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PIOTR PRAGACZ

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ENUMERATIVE GEOMETRY OF DEGENERACY LOCI

By Piotr PRAGACZ (1)

To the memory of Roman Kiełpiński

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Introduction

Numerical properties of degeneracy loci of morphisms of vector bundles are useful tools in many problems concerning "enumeration" in algebraic geometry and topology. The most typical is the following situation. Let E and F be vector bundles of ranks n and m on a scheme X. Let $\varphi: F \to E$ be a morphism of vector bundles.

The set:

$$D_r(\varphi) = \{x \in X,\ \operatorname{rank} \varphi(x) \leqq r\}$$

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is called the *degeneracy locus of rank r* of φ. $D_r(\varphi)$ as a subscheme of X is locally defined by the ideal generated by all $(r+1)$-order minors of φ. In [T] Thom observed that for a "general" morphism of vector bundles $\varphi: F \to E$, the fundamental class of $D_r(\varphi)$ should be a polynomial in the Chern classes of E and F independent of the morphism φ itself. This polynomial, generalizing Giambelli's formula for the degree of a projective determinantal variety, was subsequently found by Porteous (*see* [Po]).

$$[D_r(\varphi)] = \mathrm{Det}\,[c_{n-r-p+q}(E-F)], \qquad 1 \leqq p,\ q \leqq m-r,$$

and was applied in numerous situations in geometry and topology.

For some geometric purposes, however, a deeper insight into enumerative properties of degeneracy loci is required. For example, a study of the Chern numbers of degeneracy loci and their Chow groups leads in a natural way to an investigation of the following more general

Problem. — What are the polynomials in the Chern classes of E and F which describe cycles supported in $D_r(\varphi)$, in an universal way?

More precisely, let $i_r: D_r(\varphi) \to X$ be the inclusion map and let $(i_r)_*: A.(D_r(\varphi)) \to A.(X)$ be the corresponding map of the Chow groups (*cf.* [F]). Let $\mathbb{Z}[c.(A), c.(B)] = \mathbb{Z}[c_1(A), \ldots, c_n(A), c_1(B), \ldots, c_m(B)]$ be a graded polynomial $\mathbb{Z}$-algebra where $\deg c_k(A) = \deg c_k(B) = k$. Let $\mathscr{P}_r$ be the ideal of all polynomials $P \in \mathbb{Z}[c.(A), c.(B)]$ such that for every morphism $\varphi: F \to E$ of vector bundles of ranks m and n on an arbitrary scheme X, and for every $\alpha \in A(X)$

$$P(c.(E), c.(F)) \cap \alpha \in \mathrm{Im}\,(i_r)_*.$$

Here $c.(E)$, $c.(F)$ denote the Chern classes of E and F. Of course the Giambelli-Thom-Porteous polynomials describing the fundamental classes $[D_i(\varphi)]$ for $i \leqq r$ belong to $\mathscr{P}_r$, but they do not generate this ideal for $r > 0$.

In the present paper we give an explicit description of the ideal $\mathscr{P}_r$ for every r. This is done in Theorem 3.4. A proper language to achieve this goal is provided by certain class of symmetric polynomials. Maybe it is in order to recall that symmetric polynomials very often play a significant role in cohomological computations; they lead to a description of cohomology rings of many important varieties and to discovery of important formulas in cohomology rings, as well (*see* [H], [M] and [F]). In our situation these are the so called Schur S-polynomials which play a fundamental role in a description of the ideal $\mathscr{P}_r$. More precisely, we use a generalization of the usual Schur S-polynomials depending on two sets of variables. The definitions and properties of these generalized Schur S-polynomials are given in Section 1. Our new geometric applications of them are based on their factorization property stated in Lemma 1.1, and on a certain formula for Gysin push forward (*see* Proposition 2.2).

Let us notice that the ideal $\mathscr{P}_r$ has a remarkable interpretation in elimination theory as a generalization of the resultant. Let

$$A(x) = x^n + \sum_{i=1}^{n} c_i(A)\,x^{n-i},\quad B(x) = x^m + \sum_{j=1}^{m} c_j(B)\,x^{m-j}$$

be two polynomials in $\mathbb{Z}[c.(A), c.(B)][x]$. Then the ideal of all polynomials $P \in \mathbb{Z}[c.(A), c.(B)]$ which vanish if $A(x)$ and $B(x)$, specialized to a field, have $r+1$ roots in common, is equal to $\mathcal{P}_r$ (*see* [P₂]). A geometric interpretation of this ideal allows us to study its algebraic properties by methods of intersection theory, and especially of Schubert Calculus. We obtain in this way a certain "small", finite set of generators of $\mathcal{P}_r$ as well as a certain $\mathbb{Z}$-basis of it (*see* Propositions 6.1 and 6.2).

The main Theorem 3.4 and methods used to prove it, allow us to give an explicit description of the Chow groups of universal degeneracy loci (*see* Propositions 4.2, 4.3). Moreover, as a by-product of our considerations we obtain a simple rule for the computation of the Chern numbers of kernel and cokernel bundles (*see* Proposition 5.3) and an algorithm which gives the Chern numbers of smooth degeneracy loci themselves. In particular we arrive at a closed expression for the Euler-Poincaré characteristic of a smooth degeneracy loci in an arbitrary dimension (*see* Proposition 5.7).

For some geometric aims it is important also to investigate the two cases when $F = E^\vee$ and $\varphi: E^\vee \to E$ is symmetric or antisymmetric (in the last case we assume that r is even and that the subscheme structure imposed on $D_r(\varphi)$ is defined by the ideal generated by all $(r+2)$-order subpfaffians of φ). The formulas for fundamental classes of $D_r(\varphi)$ in these cases were found in [J-L-P]. If φ is symmetric and "sufficiently general", then

$$[D_r(\varphi)] = 2^{n-r} \operatorname{Det}[c_{n-r-2p+q+1}(E)], \qquad 1 \leqq p, q \leqq n-r.$$

If φ is antisymmetric and "sufficiently general", then

$$[D_r(\varphi)] = \operatorname{Det}[c_{n-r-2p+q}(E)], \qquad 1 \leqq p, q \leqq n-r-1.$$

A description of the ideal in $\mathbb{Z}[c_1(A), \ldots, c_n(A)]$, which corresponds to the ideal $\mathcal{P}_r$, requires another family of symmetric polynomials. It turns out that a family of the so called Schur Q-polynomials provides a good tool for investigation of symmetric and antisymmetric degeneracy loci from the above point of view. These polynomials were introduced by Schur in [Sch] in order to describe projective characters of the symmetric group. Schur Q-polynomials satisfy the corresponding factorization property (*see* Lemma 1.13) and, specialized to the Chern classes of some vector bundles, behave nicely when pushing forward in the Chow groups of Grassmannian bundles (*cf.* Proposition 2.8). This, with some little modifications, makes it possible to carry out in Section 7 the previous program in the case of symmetric and antisymmetric degeneracy loci. To compute the Chern numbers of these loci we need formulas for the Segre classes of the second symmetric and exterior power of a vector bundle. Such a formula, involving Pfaffians and binomial coefficients, is given in Proposition 7.12.

Let us notice that Propositions 5.3, 5.7, 7.9 and 7.13 give an explicit answer to questions left open in [H-T].

This paper is a unified and extended version of the author's earlier preprints "Degeneracy loci and symmetric functions I and II".

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Notations and conventions

Schemes and Chow groups

The word *scheme* means in this paper an equidimensional algebraic scheme of finite type over a field K.

The word *point* means always a closed point

If X is a scheme its Chow group graded by dimension will be denoted by $A_{.}(X)$, and graded by codimension by $A^{\cdot}(X)$. If a specification of the grading is not necessary, we will write A(X).

Matrices

A symmetric matrix $X=[x_{pq}]$, $1 \leqq p, q \leqq n$, where $x_{pq}=x_{qp}$ will be denoted by $X=[x_{pq}]$, $1 \leqq p \leqq q \leqq n$. Similarly, an antisymmetric matrix $X=[x_{pq}]$, $1 \leqq p, q \leqq n$, where $x_{pp}=0$, $x_{pq}=-x_{qp}$ will be denoted by $X=[x_{pq}]$, $1 \leqq p < q \leqq n$.

Partitions

By a *partition* we mean a weakly decreasing sequence $I=(i_1, \ldots, i_r)$ of integers where $i_1 \geqq i_2 \geqq \ldots \geqq i_r \geqq 0$.

Instead of $(i, \ldots, i)$ (r-times) we will write $(i)^r$.

If for some k $i_1 > i_2 > \ldots i_k > i_{k+1} = i_{k+2} = \ldots = i_r = 0$, then I will be called *strict*.

For given partitions $I=(i_1, \ldots, i_r)$, $J=(j_1, \ldots, j_r)$, $I \pm J$ will denote the sequence $(i_1 \pm j_1, \ldots, i_r \pm j_r)$ and $I \subset J$ will mean that $i_k \leqq j_k$ for every k.

If $I=(i_1, \ldots, i_r)$, $J=(j_1, \ldots, j_s)$ are two sequences of integers, then the juxtaposition sequence $(i_1, \ldots, i_r, j_1, \ldots, j_s)$ will be denoted by I, J.

The conjugate partition of a partition I, noted $I^{\sim}$, is the partition $(j_1, j_2, \ldots)$, where $j_k = \mathrm{card}\{h : i_h \geqq k\}$.

Finally, for a given partition I its *weight:* $\sum_{k=1}^{r} i_k$ will be denoted by $|I|$ and its *length:* $\mathrm{card}\{k, i_k \neq 0\}$ will be denoted by $l(I)$.

1. Two classical families of symmetric polynomials

Schur S-polynomials

Let $A=(a_1, \ldots, a_n)$ and $B=(b_1, \ldots, b_m)$ be two sequences of elements of a commutative ring R. For a given sequence $I=(i_1, \ldots, i_r)$ of integers, the *Schur S-polynomial*

$s_{\mathrm{I}}(\mathrm{A};\ \mathrm{B})$ is defined as the determinant of the matrix

$$[s_{i_p-p+q}(\mathrm{A};\ \mathrm{B})], \qquad 1 \leqq p,\ q \leqq r \tag{1}$$

where $s_k(\mathrm{A};\ \mathrm{B}) \in \mathrm{R}$ is given by the following identity in $\mathrm{R}[[t]]$.

$$\left[\prod_{i=1}^{n}(1-ta_i)\right]^{-1}\prod_{j=1}^{m}(1-tb_j)=\sum_{k=0}^{\infty}s_k(\mathrm{A};\ \mathrm{B})\,t^k$$

and $s_k(\mathrm{A};\ \mathrm{B})=0$ if $k<0$. By permuting the rows of the matrix (1), if necessary, we see that each different from zero Schur S-polynomial is equal $-$up to a sign$-$ to a certain Schur S-polynomial indexed by a partition. Assume for a moment that I is a partition. Notice that if $b_1=\ldots=b_m=0$, then $s_I(\mathrm{A};\ \mathrm{B})$ becomes the "usual" Schur S-polynomial $s_{\mathrm{I}}(\mathrm{A})$ (*cf.* [M], [L-S]), and if $a_1=\ldots=a_n=0$, then $s_{\mathrm{I}}(\mathrm{A};\ \mathrm{B})=(-1)^{|\mathrm{I}|}s_{\mathrm{I}^\sim}(B)$ where $\mathrm{I}^\sim$ is the conjugate partition of I. The following formula expresses $s_{\mathrm{I}}(\mathrm{A};\ \mathrm{B})$ in terms of usual Schur S-polynomials

$$s_{\mathrm{I}}(\mathrm{A};\ \mathrm{B})=\sum(-1)^{|\mathrm{I}|-|\mathrm{J}|}s_{\mathrm{J}}(\mathrm{A})\,s_{\mathrm{I}^\sim/\mathrm{J}^\sim}(\mathrm{B}), \tag{2}$$

where the sum is over all partitions J, and $s_{\mathrm{I}^\sim/\mathrm{J}^\sim}(\mathrm{B})$ denotes the corresponding skew Schur polynomial (*see* [M], I.5). The formula (2) is a simple consequence of a general λ-ring calculus. Recall that for every element x in an arbitrary λ-ring one defines $s_{\mathrm{I}}(x)$ as the determinant of the matrix

$$[s_{i_p-p+q}(x)], \qquad 1 \leqq p,\ q \leqq r$$

where $s_k(x)=(-1)^k\lambda^k(-x)$ for any k. Then the following *Linearity Formula* holds

$$s_{\mathrm{I}}(x-y)=\sum(-1)^{|\mathrm{I}|-|\mathrm{J}|}s_{\mathrm{J}}(x)\,s_{\mathrm{I}^\sim/\mathrm{J}^\sim}(y), \tag{3}$$

where the sum is over all partitions J (*see* [M] Remark I.5.3, and [L-S]). Let $\underline{\mathrm{Sym}}(\mathrm{A})$ be the ring of symmetric polynomials in A. Recall that $\underline{\mathrm{Sym}}(\mathrm{A})$ has a natural λ-ring structure (*see* [M] Remark I.2.15). Then in the λ-ring $\underline{\mathrm{Sym}}(\mathrm{A})\otimes_{\mathrm{Z}}\underline{\mathrm{Sym}}(\mathrm{B})=\underline{\mathrm{Sym}}(\mathrm{A},\ \mathrm{B})$ the formula (2) is a consequence of the formula (3) with $x=\sum_{i=1}^{n}a_i$, $y=\sum_{j=1}^{m}b_j$. We refer to [L-S] for the theory of Schur S-polynomials in the λ-ring set-up. Another consequence of the linearity formula (3) is

$$s_{\mathrm{I}}(\mathrm{A})=\sum_{\mathrm{J}}s_{\mathrm{J}}(\mathrm{A};\ \mathrm{B})\,s_{\mathrm{I}/\mathrm{J}}(\mathrm{B}). \tag{4}$$

Moreover, inspired by the notation in λ-ring calculus, from now on we will use the following more suggestive notation for Schur S-polynomials

$$s_{\mathrm{I}}(\mathrm{A}-\mathrm{B})=s_{\mathrm{I}}(\mathrm{A};\ \mathrm{B}). \tag{5}$$

The following Lemma will be constantly used in this article.

LEMMA 1.1. — (Factorization Formula) *Let* $I=(i_1, \ldots, i_n)$ *and* $J=(j_1, \ldots, j_p)$, $j_1 \leqq m$ *be two partitions. Then*

$$s_{(m)^n+I,\,J}(A-B)=s_I(A)\,s_{(m)^n}(A-B)\,s_J(-B).$$

For a proof (*see* [B−R] 6.20 or [L-S] 7.6).

For the partition $(m)^n$ we have the following expression of the corresponding Schur S-polynomial in terms of $\{a_i\}$ and $\{b_j\}$.

LEMMA 1.2:

$$s_{(m)^n}(A-B)=\prod_{i,\,j}(a_i-b_j), \qquad i=1, \ldots, n;\quad j=1, \ldots, m.$$

For a proof *see* for example [J-L-P] Proposition 3 or [L-S] 7.6.

If E and F are two vector bundles of rank n and m respectively then we define $s_I(E-F)$ as $s_I(A-B)$ where A (resp. B) is the set of the Chern roots of E (resp. F) (*cf.* [F] Remark 3.2.3 for this last notion).

By the splitting principle, Lemma 1.2 can be rewritten as

LEMMA 1.3. — *Let* E *and* F *be two vector bundles of ranks* n *and* m *on a scheme* X. *Then* $c_{\text{top}}(E\otimes F^\vee)=s_{(m)^n}(E-F)$.

Remark 1.4. – The polynomials denoted here by $s_I(A-B)$ appear in the literature also under the names "Hook Schur functions" [B-R], "Super-Schur functions" or "Schur bisymmetric functions".

The reader who is interested mainly in the case of degeneracy loci associated with the generic morphism $\varphi: F \to E$, can omit the next (sub)section in the first reading.

SCHUR Q-POLYNOMIALS

The rest of this Section will be devoted to description of another important family of symmetric polynomials introduced by Schur in [Sch], which are less well known than the Schur S-polynomials.

Let $A=(a_1, \ldots, a_n)$ be a sequence of elements of a commutative ring R. Define $Q(t)\in R[[t]]$ and $q_k(A)\in R$ by

$$Q(t)=\prod_{i=1}^{n}(1+a_i t)\left[\prod_{i=1}^{n}(1-a_i t)\right]^{-1}=\sum_{k=0}^{\infty} q_k(A)\,t^k \tag{6}$$

and $q_k(A)=0$ if $k<0$. Thus for every k, $q_k(A)\in R$ is symmetric with respect to $a_1, \ldots, a_n$. For given nonnegative integers i, j define

$$Q_{i,\,j}(A)=q_i(A)\,q_j(A)-2\,q_{i+1}(A)\,q_{j-1}(A)+\ldots+(-1)^j\,2\,q_{i+j}(A).$$

Since $Q(t)\,Q(-t)=1$, we have for $k>0$

$$q_k(A)-q_1(A)\,q_{k-1}(A)+q_2(A)\,q_{k-2}(A)+\ldots+(-1)^k q_k(A)=0$$

and therefore

$$Q_{i,j}(A) = -Q_{j,i}(A) \text{ for } i, j \geqq 0 \text{ and } i+j > 0.$$

In particular, $Q_{i,0}(A) = q_i(A) = -Q_{0,i}(A)$ if $i > 0$. Finally, let $I = (i_1, \ldots, i_r)$ be a sequence of nonnegative integers. If r is even, we define the *Schur Q-polynomial* $Q_I(A)$ as the Pfaffian of the antisymmetric matrix

$$[Q_{i_s, i_t}(A)] \qquad 1 \leqq s < t \leqq r \tag{7}$$

and if r is odd we put $Q_I(A) = Q_{(i_1, \ldots, i_r, 0)}(A)$.

The following properties of Schur Q-polynomials follow from standard properties of Pfaffians

LEMMA 1.5. — (i) *For any sequence* $I = (i_1, \ldots, i_r)$ *of nonnegative integers,*

$$Q_{(i_1, \ldots, i_r, 0, \ldots, 0)}(A) = Q_I(A).$$

(ii) *For any* $I = (i_1, \ldots, i_r)$, $Q_I(A)$ *is a symmetric polynomial in* $a_1, \ldots, a_n$ *of degree* $i_1 + \ldots + i_r$.

