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CONJUGACY CLASSES OF FINITE SOLVABLE SUBGROUPS IN LIE GROUPS

BY ERIC M. FRIEDLANDER (1) AND GUIDO MISLIN

In Memory of Alex Zabrodsky

1. Introduction

In two recent papers ([5], [6]), we introduced the concept of a locally finite approximation Γ_G of a compact Lie group G. The group Γ_G, by definition a countable union of its finite subgroups, offers a useful tool in the homotopy theoretic analysis of the classifying space BG of the Lie group G. Our study required the establishment of certain connections between the finite subgroups of Γ_G and G. In this paper, we concentrate on the algebraic problem of comparing conjugacy classes of finite subgroups of Γ_G and G.

Specializing our Main Theorem (*cf.* 5.2 below) to the case in which G is connected, we have the following theorem. We adopt the following notational conventions, maintained throughout: p is a fixed prime, $\mathbf{F}$ the algebraic closure of the prime field $\mathbf{F}_p$, and $\mathbf{C}$ the field of complex numbers. Furthermore, for any pair of groups H and G, we denote by $\mathrm{Hom}_c(\mathrm{H}, \mathrm{G})$ the set of conjugacy classes of homomorphisms from H to G.

(1.1) THEOREM. — *Let* G *be a compact, connected Lie group and let* $\Gamma_G = G_{\mathbf{Z}}(\mathbf{F})$ *denote the group of* $\mathbf{F}$*-valued points of the reductive integral group scheme* $G_{\mathbf{Z}}$ *whose* $\mathbf{C}$*-valued points constitute a complex form* $G\mathbf{C}$ *for* G. *For any finite solvable group* π *of order prime to* p $(=\mathrm{char}(\mathbf{F}))$, *a choice of embedding of the the Witt vectors of* $\mathbf{F}$ *into* $\mathbf{C}$ *determines a natural bijection*

$$\Psi: \quad \mathrm{Hom}_c(\pi, \Gamma_G) \to \mathrm{Hom}_c(\pi, G) \simeq \mathrm{Hom}_c(\pi, G\mathbf{C}).$$

In the special case in which $G = U_n$, we have $G_{\mathbf{Z}} = GL_n$ and classical Wedderburn theory implies the above theorem not only for solvable groups π but for any finite group of order prime to p. We conjecture that Theorem 1.1 should remain valid whenever π is a finite group whose order is not divisible by p, although we know only fragmentary results suggesting this generalization.

(1) Partially supported by the N.S.F.

Our paper is organized as follows. Using results of [6], section 2 defines the map Ψ of (1.1) and proves its surjectivity in the generality of "generalized reductive groups" and finite super-solvable groups. We find this generality necessary for the proof of (1.1), as well as its non-connected generalization, Theorem 5.2. In section 3, we use a comparison of cohomology rings of the classifying spaces of BG and $B\Gamma_G$ to prove bijectivity of Ψ in the special case in which π is an elementary abelian group. This special case, together with a discussion in section 4 of centralizers in generalized reductive groups, constitutes the basis for our inductive proof of Theorem 5.2 presented in section 5.

While this work was in progress, the first-named author enjoyed the hospitality of the Institute for Advanced Study and E.T.H., Zurich.

2. Generalized reductive groups

In our previous papers concerning locally finite approximations ([5], [6]), we began with a compact Lie group G and obtained a "generalized reductive group" over a suitable ring of Witt vectors which related G to a corresponding algebraic group over a field of finite characteristic. In order to prove Theorem 1.1 (and its non-connected generalization Theorem 5.2), we shall consider centralizers which are generalized reductive groups over rings of Witt vectors but which are not presented as arising from compact Lie groups. We begin this section with the definition of an abstract generalized reductive group G_S over a base scheme S. Using the results of [6], we exhibit a map Ψ: $\mathrm{Hom}_c(\pi, G_S(\mathbf{F})) \to \mathrm{Hom}_c(\pi, G_S(\mathbf{C}))$ for any finite group π of order prime to p and any generalized reductive group G_S whenever S equals the spectrum of the Witt vectors of $\mathbf{F}$ (discussed immediately below). We conclude this section by proving that Ψ is surjective whenever π is super-solvable.

We recall that $\mathbf{W}$, the ring of Witt vectors of $\mathbf{F}$, is a complete discrete valuation ring of characteristic 0 with residue field $\mathbf{F}$. One can construct $\mathbf{W}$ by completing at some prime above p the ring obtained by adjoining to $\mathbf{Z}$ all roots of unity of order prime to p. For our purposes, $\mathbf{W}$ is particularly useful. Namely, $\mathbf{W}$ maps to $\mathbf{F}$ by the canonical quotient map, can be embedded into $\mathbf{C}$ using the fact that the cardinality of the quotient field of $\mathbf{W}$ equals that of $\mathbf{C}$, and is contractible in the etale topology. *We fix an embedding of* $\mathbf{W}$ *into* $\mathbf{C}$ *for all that follows.* Many of our constructions [e. g., that of Ψ in (2.3)] depend on this choice of embedding.

A group scheme G_S over a base scheme S is said to be a reductive group over S if G_S is smooth, affine over S and satisfies the property that each of its geometric fibres is a connected, reductive algebraic group.

(2.1) Definition. — A group scheme G_S over S is said to be a generalized reductive group over S if it fits into an extension of group schemes over S

$$G_S^0 \to G_S \to \tau_S$$

in which τ_S is finite etale over S and G_S^0 is a reductive group over S. (Thus, G_S^0 is a normal subgroup scheme of G_S and τ_S represents the sheaf-theoretic quotient G_S/G_S^0.)

In much of what follows, S will be the spectrum of **F**, **C**, or **W**. In these cases, any finite etale group scheme τ_S over S must be of the form $\tau \otimes S$ for some discrete group τ (i. e., a disjoint union of copies of S indexed by τ). For G_S generalized reductive over such a scheme S, $G_S/G_S^0 \simeq \tau \otimes S$, where τ is the group of scheme-theoretic connected components $\pi_0 G_S$ of G_S.

