ANNALES SCIENTIFIQUES DE L'É.N.S.

DETLEF GROMOLL
KARSTEN GROVE

A generalization of Berger's rigidity theorem for positively curved manifolds

Annales scientifiques de l'É.N.S. 4e série, tome 20, nº 2 (1987), p. 227-239
<http://www.numdam.org/item?id=ASENS_1987_4_20_2_227_0>

Ann. scient. Éc. Norm. Sup.,
4 série, t. 20, 1987, p. 227 à 239.

« dedicated to Marcel Berger on his sixtieth birthday »

A GENERALIZATION OF BERGER'S RIGIDITY THEOREM FOR POSITIVELY CURVED MANIFOLDS (1)

By Detlef GROMOLL and Karsten GROVE

In this paper we consider a rigidity problem for compact connected riemannian manifolds M of dimension $n \geqq 2$, with positive sectional curvature K. For convenience we normalize the metric so that $K \geqq 1$. By the classical result of Bonnet-Myers, the diameter of M satisfies $\operatorname{diam}(M) \leqq \pi$, and by a theorem of Toponogov equality holds if and only if M is isometric to the unit sphere $S^n(1)$ in $\mathbb{R}^{n+1}$. The question we are concerned with here arises from the following result of [GS], a homotopy version of which was first proved by Berger (*cf.* [CE] or [GKM]).

Diameter sphere theorem. — *A complete riemannian manifold* M *of dimension* $n \geqq 2$ *with* $K \geqq 1$ *and* $\operatorname{diam}(M) > \pi/2$ *is homeomorphic to the sphere* S^n.

This conclusion is no longer true if the condition $\operatorname{diam}(M) > \pi/2$ is relaxed to $\operatorname{diam}(M) \geqq \pi/2$. Real projective space, as well as lens spaces in general, with metrics of constant curvature 1 provide simple counterexamples. The other projective spaces with their standard metrics of curvature $1 \leqq K \leqq 4$ are simply connected examples with diameter $\pi/2$.

In this paper we give an essentially complete classification of the manifolds M with $K \geqq 1$ and $\operatorname{diam}(M) = \pi/2$.

Theorem A. — *Let* M *be a complete riemannian manifold of dimension* $n \geqq 2$ *with* $K \geqq 1$ *and* $\operatorname{diam}(M) = \pi/2$. *Then either*

(i) M *is a twisted sphere, or*

(ii) $\tilde{M}$, *the universal covering of* M, *is isometric to a rank* 1 *symmetric space, except possibly when* M *has the integral cohomology ring of the Cayley plane* $\mathbb{C}aP^2$.

Moreover in the non-simply connected case we have:

(1) This work was partially supported by the National Science Foundation (Grant MCS 7802679A03) and the Danish Natural Science Research Council (511-8169, 10060).

THEOREM B. — *Let* M *be as above with non-trivial fundamental group* $\pi_1(M)=\Gamma$. *Then either*

(*i*) $\tilde{M}$ *is isometric to* $S^n(1)$, *and the action of* Γ *on* $S^n(1)$ *has a proper totally geodesic invariant subsphere* $S^k(1)$ *in* $S^n(1)$, *or*

(*ii*) $\tilde{M}$ *is isometric to* $\mathbb{C}P^{2d-1}$ *with its standard metric of curvature* $1 \leqq K \leqq 4$, *and* Γ *is isomorphic to* $\mathbb{Z}_2$ *acting on* $\mathbb{C}P^{2d-1}$ *by the involution* I,

$$I[z_1, \ldots, z_{2d}]=[\bar{z}_{d+1}, \ldots, \bar{z}_{2d}, -\bar{z}_1, \ldots, -\bar{z}_d],$$

in homogeneous coordinates of $\mathbb{C}P^{2d-1}$.

These results were announced in [GG_1]. Special cases of Theorem B were discussed in [SS] and [Sa]. For a classification of spherical space forms we refer to [W].

As the diameter sphere theorem generalizes the classical sphere theorem of Rauch, Berger, and Klingenberg, Theorem A above extends the following well-known rigidity result.

THEOREM (Berger). — *Let* V *be a complete simply connected riemannian manifold with* $1 \leqq K \leqq 4$. *Then* V *is a twisted sphere, or* V *is isometric to a rank* 1 *symmetric space.*

In fact, the assumptions $1 \leqq K \leqq 4$ and V simply connected imply, although non-trivially, that the injectivity radius of V satisfies $\operatorname{inj}(V) \geqq \pi/2$ (*cf.* [CG_1] or [KS]), in particular $\operatorname{diam}(V) \geqq \pi/2$.