(iii) *For any nonnegative integers i, j such that* $i+j > 0$

$$Q_{(\ldots, i, j, \ldots)}(A) = -Q_{(\ldots, j, i, \ldots)}(A).$$

In particular $Q_{(\ldots, i, \ldots, i, \ldots)}(A) = 0$ for $i > 0$.

It follows from (iii) that the only nonzero Q-polynomials are given —up to a sign— by $Q_I(A)$ where I is a strict partition, (*i.e.* $i_1 > i_2 > \ldots > i_k > i_{k+1} = \ldots = i_r = 0$ for some k).

Example 1.6. — $q_k(A) = 2 \sum s_I(A)$, where the summation ranges over all hook partitions I of length k. It follows from the formula (6) that $q_k(A) = \sum_{i=0}^{k} s_i(A)\, s_{(1)^{k-i}}(A)$. Then Pieri's formula for Schur S-polynomials (*see* [M] 1.5.17) yields the desired identity.

Assume for a moment that $a_1, \ldots, a_n$ are algebraically independent over $\mathbb{Z}$. Let I be a strict partition. Then, by the (7) $Q_I(A)$ is a sum of monomials of the form $z q_{k_1}(A) \ldots q_{k_s}(A)$, where $z \in \mathbb{Z}$ and $k_1 + \ldots + k_s = l(I)$. Therefore Example 1.6 implies that there exists in $\mathbb{Z}[a_1, \ldots, a_n]$ a polynomial $P_I(A)$ such that $Q_I(A) = 2^{l(I)} P_I(A)$. We will call $P_I(A)$ the *Schur P-polynomial* and use it interchangeably with the $Q_I(A)$.

The following fact proved by Schur (*see* [Sch] p. 225) will be crucial for applications of Schur Q-polynomials for our purposes.

PROPOSITION 1.7. — *Let* $I = (i_1, \ldots, i_k)$, $k \leqq n$ *be a strict partition of length k. Then the following equality holds*

$$P_I(A) = \sum_{w \in S_n/(S_1)^k \times S_{n-k}} w \Big[a_1^{i_1} a_2^{i_2} \ldots a_k^{i_k} \prod_{\substack{1 \leqq i < j \leqq n \\ i \leqq k}} (a_i + a_j)(a_i - a_j)^{-1} \Big],$$

where for a given polynomial $f \in \mathbb{Z}[a_1, \ldots a_n]$, $wf(a_1, \ldots, a_n)$ *means* $f(a_{w(1)}, \ldots, a_{w(n)})$.

For example,

$$P_{(i)}(A) = \sum_{s=1}^{n} a_s^i \prod_{s' \neq s} (a_s + a_{s'})(a_s - a_{s'})^{-1},$$

$$P_{(i, j)}(A) = \sum_{s, t=1}^{n} a_s^i a_t^j \frac{a_t - a_s}{a_t + a_s} \prod_{s' \neq s} \frac{a_s + a_{s'}}{a_s - a_{s'}} \prod_{t' \neq t} \frac{a_t + a_{t'}}{a_t - a_{t'}}.$$

For the reader's convenience we will give a proof of Proposition 1.7 in the Appendix to this paper. Comparing Proposition 1.7 with definition 2.2 of the Hall-Littlewood polynomials in [M] p. 104 we obtain

COROLLARY 1.8. — $P_I(A)$ *is a specialization of the Hall-Littlewood polynomial* $P_I(A, t)$ *for* $t = -1$.

Let I, J be partitions. Let $Q_{I/J}(A; t)$ be the skew Hall-Littlewood polynomial as defined in [M].III.5. Define the skew Q-polynomial $Q_{I/J}(A)$ by

$$Q_{I/J}(A) = Q_{I/J}(A; t = -1).$$

As a consequence of the formulas III.5.2 and III.5.5 in [M] satisfied by Hall-Littlewood polynomials, we obtain

LEMMA 1.9. — (i) *For any partitions* I, J $Q_{I/J}(A)$ *is a* $\mathbb{Z}$*-linear combination of the Schur Q-polynomials* $Q_L(A)$.

(ii) *Let* I *be a partition and let* $A = (a_1, \ldots, a_n)$ $B = (b_1, \ldots, b_m)$ *be two sequences of elements in a commutative ring. Let* A, B *be the sequence* $(a_1, \ldots, a_n, b_1, \ldots, b_m)$. *Then we have*

$$Q_I(A, B) = \sum Q_J(A) Q_{I/J}(B),$$

where the sum is over all (*strict*) *partitions* J.

For a given sequence $A = (a_1, \ldots, a_n)$ of elements in a commutative ring we write $A^- = (-a_1, \ldots, -a_n)$ and $\{A\}$—for the set $\{a_1, \ldots, a_n\}$.

LEMMA 1.10. — *Assume that* $\{A\} = \{A^-\}$. *Then* $Q_{I/J}(A) = 0$ *for every* I, J *such that* $I \neq J$.

Proof. — By Lemma 1.9 (i) it suffices to show that for every I $Q_I(A) = 0$ if $\{A\} = \{A^-\}$. Since $Q_I(A)$ is a polynomial in the $q_k(A)$, $k = 1, 2, \ldots$, the assertion is reduced to showing that $q_k(A) = 0$ if $\{A\} = \{A^-\}$. But this follows immediately from the formula (6). ■

Let ρ_k denote the partition $(k, k-1, \ldots, 2, 1)$.

LEMMA 1.11:

$$s_{\rho_{n-1}}(\mathrm{A})=\prod_{i<j}(a_i+a_j), \qquad s_{\rho_n}(\mathrm{A})=\prod_i a_i \prod_{i<j}(a_i+a_j)$$

(for example *see* [M] p. 31).

If E is a vector bundle on a scheme X, then by $Q_I(E)$ (resp. $P_I(E)$) we will denote $Q_I(A)$ (resp. $P_I(A)$) where A is the set of Chern roots of E. By the splitting principle the above Lemma can be rewritten as

LEMMA 1.12. — *Let* E *be a vector bundle of rank* n *on a scheme* X. *Then* $c_{\mathrm{top}}(S_2 E)=2^n s_{\rho_n}(E)$, $c_{\mathrm{top}}(\Lambda^2 E)=s_{\rho_{n-1}}(E)$.

The following factorization property of Schur Q-polynomials (and its proof) is due to R. Stanley.

LEMMA 1.13 ([St]). — *Let* $\mathrm{A}=(a_1, \ldots, a_n)$ *and let* $\mathrm{I}=(i_1, \ldots, i_n)$ *be a partition. Then*

$$P_{\rho_{n-1}+\mathrm{I}}(\mathrm{A})=s_{\rho_{n-1}}(\mathrm{A})\, s_{\mathrm{I}}(\mathrm{A}).$$

Proof. — By Proposition 1.7 we have

$$P_{\rho_{n-1}+\mathrm{I}}(\mathrm{A}) \sum_{w\in S_n} w[a_1^{i_1+n-1} a_2^{i_2+n-2}\ldots a_n^{i_n}\prod_{i<j}(a_i+a_j)\prod_{i<j}(a_i-a_j)^{-1}]$$

$$=\prod_{i<j}(a_i+a_j)\sum_{w\in S_n} w[a_1^{i_1+n-1} a_2^{i_2+n-2}\ldots a_n^{i_n}\prod_{i<j}(a_i-a_j)^{-1}]=s_{\rho_{n-1}}(\mathrm{A})\, s_{\mathrm{I}}(\mathrm{A}).$$

The last equality follows from Lemma 1.11 and the Jacobi definition of Schur polynomial (*see* [M] (3.1) p. 24). ■

COROLLARY 1.14:

$$P_{\rho_{n-1}}(\mathrm{A})=s_{\rho_{n-1}}(\mathrm{A}),\ Q_{\rho_n}(\mathrm{A})=2^n s_{\rho_n}(\mathrm{A}).$$

2. Formulas for Gysin push forwards in Grassmannian and flag bundles

In this chapter E will denote a vector bundle of rank n on a scheme X and $\pi: G=G_r(E)\to X$ will be the Grassmannian bundle parametrizing the rank r subbundles of E. Let $0\to R\to E_G\to Q\to 0$ be the tautological sequence of vector bundles on $G_r(E)$, where rank $R=r$. Letting $q=n-r$ we will also treat G as the Grassmannian bundle of q-quotients of E and write $G=G^q(E)$.

LEMMA 2.1. — *With the above notation, assume that* $q=1$. *Then for every vector bundle* H *on* X, *and any* $\alpha\in A(X)$

$$\pi_*[s_{\mathrm{I}}(R-H_G)\, s_j(Q-H_G)\cap \pi^*\alpha]=s_{j-n+1}(E-H)\cap\alpha.$$

Proof. — Let $\xi = c_1(Q)$. It follows from the identity

$$s_i(R-H_G) = s_i(E_G-H_G) - \xi\, s_{i-1}(E_G-H_G)$$

that

$$s_I(R-H_G) = \begin{vmatrix} 1 & \xi & \ldots & \xi^{n-1} \\ s_{i_1-1}(E_G-H_G) & s_{i_1}(E_G-H_G) & \ldots & s_{i_1+n-2}(E_G-H_G) \\ s_{i_2-2}(E_G-H_G) & s_{i_2-1}(E_G-H_G) & \ldots & s_{i_2+n-3}(E_G-H_G) \\ \vdots & \vdots & \ddots & \vdots \end{vmatrix}$$

It is well known that $\pi_*(\xi^k) = s_{k-n+1}(E)$. Therefore by the linearity formula $\pi_*(s_k(Q-H_G)) = \sum_i s_{k-i}(-H)\,\pi_*(\xi^i) = s_{k-n+1}(E-H)$. Since $rk\,Q=1$, we have also $\xi^p s_j(Q-H_G) = s_{j+p}(Q-H_G)$. Hence

$$\pi_*(s_I(R-H_G)\, s_j(Q-H_G) \cap \pi^*\alpha)$$

$$= \begin{vmatrix} s_{j-n+1}(E-H) & s_{j-n+2}(E-H) & \ldots & s_j(E-H) \\ s_{i_1-1}(E-H) & s_{i_1}(E-H) & \ldots & s_{i_1+n-1}(E-H) \\ \vdots & \vdots & \ddots & \vdots \end{vmatrix} \cap \alpha$$

$$= s_{j-n+1}(E-H) \cap \alpha. \quad \blacksquare$$

An induction procedure described in the proof of Proposition 1 in [J-L-P] allows us easily to generalize the above Lemma for any q.

PROPOSITION 2.2. — *With the above notation, for every vector bundle* H *on* X *and any* $\alpha \in A(X)$

$$\pi_*[s_I(R-H_G)\, s_J(Q-H_G) \cap \pi^*\alpha] = s_{J-(r)^q,\, I}(E-H) \cap \alpha.$$

For $0 < k \leqq n$, let $\tau^k : Fl^k(E) \to X$ be the flag bundle parametrizing the flags of consecutive quotients of E of ranks $k, k-1, \ldots, 2, 1$. Let

$$E \twoheadrightarrow Q^k \twoheadrightarrow Q^{k-1} \twoheadrightarrow \ldots \twoheadrightarrow Q^2 \twoheadrightarrow Q^1$$

be the tautological sequence on $Fl^k(E)$. Define the line bundles $L_1, \ldots, L_k$ on $Fl^k(E)$ by $L_i = \mathrm{Ker}(Q^i \twoheadrightarrow Q^{i-1})$. Let $a_i = c_1(L_i)$. In particular if $k = n = \mathrm{rank}\, E$ we obtain the flag bundle $Fl(E)$ parametrizing the complete quotient flags of E. Recall that the consecutive projections

$$Fl(E) \to Fl^k(E) \to G^k(E) \to X$$

induce the following chain of injections of the corresponding Chow groups

$$(8) \qquad A(X) \to A(G^k(E)) \to A(Fl^k(E)) \to A(Fl(E)).$$

Let $A = (a_1, \ldots, a_n)$, $A^k = (a_1, \ldots, a_k)$, $A_{n-k} = (a_{k+1}, \ldots, a_n)$. The sequence (8) allows us to identify $A(X)$ as the subring in $A(Fl(E))$ of all symmetric polynomials in A, $A(G^k(E))$ as the subring in $A(Fl(E))$ of all symmetric polynomials in A^k and in A_{n-k}, and finally, $A(Fl^k(E))$ as the subring of polynomials in $a_1, \ldots, a_n$, which are symmetric in A^k.

Using the presentation of $Fl(E)$ as a composition of successive projective bundles and the well known description of the Gysin push forward for a projective bundle (*see* Lemma 2.1) one proves easily

LEMMA 2.3. — *For* $\alpha \in A(X)$, *the following equality holds* ($\tau = \tau^n$):

$$\tau_*(a_1^{i_1} a_2^{i_2} \ldots a_n^{i_n} \cap \tau^* \alpha) = s_{i_1-n+1,\, i_2-n+2,\, \ldots,\, i_n}(E) \cap \alpha$$

(*see* also [H-T] Proposition 2.3).

The following two Lemmas give some alternative formulas for Gysin push forward for (total) flag bundle and Grassmannian bundle. We will prove only the first formula. A proof of the second one is similar, and is left to the reader.

LEMMA 2.4. — *For any polynomial* P *in n variables and any* $\alpha \in A(X)$

$$\tau_*(P(a_1, \ldots, a_n) \cap \tau^* \alpha) = \sum_{w \in S_n} w[P(a_1, \ldots, a_n) \prod_{i<j} (a_i - a_j)^{-1}] \cap \alpha. \tag{9}$$

Proof. – Denote the morphism $A(Fl(E)) \to A(X)$ defined by the right hand side of (9) by τ'. Recall that τ_* is an $A(X)$-morphism and that $a_1^{i_1} \ldots a_n^{i_n}$, $i_k \leqq n-k$, $k=1, \ldots, n$ are generators of $A^{\cdot}(Fl(E))$ over $A(X)$ (*see* [H]). Therefore it suffices to prove that τ' is an $A(X)$-morphism and $\tau'(a_1^{i_1} \ldots a_n^{i_n}) = \tau_*(a_1^{i_1} \ldots a_n^{i_n})$, $i_k \leqq n-k$, $k=1, \ldots, n$. Indeed, it follows from the definition of τ' that $\tau'(P_1 \cdot P_2) = \tau'(P_1) \cdot P_2$ if P_2 is symmetric in A. Furthermore, τ' sends a polynomial P to a certain polynomial of the degree $\binom{n-1}{2}$ less than the degree of P. In particular $\tau'(a_1^{i_1} \ldots a_n^{i_n}) = 0$ if $i_k \leqq n-k$, $k=1, \ldots, n$ and $i_k < n-k$ for some k. One checks readily that $\tau'(a_1^{n-1} \ldots a_{n-1}) = 1$. But by Lemma 2.3 the same equalities hold with τ_* used instead of τ'. ■

LEMMA 2.5. — *With the above notation the Gysin morphism* $\pi_*: A(G^k(E)) \to A(X)$ *is induced by the following operation on polynomials*

$$P(a_1, \ldots, a_n) \mapsto \sum_{w \in S_n/S_k \times S_{n-k}} w[P(a_1, \ldots, a_n) \prod_{\substack{1 \leqq i \leqq k \\ k+1 \leqq j \leqq n}} (a_i - a_j)^{-1}].$$

As a consequence of these two facts we have

LEMMA 2.6. — *With the above notation the Gysin morphism* $\tau_*^k: A(Fl^k(E)) \to A(X)$ *is induced by the following operation on polynomials*

$$P(a_1, \ldots, a_n) \mapsto \sum_{w \in S_n/(S_1)^k \times S_{n-k}} w[P(a_1, \ldots, a_n) \prod_{\substack{1 \leqq i < j \leqq n \\ i \leqq k}} (a_i - a_j)^{-1}].$$

Proof. – The flag bundle $\tau^k: Fl^k(E) \to X$ can be presented as the composition

$$\tau^k: Fl(Q) \xrightarrow{\tau_Q} G^k(E)) \xrightarrow{\pi} X$$

where Q is the tautological quotient vector bundle of rank k on $G^k(E)$. Therefore the assertion follows from a presentation of τ^k_* as the composition

$$\tau^k_* : A(F l^k(E)) = A.(F l(Q)) \overset{(\tau_Q)_*}{\to} A(G^k(E)) \overset{\pi_*}{\to} A(X)$$

and from Lemmas 2.4, 2.5. ■

Proposition 1.7 combined with Lemma 2.6 gives

COROLLARY 2.7. — *For every strict partition* $I = (i_1, \ldots, i_k)$ *of length* k *where* $k \leqq n$, *and for any* $\alpha \in A(X)$

$$P_I(E) \cap \alpha = \tau^k_* \, [a_1^{i_1} \ldots a_k^{i_k} \prod_{\substack{1 \leqq i < j \leqq n \\ i \leqq k}} (a_i + a_j) \cap (\tau^k)^* \alpha].$$

The following fact extends the main calculation in [J-L-P].

PROPOSITION 2.8. — *With the above notation, for every strict partition* $I = (i_1, \ldots, i_q)$ *of length* $\geqq q-1$, *and for every* $\alpha \in A(X)$

$$\pi_* [c_{\text{top}}(R \otimes Q) P_I(Q) \cap \pi^* \alpha] = P_I(E) \cap \alpha.$$

Proof. — Observe that if $l(I) = q$ or $q-1$, then by Corollary 2.7, for $\beta \in A(G^q(E))$

$$P_I(Q) \cap \beta = (\tau_Q)_* [a_1^{i_1} \ldots a_q^{i_q} \prod_{1 \leqq i < j \leqq q} (a_i + a_j) \cap (\tau_Q)^* \beta].$$

Indeed, $\tau_Q^{q-1} = \tau_Q^q = \tau_Q$. Therefore we have

$$\pi_* [c_{\text{top}}(R \otimes Q) P_I Q \cap \pi^* \alpha]$$
$$= \pi_* (\tau_Q)_* [a_1^{i_1} \ldots a_q^{i_q} \prod_{1 \leqq i < j \leqq q} (a_i + a_j) (\tau_Q)^* c_{\text{top}}(R \otimes Q) \cap (\tau^q)^* \alpha]$$

(by the projection formula for τ_Q)

$$= (\tau^q)_* [a_1^{i_1} \ldots a_q^{i_q} \prod_{\substack{1 \leqq i < j \leqq n \\ i \leqq q}} (a_i + a_j) \cap (\tau^q)^* \alpha]$$

(by the splitting principle for $R \otimes Q$)

$$= P_I(E) \cap \alpha \qquad \text{(by Corollary 2.7).}$$ ■

Remark 2.9. — The results in this Section were stated and proved for Chow groups. They remain valid, however, also for other (co)homology theories — in particular for complex manifolds and singular cohomology. The proofs are the same.