The following theorem, essentially proved in [6; 1.5], is a major step in proving Theorem 1.1 and its generalization Theorem 5.2.

(2.2) THEOREM. — *Let* $S = \operatorname{Spec} \mathbf{W}$ *and let* G_S *be a generalized reductive group over* S. *Then for any finite group* π *of order prime to p, the quotient map* $\mathbf{W} \to \mathbf{F}$ *induces a bijection*

$$\alpha: \quad \operatorname{Hom}_c(\pi, G_S(\mathbf{W})) \to \operatorname{Hom}_c(\pi, G_S(\mathbf{F})).$$

Proof. — Let $\mathbf{W}_n$ denote the ring of Witt vectors of length n of **F**. Then $\mathbf{W}_n$ is the quotient of **W** by the n-th power of the maximal ideal and $\mathbf{W} \simeq \varprojlim \mathbf{W}_n$. As argued in the proof of [6; 1.5], the quotient map $\mathbf{W}_n \to \mathbf{W}_{n-1}$ for any $n > 1$ induces a surjection

$$\alpha_n: \operatorname{Hom}(\pi, G_S(\mathbf{W}_n)) \to \operatorname{Hom}(\pi, G_S(\mathbf{W}_{n-1}))$$

which immediately implies the asserted surjectivity of α. Moreover, the lifting of maps via α_n is unique up to conjugation by an element in $G_S(\mathbf{W}_n)$ which maps to the identity in $G_S(\mathbf{W}_{n-1})$. This easily implies the asserted injectivity of α. □

Using the map α of (2.2), we now define (in a much more general context) the map asserted to be a bijection in Theorem 1.1. In (2.3) below and in subsequent discussions as well, we let $G\mathbf{C}$ denote the complex Lie group with underlying discrete group $G_S(\mathbf{C})$.

(2.3) DEFINITION. — Let $S = \operatorname{Spec} \mathbf{W}$ and let G_S be a generalized reductive group over S. Then for any finite group π of order prime to p, there is a natural map

$$\Psi: \quad \operatorname{Hom}_c(\pi, G_S(\mathbf{F})) \to \operatorname{Hom}_c(\pi, G\mathbf{C})$$

defined as the composition of the inverse of α occuring in (2.2) and the map $\beta: \operatorname{Hom}_c(\pi, G_S(\mathbf{W})) \to \operatorname{Hom}_c(\pi, G\mathbf{C})$ induced by the inclusion $\mathbf{W} \subset \mathbf{C}$. Moreover, if K is a compact form of $G\mathbf{C}$, then we also denote by Ψ the natural function

$$\Psi: \quad \operatorname{Hom}_c(\pi, G_S(\mathbf{F})) \to \operatorname{Hom}_c(\pi, K)$$

obtained by composing the preceding map with the inverse of the bijection $\operatorname{Hom}_c(\pi, K) \to \operatorname{Hom}_c(\pi, G\mathbf{C})$ induced by the inclusion $K \subset G\mathbf{C}$ (*cf.* [2]).

The next proposition is a generalization of [6; 1.5], which considered only finite groups of prime power order. We recall that a finite group π is called super-solvable if it admits a nested sequence of normal subgroups $\pi_i \subset \pi$ whose successive quotients π_i/π_{i-1} are cyclic. In particular, any finite nilpotent group is super-solvable. In Theorem 5.2, the reader will find another proof of Proposition 2.4 applicable to any finite solvable group of order prime to p.

(2.4) PROPOSITION. — *Let* $S = \operatorname{Spec} \mathbf{W}$ *and let* G_S *be a generalized reductive group over* S *and let* $\pi \subset G_S(\mathbf{C})$ *be a finite, super-solvable subgroup of order prime to p. Then there exists some element* $x \in G_S(\mathbf{C})$ *such that the conjugate of* π *by* x, π^x, *is a subgroup of* $G_S(\mathbf{W}) \subset G_S(\mathbf{C})$.

Proof. — By [11; II.5.17], π normalizes a maximal torus of $G\mathbf{C}$. Because all maximal tori of $G\mathbf{C}$ are conjugate and because every maximal torus of G_S is split, we may assume that π is a subgroup of the $\mathbf{C}$-valued points of the closed S-subgroup scheme N_S of G_S defined as the normalizer of some S-split, maximal torus T_S of G_S. Because $N_S \subset G_S$ is a smooth, closed subgroup (*cf.* [6; 1.6], [10; VIII.6.5]), the smooth closed map $G_S \to \tau_S = G_S/G_S^0$ is such that the image of N_S is open and closed in τ_S. Since τ_S is a constant group over S, this image of N_S is a constant subgroup $\eta_S \subset \tau_S$. We recall that M_S, the normalizer of T_S in G_S^0, is generalized reductive (given as an extension of T_S by the Weyl group over S of the reductive group G_S^0). Consequently, N_S is likewise generalized reductive, given as an extension of M_S by η_S.

Let $f: \pi \to N_S(\mathbf{C})$ be a given homomorphism, let $\sigma = \pi \cap f^{-1}(T_S(\mathbf{C}))$, and consider the resulting maps of extensions

$$\begin{array}{ccccc} \sigma & \to & \pi & \to & \pi/\sigma \\ \downarrow & & \downarrow & & \downarrow \\ T_S(\mathbf{C}) & \to & N_S(\mathbf{C}) & \to & W_S(\mathbf{C}) \\ \uparrow & & \uparrow & & \uparrow \\ T_S(\mathbf{W}) & \to & N_S(\mathbf{W}) & \to & W_S(\mathbf{W}) \end{array}$$

where $W_S = N_S/T_S$. The homomorphism $\sigma \to T_S(\mathbf{C})$ factors through $T_S(\mathbf{W}) \to T_S(\mathbf{C})$, so that a straight-forward cohomological argument implies that $\pi \to N_S(\mathbf{C})$ factors up to conjugation through $N_S(\mathbf{W}) \to N_S(\mathbf{C})$ (*cf.* [6; 1.7]). $\square$

Interpreting Theorem 2.2 and Proposition 2.4 in terms of the map Ψ of Definition 2.3, we immediately conclude the following corollary.