The paper is divided into five sections. First we construct a "dual" pair of convex sets A and A′ in M at maximal distance (*cf.* also [SS], [Sa], and [S]). In Section 2, using ideas of [GS], we show that the complement of a "tubular" neighborhood of $A \cup A'$ in M is topologically a product. This implies in particular: If A and A′ are both contractible then M is a twisted sphere (Theorem 2.5). In the remaining cases we prove, in Section 3, that A and A′ have no boundary (one of them is possibly a point), and that any geodesic perpendicular to A gives rise to a minimal connection from A to A′, and vice versa. This leads to the construction of a riemannian submersion from the unit normal sphere at any point of A′ to A, and vice versa. Our study of metric fibrations in [GG_2]and [GG_3] is then used in Section 4 to deal with the simply connected case (Theorem 4.3 and Remark 4.4). In Section 5 we give a similar analysis for the covering space $\tilde{M}$ of M, when M is not simply connected (Theorem 5.1, 5.2, 5.3). Theorems A and B are immediate consequences of 2.5, 3.2, 4.3, 5.1, 5.2 and 5.3.

We refer to [GKM] and [CE] for basic tools and results in riemannian geometry that will be used freely.

1. Dual convex sets

The metric distance between points $x, y \in M$ is denoted by $d(x, y)$, and $d(x, B)$ is the distance from x to a subset $B \subset M$. For a smooth submanifold $V \subset M$ and $x \in V$, $T_x V$ and $T_x^{\perp}$ are the tangent and normal spaces of V at x, respectively. As usual $\exp_V : T^{\perp} V \to M$ is the (normal) exponential map, and $C(V)$ is the cutlocus of V in M.

All geodesics are parametrized by arc length on [0,], unless otherwise stated, and L denotes the arc length functional. To specify the initial direction $u \in T_x M$ of a geodesic emanating from x, we sometimes write c_u, i. e. $c_u(t) = \exp_x(tu)$ or $\dot{c}_u(0) = u$.

Recall that a *hinge* at p in M is a triple (c_1, c_2, α) where c_1 and c_2 are geodesics in M with $c_1(0) = c_2(0) = p$ and and $\measuredangle(\dot{c}_1(0), \dot{c}_2(0)) = \alpha$, The following version of the basic triangle comparison theorem of Toponogov will be most important to us throughout this paper:

THEOREM 1.1 (Toponogov). — *Suppose* $K \geqq 1$. *Let* (c_1, c_2, α) *be a hinge in* M *and* $(\bar{c}_1, \bar{c}_2, \alpha)$ *a hinge in* $S^2(1)$ *with* $L(c_i) = L(\bar{c}_i) = l_i$, $i = 1,2$.

(i) *If* c_1 *is minimal and* $l_2 < \pi$ *then*

$$d(c_1(l_1), c_2(l_2)) \leqq d(\bar{c}_1(l_1), \bar{c}_2(l_2)).$$

(ii) *If in addition* $0 < \alpha < \pi$ *and equality holds in* (i), *then there is a minimal geodesic* c_3 *in* M *from* $c_1(l_1)$ *to* $c_2(l_2)$ *such that: The triangle* (c_1, c_2, c_3) *spans an immersed totally geodesic surface in* M, *of constant curvature* 1, *in which the minimal connections from* $c_1(l_1)$ *to* $c_2(t)$, $0 \leqq t \leqq l_2$, *are also minimal in* M.

DEFINITION 1.2. — *A subset* $B \subset M$ *is totally* a-*convex*, $0 < a \leqq \infty$, *if for any pair* p_1, $p_2 \in B$ *and any geodesic* $c : [0, l] \to M$ *with* $c(0) = p_1$, $c(l) = p_2$ *and* $l < a$, *one has* $c[0, l] \subset B$.

Obviously, B is totally convex in the sense of [CG_2] if and only if it is totally ∞-convex.

From now on we always assume that $K \geqq 1$ and $\operatorname{diam}(M) = \pi/2$.

For any subset $B \subset M$ let B' denote the set of points in M at maximal distance $\pi/2$ from B, i. e.

$$B' = \left\{ x \in M \mid d(x, B) = \frac{\pi}{2} \right\}.$$

We refer to B' as the *dual* set of B in M.