3. The ideal of universal polynomials describing cycles supported in a degeneracy locus

Let us fix integers $m > 0$, $n > 0$ and $r \geqq 0$. Let $\varphi : F \to E$ be a morphism of vector bundles on a scheme X. Assume that rank $E = n$, rank $F = m$. Consider the degeneracy locus

$$D_r(\varphi)=\{x\in X,\ \operatorname{rank}\varphi(x)\leqq r\}$$

where a subscheme structure is determined by the ideal generated by all $(r+1)$-order minors of φ. The aim of this chapter is to study the following set $\mathscr{P}_r$ of polynomials in the graded polynomial $\mathbb{Z}$-algebra

$$\mathbb{Z}[c.(A),\ c.(B)]=\mathbb{Z}[c_1(A),\ \ldots,\ c_n(A),\ c_1(B),\ \ldots,\ c_m(B)],$$

where $c_1(A), \ldots, c_n(A), c_1(B), \ldots, c_m(B)$ are two sets of independent variables and $\deg c_k(A)=\deg c_k(B)=k$ for every k. Let $i_r: D_r(\varphi)\to X$ be the inclusion, and let $(i_r)_*: A.(D_r(\varphi))\to A.(X)$ be the induced push forward map of the Chow groups. Define $\mathscr{P}_r$ to be the set of all polynomials $P\in\mathbb{Z}[c.(A), c.(B)]$ such that for every morphism $\varphi: F\to E$ of vector bundles on an arbitrary scheme X (rank $E=n$, rank $F=m$), and every $\alpha\in A.(X)$

$$P(c_1(E),\ \ldots,\ c_n(E),\ c_1(F),\ \ldots,\ c_m(F))\cap\alpha\in\operatorname{Im}(i_r)_*.$$

($c_k(E)$, $c_k(F)$ denote here the k-th Chern classes of the vector bundles E and F). It is not difficult to see that $\mathscr{P}_r$ is an ideal in $\mathbb{Z}[c.(A), c.(B)]$.

We start with a computational Lemma which will be frequently used in this work. Let E, F be two vector bundles on a scheme X of ranks n and m respectively. Let $\pi_E: G_r(E)\to X$ (resp. $\pi_F: G^r(F)\to X$) be the Grassmannian bundle parametrizing r-subbundles of E (resp. r-quotients of F). Moreover, let

$$0\to R_E^{(r)}\to E_{G_r(E)}\to Q_E^{(n-r)}\to 0$$

$$0\to R_F^{(m-r)}\to F_{G^r(F)}\to Q_F^{(r)}\to 0$$

be the tautological sequences on $G_r(E)$ and $G^r(F)$ involving bundles of the indicated ranks. Consider the following product of Grassmannian bundles

$$\pi: G=G^r(F)\times_X G_r(E)\xrightarrow{\pi_F\times 1} G_r(E)\xrightarrow{\pi_E} X.$$

In the sequel instead of $(R_E)_G$, $(Q_F)_G$, $\ldots$, we will write R_E, Q_F, $\ldots$, for short.

LEMMA 3.1. — *For any partitions* I, J *such that* $l(I)\leqq n-r$, $l(\tilde{J})\leqq m-r$ *and any* $\alpha\in A(X)$ *the following equality holds*

$$\pi_*[s_I Q_E\, s_J(-R_F)\, c_{top}(\operatorname{Hom}(F, E)_G/\operatorname{Hom}(Q_F, R_E))\cap\pi^*\alpha]=s_{(m-r)^{n-r}+I,\ J}(E-F)\cap\alpha.$$

Proof. — First, let us record the following simple consequence of Proposition 2.2. For every vector bundle H on X and every $\beta\in A(X)$ we have

$$(10)\qquad (\pi_F)_*[s_{(m-r)^r,\ J}(H_{G^r(F)}-R_F)\cap\pi_F^*\beta]=s_J(H-F)\cap\beta.$$

Indeed, $G^r(F)$ is isomorphic to the Grassmannian bundle $\pi_F^\vee : G_r(F^\vee) \to X$. The tautological exact sequence on $G_r(F^\vee)$ can be written as

$$0 \to Q_F^\vee \to F^\vee_{G_r(F^\vee)} \to R_F^\vee \to 0.$$

Then we have

$$\begin{aligned}
&(\pi_F)_* [s_{(m-r)^r, J}(H_{G^r(F)} - R_F) \cap \pi_F^* \beta] \\
&\qquad = (\pi_F^\vee)_* [s_{J^\sim + (r)^{m-r}}(R_F^\vee - H^\vee_{G_r(F^\vee)}) \cap (\pi_F^\vee)^* \beta] \\
&\qquad = s_{J^\sim}(F^\vee - H^\vee) \cap \beta \qquad \text{(by Proposition 2.2)} \\
&\qquad = s_J(H - F) \cap \beta
\end{aligned}$$

and (10) is proved. Next, notice that in the Grothendieck group

$$[\mathrm{Hom}(F, E)_G/\mathrm{Hom}(Q_F, R_E)] = [R_F^\vee \otimes R_E + F^\vee \otimes Q_E].$$

Therefore we have to evaluate

$$\begin{aligned}
&\pi_* [s_I Q_E s_J(-R_F) c_{\mathrm{top}}(R_F^\vee \otimes R_E) c_{\mathrm{top}}(F^\vee \otimes Q_E) \cap \pi^* \alpha] \\
&\qquad = \pi_* [s_J Q_E s_J(-R_F) s_{(m-r)^r}(R_E - R_F) s_{(m-r)^n}(Q_E - F) \cap \pi^* \alpha] \\
&\qquad\qquad \text{(by Lemma 1.3)} \\
&\qquad = \pi_* [s_{(m-r)^r, J}(R_E - R_F) s_{(m-r)^n + I}(Q_E - F) \cap \pi^* \alpha] \\
&\qquad\qquad \text{(by the factorization formula)} \\
&\qquad = (\pi_E)_* [s_J(R_E - F) s_{(m-r)^n + I}(Q_E - F) \cap \pi_E^* \alpha] \\
&\qquad\qquad \text{(by (10))} \\
&\qquad = s_{(m-r)^{n-r} + I, J}(E - F) \cap \alpha
\end{aligned}$$

(by Proposition 2.2 applied to π_E). ■

PROPOSITION 3.2. — *Let* I, J *be two partitions such that* $l(I) \leqq n-r$, $l(J^\sim) \leqq m-r$. *Then the Schur S-polynomial* $s_{(m-r)^{n-r}+I, J}(A-B)$ *belongs to* $\mathscr{P}_r$.

Proof. — Let $\varphi : F \to E$ be a morphism of vector bundles on a scheme X, where rank $E = n$, rank $F = m$. Preserving the above notation, consider the following geometric construction. The morphism φ induces the section s_φ of Hom(F, E) and thus the section $\overline{s_\varphi}$ of the vector bundle $\mathrm{Hom}(F, E)_G/\mathrm{Hom}(Q_F, R_E)$ on G. Let Z be the subscheme of zeros of $\overline{s_\varphi}$. It follows from the definition of Z that the restriction ρ of π to Z factorizes through $D_r(\varphi)$; in other words we have a commutative diagram of schemes

(11)
$$\begin{array}{ccc}
Z & \xrightarrow[i_r']{} & G \\
\downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\pi} \\
D_r(\varphi) & \xrightarrow[i_r]{} & X
\end{array}$$

where i'_r is the inclusion. Let $P=P(E, F)=s_{(m-r)^{n-r}+I, J}(E-F)$. To see that $P \cap \alpha \in \operatorname{Im}(i_r)_*$ we can pass to generic case. Let $\bar{X}=\operatorname{Hom}(F, E)$ and denote by $p: \bar{X} \to X$ the canonical projection. Recall that there exists on $\bar{X}$ the canonical (tautological) morphism $\bar{\varphi}: \bar{F} \to \bar{E}$ where $\bar{E}=E_{\bar{X}}$, $\bar{F}=F_{\bar{X}}$ and we have $D_r(\bar{\varphi}) \subset \bar{X}$ – the corresponding degeneracy locus. The morphism φ induces a section $s_\varphi: X \to \bar{X}$ such that $p \circ s_\varphi = id$, $(s_\varphi)^*(\bar{E})=E$ and $(s_\varphi)^*(\bar{F})=F$. Thus $s_\varphi^*[P(\bar{E}, \bar{F}) \cap p^* \alpha]=P(E, F) \cap \alpha$. Now from the cartesian square

$$\begin{array}{ccc} D_r(\bar{\varphi}) & \xrightarrow[\bar{i}_r]{} & \bar{X} \\ \uparrow & & \uparrow {\scriptstyle s_\varphi} \\ D_r(\varphi) & \xrightarrow[i_r]{} & X \end{array}$$

we obtain the commutative diagram of the Chow groups (*cf.* [F] Proposition 1.7)

$$\begin{array}{ccc} A.(D_r(\bar{\varphi})) & \xrightarrow[(\bar{i}_r)_*]{} & A.(\bar{X}) \\ \downarrow & & \downarrow {\scriptstyle (s_\varphi)^*} \\ A.(D_r(\varphi)) & \xrightarrow[(i_r)_*]{} & A.(X) \end{array}$$

In particular we infer that $P(\bar{E}, \bar{F}) \cap p^* \alpha \in \operatorname{Im}(\bar{i}_r)_*$ implies $P(E, F) \cap \alpha \in \operatorname{Im}(i_r)_*$. Therefore we can assume that the morphism $\varphi: F \to E$ in question is generic in the above sense. But, then by looking at the local coordinates we see that $\operatorname{codim}_G(Z)=mn-r^2$ and thus

$$(i'_r)_*[Z]=c_{top}[\operatorname{Hom}(F, E)_G/\operatorname{Hom}(Q_F, R_E)] \cap [G]. \tag{12}$$

Now, let $\alpha \in A(X)$. Since the diagram

$$\begin{array}{ccc} A(Z) & \xrightarrow[(i'_r)_*]{} & A(G) \\ \downarrow {\scriptstyle \rho_*} & & \downarrow {\scriptstyle \pi_*} \\ A(D_r(\varphi)) & \xrightarrow[(i_r)_*]{} & A(X) \end{array}$$

of the Chow groups is commutative (*see* [F] chap. 1) the Proposition will be proved if we find elements $z_{I, J} \in A(Z)$ such that $\pi_*(i'_r)_*(z_{I, J})=P(E, F) \cap \alpha$. Indeed, for the element $\rho_* z_{I, J}$ in $A(D_r(\varphi))$, we then have $(i_r)_*(\rho_* z_{I, J})=P(E, F) \cap \alpha$. Define $z_{I, J}$ as $(i'_r)^*[s_I Q_E s_J(-R_F) \cap \pi^* \alpha]$. We have

$$\begin{aligned} \pi_*(i'_r)_*(z_{I, J}) &= \pi_*(i'_r)_*\{(i'_r)^*[s_I Q_E s_J(-R_F) \cap \pi^* \alpha]\} \\ &= \pi_*\{s_I Q_E s_J(-R_F)\, c_{top}[\operatorname{Hom}(F, E)_G/\operatorname{Hom}(Q_F, R_E)] \cap \pi^* \alpha\} \end{aligned}$$

by the projection formula for i'_r and by (12). The final assertion now follows from Lemma 3.1. ■

Remark 3.3. — Let $M_{m\times n}(K)$ be the affine space of $m\times n$ matrices over a field K. Let $D_r\subset M_{m\times n}(K)$ be the determinantal subscheme of matrices of rank $\leqq r$. Then the construction (11) is a desingularization of D_r. The morphism ρ restricted to $\rho^{-1}(D_r-D_{r-1})$ is an isomorphism; if we identify $M_{m\times n}(K)$ with $\mathrm{Hom}(K^m, K^n)$, then the inverse morphism to ρ on D_r-D_{r-1} is given by $f\mapsto(f, K^m\twoheadrightarrow \mathrm{Im}\, f, \mathrm{Im}\, f\hookrightarrow K^n)$.

Assume that $A=(a_1, \ldots, a_n)$, $B=(b_1, \ldots, b_m)$ are two disjoint sequences of algebraically independent elements over $\mathbb{Z}$. Let

$$\mathbb{Z}[c.(A), c.(B)]=\mathbb{Z}[c_1(A), \ldots, c_n(A), c_1(B), \ldots, c_m(B)]$$

be a graded polynomial $\mathbb{Z}$-algebra where $\deg c_k(A)=\deg c_k(B)=k$. The assignment

$$c_k(A)\mapsto(k\text{-th elementary symmetric function in A})$$

and likewise for $c_k(B)$, defines an isomorphism $\mathbb{Z}[c.(A), c.(B)]\xrightarrow{\simeq}\underline{\mathrm{Sym}}(A, B)$ and allows us to treat $\mathscr{P}_r$ as an ideal in $\underline{\mathrm{Sym}}(A, B)$.

The main result of this paper is the following

THEOREM 3.4. — *The ideal* $\mathscr{P}_r$ *of* $\underline{\mathrm{Sym}}(A, B)$ *is generated by Schur* S-*polynomials* $s_I(A-B)$, *where* I *ranges over all partitions such that* $I\supset(m-r)^{n-r}$.

Let us denote by $\mathscr{I}_r$ the ideal of $\underline{\mathrm{Sym}}(A, B)$ generated by all Schur S-polynomials $s_I(A-B)$, where $I\supset(m-r)^{n-r}$.

First we prove that $\mathscr{I}_r\subset\mathscr{P}_r$. It suffices to show that the Schur S-polynomials $s_{(m-i)^{n-i}+I, J}(A-B)$ belong to $\mathscr{P}_r$, where $i=0, \ldots, r$ and $l(I)\leqq n-i$, $l(J^\sim)\leqq m-i$. By Proposition 3.2 $s_{(m-i)^{n-i}+I, J}(A-B)$ belongs to $\mathscr{P}_i$. Since clearly $\mathscr{P}_i\subset\mathscr{P}_r$ $i=0, \ldots, r$, the assertion follows.

Now we will prove that $\mathscr{P}_r\subset\mathscr{I}_r$. Consider the following situation. Let V, W be two vector spaces over a field K. Assume that $v=\dim(V)>m$, $w=\dim(W)>n$. Let $G^m=G^m(V)$ be the Grassmannian of m-quotients of V and let $G_n=G_n(W)$ be the Grassmannian of n-subspaces of W. Let Q^m_V be the tautological m-quotient bundle on G^m and let R^n_W be the tautological n-subbundle on G_n. Finally, let

(13) $X=X_{v, w}=\mathrm{Hom}((Q^m_V)_{G^m\times G_n}, (R^n_W)_{G^m\times G_n})$, $F=F_{v, w}=(Q^m_V)_X$, $E=E_{v, w}=(R^n_W)_X$.

We have on X the canonical (tautological) morphism $\varphi: F\to E$. Let $D_r=D_r(v, w)$ denote the degeneracy locus $D_r(\varphi)$. Observe that by Thom isomorphism (*see* [F] Theorem 3.3) we have $A^{\cdot}(X)\simeq A^{\cdot}(G^m\times G_n)$ because X is a vector bundle on $G^m\times G_n$. Let us notice two features of this situation.

1). The morphism φ is given locally by $m\times n$ matrix of variables.

2). By Schubert Calculus all elements of the form $s_I E\cdot s_J F$, $l(J)\leqq m$, $l(I)\leqq n$ are non-zero for $v, w\gg 0$, and every finite set $\{s_{I_1}E\cdot s_{J_1}F, \ldots, s_{I_k}E\cdot s_{J_k}F\}$, $(I_p, J_p)\neq(I_q, J_q)$ if $p\neq q$, becomes a family of $\mathbb{Z}$-linearly independent elements for $v, w\gg 0$.

Let $\mathscr{I}_r(E, F)$ be the ideal in $A(X)$ generated by all Schur S-polynomials $s_I(E-F)$ where $I\supset(m-r)^{n-r}$. Our aim is to prove the following

PROPOSITION 3.5. — *For the degeneracy locus* $D_r = D_r(v, w)$ *described above and for any* $v > m$, $w > n$, *we have* $\operatorname{Im}(i_r)_* = \mathscr{I}_r(E, F)$.

Notice that this Proposition implies that $\mathscr{P}_r \subset \mathscr{I}_r$. Indeed, the property 2) of the construction (13) guarantees that letting $v, w \to \infty$, we do not lose any of the polynomials from $\mathscr{P}_r$ in this counting.