(2.5) COROLLARY. — *The map* $\Psi: \operatorname{Hom}_c(\pi, G_S(\mathbf{F})) \to \operatorname{Hom}_c(\pi, G\mathbf{C})$ *of* (2.3) *is a surjection whenever* π *is a finite, super-solvable group of order prime to p.* $\square$

3. The case of elementary abelian groups

The purpose of this section is to prove the bijectivity of the map $\Psi: \operatorname{Hom}_c(\pi, G_S(\mathbf{F})) \to \operatorname{Hom}_c(\pi, G\mathbf{C})$ of (2.3) in the special case in which π is an elementary abelian q-group (i.e., $\pi \simeq \mathbf{Z}/q\mathbf{Z}^{\oplus r}$, some $r > 0$) for some prime q different from p. Our proof is cohomological in nature, relying on a point of view first espoused by Quillen [9] which leads to the result that conjugacy classes of maps from elementary abelian q-groups into a compact Lie group G are determined by the cohomology ring $H^*(BG, \mathbf{Z}/q\mathbf{Z})$ (*cf.* [7; 4.3.1]). Thus, our first order of business is to compare the cohomology of $BG_S(\mathbf{F})$ and $BG\mathbf{C}$, generalizing slightly a result in [5] that $BG_S(\mathbf{F})$ is a "locally finite approximation away from p" of $BG\mathbf{C}$. We then observe that a recent result in [4] enables one to apply Quillen's point of view to the locally finite group

$G_S(\mathbf{F})$. The proof of injectivity (for π an elementary abelian q-group) is then straight forward.

Throughout this section we set S equal Spec $\mathbf{W}$. Any reductive group G_S over S is split over S, and arises by base change from a reductive group $G_{\mathbf{Z}}$ over $\mathbf{Z}$. More generally, any generalized reductive group G_S over S is defined and split over the spectrum of some subring $A \subset W$ which is finitely generated over $\mathbf{Z}$. Consequently, for any "sufficiently large" p^d (i. e., for d divisible by sufficiently high powers of sufficiently many primes), we may by an abuse of notation use $G_S(\mathbf{F}_{p^d})$ to denote the $\mathbf{F}_{p^d}$-valued points of $G_{\mathrm{Spec}\,A}$, where G_S is obtained by base change from the split group $G_{\mathrm{Spec}\,A}$ for $A \subset \mathbf{W}$ finitely generated over $\mathbf{Z}$. In particular, $G_S(\mathbf{F})$ is a "locally finite group", the countable union of its finite subgroups $G_S(\mathbf{F}_{p^d})$ for sufficiently large p^d.

(3.1) Proposition. — *Let* $S = \mathrm{Spec}\,\mathbf{W}$ *and let* G_S *be a generalized reductive group over* S. *There exists a naturally determined homotopy class of maps (a "locally finite approximation")*

$$\Phi:\ B\,G_S(\mathbf{F}) \to BG\,\mathbf{C},$$

such that the following properties are satisfied.

(i) $\pi_1(\Phi)$ *induces an isomorphism*

$$\Phi_*:\ G_S(\mathbf{F})/G_S^0(\mathbf{F}) \to G\,\mathbf{C}/G\,\mathbf{C}^0.$$

(ii) *For every finite* $\pi_0(G\,\mathbf{C})$*-module* A *of order prime to* p, Φ *induces an isomorphism*

$$\Phi^*:\ H^*(BG\,\mathbf{C}, A) \to H^*(B\,G_S(\mathbf{F}), A).$$

(iii) Φ *is natural with respect to morphisms of group schemes over* S.

Proof. – The construction of [5; 2.4] yields a map Φ: $B\,G_S(\mathbf{F}) \to BG\,\mathbf{C}$ fitting into a map of fibration sequences

$$\begin{array}{ccccc} BG_S^0(\mathbf{F}) & \to & BG_S(\mathbf{F}) & \to & B\,\tau_S(\mathbf{F}) \\ \downarrow \Phi^0 & & \downarrow \Phi & & \downarrow \Phi_\tau \\ BG\,\mathbf{C}^0 & \to & BG\,\mathbf{C} & \to & B\,\tau_S(\mathbf{C}) \end{array} \tag{3.1.1}$$

where Φ_τ is a homotopy equivalence induced by the isomorphisms

$$\tau_S(\mathbf{F}) \simeq \tau_S(\mathbf{W}) \simeq \tau_S(\mathbf{C}).$$

Because G_S^0 is a reductive group over S, the left vertical map is a locally finite approximation away from p in the sense of [5; 1.1], and in particular induces an isomorphism Φ^{0*}: $H^*(BG\,\mathbf{C}^0, A) \to H^*(BG_S^0(\mathbf{F}), A)$. Thus, (ii) follows using the induced map of Serre spectral sequences associated to (3.1.1). Moreover, (i) follows by identifying $\pi_1(\Phi)$ with the isomorphism on fundamental groups induced by Φ_τ. Finally, (iii) follows from the naturality of the construction of [5; 2.4]. $\square$

The following proposition identifies the isomorphism Φ^* of (3.1. (ii)).

