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Divisor classes associated to families of stable varieties, with applications to the moduli space of curves

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DIVISOR CLASSES ASSOCIATED TO FAMILIES OF STABLE VARIETIES, WITH APPLICATIONS TO THE MODULI SPACE OF CURVES

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1. Introduction and statement of the main results

The basic objects we will be concerned with in this paper are families of polarized complex algebraic varieties. By this we mean an algebraic family of pairs (X_t, L_t), where X_t is an algebraic variety and L_t a line bundle on X_t; or, more precisely, a proper flat morphism $\pi: X \to T$ and a line bundle L on X modulo pullbacks of line bundles on T. We will always assume that X and T are separated, that T is irreducible and X pure-dimensional; on the other hand, X and T need not be reduced. We let k be the dimension of T and d the relative dimension of π.

What sort of cohomological invariants can one associate to such a family? Normally, given a line bundle L on a space X, we could take the first Chern class of L; but since

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here L is only defined up to twists by line bundles from T, this is not *a priori* well-defined. Another invariant we can look at is the first Chern class of the direct image sheaf $\pi_* L$, when this is locally free; but here again this class is not well-defined: if $L' = L \otimes \pi^* M$ for some line bundle M on T, we will have $\pi_* L' = M \otimes \pi_* L$, and hence $c_1(\pi_* L')$ equals $c_1(\pi_* L') + rc_1(M)$, where r is the rank of $\pi_* L$. There is, however, a linear combination of these two classes, or, rather, of the first one with the pullback of the second one to X, that is invariant under twists of L by pullbacks of line bundles from T, namely the divisor class $rc_1(L) - \pi^* c_1(\pi_* L)$ in the cycle group $A^1(X)$. Instead of working with it directly, we will find it more convenient to use the class

$$\tilde{\mathscr{E}}(L) = (rc_1(L) - \pi^* c_1(\pi_* L)) \cap [X] \in A_{k+d-1}(X),$$

where [X] stands for the fundamental class of X and "$\cap$" denotes cap product. We can also define a divisor class on the base T by taking a power of the class $rc_1(L) - \pi^* c_1(\pi_* L)$ and pushing it forward: we set

$$\mathscr{E}(L) = \pi_*((rc_1(L) - \pi^* c_1(\pi_* L))^{d+1} \cap [X]) \in A_{k-1}(T).$$

What can we say about the classes $\tilde{\mathscr{E}}$ and $\mathscr{E}$ in general? Apparently, not much. If, however, we make a suitable positivity and stability assumption about the line bundle $L_{|\pi^{-1}(t)}$ for general t, we find that $\mathscr{E}$ lies in the closure of the cone of effective divisor classes. This is the content of Theorem (1.1) below, which is the main result of this paper.

Before stating the theorem, we explain our terminology and assumptions. We begin by generalizing slightly the definition of $\tilde{\mathscr{E}}$ and $\mathscr{E}$. Let F be a locally free coherent sheaf of rank r on T. We set

$$\tilde{\mathscr{E}}(L, F) = (rc_1(L) - \pi^*(c_1(F))) \cap [X] \in A_{k+d-1}(X)$$
$$\mathscr{E}(L, F) = \pi_*((rc_1(L) - \pi^* c_1(F))^{d+1} \cap [X]) \in A_{k-1}(T).$$

By the push-pull formula

$$\mathscr{E}(L, F) = r^{d+1} \pi_*(c_1(L)^{d+1} \cap [X]) - (d+1) r^d c_1(F) \cap \pi_*(c_1(L)^d \cap [X]).$$

Notice that $\mathscr{E}(L, F)$ and $\tilde{\mathscr{E}}(L, F)$ are left unchanged if we tensor F by a line bundle M and L by $\pi^*(M)$. Also, our old $\mathscr{E}(L)$ and $\tilde{\mathscr{E}}(L)$ are just $\mathscr{E}(L, \pi_* L)$ and $\tilde{\mathscr{E}}(L, \pi_* L)$.

One can define $\mathscr{E}(L, F)$ [and $\tilde{\mathscr{E}}(L, F)$] also when F is just a coherent sheaf, provided it is locally free on an open subset U of T such that $T - U$ has codimension two or greater (notice that this is always the case when F is torsion-free and T is normal). In fact, $A_{k-1}(T)$ equals $A_{k-1}(U)$, and one merely defines $\mathscr{E}(L, F)$ to be the image of $\mathscr{E}(L_{|\pi^{-1}(U)}, F_{|U})$ in $A_{k-1}(T)$.

The statement of Theorem (1.1) involves the notion of stability, whose meaning in our context we now explain. Let Z be a projective variety, M a line bundle on Z, V a vector subspace of $H^0(Z, M)$. Suppose that V has no base points and is very ample. Let

$$j: \quad Z \to \mathbb{P}(V^\vee)$$

be the embedding defined by V. Then, for large enough n, the natural map

$$\varphi_n: \quad \mathrm{Sym}^n(V) \to H^0(Z, M^n)$$

is onto. Thus, setting $N = h^0(Z, M^n)$,

$$\Lambda^N \varphi_n: \quad \Lambda^N \mathrm{Sym}^n(V) \to \Lambda^N H^0(Z, M^n)$$

is a nonzero element of the vector space $\Lambda^N \mathrm{Sym}^n(V)^{\check{}} \otimes (\Lambda^N H^0(Z, M^n))$. We shall say that j is a (Hilbert) stable or semistable embedding if $\Lambda^N \varphi_n$ is stable or semistable, in the sense of geometric invariant theory, under the action of $SL(V)$, for arbitrarily large values of n.

We then have:

THEOREM (1.1). — *Let* X *and* T *be separated, with* T *irreducible of dimension* k *and* X *of pure dimension* $k+d$. *Let* $\pi: X \to T$ *be a flat proper morphism. Let* L *be a line bundle on* X, *and* F *a coherent subsheaf of* $\pi_*(L)$ *that is locally free off a subvariety of* T *of codimension two or greater. Suppose that the following conditions are satisfied:*

(i) *If* t *is a general point of* T, *then* $F_t \otimes \mathbb{C} \subset H^0(\pi^{-1}(t), L_{|\pi^{-1}(t)})$ *is base-point-free, very ample, and yields a semi-stable embedding of* $\pi^{-1}(t)$.

(ii) L *is relatively ample.*

Then $\mathscr{E}(L, F)$ *lies in the closure of the cone in* $A_{k-1}(T) \otimes \mathbb{Q}$ *generated by the effective Weil divisors; if* F *is locally free* $\mathscr{E}(L, F)$ *lies in the closure of the cone generated by the effective Cartier divisors.*

How we topologize $A_{k-1}(T) \otimes \mathbb{Q}$ is immaterial: any linear topology will do, as will be apparent from the proof. In most applications of the theorem, F will be equal to $\pi_*(L)$. We mention a simple consequence of (1.1).

COROLLARY (1.2). — *Suppose the hypotheses of Theorem* (1.1) *are satisfied. Assume moreover that condition* (i) *holds outside of a finite number of points of* T, *that* T *is projective, and that* F *is locally free. Then the class* $\mathscr{E}(L, F)$ *lies in the closure of the ample cone in* $A_{k-1}(T) \otimes \mathbb{Q}$.

We will give a proof of the basic theorem, of a variant of it, and of the corollary, in the next section. In section 3 we will give an example, due to Ian Morrison, that shows that the hypothesis of semistability on the general fiber is a crucial one. In section 4 we will apply the basic theorem to the case of a family of curves polarized by their canonical line bundles, to obtain some inequalities among divisor classes on the moduli spaces of curves. In particular we will prove the

THEOREM (1.3). — *Let* $\bar{M}_g$ *be the moduli space of stable genus* g *curves, with* $g \geqq 2$, *and let* λ, $\delta \in \mathrm{Pic}(\bar{M}_g) \otimes \mathbb{Q}$ *be the Hodge class and the boundary class. Then the class* $a\lambda - b\delta$ *has non-negative degree on every curve in* $\bar{M}_g$ *not contained in the boundary* $\Delta = \bar{M}_g - M_g$ *if and only if*

$$a \geqq (8 + 4/g)\, b,$$

and is ample if and only if

$$a > 11 . b > 0.$$

What was previously known [12] was that $a\lambda - b\delta$ is not ample if $a < 11 . b$ and is ample for $a \geqq (11.2) . b > 0$. The first part of Theorem (1.3) has also been independently proved by Xiao Gang [15], using somewhat different techniques.