PROPOSITION 1.3. — *B' is totally π-convex.*

Proof. — Since clearly $B' = \bar{B}'$ and $\bar{B}' = \bigcap_{p \in B} \{p\}'$, we need only consider the case $B = \{p\}$. Suppose $d(x_1, p) = d(x_2, p) = \pi/2$, and let c be a geodesic from x_1 to x_2 with $L(c) < \pi$. Since x_1 is at maximal distance from p, we can choose a minimal geodesic c_1 from x_1 to p such that $\alpha = \measuredangle(\dot{c}_1(0), \dot{c}(0)) \leqq \pi/2$. But $L(c) < \pi$ and $d(p, x_2) = \pi/2$, and we obtain $\alpha = \pi/2$ from (i) and $d(p, c(t)) = \pi/2$ for any $0 \leqq t \leqq L(c)$ from (ii) in 1.1.

The following properties of dual sets are obvious:

(i) $B \subset B''$,

(ii) if $B_1 \subset B_2$ then $B'_1 \supset B'_2$,

and in particular,

(iii) $B' = B'''$.

From now on we fix a non-empty set $B \subset M$ with $B' \neq \varnothing$ and put $A = B'$ (for B we could choose a suitable point). Then $A'' = A$, i. e.

$$A' = \left\{ x \in M \,\middle|\, d(x, A) = \frac{\pi}{2} \right\}, \qquad A = \left\{ x \in M \,\middle|\, d(x, A') = \frac{\pi}{2} \right\}.$$

We refer to A and A′ as a *dual pair* in M. By the structure theorem for convex sets in riemannian manifolds, we know in particular that A and A′ are topological manifolds with (possibly empty) boundary whose interior is smooth and totally geodesic (*cf.* [CG_2]).

Let $a = \dim A$ and $a' = \dim A'$.

PROPOSITION 1.4. — $a + a' \leqq n - 1$.

Proof. — If A or A′ is a point the claim is obvious. Assume $a, a' \geqq 1$ and choose interior points $x \in A$, $x' \in A'$. Let c be a minimal geodesic from x to x'. Now if $a + a' \geqq n$ then there exists a unit parallel field X along c which is tangent to A at x and tangent to A′ at x'. Since M has positive curvature, X gives rise to shorter curves connecting A and A′, which is a contradiction.

Aside from the intrinsic structure of A and A′, their extrinsic properties are equally important. For any subset $B \subset M$ and any $\varepsilon \geqq 0$, consider

$$^{\varepsilon}B = \{ x \in M \mid d(x, B) \leqq \varepsilon \}.$$

Using the compactness of M, one easily obtains the following facts, *cf.* also [Wa].

LEMMA 1.5. — *There exists an $\varepsilon_0 > 0$ such that for any ε with $0 < \varepsilon \leqq \varepsilon_0$ and any closed convex set $C \subset M$:*

(i) *For each $q \in {}^{\varepsilon}C$ there is a unique $q^* \in C$ with $d(q, q^*) = d(q, C)$, and there is a unique minimal geodesic segment qq^* from q to q^*.*

(ii) *The map $q \to q^*$ from ${}^{\varepsilon}C$ to C is Lipschitz. Deforming ${}^{\varepsilon}C$ along the geodesic segments qq^* to q^* defines a strong deformation retract of ${}^{\varepsilon}C$ onto C.*

(ii) $\partial^{\varepsilon} C = \{ x \in M \mid d(x, C) = \varepsilon \}$ *is a codimension* 1 *submanifold in* M *of class* C^1 *and a strong deformation retract of* ${}^{\varepsilon}C \setminus C$.

If in the above lemma C has no boundary, everything is of course smooth, and ${}^{\varepsilon}C$ is nothing but a tubular neighborhood of C in M.

2. Topological duality and the non-rigid case

Following the ideas in [GS] we can now smooth either one of the distance functions $d(\ , A) = d_A$, $d(\ , A') = d_{A'}$, and construct a smooth gradient vector field on M, which for sufficiently small $\varepsilon > 0$ is transversal to the boundary components $\partial^{\varepsilon} A$ and $\partial^{\varepsilon} A'$ of $M_{\varepsilon} = M \setminus ({}^{\varepsilon}A \cup {}^{\varepsilon}A')$, with no zeros in M_{ε}. For many purposes it is sufficient to have such a "gradient-like" vector field, which was observed in [G]. This simplifies some arguments. As these constructions are rather straight-forward extensions of those in [GS] (*cf.* also [G]) we only give a brief discussion.