(3.2) Proposition. — *Let* $S = \operatorname{Spec} \mathbf{W}$ *and let* G_S *be a generalized reductive group over* S. *Denote by* $\rho : G_S(\mathbf{W}) \to G_S(\mathbf{F})$ *the map induced by the quotient map* $\mathbf{W} \to \mathbf{F}$ *and denote by* $i : G_S(\mathbf{W}) \to G\,\mathbf{C}$ *the map induced by the chosen embedding* $\mathbf{W} \subset \mathbf{C}$. *Then for any prime* $q \neq p$, *the two maps* i, $\Phi \circ \rho : BG_S(\mathbf{W}) \to BG\,\mathbf{C}$ *induce the same map*

$$i^* = (\Phi \circ \rho)^* : \quad H^*(BG\,\mathbf{C}, \mathbf{Z}/q\,\mathbf{Z}) \to H^*(BG_S(\mathbf{W}), \mathbf{Z}/q\,\mathbf{Z}).$$

Proof. — We employ the following homotopy commutative diagram

$$(3.2.1)\qquad \begin{tikzcd}
BG_S(\mathbf{F}) \arrow[d] & BG_S(\mathbf{W}) \arrow[l] \arrow[r] \arrow[d] & BG_S(\mathbf{C}) \arrow[d] \\
\operatorname{Hom}_{\mathbf{F}}(\operatorname{Spec} \mathbf{F}, BG_{\mathbf{F}}) \arrow[d] & \operatorname{Hom}_S(S, BG_S) \arrow[l] \arrow[r] \arrow[d] & \operatorname{Hom}_{\mathbf{C}}(\operatorname{Spec} \mathbf{C}, BG_{\mathbf{C}}) \arrow[d] \\
\operatorname{Hom}((\operatorname{Spec} \mathbf{F})_{et}, (BG_{\mathbf{F}})_{et}) \arrow[d] & \operatorname{Hom}((S)_{et}, (BG_S)_{et}) \arrow[d, shift left] \arrow[d, shift right] & \operatorname{Hom}((\operatorname{Spec} \mathbf{C})_{et}, (BG_{\mathbf{C}})_{et}) \arrow[d] \\
(BG_{\mathbf{F}})_{et} \arrow[r] & (BG_S)_{et} & (BG_{\mathbf{C}})_{et} \arrow[l]
\end{tikzcd}$$

whose horizontal arrows are determined by base change. The middle vertical arrows of (3.2.1) are the canonical maps from algebraic function complexes to etale topological function complexes [3; 13.2]. The lower vertical arrows of (3.2.1) are given by evaluation at the canonical points of the contractible pro-spaces $(\operatorname{Spec} \mathbf{F})_{et}$, $(\operatorname{Spec} \mathbf{C})_{et}$. Since $(S)_{et}$ is also contractible, the two middle lower vertical arrows are homotopic. We recall that $H^*(BG\,\mathbf{C}, \mathbf{Z}/q\,\mathbf{Z})$ may be identified with $H^*((BG_{\mathbf{C}})_{et}, \mathbf{Z}/q\,\mathbf{Z}) \simeq H^*((BG_S)_{et}, \mathbf{Z}/q\,\mathbf{Z})$. Using this identification, $\Phi^* : H^*(BG\,\mathbf{C}, \mathbf{Z}/q\,\mathbf{Z}) \to H^*(BG(\mathbf{F}), \mathbf{Z}/q\,\mathbf{Z})$ is the map in cohomology induced by the left vertical and bottom horizontal maps of (3.2.1). Hence, the corollary follows from the homotopy commutativity of (3.2.1), which implies the commutativity of the induced diagram in cohomology with $\mathbf{Z}/q\,\mathbf{Z}$-coefficients, together with observation that the right vertical arrow induces the same map $H^*(BG\,\mathbf{C}, \mathbf{Z}/g\,\mathbf{Z}) \to H^*(BG_S(\mathbf{C}), \mathbf{Z}/g\,\mathbf{Z})$ as the tautological map $G_S(\mathbf{C}) \to G\,\mathbf{C}$. □

The following proposition is an extension of a well known result of Quillen [9] concerning conjugacy classes of elementary abelian subgroups of compact Lie groups.

(3.3) Proposition. — *Let* $S = \operatorname{Spec} \mathbf{W}$ *and let* G_S *be a generalized reductive group over* S. *Let* π *be an elementary abelian* q-*group for some prime* q *different from* p. *If* f, $g : \pi \to G_S(\mathbf{F})$ *are two homorphisms inducing the same map*

$$f^* = g^* : H^*(BG_S(\mathbf{F}), \mathbf{Z}/q\,\mathbf{Z}) \to H^*(B\pi, \mathbf{Z}/q\,\mathbf{Z}),$$

then f *and* g *are conjugate.*

Proof. — As discussed prior to (3.1), we may write $G_S(\mathbf{F})$ as a union of finite subgroups $G_S(\mathbf{F}_{p^d})$ for d sufficiently large. Moreover, for d sufficiently large, both f and g factors as maps f, $g : \pi \to G_S(\mathbf{F}_{p^d})$. We consider the maps of cohomology algebras associated to f and g :

$$H^*(G_S(\mathbf{F}), \mathbf{Z}/q) \to H^*(G_S(\mathbf{F}_{p^{de}}), \mathbf{Z}/q) \to H^*(G_S(\mathbf{F}_{p^d}), \mathbf{Z}/q) \to H^*(\pi, \mathbf{Z}/q).$$

These sequences for f and g, our hypothesis $f^* = g^*$, and an easy generalization of [4; Thm 1 (b)] to $G_S(\mathbf{F})$ (valid for any locally finite group which is a finite extension of

$G_S{}^0(\mathbf{F})$) imply the equality of the associated maps $H^*(G(\mathbf{F}_{p^{de}}), \mathbf{Z}/q) \to H^*(\pi, \mathbf{Z}/q)$ for e sufficiently large. Applying Lanne's theorem [7; 4.3.1], we conclude that $f, g: \pi \to G_S(\mathbf{F}_{p^d})$ are conjugate, thereby also proving that $f, g: \pi \to G_S(\mathbf{F})$ are conjugate. □

We now prove the bijectivity of the map Ψ of (2.3) for π an elementary abelian q-group.