We thank the referee for a number of useful comments and suggestions of improvements to the first version of the present work.

2. Proof of the main theorem

We shall now give a proof of Theorem (1.1). Clearly, it suffices to deal with the case when F is locally free. For large enough n the higher direct images of L^n vanish and the inclusion of F in $\pi_*(L)$ induces generically surjective maps of locally free sheaves

$$\varphi_n : \quad \mathrm{Sym}^n(F) \to \pi_*(L^n),$$
$$\Lambda^N \varphi_n : \quad \Lambda^N \mathrm{Sym}^n(F) \to \Lambda^N \pi_*(L^n),$$

where N stands for the rank of $\pi_*(L^n)$. By condition (i) of the theorem, for arbitrarily large values of n there is an SL-invariant homogeneous polynomial P that does not vanish at $\Lambda^N \varphi_{n \mid t}$, where t is a general point of T. Choosing local trivializations for F and $\Lambda^N \pi_*(L^n)$, we get a local regular function f by evaluating P on $\Lambda^N \varphi_n$. Since P is SL-invariant, changing trivialization of F by a matrix A changes f by a factor $(\det A)^{-N n/r}$, where r is the rank of F. Thus if, as we may, we choose P to have degree rm, the f's give a non-zero global section of the line bundle

$$\mathscr{F}_n = \mathscr{H}om((\det F)^{nNm}, (\Lambda^N \pi_*(L^n))^{rm}). \tag{2.1}$$

We may evaluate the Chern class of this line bundle by applying the Riemann-Roch theorem for singular varieties (*cf.* [4]) to L^n; this says that

$$\mathrm{ch}(\pi_*(L^n)) \cap \tau_T(\mathcal{O}_T) = \tau_T(\pi_!(L^n)) = \pi_*(\tau_X(L^n)) = \pi_*(\mathrm{ch}(L^n) \cap \tau_X(\mathcal{O}_X)).$$

Recalling that, for any Y,

$$\tau_Y(\mathcal{O}_Y) = [Y] + \text{terms of dimension } < \dim(Y),$$

and equating terms of degree $k-1$, we find that $c_1(\pi_*(L^n)) \cap [T]$ is a polynomial in n with leading term

$$(1/(d+1)!)\, n^{d+1} \pi_*(c_1(L)^{d+1} \cap [X]).$$

Thus $c_1(\mathscr{F}_n) \cap [T]$ is a polynomial in n with leading term

$$(m/(d+1)!)\, n^{d+1} \{ r \pi_*(c_1(L)^{d+1} \cap [X]) - (d+1) c_1(F) \cap \pi_*(c_1(L)^d \cap [X]) \}$$
$$= (m/(r^d (d+1)!))\, n^{d+1} \mathscr{E}(L, F).$$

In other words,

$$c_1(\mathscr{F}_n)\cap[\mathrm{T}]=(m/(r^d(d+1)!))\,n^{d+1}\,\mathscr{E}(\mathrm{L},\mathrm{F})+\mathrm{Q}(n),$$

where Q is a polynomial with coefficients in $\mathrm{A}_{k-1}(\mathrm{T})\otimes\mathbb{Q}$ of degree at most d. Thus, if E is any effective Cartier divisor class on T,

$$\mathscr{E}(\mathrm{L},\mathrm{F})=(\mathrm{E}+(r^d(d+1)!/m).\,c_1(\mathscr{F}_n)\cap[\mathrm{T}])/n^{d+1}+\mathrm{R}(n)/n^{d+1}, \tag{2.2}$$

where R is a polynomial of degree at most d. Since the divisor class $\mathrm{E}+(r^d(d+1)!/m).\,c_1(\mathscr{F}_n)\cap[\mathrm{T}]$ is effective, letting n go to infinity concludes the proof of (1.1).

To prove Corollary (1.2), notice that its hypotheses imply not only that $\mathscr{F}_n$ has a non-zero section, but also that, for all but a finite number of points $t\in\mathrm{T}$, it has a section that does not vanish at t. In particular, the intersection number of $\mathscr{F}_n$ with any irreducible curve in T is non-negative, so Seshadri's criterion of ampleness [9] implies that, for any ample line bundle M, $\mathrm{M}\otimes\mathscr{F}_n$ is ample. Thus if, in (2.2), we choose E to be ample, the conclusion of the corollary follows.

It should be observed that our methods of proof are very similar to those used by Mumford in [12] to show that $a\lambda-\delta$ is ample if $a\geqq 11.2$. It has also been brought to our attention by the referee that our proof of (1.1) is essentially the same as the proof of Theorem 8.1 in Viehweg's paper [14].

Theorem (1.1) can be sharpened somewhat; in particular hypothesis (ii) can be slightly relaxed. To exemplify this, we shall look at a proper flat family $\pi:\mathrm{X}\to\mathrm{T}$ of noded curves over a smooth complete one-dimensional base (here, and in the sequel, by noded curve we mean a complete reduced curve that is either smooth or has at most nodes as singularities). We let L be a line bundle on X and F a (necessarily locally free) coherent subsheaf of $\pi_*\mathrm{L}$. As in Theorem (1.1), we assume that F stably embeds a general fiber of π. In particular, the restriction of L to a general fiber is ample; we shall not require, however, that this be true for every fiber, but merely that the restriction of L to any component of any fiber have non-negative degree.

Now, let's analyse the proof of (1.1). This is based on the fact that

$$\mathscr{F}_n=\mathscr{H}om((\det\mathrm{F})^{n\mathrm{N}m},(\Lambda^{\mathrm{N}}\pi_*(\mathrm{L}^n))^{rm})$$

has a nonzero section for large n. To be more precise, this is also true of

$$\mathscr{G}_n=\mathscr{H}om((\det\mathrm{F})^{n\mathrm{N}m},(\Lambda^{\mathrm{N}}\mathscr{L}_n)^{rm}),$$

where $\mathscr{L}_n$ is the image of

$$\varphi_n:\quad \mathrm{Sym}^n(\mathrm{F})\to\pi_*(\mathrm{L}^n).$$

Thus $\mathscr{G}_n$ has non-negative degree. Notice, incidentally, that $\mathscr{L}_n$ equals $\pi_*(\mathrm{L}^n)$ except at a finite set of points. On the other hand, under our hypotheses, $\mathrm{R}^1\pi_*\mathrm{L}^n$ is not necessarily zero for large n, but is concentrated at a finite set of points, so that the Grothendieck

Riemann-Roch theorem gives

$$(2.3)\quad \deg(\mathscr{G}_n)=\deg(\mathscr{F}_n)-rm\,.\,h^0(\mathrm{T},\pi_*(\mathrm{L}^n)/\mathscr{L}_n)$$
$$=(m/2r)\,n^2\deg\mathscr{E}(\mathrm{L},\mathrm{F})-rm\,.\,h^0(\mathrm{T},\pi_*(\mathrm{L}^n)/\mathscr{L}_n)+rm\,.\,h^0(\mathrm{T},\mathrm{R}^1\pi_*\mathrm{L}^n)+O(n).$$

The sheaf $\mathscr{L}_1$ is of the form $\mathscr{I}\mathrm{L}$ for a suitable ideal sheaf $\mathscr{I}$. Let $e_{\mathrm{L}}(\mathscr{I})$ be the multiplicity of $\mathscr{I}$ measured via L as defined in [12]. We claim that

LEMMA (2.4). — *With the above hypotheses we have:*

(i) $h^0(\mathrm{T},\mathrm{R}^1\pi_*\mathrm{L}^n)=O(n)$.

(ii) $h^0(\mathrm{T},\pi_*(\mathrm{L}^n)/\mathscr{L}_n)=e_{\mathrm{L}}(\mathscr{I})\,.\,(n^2/2)+O(n)$.