Let H_x denote the set of all hinges $(c.\ c', \alpha)$ at $x \in M_0 = M \setminus (A \cup A')$, where c, c' are minimal geodesics from x to A, A'. We define the function $\alpha : M_0 \to \mathbb{R}$ by

$$\alpha(x) = \min \{ \alpha \mid (c,\ c',\ \alpha) \in H_x \}$$

and conclude

$$\frac{\pi}{2} < \alpha \leqq \pi \tag{2.1}$$

from 1.1 (i).

We say that $x \in M \setminus A$ (resp. $M \setminus A'$) is a *critical point* for the (non-smooth) function d_A (resp. $d_{A'}$) if for any non-zero $u \in T_x M$ there is a minimal geodesic c from x to A (resp. A') such that $\measuredangle (u,\ \dot{c}(0)) \leqq \pi/2$. By 2.1 both d_A and $d_{A'}$ have no critical points in M_0. As a consequence we obtain the following topological duality between A and A' in M.

THEOREM 2.2. — *Let* $0 < \varepsilon \leqq \varepsilon_0$ *as in Lemma* 1.5. *Then* M *is* C^1*-diffeomorphic to* ${}^{\varepsilon}A \cup_F {}^{\varepsilon}A'$, *for some diffeomorphism* $F : \partial^{\varepsilon} A \to \partial^{\varepsilon} A'$. *In particular* A (*resp.* A') *is a strong deformation retract of* $M \setminus A'$ (*resp.* $M \setminus A$).

Proof. — By 2.1, for any $x \in \bar{M}_{\varepsilon}$ we find a smooth non vanishing vector field U_x defined in a neighborhood N_x of x in M_0 with

$$\measuredangle (U_x(y),\ \dot{c}(0)) < \frac{\pi}{2} \qquad \text{and} \qquad < (U_x(y),\ \dot{c}'(0)) > \frac{\pi}{2}, \tag{2.3}$$

for all $y \in N_x$ and all minimal geodesics c, c' from y to A, A'. Let f_i be a partition of unity subordinate to the covering $\{ N_x \}$ of $\bar{M}_{\varepsilon}$. Then $U = \sum f_i U_{x_i} / \| \sum f_i U_{x_i} \|$ is a smooth unit vector field on $\bar{M}_{\varepsilon}$ satisfying 2.3 for all $y \in \bar{M}_{\varepsilon}$. Notice that by 1.5, U is traversal to both components $\partial^{\varepsilon} A$, $\partial^{\varepsilon} A'$ of ∂M_{ε}. Using 2.3 and the first variation formula, d_A is strictly decreasing on any integral curve of U. Now there is a constant $T > 0$ such that any maximal integral curve φ with $\varphi(0) \in \partial^{\varepsilon} A'$ will reach $\partial^{\varepsilon} A$ before time T. Otherwise, by a limiting argument, we would find a local integral curve on which d_A is constant. Therefore we have shown $\bar{M}_{\varepsilon}$ is C^1-diffeomorphic with $\partial^{\varepsilon} A' \times [0,1]$ and this is enough to complete the proof.

Remark 2.4. — It is straightforward to modify the construction of the vector field U in the proof of 2.2 so that

$$U(c(t)) = \dot{c}(t), \qquad \varepsilon \leqq t \leqq \frac{\pi}{2} - \varepsilon,$$

along all minimal geodesics c from A' to A. This will be used in 3.5.

THEOREM 2.5. — *A* (*resp.* A') *is contractible iff it is a point or has non-empty boundary. If both* A *and* A' *are contractible, then* M *is homeomorphic to the sphere* S^n.

This is a direct consequence of Theorem 2.2 and the following

LEMMA 2.6. — *Let* $C \subset M$ *be a closed convex set and let* $\varepsilon > 0$ *be as in Lemma* 1.5. *Then* ${}^{\varepsilon}C$ *is homeomorphic (in fact* C^1*-diffeomorphic) to the disc* D^n *if* $\partial C \neq \varnothing$.

Proof. — First note that if $\dim C = k$ then C is homeomorphic to the disc D^k. This follows from the fact the fact that the function $\psi : C \to \mathbb{R}$ given by $\psi(x) = d(x, \partial C)$ is strictly concave since K is positive, *cf.* Theorem 1.10 of [CG_2]. In particular for each $a \geqq 0$, the set

$$C^a = \{ x \in C \mid d(x, \partial C) \geqq a \}$$

is convex, and the intersection of all $C^a \neq \varnothing$ *is the unique point* s_0 where ψ has its maximum. In the language of [CG_2], s_0 is the soul of C. Now according to Lemma 2.4 of [CG_2] there is $0 < \delta \leqq \varepsilon/2$ such that