We recall the following fact

LEMMA 3.6. — *Let* $i: H' \subsetneq H$ *be a monomorphism of vector bundles on a scheme* Y. *Then the following two exact sequences are isomorphic*

$$\begin{array}{ccccccc} A^{\cdot}(H') & \xrightarrow{\quad i_* \quad} & A^{\cdot}(H) & \to & A^{\cdot}(H-H') & \to & 0 \\ \| & & \| & & \| & & \\ A^{\cdot}(Y) & \xrightarrow[c_{\mathrm{top}}(H/H') \cap -]{} & A^{\cdot}(Y) & \to & A^{\cdot}(H-H') & \to & 0 \end{array}.$$

Proof. — The assertion follows easily from the Thom isomorphism $A^{\cdot}(H') \simeq A^{\cdot}(H) \simeq A^{\cdot}(Y)$, from the self-intersection formula $i^* i_*(h) = c_{\mathrm{top}} N_H(H') \cdot h$, where $h \in A(H')$, and from the well known identification $N_H(H') \cong (H/H')_{H'}$. ■

In particular the exact sequence

$$A(\text{Zero section of H}) \to A(H) \twoheadrightarrow A(\text{H-Zero section of H})$$

can be identified with

$$A(X) \xrightarrow[c_{\mathrm{top}} H \cap -]{} A(X) \to A(\text{H-Zero section of H}) \to 0.$$

The following Lemma will be frequently used in this paper

LEMMA 3.7. — *Let* $D = D_r \supset D_{r-1} \supset \ldots \supset D_1 \supset D_0 \supset D_{-1} = \varnothing$ *be a sequence of irreducible and closed subschemes of a scheme* D *over a field* K. *Let* $\pi: Z \to D$ *be a proper, surjective morphism of schemes. Assume that for every* $k = 0, \ldots, r$ there exists an open covering $\{U_\alpha^k\}_{\alpha \in \Lambda}$ *of* $D_k - D_{k-1}$ *and a scheme* G_k, *such that for every* $\alpha \in \Lambda$, $\pi^{-1}(U_\alpha^k)$ *is isomorphic to* $U_\alpha^k \times G_k$, *and the restricted morphism* $\pi: \pi^{-1}(U_\alpha^k) \to U_\alpha^k$ *is equal to the projection* $U_\alpha^k \times G_k \to U_\alpha^k$ *onto the first factor. Then the induced map* $\pi_*: A.(Z) \to A.(D)$ *of the Chow groups is surjective.*

Proof. — Let $Z_k = \pi^{-1}(D_k)$, $k = 0, \ldots, r$. There is a commutative diagram

$$\begin{array}{ccccccc} A.(Z_{k-1}) & \to & A.(Z_k) & \to & A.(Z_k - Z_{k-1}) & \to 0 \\ \downarrow{\scriptstyle (\pi|_{Z_{k-1}})_*} & & \downarrow{\scriptstyle (\pi|_{Z_k})_*} & & \downarrow{\scriptstyle (\pi|_{Z_k - Z_{k-1}})_*} & \\ A.(D_{k-1}) & \to & A.(D_k) & \to & A.(D_k - D_{k-1}) & \to 0 \end{array}$$

with exact rows (the commutativity of the diagram on the right hand side follows from [F] Proposition 1.7). By a diagram chase we see that it suffices to prove that $(\pi|_{Z_k - Z_{k-1}})_*$ is surjective for $k = 0, \ldots, r$ and induct on k. Write D^0 for $D_k - D_{k-1}$ and π for $\pi|_{Z_k - Z_{k-1}}: Z_k - Z_{k-1} \to D^0$. Choose an open subscheme U in D^0 where

$\pi: \pi^{-1}(U) \to U$ is equal to the projection $p: U \times G_k \to U$. Similarly as above, there is a commutative diagram with exact rows

$$\begin{array}{ccccccc} A.(\pi^{-1}(D^0-U)) & \to & A.(Z_k-Z_{k-1}) & \to & A.(\pi^{-1}(U)) & \to & 0 \\ \downarrow{\scriptstyle (\pi|_{D^0-U})_*} & & \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle (\pi|_{\pi^{-1}(U)})_*} & & \\ A.(D^0-U) & \to & A.(D^0) & \to & A.(U) & \to & 0 \end{array}$$

By Noetherian induction, *i. e.* repeating the process on D^0-U, we can assume that the left vertical map is surjective. Then the assertion follows by a diagram chase, because the surjectivity of $(\pi|_{\pi^{-1}(U)})_* = p_*: A.(U \times G_k) \to A.(U)$ is obvious. This proves the Lemma. ■

The proof of Proposition 3.5 will be carried out in such a way that as a by-product we obtain a certain finite set of generators of $\mathscr{I}_r$. The following construction will lead to a particularly simple set of generators of $\mathscr{I}_r$. Recall that to a given morphism $\varphi: F \to E$ of vector bundles on X one can associate the following geometric construction (*cf.* [J-L-P] or [F] Ex. 14.4.10).

$$(14) \qquad \begin{array}{ccc} Z = \mathrm{Zeros}(F_G \to E_G \to Q) & \underset{i'_r}{\to} & G = G_r(E) \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \pi} \\ D_r(\varphi) & \underset{i_r}{\longrightarrow} & X \end{array}$$

Here i'_r is the inclusion, π is the canonical projection and ρ is the restriction of π to Z. Let us apply the construction (14) to the generic situation (13). Let U^α, where $\alpha=(\alpha_1, \ldots, \alpha_r)$, $1 \leqq \alpha_1 < \ldots < \alpha_r \leqq m$, and U_β, where $\beta=(\beta_1, \ldots, \beta_r)$, $1 \leqq \beta_1 < \ldots < \beta_r \leqq n$ be the standard coverings of $G^m(V)$ (respectively of $G_n(W)$) which trivialize the bundles Q_V and R_W. Let Λ be the set of all pairs (α, β) with α and β as above. For $(\alpha, \beta) \in \Lambda$ define $U_{(\alpha, \beta)}$ as the inverse image of $U^\alpha \times U_\beta$ with respect to the projection $X \to G^m(V) \times G_n(W)$. Let

$$D = D_r \supset D_{r-1} \supset \ldots \supset D_1 \supset D_0 \supset D_{-1} = \varnothing$$

be the sequence of determinantal varieties. Define an open covering $\{U^k_{(\alpha, \beta)}\}$, $k=0, 1, \ldots, r$, $(\alpha, \beta) \in \Lambda$ of the variety $D_r - D_{r-1}$ by $U^k_{(\alpha, \beta)} = U_{(\alpha, \beta)} \cap (D_k - D_{k-1})$. Then $\rho^{-1}(U^k_{(\alpha, \beta)}) = U^k_{(\alpha, \beta)} \times G_k(K^n)$. Since the assumptions of Lemma 3.7 are satisfied, we infer

LEMMA 3.8. — *With the above notation, the induced map $\rho_*: A.(Z) \to A.(D_r)$ is surjective.*

Consider now the commutative diagram of the Chow groups, induced by (14)

$$(15) \qquad \begin{array}{ccc} A.(Z) & \overset{(i'_r)_*}{\to} & A.(G) \\ \downarrow{\scriptstyle \rho_*} & & \downarrow{\scriptstyle \pi_*} \\ A.(D_r) & \overset{(i_r)_*}{\to} & A.(X) \end{array}$$

This diagram can be treated as a commutative diagram of A (X)-modules. Indeed A (G) is a (free) A (X)-module by the Schubert Calculus for Grassmannian bundles (*cf.* [F] chap. 14); then the Gysin morphism $(i'_r)^*$ allows to define a A (X)-module structure on A.(Z) in such a way that $(i'_r)_*$ is a A (X)-morphism because of the projection formula for i'_r. Lemma 3.8 and the commutativity of the diagram (15) imply

LEMMA 3.9. — $\operatorname{Im}(i_r)_* = \pi_* [\operatorname{Im}(i'_r)_*]$.

In order to compute $\operatorname{Im}(i'_r)_*$ we will describe now the geometry of Z in an explicit way. Recall that Z is the scheme of zeros of the section $\mathcal{O}_G \to F_G^\vee \otimes Q$ induced by the composition: $F_G \xrightarrow{\varphi_G} E_G \to Q$. We can identify $G = G_r(E)$ with the vector bundle

$$H = \operatorname{Hom}[(Q_V^m)_{G^m \times F\,l_{r,n}}, (R_W^n)_{G^m \times F\,l_{r,n}}]$$

on the scheme $G^m \times F\,l_{r,n}$ ($F\,l_{r,n} = F\,l_{r,n}(W)$ is the scheme parametrizing flags of subspaces in W of dimensions r and n). Under this identification the subscheme $Z \subset G$ becomes the subbundle

$$H' = \operatorname{Hom}[(Q_V^m)_{G^m \times F\,l_{r,n}}, (R_W^r)_{G^m \times F\,l_{r,n}}].$$

Therefore the exact sequence

$$\text{(16)} \qquad A(Z) \xrightarrow{(i'_r)_*} A(G) \to A(G-Z) \to 0$$

is equal to

$$\text{(17)} \qquad A(H') \xrightarrow{(i'_r)_*} A(H) \to A(H-H') \to 0.$$

By Lemma 3.7 the exact sequence (17) corresponds via Thom isomorphism to the sequence

$$A(Y) \xrightarrow{c_{\mathrm{top}}(H/H') \cap -} A(Y) \to A(H-H') \to 0,$$

where $Y = G^m \times F\,l_{r,n}$ and

$$H/H' = (Q_V^m)^\vee_{G^m \times F\,l_{r,n}} \otimes (R_W^n / R_W^r)_{G^m \times F\,l_{r,n}}.$$

Therefore, by expressing the assertion of Lemma 3.4 in terms of the exact sequence (16) we infer the following fact.

LEMMA 3.10. — $\operatorname{Im}(i'_r)_*$ *is a principal ideal in* A (G) *generated by* $c_{\mathrm{top}}(F_G^\vee \otimes Q)$.

LEMMA 3.11. — *The* A (X)-*module* $\pi_* [\operatorname{Im}(i'_r)_*]$ *is generated by Schur* S-*polynomials* $s_{(m-r)^{n-r}+I}(Q - F_G)$, *where* $I \subset (r)^{n-r}$.

Proof. — By the Schubert Calculus for Grassmannian bundles the A(X)-module A(G) is generated by $s_I(Q)$ where $I \subset (r)^{n-r}$. Since $c_{top}(F_G^\vee \otimes Q) = s_{(m)^{n-r}}(Q - F_G)$, the A(X)-module $\mathrm{Im}(i_r')_*$ is generated by $s_{(m)^{n-r}+I}(Q - F_G)$, where $I \subset (r)^{n-r}$, by the factorization formula. The assertion now follows from Proposition 2.2. ■

Finally, Lemmas 3.9 and 3.11 imply Proposition 3.5. This finishes the proof of Theorem 3.4.

Remark 3.12. — It is possible to give a more "down-to-earth" proof of inclusion $\mathscr{P}_r \subset \mathscr{I}_r$ by showing by induction on r that in the generic situation (13) a somewhat weaker assertion holds: if $v, w \gg 0$, then $\mathrm{Im}(i_r)_* = \mathscr{I}_r(E, F)$. We will demonstrate this alternative method for a symmetric morphism in Section 7.

COROLLARY 3.13. — *Consider the generic situation* (13) *where* $m = n = s$, $r = s - 1$. *Then* $\mathrm{I}m(i_s)_*$ *is generated by all Schur S-polynomials* $s_I(E - F)$, *where* I *ranges over all partitions of positive weight. By the linearity formula* (4), *we easily see that* $\mathrm{Im}(i_s)_*$ *is also generated by the elements of the form* $s_I E - s_I F$, *where* I *ranges over all partitions of positive weight.*

4. Chow groups of determinantal schemes

As a by-product of considerations in Section 3, we will obtain here an explicit description of the Chow groups of determinantal schemes.

Let $M = M_{m \times n}(K)$ be the affine space of $m \times n$ matrices over a field K. Let $D_r \subset M_{m \times n}(K) = \mathrm{Hom}_K(V, W)$, where $V = K^m$, $W = K^n$ be the determinantal subscheme determined by the vanishing of all $(r+1)$-order minors. Before going further we recall the following fundamental fact from Schubert Calculus. Let $e_i = (0, \ldots, 0, \underset{i}{1}, 0, \ldots, 0)$, $i = 1, \ldots, n$, be the standard basis in K^n. For each sequence $\alpha = (\alpha_1, \ldots, \alpha_r)$ where $0 < \alpha_1 < \ldots < \alpha_r \leqq n$, let A_i be the space spanned by $e_1, \ldots, e_{\alpha_i}$ and let $\Omega(\alpha) = \{L \in G_r(K^n), \dim(L \cap A_i) \geqq i, 1 \leqq i \leqq r\}$. Then the fundamental classes of $\Omega(\alpha)$, $0 < \alpha_1 < \ldots < \alpha_r \leqq n$ form a $\mathbb{Z}$-basis of $A.(G_r(K^n))$. Let $\Omega'(\alpha)$ be the subset of all linear homomorphisms f of rank r in $D_r - D_{r-1}$ such that $\mathrm{Im}(f) \in \Omega(\alpha)$.

LEMMA 4.1. — *Assume that* $m \geqq n$. *Then the assignment* $[\Omega(\alpha)] \mapsto [\Omega'(\alpha)]$ *defines an isomorphism of* $A^{\cdot}(G_r(K^n))$ *and* $A^{\cdot}(D_r - D_{r-1})$.

Proof. — It is not difficult to see that the map $D_r - D_{r-1} \to G^r(V) \times G_r(W)$ such that $f \mapsto (V/\mathrm{Ker} f, \mathrm{Im} f)$ is a locally trivial fibration with the fiber isomorphic to $GL(r, K)$. More precisely, it defines an isomorphism

$$D_r - D_{r-1} \simeq X - D_{r-1}(\varphi)$$

where for $G = G^r(V) \times G_r(W)$ and $F = (Q_V^r)_G$, $E = (R_W^r)_G$, X denotes the bundle $\mathrm{Hom}(F, E)$ on G and $\varphi: F \to E$ is the tautological morphism on X. It follows from Corollary 3.13 that

$$A^{\cdot}(X - D_{r-1}(\varphi)) \simeq A^{\cdot}(G)/\mathscr{I}$$

where $\mathscr{J}$ is the ideal in $A^{\cdot}(G)$ generated by all elements of the form $s_I(E)-s_I(F)$, where $|I|\geqq 1$. By Schubert Calculus $A(G)$ is isomorphic to $\mathbb{Z}[c.(E), c.(F)]$ modulo the ideal generated by $s_I E$, $I \not\subset (n-r)^r$ and $s_J F$, $J \not\subset (m-r)^r$. This implies that the ring $A^{\cdot}(G)/\mathscr{J}$ is isomorphic to $\mathbb{Z}[c.(E)]$ modulo the ideal generated by $s_I(E)$, $I \not\subset (n-r)^r$ *i. e.* isomorphic to $A(G_r(W))$. Therefore $A^{\cdot}(D_r-D_{r-1})$ can be identified with $A^{\cdot}(G_r(K^n))$ via the isomorphism described above. ■

In above notation let $\bar{\Omega}(\alpha)$ be the closure of $\Omega(\alpha)$ in D_r.

PROPOSITION 4.2. — *Assume that $m \geqq n$. Then the assignment $[\Omega(\alpha)]\mapsto[\bar{\Omega}(\alpha)]$ defines an isomorphism of $A^{\cdot}(G_r(K^n))$ and $A^{\cdot}(D_r)$. In particular for every k, $A^k(D_r)\simeq \bigoplus_I \mathbb{Z}$, the sum over all partitions $I\subset(r)^{n-r}$, $|I|=k$.*

Proof. — Apply the geometric construction (14) to the above situation. Recall that $Z\subset G_r(W_M)$ is the subscheme of zeros of the section

$$G_r(W_M)\to V^{\vee}_{G_r(W_M)}\otimes Q$$

where Q is the tautological quotient bundle on $G_r(W_M)$. Therefore we can identify Z with the vector bundle $\mathrm{Hom}(V_{G_r(W)}, R^r_W)$ on the Grassmannian $G_r(W)$. In particular $A^{\cdot}(Z)=A^{\cdot}(G_r(W))$ by Thom isomorphism. Moreover, by Lemma 3.7 the induced map $\rho_*: A(Z)\to A(D_r)$ is surjective. The Proposition now follows from Lemma 4.1 and the chain of surjections

$$A.(Z)\to A.(D_r)\overset{k_r^*}{\to} A.(D_r-D_{r-1}).\quad ■$$

This result can be generalized in the following way. Let E and F be two vector bundles of ranks n and m, on a scheme X. Let $\bar{D}_r\subset \mathrm{Hom}(F, E)$ be the r-th universal (tautological) degeneracy locus. Then, using the Schubert Calculus for Grassmannian bundles (*see* [F] chap. 14) and repeating previous arguments one proves

PROPOSITION 4.3. — *Assume that $m\geqq n$. Then*

$$A^{\cdot}(\bar{D}_r)\simeq A^{\cdot}(\bar{D}_r-\bar{D}_{r-1})\simeq A^{\cdot}(G_r(E)).$$

In particular, for every k, $A^k(\bar{D}_r)\simeq A^k(\bar{D}_r-\bar{D}_{r-1})\simeq\bigoplus_I A^{k-|I|}(X)$, the sum over all partitions $I\subset(r)^{n-r}$, $|I|\leqq k$.

For example, if E, F are trivial we get $A^k(X\times D_r)\cong\bigoplus A^{k-|I|}(X)$, the sum as above.

5. Chern numbers of kernel and cokernel bundles, Euler-Poincaré characteristic of smooth degeneracy loci

For the purposes of this chapter we assume that E and F are C^∞ complex vector bundles on a complex manifold X. Let $\varphi: F\to E$ be a morphism of vector bundles on X. Assume that rank $F=m$, rank $E=n$. The morphism φ induces the section

$s_\varphi \in H^0(X, \mathrm{Hom}(F, E))$. Let $\bar{D}_r$ be the universal (tautological) degeneracy locus of rank r in $\mathrm{Hom}(F, E)$. We say that φ is *general* if s_φ is transversal to $\bar{D}_r$ for all $r=0, \ldots, \min(m, n)-1$. For the rest of this section $\varphi: F \to E$ will always denote a general morphism of vector bundles. The morphism φ has rank exactly r over $D_r(\varphi)-D_{r-1}(\varphi)$, so we may define its *kernel* and *cokernel* bundles over $D_r(\varphi)-D_{r-1}(\varphi)$ by the exact sequence

$$0 \to K \to F \xrightarrow{\varphi} E \to C \to 0.$$

Of course rank $K=m-r$ and rank $C=n-r$. Suppose $D_{r-1}(\varphi)$ is empty. Then $D_r(\varphi)$ is smooth because it is isomorphic to the transversal intersection of the section s_φ with the universal degeneracy locus $\bar{D}_r-\bar{D}_{r-1}$ in $\mathrm{Hom}(F, E)$ which is known to be smooth (*see* [G-G]). Moreover the (complex) codimension of $D_r(\varphi)$ is $(m-r)(n-r)$. In this chapter we use the usual singular cohomology groups $H^\cdot(-)=H^\cdot(-, \mathbb{Z})$ rather than the Chow groups. In particular $(i_r)^*: H^\cdot(X) \to H^\cdot(D_r(\varphi))$ (resp. $(i_r)_*: H^\cdot(D_r(\varphi)) \to H^\cdot(X)$) denotes the multiplicative (resp. additive push forward) morphism associated with the inclusion $i_r: D_r(\varphi) \to X$. Recall that the formula for Gysin push forward of cycles established in Lemma 3.1 remains true in the category of complex manifolds and C^∞ complex vector bundles.