Proof. — Let's prove (i). Since $\mathrm{R}^1\pi_*\mathrm{L}^n$ is concentrated at a finite set of points, the statement is local on T. Thus we may replace T with an affine U and assume that $\mathrm{R}^1\pi_*\mathrm{L}^n$ is concentrated at $u\in\mathrm{U}$. By an étale base change we may also assume that π has sections $\Gamma_1,\ldots,\Gamma_k$ over U such that $\mathrm{L}^n(\sum\Gamma_i)$ is generated by its sections and $\mathrm{R}^1\pi_*(\mathrm{L}^n(\sum\Gamma_i))$ vanishes for every $n\geqq 1$. For each i, let γ_i be the point of Γ_i mapping to u. Let a_i (resp., b_i) be a section of $\pi_*(\mathrm{L}(\sum\Gamma_i))$ (resp., $\pi_*\mathrm{L}$) that does not vanish identically on Γ_i, and let α_i (resp., β_i) be the order to which its restriction to Γ_i vanishes at γ_i. Then $a_i b_i^{n-1}$ is a section of $\pi_*(\mathrm{L}^n(\sum\Gamma_i))$ whose restriction to Γ_i vanishes at γ_i to order $\alpha_i+(n-1)\,\beta_i$, so that, looking at the exact sequence

$$\pi_*(\mathrm{L}(\textstyle\sum\Gamma_i))\to\mathscr{O}_{\mathrm{U}}^k\to\mathrm{R}^1\pi_*\mathrm{L}^n\to 0,$$

we conclude that

$$h^0(\mathrm{R}^1\pi_*\mathrm{L}^n)\leqq\textstyle\sum\alpha_i+(n-1)\sum\beta_i,$$

as desired.

As for (ii), the question is again local on T, which we may hence replace with an affine. Then, by the definition of multiplicity,

$$(2.5)\qquad \chi(\mathrm{L}^n/\mathscr{I}^n\mathrm{L}^n)=e_{\mathrm{L}}(\mathscr{I})\,.\,(n^2/2)+O(n).$$

in [12] it is shown that

$$(2.6)\qquad \dim(\mathrm{H}^0(\mathscr{I}^n\mathrm{L}^n)/\mathrm{H}^0(\mathscr{L}_n))=O(n).$$

Actually, in Proposition (2.6) of [12], of which (2.6) is a part, it is assumed that L is generated by its sections; this hypothesis, however, is never used in the proof of (2.6). Now consider the exact sheaf sequence

$$0\to\pi_*(\mathscr{I}^n\mathrm{L}^n)\to\pi_*\mathrm{L}^n\to\pi_*(\mathrm{L}^n/\mathscr{I}^n\mathrm{L}^n)\to\mathrm{R}^1\pi_*(\mathscr{I}^n\mathrm{L}^n)\to\mathrm{R}^1\pi_*\mathrm{L}^n\to\mathrm{R}^1\pi_*(\mathrm{L}^n/\mathscr{I}^n\mathrm{L}^n)\to 0.$$

Part (i) of the lemma implies that

$$(2.7)\qquad h^0(\mathrm{R}^1\pi_*(\mathrm{L}^n/\mathscr{I}^n\mathrm{L}^n))=O(n).$$

On the other hand, the same argument used to prove (i), or, alternatively, the proof of (2.6) in [12], shows that

$$h^0(R^1\pi_*(\mathscr{I}^n L^n)) = h^1(\mathscr{I}^n L^n) = O(n). \tag{2.8}$$

Putting (2.5), (2.6), (2.7), (2.8) together yields (ii).

Q.E.D.

The remark that $\mathscr{G}_n$ has non-negative degree, (2.3) and (2.4) prove

PROPOSITION (2.9). — *Let $\pi : X \to T$ be a flat family of noded curves over a smooth complete curve. Let L be a line bundle on X, and F a coherent subsheaf of $\pi_*(L)$ of rank r. Let $\mathscr{I}$ be the ideal sheaf on X such that $\mathscr{I}L$ is the subsheaf of L generated by F. Suppose that the following hold:*

(i) *If t is a general point of T, then $F_t \otimes \mathbb{C} \subset H^0(\pi^{-1}(t), L_{|\pi^{-1}(t)})$ is base-point-free, very ample, and yields a semi-stable embedding of $\pi^{-1}(t)$.*

(ii) *For any $t \in T$, the restriction of L to any component of $\pi^{-1}(t)$ has non-negative degree.*

Then

$$0 \leqq (L^{\otimes r} \otimes \pi^*(\det F^{\vee}))^{\cdot 2} - r^2 . e_L(\mathscr{I}).$$

3. Morrison's counterexample

It is natural to ask whether the condition of stability is really necessary for the statement of Theorem (1.1), or just a requirement of the proof. The following example of a family of unstable varieties, suggested by Ian Morrison, shows that it is essential.

Of course, we have to start with an unstable variety. Perhaps the simplest such, from our point of view, is the cubic scroll in $\mathbb{P}^4$, a surface of degree 3 that may be described in several ways:

(i) as the image of $\mathbb{P}^2$ under the rational map given by the linear system of conics through a point $p \in \mathbb{P}^2$;

(ii) as the variety cut out by the 2×2 minors of a general 2×3 matrix of linear forms;

(iii) or, geometrically, by choosing a line L and a complementary 2-plane Λ in $\mathbb{P}^4$, a conic $C \subset \Lambda$, and an isomorphism between L and C, and taking the union of the lines joining corresponding pairs of points on L and C (*Fig.* 1).

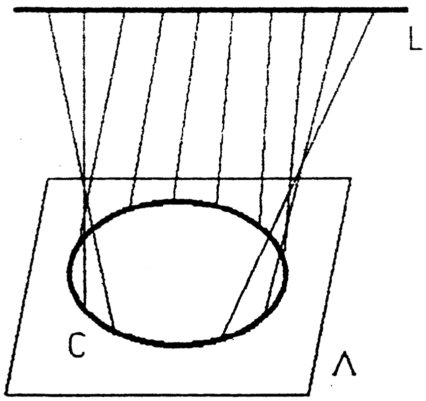

Fig. 1

We now have to construct a family of these over a one-dimensional base T, in a family of projective spaces that must be a non-trivial bundle over T. To do this, we note that the destabilizing flag for a cubic scroll consists simply of the line L. This suggests that we construct our $\mathbb{P}^4$-bundle $\mathbb{P}\,E$ over T and our family $X \subset \mathbb{P}\,E$ of scrolls in such a way that the $\mathbb{P}^1$-bundle formed by the lines L on the scrolls is relatively negative. For example, we can take $T=\mathbb{P}^1$, E the locally free sheaf

$$E=(\mathcal{O}_{\mathbb{P}^1})^{\oplus 3} \oplus (\mathcal{O}_{\mathbb{P}^1}(-1))^{\oplus 2},$$

and $\mathbb{P}\,E$ the projectivization of E (by which we mean the bundle of one-dimensional quotients of fibers of the vector bundle associated to E). Note that $\mathbb{P}\,E$ has trivial subbundles $Y \cong T \times \mathbb{P}^2$ and $Z \cong T \times \mathbb{P}^1$ corresponding to the two summands $(\mathcal{O}_{\mathbb{P}^1})^{\oplus 3}$, $(\mathcal{O}_{\mathbb{P}^1}(-1))^{\oplus 2}$ in the direct sum decomposition of E. To construct our family of scrolls, then, we will choose a conic $C \subset \mathbb{P}^2$ and an isomorphism of $\mathbb{P}^1$ with C, and take the fiber of X over each point $t \in T$ to be the union of the lines joining corresponding points in the fibers of Z and $T \times C \subset Y$.

Another way to describe X is via coordinates on $\mathbb{P}\,E$: let $[U_0 : U_1]$ be coordinates on $T=\mathbb{P}^1$; let W_0, W_1, and W_2 be a frame for $(\mathcal{O}_{\mathbb{P}^1})^{\oplus 3}$, viewed as sections of E; let W'_3 and W'_4 be sections of $(\mathcal{O}_{\mathbb{P}^1}(-1))^{\oplus 2}$ with poles at $U_0=0$ and set $W_i=U_0 W'_i$, $i=3, 4$. Then on each fiber of $\mathbb{P}\,E$ over T, $[W_0 : \ldots : W_4]$ are a system of homogeneous coordinates, in terms of which the fiber of Z is given by $W_0=W_1=W_2=0$ and the fiber of Y by $W_3=W_4=0$. We can then take X to be the locus where the matrix

$$\begin{pmatrix} W_1 & W_2 & W_4 \\ W_0 & W_1 & W_3 \end{pmatrix} \tag{3.1}$$

has rank not greater than one, that is, the subvariety defined by the 2×2 minors of (3.1).