$$C^a \subset {}^{\varepsilon/2}(C^{a'}),$$

whenever $0 \leqq a \leqq a' \leqq \max \psi$ and $a' - a < \delta$. Choose numbers $0 = a_0 < a_1 < \ldots < a_r = \max \psi$ with $a_{i+1} - a_i < \delta$ and consider the convex sets

$$C = C^{a_0} \supset C^{a_1} \supset \ldots \supset C^{a_r} = \{ s_0 \}.$$

Now $C^{a_i} \subset {}^{\varepsilon/2}(C^{a_{i+1}})$, and hence ${}^{\varepsilon/2}(C^{a_i}) \subset {}^{\varepsilon}(C^{a_{i+1}})$ for $i = 0, \ldots, r-1$. From 1.5 we then get in particular that for each $q \in \partial^{\varepsilon/2}(C^{a_i})$ there is a closest point $q^* \in C^{a_{i+1}}$ and a unique minimal geodesic from q to q^*, which intersects $\partial^{\varepsilon/2}(C^{a_{i+1}})$ at the unique closest point from q. The connecting minimal geodesics give rise to a C^1-diffeomorphism ${}^{\varepsilon/2}(C^{a_{i+1}}) \to {}^{\varepsilon/2}(C^{a_i})$. Thus we have the following sequence of diffeomorphisms

$${}^{\varepsilon}C \sim {}^{\varepsilon/2}(C^{a_0}) \sim {}^{\varepsilon/2}(C^{a_1}) \sim \ldots \sim {}^{\varepsilon/2}(C^{a_r}) = {}^{\varepsilon/2}(s_0) \sim D^n.$$

See also [S] for a slightly different proof of Theorem 2.5.

Remark 2.7. — We like to mention that the arguments used in Theorem 2.2 are by now known to also provide a simple and direct proof of the diffeomorphism statement [P] in the « Soul Theorem" of [CG_2]: The distance function d_S from a soul S in a complete noncompact manifold M with nonnegative curvature has no critical points outside S. This follows since any $x \in M \setminus S$ lies in the boundary of a compact totally convex set C with $S \subset \operatorname{int} C$, by the basic construction in [CG_2].

3. Normal holonomy of dual sets

From now on we assume that M is not homeomorphic to the sphere S^n. In view of Theorem 2.5 this means, at least one of the two dual sets, in this section say A, is not contractible. In particular $\partial A = \varnothing$ and A is not a point. We will show first that necessarily also $\partial A' = \varnothing$. Then we will prove: The cut locus C(A) of A in M is the dual set A′, and vice versa. The arguments involved will finally lead to the construction of a riemannian submersion from each unit normal sphere of one dual set onto the other. This is a crucial step toward rigidity.

We begin with the following observation, *cf.* also the proof of Proposition 1.3. For a piecewise smooth curve $\sigma:[0, l] \to M$, parallel transport along σ is denoted by $\tau_\sigma^t: T_{\sigma(0)} M \to T_{\sigma(t)} M$.

LEMMA 3.1. — *Let* $c: [0, l] \to A$ *be a geodesic and* c_u *a minimal geodesic from* $c(0)$ *to* A'. *Then* $\tau_c^t(u)$ *defines a minimal geodesic from* $c(t)$ *to* $c_u(\pi/2) \in A'$ *for* $0 \leqq t \leqq l$. *The set of these geodesics form an immersed totally geodesic surface in* M *of constant curvature* 1.

Proof. — It is sufficient to consider $l<\pi$, and since clearly $\measuredangle(u, \dot{c}(0))=\pi/2$, our claim is a direct consequence of 1.1 (ii) applied to the hinge $(c_u, c, \pi/2)$.

Now Proposition 1.4, Theorem 2.5 and Lemma 3.1 yield

THEOREM 3.2. — *If* $n=2$ *then* M *is either homeomorphic to* S^2 *or isometric to* $\mathbb{R}P^2$ *with constant curvature* 1.

Since any smooth curve is a limit of broken geodesics, another immediate consequence of 3.1 is

COROLLARY 3.3. — *Let* $\sigma:[0, l] \to A$ *be a piecewise smooth curve in* A *and suppose the unit vector* $u \in T_{\sigma(0)}^{\perp} A$ *defines a minimal geodesic* c_u *to* A', *i. e.* $c_u(\pi/2) \in A'$. *Then* $\tau_\sigma^t(u)$ *defines a minimal geodesic from* $\sigma(t)$ *to* $c_u(\pi/2)$ *for all* $0 \leqq t \leqq l$.