LEMMA 5.1. — *In the above situation, let* I, J *be two partitions such that* $l(I) \leqq n-r$, $l(J^\sim) \leqq m-r$. *Then for* $\alpha \in H^\cdot(X)$

$$(i_r)_*[s_I(C) \cdot s_J(-K) \cdot i_r^* \alpha] = s_{(m-r)^{n-r}+I, J}(E-F) \cdot \alpha.$$

Proof. — Consider the geometric construction (11):

$$\begin{array}{ccc} Z & \xrightarrow{i'_r} & G \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \pi} \\ D_r(\varphi) & \xrightarrow{i_r} & X \end{array}$$

described in Proposition 3.2. Recall that $\pi: G=G^r(F) \times_X G_r(E) \to X$ is a product of Grassmannian bundles and that Z is the set of zeros of the section of the bundle $\mathrm{Hom}(F, E)_G/\mathrm{Hom}(Q_F, R_E)$, induced by s_φ. Since $D_{r-1}(\varphi)=\varnothing$, ρ establishes an isomorphism $D_r(\varphi) \simeq Z$. This isomorphism allows us to identify K with $(i'_r)^*(R_F)$, C with $(i'_r)^*(Q_E)$ and i_r with $i'_r \circ \pi$. Therefore by the projection formula we get

$$\begin{aligned}(i_r)_*[s_I(C) \cdot s_J(-K) \cdot i_r^* \alpha] &= \pi_*(i'_r)_*\{(i'_r)^*[s_I(Q_E) \cdot s_J(-R_F) \cdot \pi^* \alpha]\} \\ &= \pi_*\{s_I(Q_E) \cdot s_J(-R_F)(i'_r)_*[Z] \cdot \pi^* \alpha\} = s_{(m-r)^{n-r}+I, J}(E-F) \cdot \alpha,\end{aligned}$$

where the last equality follows from Lemma 3.1 (*cf.* Remark 2.9). ∎

Remark 5.2. — Assume that $i: D \to X$ is a complex submanifold, say with connected components $D^1, \ldots, D^s$, then the push forward map

$$i_*: H^{\mathrm{top}}(D, \mathbb{Z}) = \mathbb{Z}^{\oplus s} \to H^{\mathrm{top}}(X, \mathbb{Z}) = \mathbb{Z}$$

assigns to $(z_1, \ldots, z_s)$ in $H^{top}(D, \mathbb{Z})$ the sum: $\sum_{i=1}^{s} z_i$ in $H^{top}(X, \mathbb{Z})$. Therefore for a given vector bundle E on D the Chern number $\prod_i c_i(E)^{\alpha_i} \cap [D] = \sum_{j=1}^{s} \prod_i c_i(E)^{\alpha_i} \cap [D^j]$, where $\sum i\alpha_i = \dim D$, may be written as $i_*(\prod_i c_i(E)^{\alpha_i})$. The same remark applies to the Chern numbers of the submanifold D itself. However, since $i_*: H^\cdot(D, \mathbb{Z}) \to H^\cdot(X, \mathbb{Z})$ is not usually injective it may be not possible to invert i_* to get the Chern classes of E and D. In [H-T] an example is given, which shows that the Chern classes of K are not in general restrictions of polynomials in the Chern classes of E and F.

Let us fix integers $\alpha_1, \alpha_2, \ldots, \alpha_{n-r} \geqq 0$ (resp. $\beta_1, \beta_2, \ldots, \beta_{m-r} \geqq 0$). Define nonnegative integers n_I (resp. m_J) by the formula

$$\prod_i c_i(C)^{\alpha_i} = \sum_I n_I s_I(C) \quad (\text{resp. } \prod_j c_j(K)^{\beta_j} = \sum_J m_J s_J(K)).$$

The numbers n_I can be evaluated from Pieri's formula (*see* [M] I.5.17), and if $\alpha = (\alpha_1, \ldots, \alpha_{n-r})$ is a partition, they are the Kostka numbers $K_{I^\sim, \alpha}$ in the notation of [M] I.6. The same remark applies to the m_I.

Lemma 5.1 yields the following closed form expression for the Chern numbers of K and C. Let $d = \dim D_r(\varphi)$.

Proposition 5.3. — *Assume that* $\sum i\alpha_i = \sum j\beta_j = \dim D_r(\varphi)$. *Then*

$$\prod_i c_i(C)^{\alpha_i} \cap [D_r(\varphi)] = \sum_I n_I s_{(m-r)^{n-r}+I}(E-F)$$

$$\prod_j c_j(K)^{\beta_j} \cap [D_r(\varphi)] = (-1)^d \sum_J m_J s_{(m-r)^{n-r}, J^\sim}(E-F).$$

Example 5.4. — $\dim D_r(\varphi) = 2$.

$$c_2(C) \cap [D_r(\varphi)] = s_{(m-r)^{n-r}+(1,1)}(E-F)$$

$$c_1^2(C) \cap [D_r(\varphi)] = s_{(m-r)^{n-r}+(2)}(E-F) + s_{(m-r)^{n-r}+(1,1)}(E-F)$$

$$c_2(K) \cap [D_r(\varphi)] = s_{(m-r)^{n-r}+(1,1)}(E-F)$$

$$c_1^2(K) \cap [D_r(\varphi)] = s_{(m-r)^{n-r},(2)}(E-F) + s_{(m-r)^{n-r},(1,1)}(E-F).$$

Remark 5.5. — The Chern numbers of K and C were originally investigated by Harris and Tu in [H-T], where they gave a certain rule for calculation of these numbers. Our approach seems to be simpler. Indeed, if $\prod c_i(K)^{\alpha_i} = \sum_I m_I s_I(K)$, then the algorithm of Harris and Tu requires one to evaluate $\sum_I m_I \dim V_I$ monomials in the Chern roots of K on $[D_r(\varphi)]$, and to perform numerous cancellations of pairwise opposite terms (V_I denotes here the irreducible polynomial representation of $Gl(m-r, \mathbb{C})$ corresponding to the weight I). On the contrary our recipe requires only the evaluation of card $\{I, m_I \neq 0\}$ expressions, which is much more economical in practice.

Let a, b be two positive integers such that $a \leqq b$. For two partitions $\mathrm{I}=(i_1, \ldots, i_a)$, $\mathrm{J}=(j_1, \ldots, j_a)$ define

$$\mathrm{D}_{\mathrm{I},\mathrm{J}}^{a,b}=\mathrm{Det}\left[\begin{pmatrix} i_p+j_q+a+b-p-q \\ i_p+a-p \end{pmatrix}\right], \qquad 1 \leqq p,\, q \leqq a.$$

LEMMA 5.6 ([L-S] 4.2). — *Let* A, B *be two vector bundles of ranks a and b, $a \leqq b$. Then the total Segre class of* A ⊗ B *is given by*

$$s.(\mathrm{A} \otimes \mathrm{B}) = \sum_{p=0}^{\infty} s_p(\mathrm{A} \otimes \mathrm{B}) = \sum_{\mathrm{I},\mathrm{J}} \mathrm{D}_{\mathrm{I},\mathrm{J}}^{a,b}\, s_{\mathrm{I}}(\mathrm{A})\, s_{\mathrm{J}}(\mathrm{B}),$$

where the sum is over all partitions $\mathrm{I}=(i_1, \ldots, i_a)$, $\mathrm{J}=(j_1, \ldots, j_a)$.

Lemmas 5.1 and 5.6 yield an algorithm for computation of the Chern numbers of $\mathrm{D}=\mathrm{D}_r(\varphi)$. Let $\mathrm{T_X}$ be the tangent bundle on X and $\mathrm{T_D}$ be the tangent bundle on D. According to Remark 5.2 we want to calculate

$$(i_r)_*[\prod_i c_i(\mathrm{T_D})^{\alpha_i}].$$

This expression can be rewritten as

$$(i_r)_*[\sum_{\mathrm{I}} d_{\mathrm{I}}\, s_{\mathrm{I}}(\mathrm{T_D})] \tag{18}$$

where d_{I} come from Pieri's formula. Then the exact sequence $0 \to \mathrm{T_D} \to \mathrm{T_{X\,|D}} \to \mathrm{N_X(D)} \to 0$ allows us to rewrite (18) as

$$(i_r)_*[\sum d'_{\mathrm{K,L}}\, s_{\mathrm{K}}(\mathrm{T_{X\,|D}})\, s_{\mathrm{L}}\, \mathrm{N_X(D)}] \tag{19}$$

where $d'_{\mathrm{K,L}}$ can be obtained from d_{I} and from the universal coefficients appearing in the linearity formula for $s_{\mathrm{I}}(\mathrm{T_{X\,|D}}-\mathrm{N_X(D)})$. Since $\mathrm{N_X(D)}=\mathrm{K}^{\vee} \otimes \mathrm{C}$ (*see* [G-G] p. 145) the expression (19) can be replaced by

$$(i_r)_*[\sum d''_{\mathrm{K,M,N}}\, s_{\mathrm{K}}(\mathrm{T_{X\,|D}})\, s_{\mathrm{M}}(-\mathrm{K})\, s_{\mathrm{N}}(\mathrm{C})]$$

where $d''_{\mathrm{K,M,N}}$ are computable from $d'_{\mathrm{K,L}}$ with the help of Lemma 5.6. Finally, by Lemma 5.1 we obtain the equality

$$(i_r)_*(\prod_i c_i(\mathrm{T_D})^{\alpha_i}) = \sum d''_{\mathrm{K,M,N}}\, s_{(m-r)^{n-r}+\mathrm{N,M}}(\mathrm{E}-\mathrm{F})\, s_{\mathrm{K}}(\mathrm{T_X})$$

A particularly simple case is the computation of the Euler-Poincaré characteristic of $\mathrm{D}_r(\varphi)$.

PROPOSITION 5.7. — *Assume that $m \geqq n$. Then the* (*topological*) *Euler-Poincaré characteristic of the smooth degeneracy locus* $\mathrm{D}_r(\varphi)$ *is given by the expression*

$$\sum_{\mathrm{I},\mathrm{J}} (-1)^{|\mathrm{I}|+|\mathrm{J}|}\, \mathrm{D}_{\mathrm{I},\mathrm{J}}^{n-r,\, m-r}\, s_{(m-r)^{n-r}+\mathrm{I},\,\mathrm{J}^{\sim}}(\mathrm{E}-\mathrm{F})\, c_{d-|\mathrm{I}|-|\mathrm{J}|}(\mathrm{X})$$

where the sum is over all partitions $I=(i_1, \ldots, i_{n-r})$, $J=(j_1, \ldots, j_{n-r})$. *Here* $c_k(X)$ *denotes the k-th Chern class of* TX *and if* $k<0$ *we define* $c_k(X)$ *to be zero.*

Proof. — By the Gauss-Bonnet theorem the Euler-Poincaré characteristic of $D=D_r(\varphi)$ is equal to $c_{top}(T_D) \cap [D]$. Therefore we have to calculate $(i_r)_*(c_{top} T_D)$. We follow the notation and the strategy described above. The expression in question is equal to

$$(i_r)_*[c_d(TX|_D - K^\vee \otimes C)] = (i_r)_*\left[\sum_{i=0}^{d} (-1)^i s_i(K^\vee \otimes C)\, i_r^* c_{d-i}(X)\right)$$

$$= (i_r)_*[\sum_{I,J} (-1)^{|I|+|J|} D_{I,J}^{n-r, m-r} s_I(C)\, s_{J^\sim}(-K)\, i_r^* c_{d-|I|-|J|}(X)]$$

$$= \sum_{I,J} (-1)^{|I|+|J|} D_{I,J}^{n-r, m-r} s_{(m-r)^{n-r}+I, J^\sim}(E-F)\, c_{d-|I|-|J|}(X)$$

where the sum is over all partitions $I=(i_1, \ldots, i_{n-r})$ and $J=(j_1, \ldots, j_{n-r})$. ■

Example 5.8. — (i) The Euler-Poincaré characteristic of a smooth determinantal curve is given by the expression:

$$s_{(m-r)^{n-r}}(E-F)\, c_1(X) - (n-r)\, s_{(m-r)^{n-r}, (1)}(E-F) - (m-r)\, s_{(m-r)^{n-r}+(1)}(E-F).$$

(ii) The Euler-Poincaré characteristic of a smooth determinantal surface is equal to:

$$s_{(m-r)^{n-r}}(E-F)\, c_2(X) - [(n-r)\, s_{(m-r)^{n-r}, (1)}(E-F) + (m-r)\, s_{(m-r)^{n-r}+(1)}(E-F)]\, c_1(X)$$
$$+\binom{n-r+1}{2} s_{(m-r)^{n-r}, (1, 1)}(E-F) + \binom{m-r+1}{2} s_{(m-r)^{n-r}+(2)}(E-F)$$
$$+\binom{n-r}{2} s_{(m-r)^{n-r}, (2)}(E-F) + \binom{m-r}{2} s_{(m-r)^{n-r}+(1, 1)}(E-F)$$
$$+[(m-r)(n-r)+1]\, s_{(m-r)^{n-r}+(1), (1)}(E-F).$$

6. The structure of the ideal $\mathscr{I}_r$

The ideal $\mathscr{I}_r$ of $\mathbb{Z}[c.(A), c.(B)]$ admits various interpretations. In Section 3 a geometric interpretation of $\mathscr{I}_r$ was discussed. In [P2] we have interpreted the ideal $\mathscr{I}_r$ as a generalization of the resultant in elimination theory. Therefore $\mathscr{I}_r$ seems to be an interesting object and its algebraic structure is worth studying. In *loc. cit.* we proved that $\mathscr{I}_r$ is a prime ideal. In this section, as a by-product of the previous geometrical considerations we obtain some informations concering sets of generators and a $\mathbb{Z}$-basis of the ideal $\mathscr{I}_r$.

PROPOSITION 6.1. — (a) *The ideal* $\mathscr{I}_r$ *is generated by Schur S-polynomials* $s_{(m-r)^{n-r}+I}(A-B)$, *where* $I \subset (r)^{n-r}$.

(b) *The ideal* $\mathscr{I}_r$ *is generated by Schur S-polynomials* $s_{(m-r)^{n-r}, J}(A-B)$, *where* $J \subset (m-r)^r$.

Proof. — (*a*) This follows immediately from $\mathscr{I}_r \subset \mathscr{P}_r$, Lemmas 3.9 and 3.11, by letting $v, w \to \infty$. Indeed if v, w tend to the infinity then $A^{\cdot}(X)$ becomes $\underline{\mathrm{Sym}}(A, B)$ and $\mathscr{I}_r(E, F)$ becomes $\mathscr{I}_r$.

(*b*) The proof can be carried out in a similar way but instead of the construction (14) one needs to consider the following one (*cf.* [F] chap. 14)

$$\begin{array}{ccc} Z = \mathrm{Zeros}(R \to F_G \to E_G) & \to & G = G_{n-r}(F) \\ \downarrow & & \downarrow \\ D_r(\varphi) & \xrightarrow{i_r} & X \end{array} \qquad \blacksquare$$

We use the notation introduced to describe the construction (13). By Lemma 4.1 we obtain the following presentation of $A^{\cdot}(D_r - D_{r-1})$ in the generic situation (13):

$$(20) \qquad A^{\cdot}(D_r - D_{r-1}) \simeq A^{\cdot}(F\,l^{m,r}(V) \times F\,l_{r,n}(W))/\mathscr{J}_r$$

where $F\,l^{m,r}(V)$ is the flag manifold parametrizing the flags of rank m, rank r quotients of V, $F\,l_{r,n}(W)$ is the flag manifold parametrizing the flags of rank r, rank n subspaces of W. Moreover $\mathscr{J}_r$ is the ideal generated by elements of the form $s_I(R^r_W) - s_I(Q^r_V)$, where $|I| \geqq 1$ and R^r_W, Q^r_V denote the corresponding tautological bundles on $F\,l_{r,n}(W)$ and $F\,l^{m,r}(V)$.

Let A_r, A^{n-r}, B_{m-r}, B^r be four sets of algebraically independent elements over $\mathbb{Z}$ of cardinality r, $n-r$, $m-r$ and r respectively.

PROPOSITION 6.2. — *The polynomials* $s_{I_i}(A-B)\,s_{J_i}(A)$ *where* I_i *contains the partition* $(m-i)^{n-i}$ *and does not contain the partition* $(m-i+1)^{n-i+1}$, $l(J_i) \leqq i$, $i=0, 1, \ldots, r$, *form a* $\mathbb{Z}$*-basis of* $\mathscr{I}_r$. *Another* $\mathbb{Z}$*-basis of* $\mathscr{I}_r$ *is given by* $s_{I_i}(A-B)\,s_{J_i}(B)$ *for the same* I_i, $l(J_i) \leqq i$, $i=0, 1, \ldots, r$.

Proof. — Consider the exact sequence $(\gamma = (m-r)(n-r))$

$$(21) \quad A^{k-\gamma}(D_r(v, w) - D_{r-1}(v, w)) \to A^k(X_{v,w} - D_{r-1}(v, w)) \to A^k(X_{v,w} - D_r(v, w)) \to 0.$$

Let us identify A_r, A^{n-r}, B_{m-r} and B^r with the Chern roots of the following vector bundles R^r_W, R^n_W/R^r_W, $\ker(Q^m_V \to Q^r_V)$ and Q^r_V, if v, w tend to the infinity. By (20), if $v, w \to \infty$, then (21) gives the exact sequence

$$S/\mathscr{J}_r \xrightarrow{\alpha} R/\mathscr{I}_{r-1} \to R/\mathscr{I}_r \to 0$$

where $R = \underline{\mathrm{Sym}}(A, B)$, $S = \underline{\mathrm{Sym}}(B_{m-r}, B^r, A_r, A^{n-r})$ and $\mathscr{J}_r$ is the ideal generated by all elements of the form $s_I(A_r) - s_I(B_r)$, $|I| \geqq 1$.

Claim. — α is a monomorphism.