Now, the Chow ring of the projective bundle $\mathbb{P}\,E$ is generated by two classes: the pullback η to $\mathbb{P}\,E$ of the class of a point in $T=\mathbb{P}^1$, and the first Chern class $\xi=c_1(\mathcal{O}_{\mathbb{P}\,E}(1))$

of the tautological bundle. These classes satisfy the relations

$$\eta^2=0, \qquad \eta\xi^4 \cap [\mathbb{P}\,E]=1, \qquad \xi^5 \cap [\mathbb{P}\,E]=c_1(E) \cap [T]=-2.$$

Note that the class of the subvariety Y is $\xi^2+2\eta\xi$, since it is the complete intersection of the two divisors (W_3) and (W_4), each of which has class $\xi+\eta$; similarly, Z, being the intersection $W_0=W_1=W_2=0$ of three divisors linearly equivalent to ξ, has class ξ^3. Given this, it is not hard to determine the class of the threefold X: for example, the hypersurfaces defined by the two minors

$$W_0 W_4 - W_1 W_3, \qquad W_1 W_4 - W_2 W_3$$

of the matrix (3.1) each have class $2\xi+\eta$, and so their intersection has class $4\xi^2+4\xi\eta$. But the intersection of these two hypersurfaces consists exactly (and with multiplicity one) of the union of X and Y. We deduce that X has class $3\xi^2+2\xi\eta$.

Alternatively, we could also find the class of X by interpreting (3.1) as the matrix representative of a bundle map $\varphi: F \to G$, where F is the pullback to $\mathbb{P}\,E$ of the bundle $\mathcal{O}^{\oplus 2} \oplus \mathcal{O}(-1)$ on $\mathbb{P}^1$ and G is $\mathcal{O}_{\mathbb{P}\,E}(1)^{\oplus 2}$, and applying Porteous' formula. We find the class of X is the second graded piece of the quotient $c(\check{F})/c(\check{G})$, that is,

$$\begin{aligned}[X] &= [(1+\eta)(1-\xi)^{-2}]_2 \\ &= [(1+\eta)(1+2\xi+3\xi^2)]_2 \\ &= 3\xi^2+2\xi\eta.\end{aligned}$$

Now, taking the line bundle L on X to be the restriction of $\mathcal{O}_{\mathbb{P}\,E}(1)$, we have of course

$$\pi^*(c_1(\pi_* L)) = \pi^*(c_1(E)) = -2\eta,$$

so the divisor class $\tilde{\mathscr{E}}(L, \pi_* L)$ associated to L is

$$\tilde{\mathscr{E}}(L, \pi_* L) = \operatorname{rank}(\pi_* L)\, c_1(L) - \pi^*(c_1(\pi_* L)) = (5\xi+2\eta)_{|X},$$

and we have

$$\begin{aligned}\tilde{\mathscr{E}}(L, \pi_* L)^3 \cap [X] &= (5\xi+2\eta)^3 (3\xi^2+2\xi\eta) \cap [\mathbb{P}\,E] \\ &= (125.3.\xi^5+125.2.\xi^4\eta+3.25.2.3.\xi^4\eta) \cap [\mathbb{P}\,E] \\ &= -750+250+450=-50,\end{aligned}$$

so that $\mathscr{E}(L, \pi_* L) = \pi_*(\tilde{\mathscr{E}}(L, \pi_* L)^3)$ cannot lie in the closure of the effective cone.

4. Applications to moduli of curves

a. THE BASIC INEQUALITY FOR NON-HYPERELLIPTIC CURVES. — As indicated in section 1 above, one of the main reasons for proving Theorem (1.1) was the hope of applying it to obtain informations about families of stable curves. In order to describe our results we need to recall the structure of the Picard groups of the moduli spaces of curves. This we shall do rather sketchily, referring to [12], [8], or [2] for details.

Let $\bar{M}_g$ be the moduli space of stable curves of genus g. As we shall see in a moment, one can define natural classes λ (the "Hodge class") and $\delta_0, \ldots, \delta_{[g/2]}$ (the "boundary classes") in the rational Picard group $\mathrm{Pic}(\bar{M}_g) \otimes \mathbb{Q}$. It is a fundamental result of Harer that these classes generate $\mathrm{Pic}(\bar{M}_g) \otimes \mathbb{Q}$ (*cf.* [5], [6]); furthermore it is not hard to see that they are independent if $g \geqq 3$, while they satisfy one linear relation for $g=1,2$ (*cf.* section 4 *b* below). It should be observed that they are not classes of line bundles on $\bar{M}_g$, but rather of "line bundles on the moduli stack of genus g curves" [11]. Roughly speaking, a line bundle on the moduli stack is the datum, for each flat proper morphism $f: X \to S$ with stable curves as fibers, of a line bundle L_f on S, natural under base change. There is an obvious notion of isomorphism for these objects, which makes it possible to define a "Picard group of the moduli stack", to be denoted $\mathrm{Pic}(\bar{\mathscr{M}}_g)$.

Clearly any line bundle on $\bar{M}_g$ gives, by pullback, a line bundle on the moduli stack. This yields a homomorphism from $\mathrm{Pic}(\bar{M}_g)$ into $\mathrm{Pic}(\bar{\mathscr{M}}_g)$ which is easily seen to have finite cokernel. It has been shown by Mumford [12] that, for $g \geqq 3$, this is in fact an inclusion and $\mathrm{Pic}(\bar{\mathscr{M}}_g)$ has no torsion, so that we may regard both groups as lattices in $\mathrm{Pic}(\bar{M}_g) \otimes \mathbb{Q}$. If L is a line bundle on $\bar{\mathscr{M}}_g$ we shall write $\mathrm{Cl}(L)$ to denote the corresponding class in $\mathrm{Pic}(\bar{\mathscr{M}}_g)$.

More specifically, the line bundle L giving rise to λ is defined by setting

$$L_f = \Lambda^g f_*(\omega_f)$$

for each family $f: X \to S$ of stable curves, where $\omega_f = \omega_{X/S}$ is the relative dualizing sheaf. Instead, the line bundle M corresponding to δ_i is

$$M_f = \mathscr{O}_S(D_i),$$

where D_i is the effective Cartier divisor in S defined as follows. We say that a stable curve has a singular point of type i at p if its partial normalization at p consists of two connected components of genera i and $g-i$, for $i>0$, and is connected for $i=0$. Let q be a point of S and let $p_1, \ldots, p_h$ be the singular points of type i in $f^{-1}(q)$; thus X is of the form $xy=\gamma_j$ near p_j, where γ_i is a function on S. Then, locally near q, D_i is defined by the equation $\Pi \gamma_j = 0$.

All this assuming, of course, that D_i does not contain a component of S. Otherwise, the definition of M_f is slightly more complicated. The only case that we will need in the sequel is the one when S is a smooth curve and, in addition, the locus of singular points of type i consists of isolated distinct points $p_1, \ldots, p_m$ plus disjoint sections $\Sigma_i, \ldots, \Sigma_n$ of $f: X \to S$ (we can always reduce to this case by a finite base change). Thus, $f: X \to S$ can be thought of as arising from a family $\varphi: Y \to S$ of (not necessarily connected) noded curves by pairwise identification of disjoint sections of smooth points $S_1, T_1, \ldots, S_n, T_n$. We also let n_k be the multiplicity of p_k; in other words, near p_k X is of the form $xy=t^{n_k}$, where t is a suitable local coordinate on S. With these notations, the formula for M_f is

$$(4.1)\qquad M_f = \bigotimes_j (\varphi_*(N_{S_j}) \otimes \varphi_*(N_{T_j}))\left(\sum n_k f(p_k)\right),$$

where N_Z stands for the normal bundle to Z.

One normally writes δ for $\sum \delta_i$; the locus of points in $\bar{M}_g$ with a singular point of type i is usually denoted Δ_i.