Pick a unit normal vector $u \in T_p^{\perp} A$ with $c_u(\pi/2)=p' \in A'$. Consider the closure E of the set of unit normal vectors to A obtained from u by parallel translation along piecewise smooth paths σ in A with $\sigma(0)=p$. Then $\pi: E \to A$ is a sub bundle of the unit normal bundle $\pi: T_1^{\perp} A \to A$ of A in M. To see this let $E_q=\{u \in E \mid \pi(u)=q\}$ and observe that E_p is the orbit through u of the closure of the normal holonomy group Φ_p at p. Furthermore, we have $E_q=\tau_\sigma^l(E_p)$ for any $\sigma:[0, l] \to A$ from p to q. Note in particular that the fiber E_q is a compact homogeneous space.

As a first application of this construction we obtain

PROPOSITION 3.4. — *If* A' *is contractible then* $A'=\{p'\}$ *is a point and* $C(A)=\{p'\}$, $C(p')=A$.

Proof. — Let $p' \in A'$ be arbitrary and c_u a minimal geodesic from some $p \in A$ to p'. Consider the bundle $\pi: E \to A$ constructed from u as above. We now claim that $E=T_1^{\perp} A$. Since A' is contractible the total space $T_1^{\perp} A$ is homeomorphic to S^{n-1} by 2.6 and 2.2. Thus if $T_1^{\perp} A \setminus E \neq \varnothing$ we see that $\pi: E \subsetneq T_1 A \to A$ is homotopic to a constant, i.e. there is a homotopy $H: E \times [0, 1] \to A$ with $H_0=\pi$ and $H_1(E)=\{q\}$. Now $\pi \circ \mathrm{id}_E=H_0$, so by the homotopy lifting property of $\pi: E \to A$ we obtain a homotopy $\tilde{H}: E \times [0, 1] \to E$ with $\tilde{H}_0=\mathrm{id}_E$ and $\tilde{H}_1(E) \subset E_q$. This is clearly impossible since E is a closed manifold, which is not a point. Now by 3.3 any $v \in E$ defines a minimal geodesic to p'. This shows that any unit normal vector to A defines a minimal geodesic to p'. Thus $A'=\{p'\}$, $C(A)=\{p'\}$, and then clearly also $C(p')=A$.

As a second application of the holonomy construction above we get similarly

PROPOSITION 3.5. — *If* A' *is not contractible, in particular* $\partial A'=\Phi$, *then* $C(A)=A'$ *and* $C(A')=A$.

Proof. — Consider a bundle $\pi: E \to A$ as before. The diffeomorphism $F: \partial^{\varepsilon} A \to \partial^{\varepsilon} A'$ in 2.2 obtained from the flow of the vector field U of 2.4 induces a diffeomorphism $\hat{F}: T_1^{\perp} A \to T_1^{\perp} A'$ of the unit normal bundles of A and A', via the normal exponential maps. By construction, $\hat{F}$ imbeds E into the normal sphere $S_{p'}$ of A' at p'. Thus $\pi: E \to A$ factors through the sphere $S_{p'}$, and we conclude as in 3.4 that $\hat{F}(E) = S_{p'}$. This proves that $C(A') = A$, and then also $C(A) = A'$.

We have seen in 3.4 and 3.5 that for any $p' \in A'$ there is a fibration

$$\pi_A: \quad S_{p'} \to A, \qquad u' \to \exp\left(\frac{\pi}{2}.u'\right)$$

of the unit normal sphere $S_{p'}$ of A' at p' over A. These fibrations impose additional geometric restrictions on A, and by symmetry also on A' if $A' \neq \{p'\}$. The basic reason for this is contained in

THEOREM 3.6. — *For any $p' \in A'$, the fibration $\pi_A: S_{p'} \to A$ is a riemannian submersion.*