To see it, consider the ring homomorphism $\beta: R/\mathscr{I}_{r-1} \to S/\mathscr{J}_r$ induced by the injection $R \to S$. If $v, w \to \infty$ then the self-intersection formula applied to the inclusion $D_r(v, w) - D_{r-1}(v, w) \to X_{v,w} - D_{r-1}(v, w)$ gives $\beta\alpha(s) = s_{(m-r)^{n-r}}(A^{n-r} - B_{m-r}) \cdot s$, where $s \in S/\mathscr{J}_r$. The claim now follows from the easy observation that a polynomial which is not in the ideal generated by Schur S-polynomials in $A_r - B^r$ cannot belong to this ideal after multiplication by $s_{(m-r)^{n-r}}(A^{n-r} - B_{m-r})$.

In particular we obtain an isomorphism $\mathscr{I}_r/\mathscr{I}_{r-1} \simeq S/\mathscr{I}_r$ of abelian groups. Observe that $S/\mathscr{I}_r$ is a free abelian group with a $\mathbb{Z}$-basis given by polynomials of the form $s_J(-B_{m-r}) s_K(A_r) s_I(A^{n-r})$, where $l(J^{\sim}) \leqq m-r$, $l(K) \leqq r$, $l(I) \leqq n-r$.

It was essentially proved in Lemma 5.1 that

$$\alpha[s_I(A^{n-r}) s_J(-B_{m-r})] = s_{(m-r)^{n-r}+I,\,J}(A-B). \tag{22}$$

It follows easily from the projection formula applied to the inclusion $D_r(v, w) - D_{r-1}(v, w) \to X_{v,w} - D_{r-1}(v, w)$ that α is a morphism of R-modules. Thus

$$\begin{aligned}\alpha[s_J(-B_{m-r}) s_K(A_r) s_I(A^{n-r})] &= \alpha[s_J(-B_{m-r}) s_K(A - A^{n-r}) s_I(A^{n-r})]\\ &= \alpha[\sum_{L \subset K} (-1)^{|L|} s_J(-B_{m-r}) s_{K/L}(A) s_{L^{\sim}}(A^{n-r}) s_I(A^{n-r})]\\ &= \alpha[\sum_{L \subset K} \sum_{M} (-1)^{|L|} (L^{\sim}, I; M) s_J(-B_{m-r}) s_{K/L}(A) s_M(A^{n-r})]\\ &= \sum_{L \subset K} \sum_{M} (-1)^{|L|} (L^{\sim}, I; M) s_{(m-r)^{n-r}+M,\,J}(A-B) s_{K/L}(A)\end{aligned}$$

where $(L^{\sim}, I; M) \in \mathbb{Z}$. This last expression can be rewritten as

$$s_{(m-r)^{n-r}+I,\,J}(A-B) s_K(A) + \sum_{K' \subset K} d_{K'}(A-B) s_{K'}(A) \tag{23}$$

where $d_{K'}(A-B)$ are $\mathbb{Z}$-combinations of Schur S-polynomials $s_T(A-B)$ where $T \supset (m-r)^{n-r}$ and $T \not\supset (m-r+1)^{n-r+1}$.

It follows from (22) and (23) by induction on $|K|$, that $s_I(A-B) s_J(A)$, where $I \supset (m-r)^{n-r}$ and $I \not\supset (m-r+1)^{n-r+1}$, $l(J) \leqq r$, generate the image of α. Since the isomorphism in question $S/\mathscr{I}_r \xrightarrow{\simeq} \mathscr{I}_r/\mathscr{I}_{r-1}$ is induced by α, the above elements generate $\mathscr{I}_r/\mathscr{I}_{r-1}$.

To show a $\mathbb{Z}$-linear independence of these elements we use the specialization $B^r = A_r$. Indeed, by the factorization formula and the linearity formula we easily get that

$$\begin{aligned}s_{(m-r)^{n-r}+I,\,J}(A-B) s_K(A) &= s_{(m-r)^{n-r}}(A^{n-r} - B_{m-r}) s_I(A^{n-r}) s_K(A) s_J(-B_{m-r})\\ &= s_{(m-r)^{n-r}}(A^{n-r} - B_{m-r}) [s_I(A^{n-r}) s_K(A_r) s_J(-B_{m-r})\\ &\qquad + \sum_{|K'| < |K|} d_{T,K',J}\, s_T(A^{n-r}) s_{K'}(A_r) s_J(-B_{m-r})]\end{aligned}$$

where $d_{T,K',J} \in \mathbb{Z}$. Since the different elements of the form $s_I(A^{n-r}) s_K(A_r) s_J(-B_{m-r})$ are $\mathbb{Z}$-linearly independent, the $\mathbb{Z}$-linear independence in question follows.

The final assertion now follows by induction on r, the case $r=0$ being an immediate consequence of the factorization formula.

This finishes the proof of Proposition 6.2. ■

7. The symmetric and antisymmetric form case

The aim of this Section is to develop a theory similar to the one in Sections 3, 4, 5, 6 in the case of degeneracy loci associated with symmetric (resp. antisymmetric) morphisms of vector bundles. We will follow the main lines of the quoted Sections; and the arguments which are analogous to the corresponding ones given in previous sections will be just sketched.

Let $\varphi: E^\vee \to E$ be a symmetric (resp. antisymmetric) morphism of vector bundles on a scheme X. Assume that rank $E=n$. Let r be a nonnegative integer; if φ is antisymmetric we assume that r is even. Let $q=n-r$. Consider the Grassmannian bundle $\pi: G=G_r(E)=G^q(E)\to X$ endowed with the tautological sequence

$$0\to R\to E_G\to Q\to 0.$$

Apply to the morphism $\varphi: E^\vee\to E$ the geometric construction (14)

$$\begin{array}{ccc} Z=\mathrm{Zeros}(E_G^\vee \overset{\varphi_G}{\to} E_G\to Q) & \underset{i'_r}{\to} & G=G_r(E) \\ \downarrow \rho & & \downarrow \pi \\ D_r=D_r(\varphi) & \underset{i_r}{\to} & X \end{array}$$

Here i_r, i'_r are the inclusions, π is the canonical projection, ρ is the restriction of π to Z.

LEMMA 7.1. — *Let* $\alpha\in A(X)$.

(i) *If* φ *is symmetric, then for any partition* I, $l(I)\leqq n-r$,

$$Q_{\rho_{n-r}+I}(E)\cap\alpha\in \mathrm{Im}(i_r)_*.$$

(ii) *If* φ *is antisymmetric, then for any partition* I, $l(I)\leqq n-r$,

$$P_{\rho_{n-r-1}+I}(E)\cap\alpha\in \mathrm{Im}(i_r)_*.$$

(Recall that $\rho_k=(k, k-1, \ldots, 2, 1)$).

Proof. — (i) Since φ is symmetric, the section $G\to E_G\otimes Q$ induced by $E_G^\vee \overset{\varphi_G}{\to} E_G\to Q$ is in fact a section of $H=\mathrm{Ker}(E_G\otimes Q\to\Lambda^2 Q)$. Observe that in K(G) we have $[E\otimes Q-\Lambda^2 Q]=[R\otimes Q+S_2 Q]$. By Lemma 1.12, $c_{\mathrm{top}}(S_2 Q)=2^q s_{\rho_q}(Q)$. Now the proof is the same mutatis mutandis as the one of Lemma 3.2; we use factorization property from Lemma 1.13 instead of Lemma 1.1, and Proposition 2.8 instead of Proposition 2.2.

(ii) The proof is the same. ■

Let $\mathbb{Z}[c.(A)]=\mathbb{Z}[c_1(A), \ldots, c_n(A)]$ be a graded polynomial $\mathbb{Z}$-algebra where $\deg c_k(A)=k$. Let $\mathscr{P}_r^s$ (resp. $\mathscr{P}_r^{as}$, r-even) be the ideal of all polynomials in $\mathbb{Z}[c.(A)]$ such that for every symmetric (resp. antisymmetric) morphism $\varphi: E^\vee\to E$ of vector bundles on an arbitrary scheme X and any $\alpha \in A(X)$

$$P(c.(E)) \cap \alpha \in \operatorname{Im}(i_r)_*.$$

$c.(E)$ denotes here the Chern classes of the bundle E.

Now let $A=(a_1, \ldots, a_n)$ be a sequence of algebraically independent elements over $\mathbb{Z}$. The assignment

$$c_k(A) \mapsto (k\text{-th elementary symmetric function in } A)$$

allows us to identify $\mathbb{Z}[c.(A)]$ with $\underline{\mathrm{Sym}}(A)$ and to treat $\mathscr{P}_r^s$ and $\mathscr{P}_r^{as}$ as ideals of $\underline{\mathrm{Sym}}(A)$.

THEOREM 7.2. — (i) *The ideal $\mathscr{P}_r^s$ of $\underline{\mathrm{Sym}}(A)$ is generated by Schur Q-polynomials $Q_I(A)$ where I ranges over all strict partitions $I \supset \rho_{n-r}$.*

(ii) *The ideal $\mathscr{P}_r^{as}$ of $\underline{\mathrm{Sym}}(A)$ is generated by Schur P-polynomials $P_I(A)$ where I ranges over all strict partitions $I \supset \rho_{n-r-1}$ (r-even).*

Let $\mathscr{I}_r^s$ (resp. $\mathscr{I}_r^{as}$) be the ideal generated by all Schur Q-polynomials (resp. P-polynomials) $Q_I(A)$ where $I \supset \rho_{n-r}$ (resp. $P_I(A)$ where $I \supset \rho_{n-r-1}$).

It follows from Lemma 7.1 applied for $i=0, \ldots, r$ instead of r itself that $\mathscr{I}_r^s \subset \mathscr{P}_r^s$ and $\mathscr{I}_r^{as} \subset \mathscr{P}_r^{as}$. We will give now a proof of the inclusion $\mathscr{P}_r^s \subset \mathscr{I}_r^s$ which uses no geometric constructions over $D_r(\varphi)$. Consider the following situation. Let V be a vector space over a field K of dimension $v > n$. Let $G_n = G_n(V)$ be the Grassmannian of n-subspaces of V. Let R be the tautological (sub)bundle on $G_n(V)$. Let

$$X = S_2(R),\ E = R_X. \tag{24}$$

On X we have the canonical (tautological) morphism $\varphi: E^\vee \to E$. Let $D_r = D_r(\varphi)$. Observe that by Thom isomorphism we have $A^\cdot(X) \simeq A^\cdot(G_n)$; moreover

1) The morphism φ is given locally by a symmetric $n \times n$ matrix of variables.

2) Every element $s_I(E)$, $l(I) \leqq n$ is non-zero for $v \gg 0$ and every finite set $\{s_{I_1}(E), \ldots, s_{I_k}(E)\}$, $I_p \neq I_q$ if $p \neq q$, becomes a family of $\mathbb{Z}$-linearly independent elements for $v \gg 0$.

We will prove by induction on r (n can vary), that for $v \gg 0$ $\operatorname{Im}(i_r)_* = \mathscr{I}_r(E)$, where $\mathscr{I}_r(E)$ is the ideal in $A(X)$ generated by all polynomials $Q_I(E)$ where $I \supset \rho_{n-r}$. This implies our assertion, since we do not lose any of the polynomials from $\mathscr{P}_r$ in this counting because of 2).

D_0 can be identified with G_n imbedded by the zero section in X. Therefore, by Lemma 3.6 we get that $\operatorname{Im}(i_0)_*$ as an $A(X)$-module is generated by $c_{\mathrm{top}}(S_2 E) = Q_{\rho_n}(E)$.

To make the inductive step $r-1 \to r$ we consider a commutative diagram

$$\begin{tikzcd} A.(D_{r-1}) \arrow[r] \arrow[dr, "(i_{r-1})_*"'] & A.(D_r) \arrow[r, "k_r^*"] \arrow[d, "(i_r)_*"] & A.(D_r - D_{r-1}) \arrow[r] & 0 \\ & A.(X) & & \end{tikzcd} \tag{25}$$

where the row is the corresponding exact sequence and $k_r: D_r - D_{r-1} \to D_r$ is the inclusion. From Schubert Calculus we know that $A(X)$ is generated by polynomials in the Chern classes of E and F. Thanks to the theory developed in [F], which allows us to treat polynomials in Chern classes as operators on $A.(\ )$, we can treat the above diagram as a diagram of $A(X)$-modules.

PROPOSITION 7.3. — *For $v \gg 0$ there exist elements $x_I \in A(D_r)$, where* I *ranges over all partitions contained in $(r)^{n-r}$, satisfying the following conditions*

(i) $(i_r)_*(x_I) = Q_{\rho_{n-r}+I}(E)$.

(ii) *The elements $(k_r)^*(x_I)$ generate the* $A(X)$*-module* $A(D_r - D_{r-1})$.

Observe that this Proposition implies that $\operatorname{Im}(i_r)_* = \mathscr{I}_r(E)$. Indeed, reasoning by induction on r we assume that $\operatorname{Im}(i_{r-1})_*$ is generated by $Q_{\rho_{n-i}+I_i}(E)$ where $I_i \subset (i)^{n-i}$ and $i=0, \ldots, r-1$. The above Proposition applied to the exact sequence (25) gives us then, that $\operatorname{Im}(i_r)_*$ is generated by $Q_{\rho_{n-i}+I_i}(E)$ where $I_i \subset (i)^{n-i}$ and $i=0, \ldots, r$. But all these elements belong to $\mathscr{I}_r(E)$. Therefore $\operatorname{Im}(i_r)_* = \mathscr{I}_r(E)$. To prove Proposition 7.3 consider arbitrary elements $x_I \in A.(D_r)$ satisfying (i). Their existence follows from Lemma 7.1. Let K and C be the kernel and cokernel bundles of the morphism φ restricted to $D_r - D_{r-1}$. Since φ is symmetric we have $K^\vee \simeq C$.

LEMMA 7.4:

$$Q_{\rho_{n-r}}(C) \cdot [k_r^*(x_I) - s_I(C)] = 0 \quad \textit{in } A(D_r - D_{r-1}).$$

Proof. — Consider the following cartesian square

$$\begin{array}{ccc} D_r & \overset{k_r}{\hookleftarrow} & D_r - D_{r-1} \\ \downarrow{\scriptstyle i_r} & & \downarrow{\scriptstyle i'_r} \\ X & \overset{l_r}{\hookleftarrow} & X - D_{r-1} \end{array}$$

By [F] Proposition 1.7, we infer that $l_r^* i_{r*} = (i'_r)_* k_r^*$. Thus $(i'_r)^* l_r^* i_{r*}(x_I) = (i'_r)^* (i'_r)_* k_r^*(x_I)$. We have

$(i'_r)^* l_r^* i_{r*}(x_I) = (i'_r)^* l_r^*(Q_{\rho_{n-r}+I}(E))$ (by the definition of x_I)

$= Q_{\rho_{n-r}+I}(C)$ (by Lemma 1.9 (ii) and 1.10, because $E = \operatorname{Im}\varphi \oplus C$ and $\operatorname{Im}\varphi = (\operatorname{Im}\varphi)^\vee$)

$= Q_{\rho_{n-r}}(C) \cdot s_I(C)$ (by Lemma 1.13).

On the other hand denoting by N the normal bundle $N_{X-D_{r-1}}(D_r - D_{r-1}) = S_2 C$ (*cf.* [G-G]) we have

$(i'_r)^* (i'_r)_* k_r^*(x_I) = c_{\mathrm{top}} N \cdot k_r^*(x_I)$ (by the self-intersection formula)

$= c_{\mathrm{top}}(S_2 C) \cdot k_r^*(x_I)$

$= Q_{\rho_{n-r}}(C) \cdot k_r^*(x_I)$ (by Lemma 1.12).

This proves the Lemma. ■

Notice that all elements $k_r^*(x_1)$ are in codimension-graded components $A^i(D_r - D_{r-1})$ where $i \leqq \dim G_r(n) = N(n, r)$, say, *i.e.* i is bounded by a number which does not depend on v.

The next two facts require an analysis of a geometry of $D_r - D_{r-1}$. A point in $D_r - D_{r-1}$ is a pair $(N, f: N^\vee \to N)$ where N is a n-dimensional subspace of V and $f: N^\vee \to N$ is a linear, symmetric map of rank r. To this point we assign the point $(\operatorname{Im} f \subsetneq N)$ in $Fl_{r,n}(V)$ where $Fl_{r,n}(V)$ is the flag manifold parametrizing the flags of rank r, rank n subspaces of V. This makes $D_r - D_{r-1}$ a locally trivial fibration with fiber isomorphic to the set of nondegenerate symmetric $r \times r$ matrices. More precisely if R^r is the tautological bundle of rank r on $Fl_{r,n}$ then $D_r - D_{r-1}$ is the open complement of the $r-1$-th degeneracy locus $D_{r-1}(\varphi')$ associated with the tautological morphism $\varphi': (R^r)^\vee_{X'} \to (R^r)_{X'}$ on $X' = (S_2 R^r)_{Fl_{r,n}}$. In other words we have $D_r - D_{r-1} \simeq X' - D_{r-1}(\varphi')$. Consider now the Grassmannian $G_r = G_r(V)$. Denote the corresponding tautological bundle on G_r by R^r for short. Then for the tautological morphism $\varphi'': (R^r)^\vee_{X''} \to (R^r)_{X''}$ on $X'' = S_2(R^r)_{G_r}$, we know the image of the map $A.(D_{r-1}(\varphi'')) \to A.(X'')$ *by our inductive hypothesis.* Namely this image is generated by all Schur Q-polynomials in R^r. Let now

$$p: X' \to X'' \text{ and } p_D: D_{r-1}(\varphi') \to D_{r-1}(\varphi'')$$

be the natural projections. We have the following cartesian square

$$\begin{array}{ccccccc} D_{r-1}(\varphi') & \xrightarrow{i'} & X' & & D_{r-1}(\varphi') & \xrightarrow{i'} & G \\ \downarrow{\scriptstyle p_D} & & \downarrow{\scriptstyle p} & = & \downarrow{\scriptstyle p_D} & & \downarrow{\scriptstyle p} \\ D_{r-1}(\varphi'') & \xrightarrow{i''} & X'' & & D_{r-1}(\varphi'') & \xrightarrow{i''} & X'' \end{array}$$

where $p: G \to X''$ is the Grassmannian bundle

$$G_{n-r}(V_{X''}/(R^r)_{X''}).$$

Thus $A(X')$ (resp. $A(D_{r-1}(\varphi'))$) is a free-module (resp. $A(D_{r-1}(\varphi''))$-module) via p (resp. via p_D). Moreover there exist bases $\{\Omega_k^D\}$, $\{\Omega_k\}$ such that $A(X') = \bigoplus_k A(X'')\,\Omega_k$, $A(D_{r-1}(\varphi')) = \bigoplus_k A(D_{r-1}(\varphi''))\,\Omega_k^D$ and $(i')_*(\Omega_k^D) = \Omega_k$ for every k. Consider the following commutative diagram (*see* [F] Proposition 1.7)

$$\begin{array}{ccc} A(D_{r-1}(\varphi')) & \xrightarrow{(i')_*} & A(X') \\ \uparrow{\scriptstyle p_D^*} & & \uparrow{\scriptstyle p^*} \\ A(D_{r-1}(\varphi'')) & \xrightarrow{(i'')_*} & A(X'') \end{array}$$

We have $(i')_*(\sum_k p_D^*(d_k)\Omega_k^D)=\sum_k p^* i''_* \ (d_k) i'_* \ \Omega_k^D=\sum_k p^* i''_* \ (d_k)\Omega_k$ in $A(X')$. This implies that $\mathrm{Im}(i'_*)$ is generated by $p^*(\mathrm{Im}\, i''_*)$ *i.e.* by all Schur Q-polynomials $Q_I((R^r)_{X'})$, I-strict, $|I|\geqq 1$.