(3.4) THEOREM. — *Let* $S = \operatorname{Spec} \mathbf{W}$ *and let* G_S *be a generalized reductive group over* S. *Let* π *be an elementary abelian* q*-group for some prime* q *different from* p. *Then the map* $\Psi: \operatorname{Hom}_c(\pi, G_S(\mathbf{F})) \to \operatorname{Hom}_c(\pi, G\,\mathbf{C})$ *of* (2.3) *is a bijection.*

Proof. — By Corollary 2.5, it suffices to prove that Ψ is an injection. For this it suffices by Theorem 2.2 to show that the map $\beta: \operatorname{Hom}_c(\pi, G_S(\mathbf{W})) \to \operatorname{Hom}_c(\pi, G\,\mathbf{C})$ induced by the inclusion $\mathbf{W} \subset \mathbf{C}$ is an injection. Consider two maps $f, g: \pi \to G_S(\mathbf{W})$ with the property that their compositions with the inclusion $i: G_S(\mathbf{W}) \subset G_S(\mathbf{C})$ are conjugate. The conjugate maps $i \circ f$, $i \circ g$ induce the same map in cohomology

$$f^* \circ i^* = g^* \circ i^*: \quad H^*(BG\,\mathbf{C}, \mathbf{Z}/q\,\mathbf{Z}) \to H^*(B\,\pi, \mathbf{Z}/q\,\mathbf{Z}).$$

Applying Proposition 3.2, we conclude that

$$f^* \circ \rho^* \circ \Phi^* = g^* \circ \rho^* \circ \Phi^*: \quad H^*(BG\,\mathbf{C}, \mathbf{Z}/q\,\mathbf{Z}) \to H^*(B\,\pi, \mathbf{Z}/q\,\mathbf{Z}).$$

Since Φ^* is an isomorphism by Proposition 3.1, we conclude that

$$f^* \circ \rho^* = g^* \circ \rho^*: \quad H^*(BG_S(\mathbf{F}), \mathbf{Z}/q\,\mathbf{Z}) \to H^*(B\,\pi, \mathbf{Z}/q\,\mathbf{Z}).$$

By Proposition 3.3, this implies that $\rho \circ f$, $\rho \circ g: \pi \to G_S(\mathbf{F})$ are conjugate. Finally, this implies by Theorem 2.2 that $f, g: \pi \to G_S(\mathbf{W})$ are conjugate as required. □

4. Centralizers and normalizers of elementary abelian groups

In Corollary 4.3, we verify that the centralizer in a generalized reductive group G_S of an elementary abelian subgroup $A \subset G_S(\mathbf{W})$ of order prime to p is again a generalized reductive group provided that $\pi_0 G_S$ has order prime to p. This result sets the stage for our inductive proof in section 5 of the bijectivity of Ψ for finite solvable groups. We apply Corollary 4.3 to obtain Theorem 4.4 which verifies that the normalizer $N(E)_S$ of an elementary abelian q-group $E \subset G_S(\mathbf{W})$ is also a generalized reductive subgroup of the generalized reductive group G_S whenever q is a prime different from p.

We adopt the following notation of Steinberg [12] in considering an automorphism $\sigma: G \to G$ of a discrete group G. We write G_σ for the subgroup of G of elements fixed by σ, $1-\sigma: G \to G$ for the (set-theoretic) map sending an element g to $g\,\sigma(g^{-1})$, and $(1-\sigma)\,G$ for the image of the map $1-\sigma$.

(4.1) PROPOSITION. — *Let* G *be a complex algebraic group with reductive connected component. Let* $x \in G$ *be an element of order* n *and let* $Z(x) \subset G$ *denote the centralizer* *of x. Then the connected component of $Z(x)$ is likewise reductive and the component group $\pi_0(Z(x))$ is an extension of a subgroup of $\pi_0(G)$ by $\pi_0((G^0) \cap Z(x))$, an abelian group of exponent dividing n.*

Proof. — Let σ denote conjugation by x, so that $Z(x) = G_\sigma$ and $(G^0) \cap Z(x) = (G^0)_\sigma$. Recall that G^0 is a central quotient of the product $R \times H$, where R is the connected component of the center of G^0 and H is the derived group of G^0. Since σ is an algebraic automorphism of R, R_σ is an algebraic subgroup of R whose connected component is a torus; as verified in [12; 9.4], H_σ has reductive connected component. Because the kernel and cokernel of $(R \times H)_\sigma \to (G^0)_\sigma$ are both finite, we conclude that $(G^0)_\sigma$ has reductive connected component. Moreover, the inclusion $(G^0)_\sigma \subset Z(x)$ has finite quotient, so that $((G^0)_\sigma)^0 \simeq (Z(x))^0$.

We observe that $\pi_0(Z(x))$ fits in an extension

$$1 \to \pi_0((G^0)_\sigma) \to \pi_0(Z(x)) \to Z(x)/(G^0)_\sigma \to 1$$

with $Z(x)/(G^0)_\sigma$ a subgroup of $\pi_0(G)$. We proceed to show that $\pi_0((G^0)_\sigma)$ is an abelian group whose exponent divides n. Let $Q = R^{\sim} \times H^{\sim} \to G^0$ be the (not necessarily algebraic) universal covering group of G^0 with kernel the central (discrete) subgroup C, and let $\sigma: Q \to Q$ denote the unique automorphism of Q lifting $\sigma: G^0 \to G^0$. Because $(R^{\sim})_\sigma$ is a (connected) subspace of the complex vector group $R^{\sim}$ and $(H^{\sim})_\sigma$ is connected by [12; 8.1], $Q_\sigma \simeq (R^{\sim})_\sigma \times (H^{\sim})_\sigma$ is connected. Using the exact sequence $Q_\sigma \to (G^0)_\sigma \to ((1-\sigma) Q \cap C)/(1-\sigma) C \to 1$ of [12; 4.5] and the connectedness of Q_σ, we conclude the isomorphisms

$$\pi_0((G^0)_\sigma) \simeq (G^0)_\sigma/\mathrm{im}(Q_\sigma) \simeq ((1-\sigma) Q \cap C)/(1-\sigma) C.$$

In particular, $\pi_0((G^0)_\sigma)$ is abelian.