Let $f: X \to S$ be a family of stable curves. If μ is a class in $\mathrm{Pic}(\bar{\mathscr{M}}_g)$, we let $\mu_f \in A^1(S)$ be the Chern class of the corresponding line bundle on S; if S is one-dimensional, we shall write $\deg_f(\mu)$ or $\deg_S(\mu)$ to denote the degree of μ_f. In addition to λ_f and $(\delta_i)_f$, $i=0, \ldots, [g/2]$, there is another natural class in $A^1(S)$, namely the pushforward $f_*(c_1(\omega_f)^2)$ of the self-intersection of the relative dualizing sheaf. It follows from the Grothendieck Riemann-Roch formula that this is tied to λ and $\delta = \sum \delta_i$ by the relation

$$(4.2)\qquad f_*(c_1(\omega_f)^2) = 12\lambda_f - \delta_f.$$

Our first step in the proof of (1.3) is to apply Theorem (1.1), or rather Proposition (2.9), to $L=\omega_\pi$, where $\pi: X \to T$ is a family of stable genus g curves over a smooth one-dimensional base T. In order to do this, we first have to assume, of course, that the dualizing sheaf embeds the general fiber of π stably; this will be the case if the general fiber of π is smooth and non-hyperelliptic. To see this, recall that a non-degenerate curve C in $\mathbb{P}^r$ is said to the linearly stable (resp. linearly semistable) if, for any linear projection

$$\pi: \mathbb{P}^r \to \mathbb{P}^s,$$

one has

$$\frac{\deg(C)}{r} < \frac{\deg(\pi(C))}{s}$$

[resp. $\deg(C)/r \leqq \deg(\pi(C))/s$]. By Clifford's theorem, a canonical curve is linearly stable. On the other hand, it is known that linear stability implies stability [12] ([3]).

Now (1.1) or (2.9) say that the line bundle $((\omega_\pi^{\otimes g} \otimes \pi^* \det(\pi_* \omega_\pi)^{-1})$ has non-negative self-intersection. This implies that

$$\begin{aligned} 0 &\leqq g(\omega_\pi \cdot \omega_\pi) - 2(\omega_\pi \cdot \pi^* \det(\pi_* \omega_\pi)) \\ &= g(\omega_\pi \cdot \omega_\pi) - (4g-4)\deg_\pi(\lambda), \end{aligned}$$

since ω_π has degree $2g-2$ on the fibers of π. Taking (4.2) into account, this becomes

$$0 \leqq (12g - (4g-4))\deg_\pi(\lambda) - g\deg_\pi(\delta);$$

([3]) What is proved here is that linear stability implies asymptotic Chow stability; but Chow stability implies Hilbert stability [10].

we have thus proved, in the special case when the general fiber is non-hyperelliptic, the

PROPOSITION (4.3). — *Any family of genus g stable curves over a smooth one-dimensional base whose general member is smooth satisfies the inequality*

$$(8+4/g)\deg(\lambda)\geqq\deg(\delta).$$

Actually, Proposition (2.9) gives a little more. Let p be a singular point of type $i>0$ in a fiber $\pi^{-1}(t)$. Then every section of the dualizing sheaf of $\pi^{-1}(t)$ vanishes at p. Thus, if $\mathscr{I}\omega_\pi$ is the subsheaf of ω_π generated by $\pi_*(\omega_\pi)$, $\mathscr{I}$ is a proper ideal at p. In fact, if C_1 and C_2 are the two components of $\pi^{-1}(t)$ meeting at p, there is a differential φ on C_1 that does not vanish at p, or, which is the same, vanishes simply at p as a section of the restriction to C_1 of $\omega_{\pi^{-1}(t)}$; we can then find a section ψ of ω_π over a neighbourhood of $\pi^{-1}(t)$ that restricts to φ on C_1 and to zero on C_2. Therefore, if X is of the form $xy=t^n$ near p, where y vanishes on C_1 and x on C_2, then, locally, $\psi=x.\eta$, where η is a section of ω_π that does not vanish at p. Hence $\mathscr{I}_p=(x,y)$, so

$$e_L(\mathscr{I})=\sum_{i>0}\deg_\pi(\delta_i),$$

and (2.9) yields

$$(8+4/g)\deg_\pi\lambda\geqq\deg_\pi\delta_0+2\sum_{i>0}\deg_\pi\delta_i. \tag{4.4}$$

b. THE HYPERELLIPTIC CASE. — As we have announced, Proposition (4.3) still holds for families of hyperelliptic curves; indeed, it is sharp for some families, and in fact we will see these are the only examples of families of generically smooth curves for which (4.3) is sharp.

We denote by I_g the locus of hyperelliptic curves in M_g and by $\bar{I}_g$ its closure in $\bar{M}_g$. As a first step, we notice that

LEMMA (4.5). — $\mathrm{Pic}(I_g)$ *is a finite group.*

Proof. — It suffices to show that $\mathrm{Pic}(I_g)$ is a torsion group. Let Δ be the divisor in the symmetric product

$$\mathrm{Sym}^{2g+2}(\mathbb{P}^1)=\mathbb{P}^{2g+2}$$

whose points are the effective divisors in $\mathbb{P}^1$ with multiple points. Clearly I_g is the quotient of $\mathbb{P}^{2g+2}-\Delta$ by PGL (2) and is normal. Now let X be the set of $(2g-1)$-tuples $(p_1,\ldots,p_{2g-1})$ of points of $\mathbb{P}^1$ such that

$$p_i\neq p_j \quad \text{if} \quad i\neq j,$$
$$p_i\neq 0,1,\infty.$$

Notice that X is the complement of a divisor in affine $(2g-1)$-space, so $\mathrm{Pic}(X)$ vanishes. Let

$$\alpha:\ X\to I_g$$

be the morphism that sends $(p_1, \ldots, p_{2g-1})$ to the class of the divisor

$$\sum_{i=1}^{2g+2} p_i,$$

where $p_{2g}=0$, $p_{2g+1}=1$, $p_{2g+2}=\infty$. Clarly, α is a finite morphism: let k be its degree.

Now let M be any line bundle on I_g; we know that $\alpha^*(M)$ is trivial. On the other hand, since α is a k-sheeted covering and I_g is normal, there is a natural map $H^0(X, \alpha^*(M)) \to H^0(I_g, M^k)$, and any nowhere vanishing section of $\alpha^*(M)$ maps to a nowhere vanishing section of M^k.

Q.E.D.

We denote by $\mathscr{I}_g$ the moduli stack of genus g smooth hyperelliptic curves, and by $\bar{\mathscr{I}}_g$ the moduli stack of stable genus g hyperelliptic curves. Mimicking what one does for $\mathscr{M}_g$ and $\bar{\mathscr{M}}_g$, one can define Picard groups $\mathrm{Pic}(\mathscr{I}_g)$, $\mathrm{Pic}(\bar{\mathscr{I}}_g)$. Our next goal is to determine the rational Picard group

$$\mathrm{Pic}_{\mathbb{Q}}(\bar{\mathscr{I}}_g) = \mathrm{Pic}(\bar{\mathscr{I}}_g) \otimes \mathbb{Q}.$$

We begin by observing that Lemma (4.5) implies that

$$\mathrm{Pic}_{\mathbb{Q}}(\mathscr{I}_g) = 0.$$

In fact, if L is a line bundle on $\mathscr{I}_g$, a power of L descends to a line bundle M on I_g; on the other hand a power of M is trivial, so the same is true for L.