LEMMA 7.5. — *If* $v \gg 0$, *then multiplication*

$$\cdot\, Q_{\rho_{n-r}}(C): A^i(D_r-D_{r-1}) \to A^{i-\gamma^s}(D_r-D_{r-1}),$$

is a monomorphism for $i \leqq N(n, r)$, $(\gamma^s=(n-r)(n-r+1)/2)$.

Proof. — From the above description and from the exact sequence

$$A(D_{r-1}(\varphi')) \to A(X') \to A(D_r-D_{r-1}) \to 0$$

where $A(X') \simeq A(Fl_{r,n})$ by Thom isomorphism, we get

$$A(D_r-D_{r-1}) \simeq A(Fl_{r,n})/(Q_I(R^r), |I| \geqq 1). \tag{26}$$

Under the above identification $C=(R^n/R^r)_{D_r-D_{r-1}}$. Let A_r, A^{n-r} be two sets of algebraically independent elements over $\mathbb{Z}$ of the cardinality r and $n-r$ respectively. The assignment $s_I A_r . s_J A^{n-r} \mapsto s_I R^r . s_J(R^n/R^r)$, gives a ring homomorphism from the ring

$$\underline{\mathrm{Sym}}(A_r, A^{n-r})/(Q_I(A_r), |I| \geqq 1) \tag{27}$$

to (26). Here $\underline{\mathrm{Sym}}(A_r, A^{n-r})$ is the ring of partially symmetric polynomials in two distinguished sets of variables. If $v \gg 0$ then the components of degree $\leqq N(n, r)$ in (26) and (27) are isomorphic. Thus it suffices to prove that multiplication by $Q_{\rho_{n-r}}(A^{n-r})$ in (27) is a monomorphism. But the polynomial which is not in the ideal generated by Q-polynomials in A_r of positive degree, cannot belong to this ideal after multiplication by $Q_{\rho_{n-r}}(A^{n-r})$ (*e.g.* consider the specialization $\{A_r\}=\{A_r^-\}$ and use Lemma 1.10). ■

Comparing Lemma 7.4 and 7.5 we see that if $v \gg 0$ then $k_r^*(x_I)=s_I(C)$. Thus to end the proof of Proposition 7.3 we need

LEMMA 7.6. — *The elements* $s_I(C)$, *where* $I \subset (r)^{n-r}$, *generate the* $A(X)$-*module* $A(D_r-D_{r-1})$.

Proof. — Recall that the $A(X)$-module structure on $A(D_r-D_{r-1})$ is given by the action of polynomials in the Chern classes of E. Consider the following map

$$\alpha: A(X)=A(G_n) \overset{(q)^*}{\to} A(Fl_{r,n})=A(X') \overset{(l)^*}{\to} A(D_r-D_{r-1})$$

where $q: Fl_{r,n} \to G_n$ is the projection and $l: D_r-D_{r-1} \to X'$ is the injection. It is easy to see that $\alpha(c_i(E))=c_i(E_{D_r-D_{r-1}})$. Thus the above $A(X)$-structure on $A.(D_r-D_{r-1})$ is the same as the one defined as follows: for $x \in A(X)$, $d \in A(D_r-D_{r-1})$ the effect of the action of x on d is $\alpha(x)\cdot d$. Since the last $A(X)$-homomorphism $(l)^*$ is surjective, it suffices to prove that $A(X')$ as the $A(X)$-module is generated by $s_I[(Q^{n-r})_{X'}]$, where $I \subset (r)^{n-r}$. But $A(Fl_{r,n})$ as the $A(G_n)$-module is generated by $s_I(Q^{n-r})$ with the same I, by Schubert Calculus. This implies the desired assertion. ■

This completes the proof of Proposition 7.3 and thus also the proof of the inclusion $\mathscr{P}_r^s \subset \mathscr{I}_r^s$.

Remark 7.7. — By a similar method one can prove the inclusions $\mathscr{P}_r \subset \mathscr{I}_r$ and $\mathscr{P}_r^{as} \subset \mathscr{I}_r^{as}$ (r-even).

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Let $\varphi: E^\vee \to E$ be a symmetric morphism (resp. antisymmetric) of C^∞ complex vector bundles on a complex manifold X. Assume that $D_{r-1}(\varphi)=\varnothing$. Then the kernel K and the cokernel C of the morphism φ restricted to $D_r(\varphi)$ are vector bundles of rank $n-r$. It is easy to see that $K^\vee \cong C$. Assume that φ is general (*see* Section 5) and write $\dim D_r(\varphi)=d$.

PROPOSITION 7.8. — (i) *Assume that* $\varphi: E^\vee \to E$ *is symmetric. Then*

$$\operatorname{codim}_X D_r(\varphi)=(n-r)\,(n-r+1)/2=\gamma^s,$$

say. Let I *be a partition such that* $l(I) \leqq n-r$, $|I| \leqq d$. *Then for any* $\alpha \in H^{d-|I|}(X, \mathbb{Z})$

$$(i_r)_*[s_I(C)\cdot i_r^*\alpha]=Q_{\rho_{n-r}+I}(E)\,.\,\alpha.$$

(ii) *Assume that* $\varphi: E^\vee \to E$ *is antisymmetric. Then*

$$\operatorname{codim}_X D_r(\varphi)=(n-r-1)\,(n-r)/2=\gamma^{as},$$

say (r *is even*). *Let* I *be a partition such that* $l(I) \leqq n-r$, $|I| \leqq d$. *Then for any* $\alpha \in H^{d-|I|}(X, \mathbb{Z})$

$$(i_r)_*[s_I(C)\cdot i_r^*\alpha]=P_{\rho_{n-r-1}+I}(E)\cdot\alpha.$$

The proof is the same as the one of Lemma 5.1. We use the construction (14) instead of (11) and Proposition 2.8 instead of Proposition 3.1.

Let us fix integers $\alpha_1, \alpha_2, \ldots, \alpha_{n-r} \geqq 0$. Define nonnegative integers n_I by the identity

$$\prod_i c_i(C)^{\alpha_i}=\sum_I n_I\, s_I(C)$$

(compare the discussion before Proposition 5.3).

PROPOSITION 7.9. — *Assume that* $\sum i\alpha_i=d$. *With the notation as above,*

(i) *if* φ *is symmetric, then*

$$\prod c_i(C)^{\alpha_i} \cap [D_r(\varphi)]=\sum_I n_I\, Q_{\rho_{n-r}+I}(E),.$$

(ii) *if* φ *is antisymmetric, then*

$$\prod c_i(C)^{\alpha_i} \cap [D_r(\varphi)]=\sum_I n_I\, P_{\rho_{n-r-1}+I}(E).$$

The proof is analogous to the one of Proposition 5.2.

Example 7.10. — $d=2$, φ is symmetric.

$$c_2(C) \cap [D_r(\varphi)] = Q_{\rho_{n-r}+(1,1)}(E)$$
$$c_1^2(C) \cap [D_r(\varphi)] = Q_{\rho_{n-r}+(2)}(E) + Q_{\rho_{n-r}+(1,1)}(E).$$

For every sequence $(j_1, j_2, \ldots, j_a)$ where $j_1 > j_2 > \ldots j_a \geqq 0$ define the number $((j_1, j_2, \ldots, j_a))$ inductively as follows:

1) $((1, 0)) = 1$

2) $a((j_1, j_2, \ldots, j_a)) - 2\sum_k ((j_1, \ldots, j_k - 1, \ldots, j_a))$

$$= \begin{cases} 0 & \text{if } j_a > 0 \\ ((j_1 - 1, \ldots, j_{a-1} - 1)) & \text{if } j_a = 0. \end{cases}$$

We assume the terms with $j_k - 1 = j_{k+1}$ in the above summation to be zero.

Moreover, for every sequence $(j_1, j_2, \ldots, j_a)$ where $a = 2b$ is even and $j_1 > j_2 > \ldots > j_a \geqq 0$ define $[j_1, \ldots, j_a]$ inductively as follows

(28)

1) $[1, 0] = 1$

2) $b[j_1, \ldots, j_{2b}] - \sum_k [j_1, \ldots, j_k - 1, \ldots, j_{2b}]$

$$= \begin{cases} 0 & \text{if } (j_{2b-1}, j_{2b}) \neq (1, 0) \\ [j_1, \ldots, j_{2b-2}] & \text{if } (j_{2b-1}, j_{2b}) = (1, 0). \end{cases}$$

We assume the terms with $j_{k+1} = j_k - 1$ in the above summation to be zero. If a is odd, we put

$$[j_1, \ldots, j_a] = \begin{cases} [j_1, \ldots, j_{a-1}], & \text{if } j_a = 0 \\ 0, & \text{if } j_a \neq 0. \end{cases}$$

The following fact was communicated to me by A. Lascoux.

PROPOSITION 7.11 ([L-L-T]). — *Let* A *be a vector bundle of rank* a.

(i) *The total Segre class of* $S_2 A$ *is given by*

$$s.(S_2 A) = \sum_I ((i_1 + a - 1, i_2 + a - 2, \ldots, i_a))\, s_I(A)$$

where the summation ranges over all partitions $I = (i_1, \ldots, i_a)$.

(ii) *The total Segre class of* $\Lambda^2 A$ *is given by*

$$s.(\Lambda^2 A) = \sum_I [i_1 + a - 1, i_2 + a - 2, \ldots, i_a]\, s_I(A)$$

where the summation ranges over all partitions $I = (i_1, \ldots, i_a)$.

By combining the above Proposition with computations due to Schubert (*see* [S]) one can obtain the following closed form expression for the coefficients $((j_1, \ldots, j_a))$ and $[j_1, \ldots, j_a]$.

PROPOSITION 7.12. — (i) *The coefficients* $((j_1, \ldots, j_a))$ *are given by*

$$((j_1, \ldots, j_a)) = \mathrm{P}f\left[\binom{j_p+j_q}{j_p}+\binom{j_p+j_q}{j_p-1}+\ldots+\binom{j_p+j_q}{j_q}\right] \quad (1 \leqq p < q \leqq a)$$

if a is even, and

$$((j_1, \ldots, j_a)) = \sum_{p=1}^{a} (-1)^p 2^{j_p} ((j_1, \ldots, \hat{j}_p, \ldots, j_a))$$

if a is odd.

(ii) *The coefficients* $[j_1, \ldots, j_a]$, *where* a *is even, are given by*

$$[j_1, \ldots, j_a] = \mathrm{P}f\,[(j_p+j_q-1)!\,(j_p-j_q)/j_p!\,j_q!] \quad (1 \leqq p < q \leqq a).$$

Proof. — Since (i) is essentially proved in [S] p. 180, we will only prove (ii). The arguments are, however, inspired by those of [S]. Define

$$[j_1, \ldots, j_a]' = \mathrm{P}f\,[(j_p+j_q-1)!\,(j_p-j_q)/j_p!\,j_q!] \quad (1 \leqq p < q \leqq a).$$

One checks readily that $[1, 0]'=1$ and $[j_1, j_2]'=[j_1-1, j_2]'+[j_1, j_2-1]'$. Therefore $[j_1, j_2]=[j_1, j_2]'$ for every j_1, j_2. To perform an induction step it is convenient to put $[j_1, j_2, \ldots, j_{2b-2}, 1, -1]=[j_1, j_2, \ldots, j_{2b-2}]$ and $[j_1, j_2, \ldots, j_{2b-2}, j_{2b-1}, -1]=0$ if $j_{2b-1} \neq 1$. Then (28) reads

$$b\,[j_1, \ldots, j_{2b}] = \sum_k [j_1, \ldots, j_k-1, \ldots, j_{2b}]. \tag{29}$$

By Laplace-type expansion for Pfaffians we have

$$[j_1, \ldots, j_{2b}]' \doteq \sum_{k=2}^{2b} [j_1, j_k]'\,[j_2, \ldots, j_{k-1}, j_{k+1}, \ldots, j_{2b}]'. \tag{30}$$

We use double induction on b and $\sum_\alpha j_\alpha$. By applying the relation (30) to each term of the right hand side of (29) we get by induction hypothesis

$$\begin{aligned} b\,[j_1, \ldots, j_{2b}] &= \sum_{k=2}^{2b} ([j_1-1, j_k]'+[j_1, j_k-1]')\,[j_2, \ldots, \hat{j}_k, \ldots, j_{2b}]' \\ &\quad + \sum_k \sum_l [j_1, j_l]'\,[j_2, \ldots, \hat{j}_l, \ldots, j_k-1, \ldots, j_{2b}]' \\ &= \sum_{k=2}^{2b} [j_1, j_k]'\,[j_2, \ldots, \hat{j}_k, \ldots, j_{2b}]' + (b-1) \sum_{l=2}^{2b} [j_1, j_l]'\,[j_2, \ldots, \hat{j}_l, \ldots, j_{2b}] \quad \text{(by (29))} \end{aligned}$$

$$=[j_1, \ldots, j_{2b}]' + (b-1) \sum_{l=2}^{2b} [j_1, j_l]' [j_2, \ldots, \hat{j_l}, \ldots, j_{2b}]'$$

(by (30) and induction hypothesis)

$$= b [j_1, \ldots, j_{2b}]' \quad \text{(by (31) again).}$$

This implies $[j_1, \ldots, j_{2b}] = [j_1, \ldots, j_{2b}]'$, as desired. ■

Propositions 7.8 and 7.12 yield an algorithm for a calculation of the Chern numbers of smooth degeneracy loci $D_r(\varphi)$ analogous to the one described in Section 5. In particular we obtain

PROPOSITION 7.13:

(i) *The Euler-Poincaré characteristic of a smooth degeneracy locus* $D_r(\varphi)$ (*of codimension* γ^s) *associated with a symmetric morphism* φ *is given by the expression*

$$\sum_{I} (-1)^{|I|} ((i_1+n-r-1, i_2+n-r-2, \ldots, i_{n-r})) Q_{\rho_{n-r}+I}(E) c_{d-|I|}(X),$$

where the summation ranges over all partitions $I=(i_1, \ldots, i_{n-r})$.

(ii) *The Euler-Poincaré characteristic of a smooth degeneracy locus* $D_r(\varphi)$ (*of codimension* γ^{as}, *r-even*) *associated with an antisymmetric morphism* φ *equals*

$$\sum_{I} (-1)^{|I|} [i_1+n-r-1, i_2+n-r-2, \ldots, i_{n-r}] P_{\rho_{n-r-1}+I}(E) c_{d-|I|}(X),$$

where the summation ranges over all partitions $I=(i_1, \ldots, i_{n-r})$.

The proof is the same as the one of Proposition 5.7. We use Proposition 7.8 instead of Lemma 5.1 and Proposition 7.12 instead of Lemma 5.6.

Example 7.14. — If $d=1$ then the Euler-Poincaré characteristic of $D_r(\varphi)$ is

(i) $Q_{\rho_{n-r}}(E) c_1(X) - (n-r+1) Q_{\rho_{n-r}+(1)}(E)$, if φ is symmetric

(ii) $P_{\rho_{n-r-1}}(E) c_1(X) - (n-r-1) P_{\rho_{n-r-1}+(1)}(E)$, if φ is antisymmetric (and r is even).

Let us notice the following analogs of Lemma 4.1 and Proposition 4.3. Let $M_n^s(K)$ (resp. $M_n^{as}(K)$) be the affine space of all $n \times n$ symmetric (resp. antisymmetric) matrices over a field K. Let D_r^s (resp. D_r^{as}, r-even) be the determinantal subscheme in $M_n^s(K)$ (resp. $M_n^{as}(K)$) defined by the vanishing of all $(r+1)$-order minors (resp. $(r+2)$-order subpfaffians).

PROPOSITION 7.15:

(i) $A^{\cdot}(D_r^s - D_{r-1}^s) \simeq A^{\cdot}(G_r(K^n))/(Q_I(R), |I| \geqq 1)$

(ii) $A^{\cdot}(D_r^{as} - D_{r-1}^{as}) \simeq A^{\cdot}(G_r(K^n))/(P_I(R), |I| \geqq 1)$

R *denotes here the tautological subbundle on the corresponding Grassmannians.*

The proof is the same as the one of Lemma 4.1.

One can generalize this fact to the case of universal (tautological) degeneracy loci of rank r in $S_2 E$ (resp. $\Lambda^2 E$) where E is a vector bundle on X. Let $\bar{D}_r^s \subset S_2 E$ and $\bar{D}_r^{as} \subset \Lambda^2 E$ be the corresponding universal (tautological) degeneracy loci.

PROPOSITION 7.16:

(i) $A^{\cdot}(\bar{D}_r^s - \bar{D}_{r-1}^s) \simeq A^{\cdot}(G_r(E))/Q_I(R), |I| \geqq 1)$

(ii) $A^{\cdot}(\bar{D}_r^{as} - \bar{D}_{r-1}^{as}) \simeq A^{\cdot}(G_r(E))/(P_I(R), |I| \geqq 1)$

R *is the tautological subbundle on* $G_r(E)$.