Consider an element $x = g\sigma(g^{-1}) \in (1-\sigma) Q \cap C$. To prove that the exponent of $\pi_0((G^0)_\sigma)$ divides n, it suffices to prove that $x^n \in C$ is congruent to the identity modulo $(1-\sigma) C$. Observe that x is congruent to $\sigma(x)$ modulo $(1-\sigma) C$. On the other hand, $x\sigma(x) \dots \sigma^{k-1}(x) \in C$ equals $g\sigma^k(g^{-1})$. Thus, x^n is congruent modulo $(1-\sigma) C$ to $g\sigma^n(g^{-1})$ which equals the identity. $\square$

Remark. — We thank the referee for the following observation. Proposition 4.1 extends in sharper form to *connected* reductive groups G (over fields of arbitrary characteristic) provided that $x \in G$ is chosen to be semi-simple of finite order n. In this case, $\pi_0(Z(x))$ is isomorphic to a subgroup of $F \equiv \ker\{H^{\sim} \to H\}$. Namely, consider $q: R \times H^{\sim} \to G$ and choose $y \in q^{-1}(\{x\})$. Then $h \to [h, y]$ determines a map (independent of the choice of y) $r: q^{-1}(Z(x)) \to F$ which factors through the restriction of q, $q^{-1}(Z(x)) \to Z(x)$. Since $h \in q^{-1}(Z(x))$ lies in the kernel of r if and only if $h \in Z(y)$ and since $Z(y)$ is connected by [11; 3.9], we conclude that the kernel of the induced map $\bar{r}: Z(x) \to F$ is precisely $Z(x)^0$. Since $[h, y]$ is central, $[h, y]^m = [h, y^m]$ and thus the image of $\pi_0(Z(x)) = Z(x)/Z(x)^0 \to F$ has exponent dividing n.

The following theorem is a scheme-theoretic version of Proposition 4.1.

(4.2) THEOREM. — *Let* $S = \operatorname{Spec} \mathbf{W}$ *and let* G_S *be a generalized reductive group over* S *whose component group* $\pi_0(G_S)$ *has order not divisible by p. If* $x \in G_S(\mathbf{W})$ *is an element of prime order* $q \neq p$, *then the centralizer* $Z(x)_S$ *of x in* G_S *is a generalized reductive group over* S *with component group* $\pi_0(Z(x)_S)$ *of order not divisible by p.*

Proof. – Once $Z(x)_S$ has been shown to be generalized reductive, the assertion concerning the order of $\pi_0(Z(x)_S)$ is an immediate consequence of Proposition 4.1. Let σ denote conjugation by x, so that $Z(x)_S = (G_S)_\sigma$. The argument of the first paragraph of (4.1) is valid for algebraic groups over arbitrary algebraically closed fields and thereby proves that the connected component of $Z(x)_S$ is reductive.

Write $Z(x)_S$ as a disjoint union of connected components $Z(x)_S^a$ indexed by $a \in \pi_0(Z(x)_S)$, each of which is smooth over S (*cf.* [6; 1.6]). If there exists some element $z_a \in Z(x)_S^a(\mathbf{W})$, then multiplication by z_a determines an isomorphism $(Z(x)_S)^0 \to Z(x)_S^a$. Thus, surjectivity of $Z(x)_S(\mathbf{W}) \to \pi_0(Z(x)_S)$ implies that $Z(x)_S$ is generalized reductive. The smoothness of $Z(x)_S^a$ implies that if $Z(x)_S^a(\mathbf{F})$ is non-empty, then $Z(x)_S^a(\mathbf{W})$ is also non-empty (*cf.* [8; 1.3.24 *b*, 1.4.2 *d*]). Thus, it suffices to prove the surjectivity of the group homomorphism $Z(x)_S(\mathbf{W}) \to \pi_0(Z\mathbf{C})$, where $Z\mathbf{C}$ is the complex Lie group with underlying discrete group $Z(x)_S(\mathbf{C})$. This will be achieved by showing that every Sylow subgroup $L \subset \pi_0(Z\mathbf{C})$ is conjugate to a subgroup in the image of $Z(x)_S(W) \to \pi_0(Z\mathbf{C})$.

Let $L \subset \pi_0(Z\mathbf{C})$ be an l-Sylow subgroup. We may assume that $l \neq p$, since the order of $\pi_0(Z\mathbf{C})$ is prime to p by Proposition 4.1. Applying [6; 3.2], we find a finite l-subgroup M of $Z(x)_S(\mathbf{C})$ mapping onto L. By Proposition 2.4, the nilpotent subgroup $\langle x, M \rangle \subset Z(x)_S(\mathbf{C}) \subset G_S(\mathbf{C})$ generated by x and M is conjugate via some $w \in G_S(\mathbf{C})$ to a subgroup of $G_S(\mathbf{W})$. Moreover, by Theorem 3.4, there exists some $y \in G_S(\mathbf{W})$ such that $x^w = x^y$. Consequently, if $t = wy^{-1}$, then $t \in Z(x)_S(\mathbf{C})$ and $M^t \subset Z(x)_S(\mathbf{C}) \cap G_S(\mathbf{W}) = Z(x)_S(\mathbf{W})$. Since M^t maps onto an l-Sylow subgroup of $\pi_0(Z\mathbf{C})$ conjugate to L, we are done. □

An easy inductive argument on the order of an elementary abelian group E implies the following consequence of Theorem 4.2.

(4.3) COROLLARY. — *Let* $S = \operatorname{Spec} \mathbf{W}$ *and let* G_S *be a generalized reductive group over* S *with* $\pi_0 G_S$ *of order prime to p. If* $E \subset G_S(\mathbf{W})$ *is an elementary abelian q-group for some prime* $q \neq p$, *then the centralizer* $Z(E)_S$ *of* E *in* G_S *is a generalized reductive group over* S *whose component group* $\pi_0(Z(E)_S)$ *has order not divisible by p.* □

Corollary 4.3 and Theorem 3.4 provide us with the following result concerning the normalizer of an elementary abelian q-group.

(4.4) THEOREM. — *Let* $S = \operatorname{Spec} \mathbf{W}$ *and let* G_S *be a generalized reductive group over* S, *with* $\pi_0(G_S)$ *of order prime to p. If* $E \subset G_S(\mathbf{W})$ *is an elementary abelian q-group for some prime* $q \neq p$, *then the normalizer* $N(E)_S$ *of* E *in* G_S *is a generalized reductive group over* S.