Proof. — By definition π_A is clearly smooth. Now let $v \in T_p A$ be a unit tangent vector and choose $u' \in S_{p'}$ with $\pi_A(u') = c_{u'}(\pi/2) = p$. From 3.1 we see that the geodesic c_v in A together with the geodesic $c_{u'}$ determine a geodesic γ_w in $S_{p'}$ with $\pi_A \circ \gamma_w = c_v$. This shows that π_A is a submersion. Since $w \in T_{u'}(S_{p'})$ is of unit length, all we need to prove is that w is perpendicular to the fiber $\pi_A^{-1}(p)$ in $S_{p'}$ at u'. Let $\varphi: (-\varepsilon, \varepsilon) \to \pi_A^{-1}(p)$ be a smooth curve with $\dot{\varphi}(0) \in T_{u'}(S_{p'})$. Then φ gives rise to a Jacobifield Y along $c_{u'}$ with $Y(0) = 0$ and $Y'(0) = \varphi'(0)$. Similarly w determines a Jacobifield X along $c_{u'}$ with $X(0) = 0$ and $X'(0) = w$ in $T_{p'}^{\perp} A'$. In fact $X(t) = \sin t \cdot W$, where W is a parallel field along $c_{u'}$ determined by $W(\pi/2) = v$. If we let $\mathscr{W}$ denote the vector space of parallel fields W along $c_{u'}$ with $W(\pi/2) \in TA$, then since $Y(\pi/2) = 0$, clearly $\langle Y, W \rangle = 0$ for all $W \in \mathscr{W}$. Therefore also $\langle Y, X' \rangle = 0 = \langle Y, X'' \rangle$ and hence $\langle Y', X' \rangle = 0$. In particular

$$\langle Y'(0), X'(0) \rangle = \langle \dot{\varphi}(0), w \rangle,$$

i. e. w is perpendicular to any vector tangent to the fiber.

Although we will not use it in what follows, we like to point out an interesting and almost immediate consequence of the results in this section.

Remark 3.7. — Using that $S_{p'}$ is isometric to the unit sphere $S^{n-a'-1}$ we find that all geodesics in A are periodic with maximal period π. It then follows that all geodesics in M through A are periodic, and hence from [BB] that the rational cohomology ring of $\tilde{M}$ is generated by one element.

We will, however, make use of the following much stronger and more difficult result.

Remark 3.8. — Suppose A is simply connected. Topologically, the fibers of $\pi_A: S_{p'} \to A$ are homotopy spheres $\sum^k$, $k = 1,3$ or 7, and $k = 7$ can occur only when $S_{p'} = S^{15}$ [B]. Except for that case, our work in [GG_2] and [GG_3] then implies that $\pi_A: S_{p'} \to A$ is metrically congruent to a Hopf fibration. In particular, A is isometric to a rank 1 symmetric space with $1 \leqq K \leqq 4$ and $\operatorname{diam}(A) = \pi/2$.

We are now ready to complete the metric classification of non-spherical manifolds with $K \geqq 1$ and diameter $\pi/2$.

4. Rigidity in the simply connected case

In this section we assume that M is simply connected and not homeomorphic to S^n. In particular $n \geqq 3$, and the dual sets A and A′ in M are totally geodesic closed submanifolds, totally π-convex, one of them possibly a point. It is an immediate consequence of 3.1 that all geodesics in A, A′ are periodic with common (not smallest) period 2π.

LEMMA 4.1. — *Any closed geodesic of lenth $<2\pi$ in* A, A′ *is homotopically trivial in* A, A′.

Proof. — Let c be a closed geodesic of length l in A, i. e. $\dot{c}(l)=\dot{c}(0)$. If $l<2\pi$, then $l=2\pi/k$ for some integer $k \geqq 2$, in particular $l \leqq \pi$. Let $p=c(0)$, $q=c(t_0)$ for some $t_0 \in (0, l)$, and consider the two (not normal) geodesics γ_i in A defined by

$$\gamma_0(s)=c(st_0), \qquad \gamma_1(s)=c(l-s(l-t_0))$$

and $0 \leqq s \leqq 1$. Now let $\{\gamma_t\}$, $0 \leqq t \leqq 1$, be a homotopy of smooth curves in M from γ_0 to γ_1. Since any geodesic in M from p to q of length $>\pi$ has index $\geqq n-1 \geqq 2$ we conclude from (degenerate) Morse theory (*cf.* [CG_1]) that we may assume $L(\gamma_t) \leqq \pi$ for $0 \leqq t \leqq 1$. If for some t, we have $\gamma_t[0,1] \cap A' \neq \varnothing$, then clearly γ_t is a geodesic of length π and γ_t on $[0,1/2]$ is a minimal geodesic from p to A′. It then follows from 3.1 that $q=\gamma_t(1)=c(\pi)$. By choosing t_0 so that $c(\pi) \neq c(t_0)=q$ we conclude that for all $0 \leqq t \leqq 1$ necessarily $\gamma_t[0,1] \subset M \setminus A'$. Now 2.2 implies that γ_0 and γ_1 are homotopic in A, or equivalently, c is homotopically trivial in A.