What this means is that a class in $\mathrm{Pic}_{\mathbb{Q}}(\bar{\mathscr{I}}_g)$ should be a linear combination of "boundary classes". Things are slightly complicated by the fact that, while Δ_i cuts out on $\bar{I}_g$ an irreducible divisor when $i>0$, the intersection of Δ_0 with $\bar{I}_g$ breaks up into several irreducible components. To see this, let C be a stable hyperelliptic curve of genus g: then C has a semistable model $\tilde{C}$ which is a two-sheeted admissible cover (*cf.* [3] or [8]) of a stable $(2g+2)$-pointed noded curve R of arithmetic genus zero. Let $f: \tilde{C} \to R$ be the covering map, and let p be a singular point of R. The complement of p has two connected components R′ and R″, so the set of marked points of R breaks up into two subsets, those lying on R′ and those lying on R″; let α and $2g+2-\alpha \geqq \alpha$ be the orders of these two subsets. We will call α the index of the point p; notice that $\alpha \geqq 2$. Suppose that p has odd index $\alpha = 2i+1$, $i>0$; then f must be branched at p, and the unique point q lying above p is a singular point of type i, according to the terminology introduced at the beginning of this section. In particular, it follows from the irreducibility of the space of h-pointed stable curves of genus zero that the intersection of Δ_i with $\bar{I}_g$ is irreducible. Suppose instead that the index of p is even and equal to $2i+2$. Then f is unbranched at p, so $f^{-1}(p)$ consists of two points q_1 and q_2, and $f^{-1}(R')$ and $f^{-1}(R'')$ are semistable hyperelliptic curves of genera i and $g-i-1$, joined at couples of points that are conjugate under the hyperelliptic involution. In particular, q_1 and q_2 are singular points of type 0. We let Ξ_i be the locus of all curves C in $\bar{I}_g$ such that R has a singular point of index $2i+2$. The preceding discussion shows that

$$\Delta_0 \cap \bar{I}_g = \Xi_0 \cup \Xi_1 \cup \ldots \cup \Xi_{[(g-1)/2]}.$$

It is also clear that each Ξ_i is irreducible.

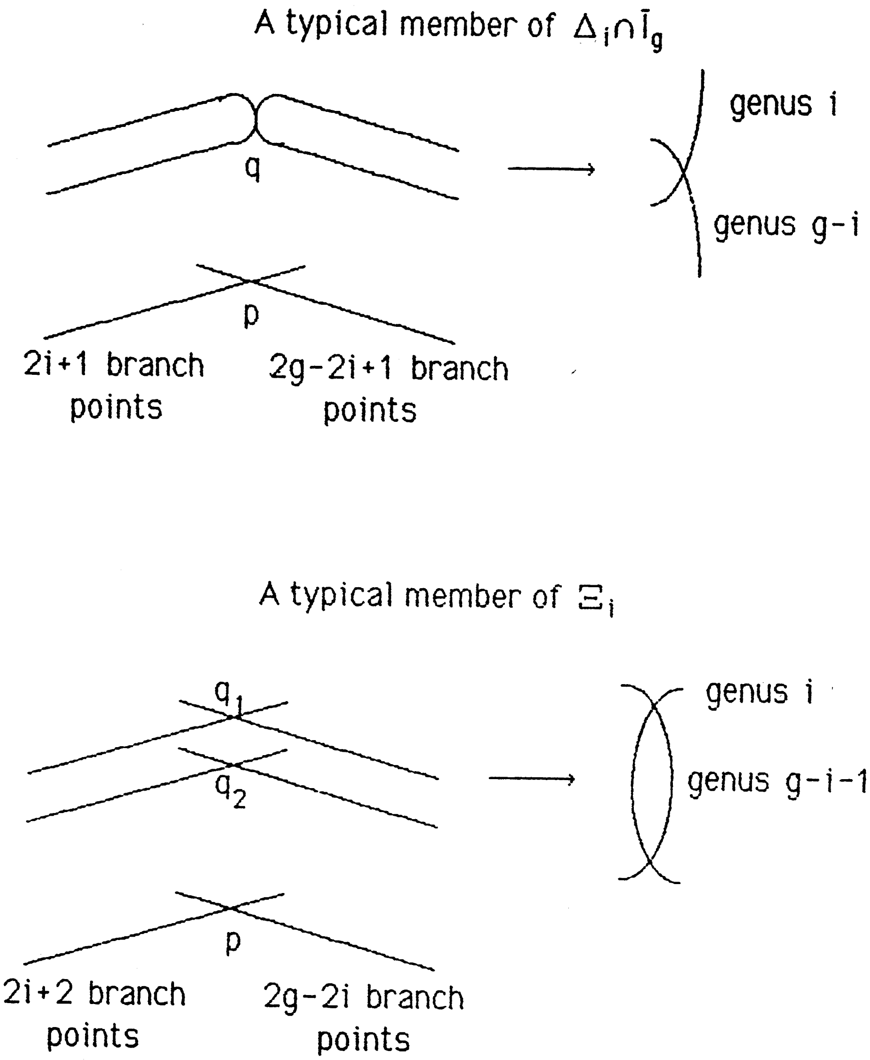

Fig. 2

Let C be a general point of Ξ_i or $\Delta_i \cap \bar{I}_g$, and $f: \tilde{C} \to R$ the corresponding admissible covering. Suppose C belongs to Ξ_0. Thus C is obtained from a smooth hyperelliptic curve of genus $g-1$ by identifying two points that are conjugate under the hyperelliptic involution, while $\tilde{C}$ is the blow-up of C at its singular point. It follows that the universal deformation space of the admissible covering $f: \tilde{C} \to R$ is a two-sheeted covering of the universal deformation space of C, branched along the locus of curves in Ξ_0. On the other hand, if C belongs to Δ_i or to Ξ_i, $i \geqq 1$, then the universal deformation spaces of $f: \tilde{C} \to R$ and of C are the same.

The divisors Ξ_i pull back to Cartier divisors on $\bar{\mathscr{I}}_g$, since the universal deformation space of a hyperelliptic curve within hyperelliptic curves is smooth; the class of Ξ_i in $\mathrm{Pic}(\bar{\mathscr{I}}_g)$ will be denoted ξ_i. In what follows, we shall improperly use the symbols λ, δ_i also to denote the restrictions of λ and δ_i to $\bar{\mathscr{I}}_g$. In view of the discussion above, the class δ_0 is related to the ξ_i by the identity

$$(4.6)\qquad \delta_0=\xi_0+2\xi_1+\ldots+2\xi_{[(g-1)/2]}.$$

If L is a line bundle on $\bar{\mathscr{I}}_g$ which is trivial on $\mathscr{I}_g$ there are integers n_i, m_i such that

$$\mathrm{Cl}(\mathrm{L})=\sum n_i\xi_i+\sum_{i>0} m_i\delta_i$$

The n_i (resp. m_i) are determined as follows. Choose a nowhere vanishing section s of L on $\mathscr{I}_g$, and let C be a curve in Ξ_i (resp. Δ_i). Then n_i (resp. m_i) is the order of zero of s along the locus of hyperelliptic curves belonging to Ξ_i (resp. Δ_i) in the universal deformation space (as a hyperelliptic curve) of C. Thus $\xi_0, \ldots, \xi_{[(g-1)/2]}$, $\delta_1, \ldots, \delta_{[g/2]}$ generate $\mathrm{Pic}_{\mathbb{Q}}(\bar{\mathscr{I}}_g)$. In particular λ is a rational linear combination of them.

Proposition (4.7). — *The classes $\xi_0, \ldots, \xi_{[(g-1)/2]}$, $\delta_1, \ldots, \delta_{[g/2]}$ freely generate $\mathrm{Pic}_{\mathbb{Q}}(\bar{\mathscr{I}}_g)$. Furthermore in $\mathrm{Pic}_{\mathbb{Q}}(\bar{\mathscr{I}}_g)$ we have:*

$$(8g+4)\lambda=g\xi_0+\sum_{i=1}^{[(g-1)/2]} 2(i+1)(g-i)\xi_i+\sum_{j=1}^{[g/2]} 4j(g-j)\delta_j.$$

Since we already know that λ is a linear combination of the ξ_i and the δ_i, to prove Proposition (4.7) it is enough to check that the degrees of the two sides of the identity in the statement are the same on sufficiently many "independent" families of hyperelliptic curves with a one-dimensional base.