Proof. – The quotient group $N(E)_S/Z(E)_S$ is a subgroup of $\operatorname{Aut}(E_S)$, so that it suffices to prove that the order of the **F**-valued points of this quotient equals the order of the **C**-valued points. This is given by Theorems 2.2 and 3.4, which imply that the inclusion $E \subset G_S(\mathbf{W})$ and this inclusion preceded by an automorphism are conjugate as maps to $G_S(\mathbf{W})$ if and only they determine conjugate maps to $G_S(\mathbf{F})$ if and only if they are conjugate as maps to $G_S(\mathbf{C})$. □

5. Finite solvable subgroups

In this section, we apply Theorem 4.2 and induction to prove in Theorem 5.2 that the natural map $\Psi: \mathrm{Hom}_c(\pi, G_S(\mathbf{F})) \to \mathrm{Hom}_c(\pi, G\,\mathbf{C})$ of (2.3) is a bijection whenever π is a finite, solvable group of order prime to p. Theorem 5.4 gives a reformulation of Theorem 5.2 in terms of categories of finite solvable subgroups.

We require the following lemma, the first assertion of which we have implicitly employed in several arguments of previous sections.

(5.1) LEMMA. — *Let G_S be a generalized reductive group over* S. *If $G_S \to K_S \to \mu_S$ is an extension of group schemes over* S *with μ_S finite etale over* S, *then K_S is also generalized reductive. If $\eta_S \subset G_S$ is a closed, normal subgroup scheme with η_S finite etale over* S *and* $\dim_S(\eta_S)$ *invertible as a global function on S, then G_S/η_S is representable by a generalized reductive group over* S.

Proof. — The first assertion is an easy consequence of the fact that the sheaf-theoretic quotient K_S/G_S^0 fits in an extension of one finite etale group scheme over S by another and is therefore representable by a finite etale group scheme over S. To prove the second assertion, we observe that the hypothesis on $\dim_S(\eta_S)$ implies that $\gamma_S = \eta_S \cap G_S^0$ is a subgroup of G_S^0 of multiplicative type, so that G_S^0/γ_S is representable by a reductive group over S ([10]; VIII.5.1; XIX.1.7). A homogeneity argument now implies that G_S/η_S is representable by a group scheme over S containing G_S^0/γ_S as an open and closed subscheme. Since the quotient $(G_S/\eta_S)/(G_S^0/\gamma_S)$ is isomorphic to the quotient of G_S/G_S^0 by η_S/γ_S, we conclude that it is finite etale and thus that G_S/η_S is generalized reductive. □

(5.2) THEOREM. — *Let* $S = \mathrm{Spec}\,\mathbf{W}$ *and let* G_S *be a generalized reductive group over* S. *Then for any finite solvable group π of order prime to p, the natural map of* (2.3)

$$\Psi: \quad \mathrm{Hom}_c(\pi, G_S(\mathbf{F})) \to \mathrm{Hom}_c(\pi, G_S(\mathbf{C})) = \mathrm{Hom}_c(\pi, G\,\mathbf{C})$$

is a bijection.

Proof. — Let $\beta : \mathrm{Hom}_c(\pi, G_S(\mathbf{W})) \to \mathrm{Hom}_c(\pi, G\,\mathbf{C})$ be the natural map. By Theorem 2.2, the injectivity (respectively, surjectivity) of Ψ is equivalent to the injectivity (resp., surjectivity) of

$$\beta: \quad \mathrm{Hom}_c(\pi, G_S(\mathbf{W})) \to \mathrm{Hom}_c(\pi, G\,\mathbf{C}).$$

To prove the injectivity of β, we consider homomorphisms $f, g: \pi \to G_S(\mathbf{W})$ for which there exists some $x \in G_S(\mathbf{C})$ with $f^x = g$ and proceed to prove the existence of some $y \in G_S(\mathbf{W})$ with $f^y = g$. Clearly, we may assume that f and g are injective. For notational simplicity, we shall successively replace f by $G_S(\mathbf{W})$-conjugates, the last of which shall be equal to g. Observe that we may assume $x \in G_S^0(\mathbf{C})$ by replacing f by f^w for some

$w \in G_S(\mathbf{W})$ with the image of w equal to that of x in $\pi_0 G\mathbf{C}$. It then follows that $f(\pi)$ and $g(\pi)$ map to the same subgroup $H \subset \pi_0 G\mathbf{C}$, so that we may replace G_S by the open-closed subgroup with connected component group equal to H. Thus, we may assume that G_S has component group of order prime to p.

We choose an elementary abelian normal subgroup A_1 of $\pi = \pi_1$ and set π_2 equal to π_1/A_1. Inductively, let A_i be an elementary abelian normal subgroup of π_i and set $\pi_{i+1} = \pi_i/A_i$. Since π is solvable, we have that $A_d = \pi_d$ for some d. Using Theorem 3.4, we may replace f by some $G_S(\mathbf{W})$-conjugate with the property that $f(A_1) = g(A_1)$, denoted by B_1 for notational convenience. By Theorem 4.4, $N(B_1)_S$, the normalizer of B_1 in G_S, is generalized reductive over S. Moreover, our x (for which $f^x = g$) lies in $N(B_1)_S(\mathbf{C})$ because $B_1 = f(A_1) = g(A_1) = f^x(A_1) = (B_1)^x$. As explained above, we may assume that $x \in N(B_1)^0_S(\mathbf{C})$ by replacing f by an $N(B_1)_S(\mathbf{W})$-conjugate.