Finally, notice that by methods analogous to those which allowed us to prove Proposition 6.1 one can obtain the following finite sets of generators of the ideals $\mathscr{I}_r^s$, $\mathscr{I}_r^{as}$.

PROPOSITION 7.17. — (i) *The ideal* $\mathscr{I}_r^s$ *is generated by all Schur Q-polynomials of the form* $Q_{\rho_{n-r}+I}(A)$, *where* $I \subset (r)^{n-r}$.

(ii) *The ideal* $\mathscr{I}_r^{as}$ *(r-even) is generated by all Schur P-polynomials of the form* $P_{\rho_{n-r-1}+I}(A)$, *where* $I \subset (r)^{n-r}$.

8. Comments and open problems

(8.1). The main theorems in Section 3 and 7 were proved in the context of Chow groups. However, one can consider an anologue of the ideal $\mathscr{P}_r$, in other cohomology theories. The proof of the inclusion $\mathscr{I}_r \subset \mathscr{P}_r$ remains valid (*see* Remark 2.9).

PROBLEM. — Is it true that $\mathscr{P}_r = \mathscr{I}_r$, for other cohomology theories?

(8.2). The Giambelli-Thom-Porteous formula is valid if X is a Cohen-Macaulay scheme and $\operatorname{codim}_X D_r(\varphi) = (m-r)(n-r)$ (*cf.* [F] 14.4). The following example shows that the equality $\operatorname{Im}(i_r)_* = \mathscr{I}_r(E, F)$ can fail, if these assumptions are satisfied. Consider the construction (13) with $m = n \geqq 2$. Then $D_{n-1}(\varphi)$ is equal to $D_0(\tilde{\varphi})$, where $\tilde{\varphi} = \Lambda^{\mathrm{top}}\varphi : \tilde{F} = \Lambda^{\mathrm{top}} F \to \tilde{E} = \Lambda^{\mathrm{top}} E$. But the ideal $\mathscr{I}_0(\tilde{E}, \tilde{F})$ generated by $s_1(\tilde{E} - \tilde{F}) = s_1(E - F)$ is not equal to the ideal $\mathscr{I}_{n-1}(E, F)$ generated by all $s_I(E-F)$, $|I| \geqq 1$.

It would be interesting to characterize a class of morphisms φ for which $\operatorname{Im}(i_r)_* = \mathscr{I}_r(E, F)$.

(8.3). Let $\varphi : F \to E$ be a morphism of vector bundles on X. Assume that the both vector bundles E and F are filtred

$$F_1 \subset F_2 \subset \ldots \subset F_r = F$$

$$E = E_1 \twoheadrightarrow E_2 \twoheadrightarrow \ldots \twoheadrightarrow E_r.$$

Consider the locus

$$\Omega = \{x \in X, \dim \operatorname{Ker}(F_i(x) \subsetneq F(x) \xrightarrow{\varphi(x)} E(x) \twoheadrightarrow E_i(x)) \geqq i, \text{ for every } i\}.$$

By generalizing the formulas in [Po], [K-L] and [L] one can prove that for "sufficiently general" φ ($n_i = \operatorname{rank} E_i$, $m_j = \operatorname{rank} F_j$)

$$[\Omega] = \operatorname{Det}[c_{n_i - m_i + j}(E_i - F_j)] \qquad 1 \leqq i, j \leqq r.$$

We plan to study these loci from the point of view of the present paper elsewhere.

(8.4). Arguing as in Section 4 one can prove the chains of surjections:

(i) $A(G_r(K^n)) \twoheadrightarrow A(D_r^s) \twoheadrightarrow A(G_r(K^n))/(Q_I(R), |I| \geqq 1)$

(ii) $A(G_r(K^n)) \twoheadrightarrow A(D_r^{as}) \twoheadrightarrow A(G_r(K^n))/(P_I(R), |I| \geqq 1)$

where R is the tautological bundle on $G_r(K^n)$. This gives us certain insight in $A(D_r^s)$, $A(D_r^{as})$ but does not describe these groups explicitly. On the other hand, from the sequences

$$A(D_{r-1}^s) \to A(D_r^s) \twoheadrightarrow A(D_r^s - D_{r-1}^s),$$
$$A(D_{r-1}^{as}) \to A(D_r^{as}) \twoheadrightarrow A(D_r^{as} - D_{r-1}^{as})$$

it follows, that if $k < \dim D_r^s - \dim D_{r-1}^s$ (resp. $k < \dim D_r^{as} - \dim D_{r-1}^{as}$) then $A^k(D_r^s) \simeq A^k(D_r^s - D_{r-1}^s)$ (resp. $A^k(D_r^{as}) \simeq A^k(D_r^{as} - D_{r-1}^{as})$). In particular $A^1(D_r^s) = \mathbb{Z}/2\mathbb{Z}$, $A^1(D_r^{as}) = 0$. As in Section 4, it is possible to generalize these considerations to the case of universal degeneracy loci $\bar{D}_r \subset S_2 E$ (resp. $\bar{D}_r \subset \Lambda^2 E$, r-even), where E is a vector bundle on X. For example if E is trivial we get

$$A^1(X \times D_r^s) = A^1(X) \oplus \mathbb{Z}/2\mathbb{Z} \qquad \text{and} \qquad A^1(X \times D_r^{as}) = A^1(X).$$

PROBLEM. — Describe the Chow groups of D_r^s, $\bar{D}_r^s$, D_r^{as}, $\bar{D}_r^{as}$ explicitly.

(8.5). Notice that Proposition 5.7 allows us to calculate the Euler-Poincaré characteristic of varieties W_d^r parametrizing the linear systems of degree d and dimension $\geqq r$ in the Jacobian of a curve, provided W_d^r is smooth. We plan to discuss this subject in more details elsewhere.

(8.6). It is known that the formula for the Euler-Poincaré characteristic of degeneracy loci (*see* Proposition 5.7) can fail if $D_r(\varphi)$ is not smooth. For, in the case when $\varphi: \mathbf{1}_X \to L$ is a section of line bundle, and the hypersurface $D_0(\varphi)$ has only one isolated singularity in the point x, then the difference between the Euler-Poincaré characteristic and formula (5.7) is measured by the Milnor number of x. It would be interesting to generalize the formula (5.7) to possibly singular degeneracy loci.

(8.7). Consider the homogenous space Sp_n/U_n. The Schubert varieties Ω_I in this space are parametrized by strict partitions $I = (i_1, \ldots, i_n)$ where $I \subset \rho_n$ (*see* [B-H]). The Schubert varieties $\Omega_p = \Omega_{(p, 0, \ldots, 0)}$ $(1 \leqq p \leqq n)$ are called special. The authors of [B-H] raised the following question: Is there a "Giambelli-formula" that expresses each Schubert class as a polynomial in the special Schubert classes? It turns out that by combining results of [Mo] and of [B-H] one can prove that the formula in question is given by the Schur Q-polynomial $Q_I(A)$, where the role of the $q_p(A)$ is played by Ω_p (recall that this polynomial is given explicitly in (7)). More precisely, in [Mo] the following "Pieri-formula" for the multiplication of Schur Q-polynomials was established. Let $I = (i_1, \ldots, i_k)$ be a strict partition of length k. Then

$$Q_I(A)\, q_r(A) = \sum 2^{m(J)} Q_J(A)$$

where the summation ranges over all strict partitions J of length k or $k+1$ such that $i_{p-1} \geqq j_p \geqq i_p$ ($i_0 = \infty$, $i_{k+1} = 0$), $|J| = r + |I|$. Moreover,

$$m(J) = \operatorname{card}\{1 \leqq p \leqq k, j_{p+1} < i_p < j_p\}.$$

Comparing it with the "Pieri-formula" for the Chow ring of Sp_n/U_n proved in [B-H], one obtains that the assignment: $Q_I(A) \mapsto \Omega_I$, defines a *ring* homomorphism

$$\text{Ring of Schur Q-polynomials in } n\text{-variables} \twoheadrightarrow A^{\cdot}(Sp_n/U_n).$$

This map allows us to identify $A^{\cdot}(Sp_n/U_n)$ with the quotient ring of the ring of Q-polynomials by the ideal generated by all $Q_I(A)$, where $I \not\subset \rho_n$. The same observation applies to the Chow ring of SO_{2n+1}/U_n (*see loc. cit.*) but instead of the polynomials $Q_I(A)$ one should use the $P_I(A)$.

For more details see a forthcoming paper [P_3].

(8.8). In Propositions 6.1 and 7.17 we have described some finite sets of generators of the ideals $\mathscr{I}_r$, $\mathscr{I}_r^s$ and $\mathscr{I}_r^{as}$.

CONJECTURE. — (i) If $m \geqq n$ then the elements $s_{(m-r)^{n-r}+I}(A)$ where I ranges over all partitions $I \subset (r)^{n-r}$, form a minimal set of generators of $\mathscr{I}_r$.

(ii) The elements $Q_{\rho_{n-r}+I}(A)$ (resp. $P_{\rho_{n-r-1}+I}(A)$) where $I \subset (r)^{n-r}$, form a minimal set of generators of $\mathscr{I}_r^s$ (resp. $\mathscr{I}_r^{as}$).

(iii) If $m \geqq n$, then the minimal number of generators of each of the ideals $\mathscr{I}_r$, $\mathscr{I}_r^s$ and $\mathscr{I}_r^{as}$ is equal to $\binom{n}{r}$.

This Conjecture was checked by the author for $n \leqq 6$.

9. Appendix: a result of Schur

We provide here a sketch of the proof of Proposition 1.7 (refering for details to [Sch], if necessary).

It is proved in ([M] III 2.3) that the Hall-Littlewood polynomial $P_{(i)}(A; t)$ satisfies:

$$P_{(i)}(A; t) = \sum_{r=0}^{i} (-1)^r s_{(i-r, 1^r)}(A).$$

where $A = (a_1, \ldots, a_n)$ is a sequence of independent variables. Thus, by Example 1.6 and Corollary 1.8, the Proposition is true for $k=1$. Then the relation

$$Q_{ij}(A) = q_i(A)\, q_j(A) - q_{i+1}(A)\, q_{j-1}(A) - Q_{i+1,\, j-1}(A)$$

allows us to prove the Proposition for $k=2$ by induction on $j=1, \ldots, i-1$. Taking into account definition (7) and Laplace-type expansion for Pfaffians, it suffices to show that the polynomials ($I=(i_1, \ldots, i_k)$, I-strict, $l(I)=k$):

$$Q'_I(a_1, \ldots, a_n)=2^k \sum_{w \in S_n/(S_1)^k \times S_{n-k}} w\Big[a_1^{i_1}\ldots a_k^{i_k} \prod_{\substack{1 \leq i<j \leq n \\ i \leq k}} (a_i+a_j)(a_i-a_j)^{-1}\Big]$$

satisfy the relations

$$Q'_I(A)=\sum_{p=2}^{k} (-1)^p Q_{i_1, i_p}(A)\, Q'_{i_2, \ldots, \hat{i}_p, \ldots, i_k}(A),$$

if k is even and

$$Q'_I(A)=\sum_{p=1}^{k} (-1)^{p-1} q_{i_p}(A)\, Q'_{i_1, \ldots, \hat{i}_p, \ldots, i_k}(A),$$

if k is odd.

Consider the following elements in $(\mathbb{Z}[A])_0$:

$$T_r=\prod_{s \neq r} (a_r+a_s)(a_r-a_s)^{-1}, \qquad r=1, \ldots, n,$$

and for $u_1, \ldots, u_k \in \mathbb{Z}[A]$,

$$w(u_1, \ldots, u_k)=\prod_{1 \leq p<q \leq k} (u_p-u_q)(u_p+u_q)^{-1}.$$

Then the above definition of $Q'_I(A)$ can be rewritten as

$$Q'_I(A)=2^k \sum_{r_1, \ldots, r_k=1}^{n} \frac{a_{r_1}^{i_1} a_{r_2}^{i_2}\ldots a_{r_k}^{i_k}}{T_{r_1} T_{r_2}\ldots T_{r_k}} w(a_{r_k}, a_{r_{k-1}}, \ldots, a_{r_1}).$$

Thus, it suffices to prove the following relations

$$w(u_1, \ldots, u_k)=\sum_{p=2}^{k} (-1)^p w(u_1, u_p)\, w(u_2, \ldots, \hat{u}_p, \ldots, u_k),$$

if k is even

$$w(u_1, \ldots, u_k)=\sum_{p=1}^{k} (-1)^p w(u_1, \ldots, \hat{u}_p, \ldots, u_k),$$

if k is odd.

The second equality follows from the first one, by letting one of variables involved to be zero. To prove the first equality it remains to show that for even k

$$w(u_1, \ldots, u_k)=\mathrm{Pf}\,[(u_p-u_q)(u_p+u_q)^{-1}], \qquad (1 \leq p<q \leq k),$$

and apply some well-known property of Pfaffians. Finally, for this last claim, notice that $Pf[(u_p-u_q)(u_p+u_q)^{-1}]\prod_{p<q}(u_p+u_q)$ vanishes if $u_p=u_q$ for some $p\neq q$ and has the same degree as $\prod_{p<q}(u_p-u_q)$. Being rational functions with integral coefficients, the above polynomials must differ by certain constant factor. Using Laplace-type expansion for Pfaffians and induction assumption one shows easily that for even k this factor is equal to 1.

This finishes the proof of Proposition 1.7. ■

REFERENCES

[B-R] A. Berele, A. Regev, *Hook Young Diagrams With Applications to Combinatorics and to Representation Theory of Lie Superalgebras* (*Adv. in Math.*, Vol. 64, pp. 118-175, 1987).

[B-H] B. Boe, H. Hiller, *Pieri Formula for* SO_{2n+1}/U_n *and* Sp_n/U_n (Adv. in Math., Vol. 62, pp. 49-68, 1986).

[F] W. Fulton, *Intersection Theory*, Springer Verlag, 1984.

[G-G] M. Golubitsky, V. Guillemin, *Stable Mappings and their Singularities* (*Graduate Texts in Math.*, Vol. 14, Springer-Verlag, 1973).

[H-T] J. Harris, L. Tu, *Chern Numbers of Kernel and Cokernel Bundles* (*Invent. Math.*, Vol. 75, pp. 467-475, 1984).

[H] F. Hirzebruch, *Topological Methods in Algebraic Geometry* (*Grundlehren der Math.*, Wissenschaften, Vol. 131, 1956, *Third Enlarged*, Springer-Verlag, 1966).

[J-L-P] T. Jòzefiak, A. Lascoux, P. Pragacz, *Classes of Determinantal Varieties Associated with Symmetric and Antisymmetric Matrices* (*Izwiestja* A.N. S.S.S.R., Vol. 45, N° 9, pp. 662-673, 1981).

[K-L] G. Kempf, D. Laksov, *The Determinantal Formula of Schubert Calculus* (*Acta Math.*, Vol. 132, pp. 153-162, 1974).

[L-L-T] D. Laksov, A. Lascoux, A. Thorup, *On Giambelli's Theorem on Complete Correlations*, to Appear in *Acta Mathematica.*

[L] A. Lascoux, *Puissances extérieures, déterminants et cycles de Schubert* (*Bull. Soc. Math., France*, Vol. 102, pp. 161-179, 1974).

[L-S] A. Lascoux, M. P. Schützenberger, *Formulaire raisonné de fonctions symétriques*, prépublication de l'Université Paris VII, 1985.

[M] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, Oxford Univ. Press, 1979.

[Mo] A. Morris, *A Note on the Multiplication of Hall Functions* (*J. London Math. Soc.*, Vol. 39, pp. 481-488, 1964).

[Po] I. R. Porteous, *Simple Singularities of Maps, Proc. Liverpool Singularities Symposium I* (*Lecture Notes in Mathematics*, Vol. 192, pp. 286-307, 1971).

[P_1] P. Pragacz, *Determinantal Varieties and Symmetric Polynomials* (*Functional Analysis and Its Applications*, Vol. 21, N° 3, pp. 89-90, 1987).

[P_2] P. Pragacz, *A Note on Elimination Theory* (*Proc. Kon. Ned. Akad. van Wetensh.*, Vol. A 90(2) (= *Indagationes Math.*, Vol. 49(2)), pp. 215-221, 1987).

[P_3] P. Pragacz, *Algebro-Geometric Applications of Schur* S- *and* Q-*Polynomials* (in preparation in Séminaire d'Algèbre Dubreil-Malliavin 1987-1988, Springer Lecture Notes in Math.).

[S] H. Schubert, *Allgemaine Anzahlfunctionen für Kegelschnitte, Flächen und Räume zweiten Grades in n Dimensionen* (*Math. Ann.*, Vol. 45, pp. 153-206, 1894).

[Sch] I. Schur, *Uber die Darstellung der Symmetrischen und den Alternierenden Gruppe durch Gebrochene Lineare Substitutionen* (*Journal für die reine u. angew. Math.*, Vol. 139, pp. 155-250, 1911).

[St] R. P. Stanley, *Problem 4 in Problem Session, Combinatorics and Algebra* (*Contemporary Mathematics*, A.M.S., Vol. 34, 1984).

[T] R. THOM, *Les ensembles singuliers d'une application différentiable et leurs propriétés homologiques*, Séminaire de Topologie de Strasbourg, (December 1957).

Note added in proof. — (1) Since submitting this paper I have learned that the following special case of our Proposition 5.7: $X=P^N_{\mathbf{C}}$, $F=\mathcal{O}_X(-d_1)+\ldots+\mathcal{O}_X(-d_p)$, $E=\mathcal{O}_X$, $r=0$, was established by other methods in the paper: V. NAVARRO AZNAR, On the Chern classes and the Euler characteristic for nonsingular complete intersections, *Proc. of the Amer. Math. Soc.*, Vol. 78, pp. 143-148, 1980.

(2) The assumption $l(I)\geqslant q-1$ in Proposition 2.8 can be dropped-see [P_3].

(3) Methods similar to those used in Section 4. allow one to study the Chow groups of projective determinantal varieties. We plane to treat this subject in some future article.

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P. PRAGACZ, Institute of Mathematics,
Polish Academy of Sciences,
Chopina 12,
87-100 Toruń, Poland.