To see all this, let's start with the simplest case of a family $\pi: X \to T$ of hyperelliptic curves, the case where T is a smooth curve and X is given simply as a double cover $\eta: X \to Y$ of $Y=T\times\mathbb{P}^1$ branched along a general curve C of type $(2g+2, 2m)$ in Y (here "general" means C is smooth and simply branched over T). In this situation, X will be smooth since C is, and all the fibers of X over T will be irreducible curves with at most one node. In particular, if singular, they will be stable and will not belong to Ξ_i or Δ_i for $i>0$. Thus the degree of ξ_i and of δ_i is zero for $i\geqq 1$, while the degree of ξ_0 equals the number of branch points of C over T, i. e.,

$$\begin{aligned}\deg\xi_0&=(\mathrm{C}.\omega_{\mathrm{Y/T}})+(\mathrm{C}.\mathrm{C})\\&=-2.2m+4m.(2g+2)\\&=8mg+4m,\end{aligned}$$

since the relative dualizing sheaf $\omega_{\mathrm{Y/T}}$ has type $(-2, 0)$. Next, to calculate the self-intersection of the relative dualizing sheaf ω_π, observe that, by the Riemann-Hurwitz formula,

$$\omega_\pi=\eta^*\omega_{\mathrm{Y/T}}(\tilde{\mathrm{C}}),$$

where $\tilde{C} \subset X$ is the ramification curve of η. We then have

$$(\eta^* \omega_{Y/T} \,.\, \eta^* \omega_{Y/T}) = 2(\omega_{Y/T} \,.\, \omega_{Y/T}) = 0,$$
$$(\eta^* \omega_{Y/T} \,.\, \tilde{C}) = (\omega_{Y/T} \,.\, C) = -2.2m,$$
$$(\tilde{C} \,.\, \tilde{C}) = (C \,.\, C) = 2m \,.\, (2g+2),$$

and so

$$(\omega_\pi \,.\, \omega_\pi) = -8m + 2m \,.\, (2g+2)$$
$$= 4mg - 4m.$$

Using formula (42) we find that the degree of the Hodge bundle is

$$\deg_\pi \lambda = \frac{4mg - 4m + 8mg + 4m}{12} = mg,$$

so that

$$(8g+4) \deg_\pi \lambda = g \,.\, \deg_\pi \xi_0,$$

as desired.

The analysis of a general family of hyperelliptic curves over a smooth one-dimensional base is of course more complicated. Since we are only interested in comparing the degrees of the two sides of the identity in (4.7), we may limit ourselves to families $\pi: X \to T$ of admissible covers; that is, double covers of families $f: Y \to T$ of stable $(2g+2)$-pointed noded curves of arithmetic genus 0, branched along the $2g+2$ distinguished sections σ_i of f and possibly at some of the nodes of fibers of f, in accordance with the local description of such covers given in [3] or in [8]. In fact, from any family of hyperelliptic curves over a smooth one-dimensional base we may get a family of admissible covers by base change and blow-up of singular points in the fibers, and these operations have the effect of multiplying all degrees by the same constant.

We begin our analysis with the base Y of our family of double covers. Let $\{p_i\}$ be the set of points of Y that are nodes of their fibers; if the local equation of Y at p_i is $xy - t^{m_i}$ we will say that p_i has multiplicity m_i. We also let α_i be the index of p_i. We have then the:

LEMMA (4.8). — $(2g+1) \sum_i (\sigma_i \,.\, \sigma_i) = -\sum_i m_i \alpha_i (2g+2-\alpha_i)$.

Proof. — First, observe that both sides of (4.8) are unchanged if we resolve the rational double points of Y; we may thus assume that Y is smooth, and thus is the blow-up of a $\mathbb{P}^1$-bundle Z over T at a sequence of points smooth in their fibers. Now, if τ_1, τ_2 are sections of a $\mathbb{P}^1$-bundle over the curve T, the difference $\tau_1 - \tau_2$ is numerically equivalent to a multiple of the fiber, and so has self-intersection zero; thus

$$(\tau_1 \,.\, \tau_1) + (\tau_2 \,.\, \tau_2) = 2(\tau_1 \,.\, \tau_2).$$

Given n sections τ_i, we can sum over all couples of indices i, j such that $i<j$ to obtain

$$(n-1) \sum_i (\tau_i \,.\, \tau_i) = 2 \sum_{i<j} (\tau_i \,.\, \tau_j). \tag{4.9}$$

Now, blowing up the bundle Z at a smooth point of a fiber through which exactly k of the sections τ_i pass, we create a node p of a fiber with index k; at the same time the left hand side of (4.9) decreases by $k(n-1)$ and the right hand side decreases by $k(k-1)$. We deduce that after any sequence of such blow-ups we will have

$$(n-1)\sum_i (\tau_i . \tau_i) = 2\sum_{i<j} (\tau_i . \tau_j) - \sum_h \alpha_h (n-\alpha_h).$$

Assuming that all the sections τ_i are disjoint and setting $n=2g+2$ we arrive at formula (4.8).

Q.E.D.

We may now start our analysis of the family $\pi : X \to T$. We denote by η the double cover $X \to Y$ and by $R \subset X$ its ramification divisor. We denote by ε_j the number of points p_i of index $2j+1$, counted according to their multiplicity, and by ν_j the number of points p_i of index $2j+2$. Clearly

$$\text{(4.10)} \qquad \begin{aligned} &\deg_\pi \xi_0 = 2\nu_0 \\ &\deg_\pi \xi_i = \nu_i, \qquad i \geqq 1 \\ &\deg_\pi \delta_i = \varepsilon_i/2, \qquad i \geqq 1. \end{aligned}$$

To determine the other invariants of π note that, by the Riemann-Hurwitz formula

$$\omega_\pi = \eta^* \omega_f (R).$$

Writing σ for $\sum \sigma_i$ and observing that, since σ consists of a bunch of disjoint sections of f, $(\sigma . \sigma)$ equals $-(\sigma . \omega_f)$, we have

$$\begin{aligned} (\omega_\pi . \omega_\pi) &= (\eta^* \omega_f . \eta^* \omega_f) + 2(\eta^* \omega_f . R) + (R . R) \\ &= 2(\omega_f . \omega_f) + 2(\omega_f . \sigma) + (\sigma . \sigma)/2 \\ &= 2(\omega_f . \omega_f) - 3(\sigma . \sigma)/2. \end{aligned}$$

Since Y is (after resolving its rational double points, which won't affect this) the blow-up of a $\mathbb{P}^1$-bundle over T a total of $\sum m_i = \sum \varepsilon_j + \sum \nu_j$ times, we have

$$(\omega_f . \omega_f) = -\sum \varepsilon_j - \sum \nu_j.$$

Next, by Lemma (4.8),

$$(2g+1)\sum_i (\sigma_i . \sigma_i) = -\sum_j (2j+2)(2g-2j)\nu_j - \sum_j (2j+1)(2g+1-2j)\varepsilon_j.$$

Putting these together, we have

$$2(2g+1)(\omega_\pi . \omega_\pi) = \sum_j [6(j+1)(2g-2j) - 4(2g+1)]\nu_j + \sum_j [3(2j+1)(2g+1-2j) - 4(2g+1)]\varepsilon_j,$$

and combining this with (4.10), (4.6), and the standard relation (4.2),

$$24(2g+1)\deg_\pi \lambda = 2(2g+1)[(\omega_\pi . \omega_\pi) + \deg_\pi \delta]$$
$$= 6g \deg_\pi \xi_0 + \sum_{j>0} 6(j+1)(2g-2j)\deg_\pi \xi_j$$
$$+ \sum_j 6[(2j+1)(2g+1-2j)-(2g+1)]\deg_\pi \delta_j.$$

We arrive finally at the relation

$$(4.11)\quad (8g+4)\deg_\pi \lambda = g.\deg_\pi \xi_0 + \sum_{i>0} 2(i+1)(g-i)\deg_\pi \xi_i + \sum_{j>0} 4j(g-j)\deg_\pi \delta_j,$$

as desired.

Proposition (4.7) now follows readily from looking at families of curves obtained by taking double covers of $T \times \mathbb{P}^1$ branched over curves of type $(2m, 2g+2)$, generic except for ordinary j-fold points: it is easy to see that, in addition to the family constructed above with all $\deg \delta_i$ and $\deg \xi_i$ zero except for $\deg \xi_0$, there exists for each $j>0$ a family with all $\deg \delta_i$ and $\deg \xi_i$ zero except for $\deg \xi_0$ and $\deg \xi_j$ (resp. $\deg \delta_j$).

Q.E.D.

Observe that formula (4.11) proves Proposition (4.3) in the hyperelliptic case; one simply has to use (4.6) and to remark that, for $1 \leqq i \leqq [(g-1)/2]$ (resp., $1 \leqq i \leqq [g/2]$), $(i+1)(g-i)$ [resp., $4i(g-i)$] is strictly larger than g. This concludes the proof of (4.3); it also shows that the families of hyperelliptic curves all of whose singular fibers are not in Δ_i or in Ξ_i for $i \geqq 1$ are the only ones for which equality holds in (4.3). In fact, these are essentially the only families of curves, hyperelliptic or not, for which this happens, as our next result indicates.