Let $M_{1,S}$ be the open-closed subgroup of $N(B_1)_S$ with component group generated by $f(\pi)$ [or, equally, $g(\pi)$], so that $\pi_0 M_{1,S}$ is of order prime to p. By Lemma 5.1, $G_{2,S} = M_{1,S}/B_1$ is also generalized reductive. Let $f_2, g_2 : \pi_2 \to G_{2,S}(\mathbf{W})$ be the maps induced by f and g respectively. Using Theorem 3.4 again, we may replace f_2 by some $G_{2,S}(\mathbf{W})$-conjugate with the property that $f_2(A_2) = g_2(A_2)$, also replacing f by a $G_S(\mathbf{W})$-conjugate which induces the new f_2. Continuing inductively, we obtain $f_i, g_i : \pi_i \to G_{i,S}$; $f_i(A_i) = g(A_i) = B_i$; f_i, g_i induced by f_{i-1}, g_{i-1}; $G_{i+1,S} = M_{i,S}/B_i$. Here, $M_{i,S}$ is an open-closed subgroup of the normalizer $N(B_i)_S \subset G_{i,S}$ with $\pi_0 M_{i,S}$ of order prime to p and $f_i(\pi_i)$, $g_i(\pi_i)$ are both contained in $M_{i,S}(\mathbf{W})$. Moreover, the projection of x (where $f^x = g$) is contained in $M^0_{i,S}(\mathbf{C})$.

Set $W_{d,S} = M_{d,S}$ and define $W_{i,S}$ by descending induction on i to be the pull-back of the quotient map $M_{i,S} \to G_{i+1,S}$ via the closed immersion $W_{i+1,S} \to G_{i+1,S}$. Thus, we obtain a tower of cartesian squares, a section of which looks as follows:

$$\begin{tikzcd}
W_{i-1,S} \arrow[rr] \arrow[d] & & M_{i-1,S} \arrow[d] \\
W_{i,S} \arrow[r] \arrow[d] & M_{i,S} \arrow[r] \arrow[d] & G_{i,S} \\
W_{i+1,S} \arrow[r] & G_{i+1,S} &
\end{tikzcd}$$

Hence, $W_{i,S} \to W_{i+1,S}$ is an etale surjective homomorphism with kernel isomorphic to the constant group scheme over S with component group B_i. By construction, the maps $f, g : \pi \to G_S(\mathbf{W})$ factor through $W_{1,S}(\mathbf{W}) \subset G_S(\mathbf{W})$ and their compositions with $W_{1,S}(\mathbf{W}) \to W_{1,S}(\mathbf{C})$ are conjugate by $x \in W_{1,S}(\mathbf{C})$. Moreover, the image groups $f(\pi), g(\pi) \subset W_{1,S}(\mathbf{W})$ are equal, given as the normal subgroup $\ker \{W_{1,S}(\mathbf{W}) \to W_{d,S}(\mathbf{W})\}$. Since $W_{1,S}$ is generalized reductive, we may replace f by a conjugate f^y, $y \in W_{1,S}(\mathbf{W})$, with y projecting to the image of x in $\pi_0 W_{1,S}$. This permits us to assume that $f^x = g$ for some $x \in W^0_{1,S}(\mathbf{C})$. Since the conjugation action of the connected group $W^0_{1,S}(\mathbf{C})$ on the finite normal subgroup $f(\pi) = g(\pi) \subset W_{1,S}(\mathbf{C})$ is necessarily trivial, we conclude that our much modified f [$G_S(\mathbf{W})$-conjugate to our original f] equals g.

We conclude the proof of (5.2) by proving the surjectivity of β. Let $A \subset \pi$ be some non-trivial normal elementary abelian subgroup of π, and consider some monomorphism $f: \pi \to G\mathbf{C}$. Without loss of generality, we may assume that $\pi_0 G_S$ has order prime to p. We proceed to verify that some $G_S(\mathbf{C})$-conjugate of f has image contained in $G_S(\mathbf{W})$. Using Theorem 3.4, we find some $w \in G_S(\mathbf{C})$ such that $f^w(A) \subset G_S(\mathbf{W})$. Applying Theorem 4.4 with $E = f^w(A)$, we may replace G_S by $N(E)_S$ and so assume that $E_S \subset G_S$ is normal. By Lemma 5.1, $H_S = G_S/E_S$ is a generalized reductive group over S. Let $g: \pi/A \to H_S(\mathbf{C})$ denote the map induced by f^w. Arguing by induction (on the cardinality of π), we may find some $t \in G_S(\mathbf{C})$ with the property that $g^t: \pi/A \to H_S(\mathbf{C})$ has image contained in $H_S(\mathbf{W})$. Since $f^{wt}: \pi \to G_S(\mathbf{C})$ has image contained in the pre-image of $H_S(\mathbf{W})$ which is contained in $G_S(\mathbf{W})$, we conclude that $f^{wt}(\pi) \subset G_S(\mathbf{W})$ as required. □

Proof of Theorem 1.1. — If G is a compact connected Lie group, then the complex form $G\mathbf{C}$ of G is the complex Lie group associated to the complex points of a reductive group over Spec $\mathbf{Z}$ (*cf.* [1]). Moreover, for any finite group π the inclusion $G \subset G\mathbf{C}$ induces a bijection $\mathrm{Hom}_c(\pi, G) \to \mathrm{Hom}_c(\pi, G\mathbf{C})$ (*cf.* [2]). Thus, Theorem 1.1 follows as a special case of Theorem 5.2. □

We conclude with a reinterpretation of Theorem 5.2 in terms of categories of solvable subgroups, the point of view taken in [6].

(5.3) Definition. — For any group H, let $\mathscr{S}_{p'}(H)$ denote the category whose objects are finite solvable subgroups of H of order prime to p and whose morphisms from $\pi \subset H$ to $\tau \subset H$ are group homomorphisms from π to τ which are the restrictions of inner automorphism of H.

We leave to the reader the straight-forward verification of the following theorem from Theorems 2.2 and 5.2.

(5.4) Theorem. — *Let* $S = \mathrm{Spec}\,\mathbf{W}$ *and let* G_S *be a generalized reductive group over* S. *Then there are natural equivalences of categories*

$$\mathscr{S}_{p'}(G_S(\mathbf{F})) \leftarrow \mathscr{S}_{p'}(G_S(\mathbf{W})) \to \mathscr{S}_{p'}(G_S(\mathbf{C}))$$

induced by the homorphisms $G_S(\mathbf{W}) \to G_S(\mathbf{F})$, $G_S(\mathbf{W}) \to G_S(\mathbf{C})$. □

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