

[Ro] V. A. Rohlin, *Proof of Gudkov Hypotheses* (*Funct. Anal. and Appl.*, 1972, pp. 136-138).

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