THEOREM (4.12). — *Let* $\pi : X \to T$ *be any non-isotrivial family of stable curves of genus* g *whose general member is smooth. Then equality holds in* (4.3) *if and only if the general fiber of* π *is hyperelliptic and the singular fibers of* π *do not belong to* Δ_i *or to* Ξ_i *for* $i \geqq 1$.

Proof. — It suffices to show that the general fiber of π is hyperelliptic if equality holds in (4.3). Assume this is not the case: to get a contradiction, we go back to the proof of Theorem (1.1), with $L = \omega_\pi$ and $F = \pi_*(L)$. The proof is based on the fact that the line bundle $\mathscr{F}_n$ [*cf.* (2.1)] has nonzero sections for large n, and thus the degree of its Chern class is non-negative. This degree, in the case at hand, is a polynomial in n of degree at most 2, and our hypotheses precisely say that its degree 2 term vanishes. Thus the coeeficient of the degree 1 term is non-negative; on the other hand, the Grothendieck Riemann-Roch formula shows that this coefficient is

$$m(g-1)\deg_\pi \lambda - \frac{gm}{2}(\omega_\pi)^{\cdot 2} = -\frac{m}{2}[(10.g+2)\deg_\pi \lambda - g.\deg_\pi \delta],$$

which is negative as soon as $g>1$ unless the family is isotrivial.

Q.E.D.

Remark (4.13). — When $g=1, 2$, Proposition (4.7) implies that, over $\mathbb{Q}$,

$$12\lambda=\delta,$$

$$10\lambda=\delta_0+2\delta_1,$$

respectively. In fact, these equalities are valid over $\mathbb{Z}$: the first follows from (4.2) by noticing that $\omega_{X/T}$ is trivial along the fibers of any family of elliptic curves, while the second is due to Mumford [13]. Thus, (4.7) can be viewed as a partial generalization of Mumford's result.

c. THE SINGULAR CASE. — It is natural to ask now whether the inequality (4.3) holds as well for families of singular stable curves. The answer, of course, is no: there is the standard example [12] of the family of curves $\{C_\mu\}$ obtained by taking a general pencil $\{E_\mu\}$ of plane cubics with base point q and attaching a fixed curve C' of genus $g-1$ to E_μ by identifying a fixed point $p\in C'$ with q. For this family (as we shall see) the ratio of $\deg\delta$ to $\deg\lambda$ is 11. To complete our discussion, then, we would like to claim that in fact this example is extremal, i. e., that for any family of stable curves we have

$$(4.14)\qquad 11.\deg\lambda\geqq\deg\delta.$$

To do this, suppose that $\pi:X\to T$ is any family of stable curves of genus g. Possibly after a finite base change, which won't affect the validity of (4.14), we can realize π as the union of families $\pi_i:X_i\to T$ flat with generically smooth fibers over T, with sections σ_α of $\pi_{i(\alpha)}$ identified with sections τ_α of $\pi_{j(\alpha)}$. We see from the exact sequence

$$0\to\oplus\,\pi_{i*}(\omega_{\pi_i})\to\pi_*(\omega_\pi)\xrightarrow{R}\mathcal{O}^h\to 0,$$

where the map R is given by residues along the sections σ_α giving rise to singular points of type 0 in the fibers, and h is the number of these sections, that the degree of the Hodge bundle of π will then be the sum of the degrees of the Hodge bundles of the π_i. As for the degree of δ, formula (4.1) gives

$$\deg_\pi\delta=\sum(\sigma_\alpha)^2+\sum(\tau_\alpha)^2.$$

We can thus write $\deg_\pi\delta$ as the sum of contributions γ_i, where γ_i is the sum of $\deg_\pi\delta_i$ and of the self-intersections of all sections σ_α and τ_α lying on X_i. We claim now that

LEMMA (4.15). — *For any i*, $11.\deg_{\pi i}\lambda\geqq\gamma_i$.

Proof. — We break this up into cases, according to the genus g_i of the general fiber of X_i. First, if $g_i\geqq 2$, then we have $(8+4/g)\deg_{\pi_i}\lambda\geqq\deg_{\pi_i}\delta$, and since any section of a family of curves of positive genus has nonpositive self-intersection (*see* [1] or [7]),

$$(4.16)\qquad \gamma_i\leqq\deg_{\pi_i}\delta\leqq(8+4/g_i)\deg_{\pi_i}\lambda\leqq 11.\deg_{\pi_i}\lambda.$$

Next, if $g_i=1$, the self-intersection of a section of π_i not passing through any singular points of fibers is just minus the degree $\deg_{\pi_i}\lambda$ of the Hodge bundle of π_i (we have in fact an isomorphism of $\pi_{i*}(\omega_{\pi_i})$ with the restriction of ω_{π_i} to σ_α). We thus have

$$\gamma_i \leqq \deg_{\pi_i}\delta - \deg_{\pi_i}\lambda = 11 \,.\, \deg_{\pi_j}\lambda. \tag{4.17}$$

Finally, if $g_i=0$, Lemma (4.8) tells us that the sum of the self-intersections of 2 or more disjoint sections of a family of noded rational curves is non-positive, which is what we need.

Q.E.D.

d. THE AMPLE CONE IN MODULI. — We simply remark here that, taking into account the inequality (4.14) above and Mumford's result that $a.\lambda-\delta$ is ample for large enough a, the remainder of Theorem (1.3) follows from Seshadri's criterion for ampleness [9].

It should also be observed that, while this settles the question of ampleness for linear combinations of λ and δ, the more general question of what divisor classes $a\lambda - b_0\delta_0 - b_1\delta_1 - \ldots$ are ample remains mysterious. To begin with, we can certainly improve Theorem (1.3), and even (4.4), if we take into account the various boundary components. For example, if a family of generically smooth curves has a reducible fiber, we don't necessarily have to apply (2.9) to the relative dualizing sheaf of the family; we can twist ω by some linear combination E of the components of the reducible fiber without affecting the hypotheses of (2.9), and, for some E, obtain a better estimate. Consider, for instance, a family $\pi: X \to T$ of stable curves over a smooth complete curve T; suppose the general fiber of π is smooth and non-hyperelliptic. For any singular point p of type $i \geqq 1$ in the fibers of π, write the corresponding fiber as the union of curves E_p and D_p of genera i and $g-i$, meeting at p, and let m_p be the multiplicity of p. Set

$$E = \sum m_p E_p,$$

where the sum is extended to all singular points of positive type, and apply (2.9) with $L=\omega_\pi(E)$ and $F=\pi_* L$: one gets

$$(8g+4)\deg_\pi\lambda \geqq g \,.\, \deg_\pi\delta_0 + \sum_{i=1}^{[g/2]} 4(g-i)\deg_\pi\delta_i,$$

which is slightly better than (4.4). We have not yet, however, been able to obtain in this way estimates that we believe are sharp.

A further problem arises when we try to look at the boundary. Specifically, one can say something on the basis of the inequalities (4.16) and (4.17); but since in particular (4.16) is known not to be sharp, we won't get an exact answer this way. Indeed, the general rule seems to be that to determine exactly the ample cone in the Picard group of $\bar{M}_g$ we have to understand what inequalities hold, not just between $\deg_\pi\lambda$ and $\deg_\pi\delta$ on families of generically smooth curves, but among $\deg_\pi\lambda$, $\deg_\pi\delta$ and $(\sigma_\alpha)^2$ for a family of generically smooth curves of genus g with sections σ_α. Put another way, we need to know what divisor classes on the moduli space $\bar{M}_{g,k}$ of stable k-pointed curves have nonnegative degree on every curve not contained in the boundary of $\bar{M}_{g,k}$.

There is some hope of getting information about such families by applying (2.9) not only to the relative dualizing sheaf, but to linear combinations of it and the sections σ_α. It is possible to give estimates on the ample cone in this way, but we do not as yet have any sharp inequalities. Indeed, to know that a given estimate was sharp, we would need a stock of examples of such families to try it on, and at present we don't know of any families that might even be suspected of being extremal.

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