PROPOSITION 4.2. — *A and* A′ *are simply connected.*

Proof. — Suppose A is not simply connected. Let $c:[0, l] \to A$ be a closed geodesic in A of minimal length in its free homotopy class. Then $l=2\pi$ by 4.1 and $\dim A = a = 1$ by standard comparison and the minimality of c, i. e. $A=c\,[0,2\pi]$ is a simple closed geodesic of length 2π. This is impossible because A is totally π-convex and $\mathrm{diam}(M)=\pi/2$.

We are now in a position to prove

THEOREM 4.3. — *If* M *is simply connected and not homeomorphic to* S^n, *then* M *is isometric to a symmetric space of rank* 1, *except possibly when* $M=M^{16}$ *has the integral cohomology ring of* $\mathbb{C}aP^2$.

Proof. — First suppose A, A′ is a dual pair, and say $A'=\{p'\}$. By 3.4, all geodesics in M starting at p' are simple loops of length π. The generalized version of the Bott-Samelson Theorem in [Be] then implies that M has the integral cohomology ring of a projective space P^m, $m \geqq 2$. Consider the riemannian submersion $\pi_A : S_{p'} \to A$ in 3.6. By the main results of [GG_2], [GG_3] (*cf.* 3.8), π_A is isometrically equivalent to a Hopf fibration $S^{km-1} \to P^{m-1}(k)$, except possibly when $S_{p'}=S^{15}$ and π_A has fibers diffeomorphic to S^7, in our case. In this last case, M^{16} has the integral cohomology ring of the Cayley plane $\mathbb{C}aP^2$. In all the other cases we will now prove that M is isometric to standard $P^m(k)$ with diameter $\pi/2$.

In the model space, for $p'_0 \in \mathrm{P}^m(k)$, we have

$$\mathrm{A}_0 = \{p'_0\}' = \mathrm{C}(p'_0) = \mathrm{P}^{m-1}(k),$$

and $\pi_{\mathrm{A}_0} : \mathrm{S}_{p'_0} \to \mathrm{A}_0$ in 3.6 is a standard Hopf fibration. According to 3.8, we have the commutative diagram

$$\begin{array}{ccccl} \mathrm{S}_{p'} & \xrightarrow{\iota_0} & \mathrm{S}_{p'_0} & = & \mathrm{S}^{km-1} \\ {\scriptstyle \pi_{\mathrm{A}}}\downarrow & & \downarrow{\scriptstyle \pi_{\mathrm{A}_0}} & & \\ \mathrm{A} & \xrightarrow[\iota]{} & \mathrm{A}_0 & = & \mathrm{P}^{m-1}(k) \end{array}$$

with isometries ι, ι_0. Clearly,

$$f: \quad \mathrm{M} \to \mathrm{P}^m(k), \qquad \exp_{p'}(tu) \to \exp_{p'_0}(t\,\iota_0\, u),$$

$0 \leqq t \leqq \pi/2$, is a well defined continuous map and $f|_{\mathrm{A}} = \iota$. By definition, f maps minimal geodesics in M from p' to $p \in \mathrm{A}$ to minimal geodesics in $\mathrm{P}^m(k)$ from p'_0 to $f(p)$. The set of all such geodesics in $\mathrm{P}^m(k)$ form a totally geodesic submanifold isometric to the sphere $\mathrm{S}^k(1/2)$ with diameter $\pi/2$. We claim that the same statement holds in M. In fact let $\mathrm{B} \subset \mathrm{A}$ be the dual set of p in A. Then $\mathrm{B} = \mathrm{A} \cap \{p\}'$ is a totally geodesic, π-convex submanifold of M isometric to standard $\mathrm{P}^{m-2}(k)$. Note that $\{p\}'' = \{p\}$. All this depends on A being isometric to standard $\mathrm{P}^{m-1}(k)$. Now $\mathrm{B}' \subset \mathrm{M}$ is a totally geodesic, totally π-convex submanifold of M which contains the set of minimal geodesics from p to p'. Moreover, for any $q \in \mathrm{B}$, $\pi_{\mathrm{B}'} : \mathrm{S}_q \to \mathrm{B}'$ is a riemannian submersion from the unit normal sphere $\mathrm{S}_q = \mathrm{S}^{2k-1}(1)$ to B′. Again by 3.8, B′ is isometric to $\mathrm{S}^k(1/2)$ and coincides with the set of minimal geodesics from p to p'. It is now an immediate consequence that f maps $\mathrm{M} \setminus \mathrm{A}$ isometrically onto $\mathrm{P}^m(k) \setminus \mathrm{A}_0$, and therefore f is globally isometric.

To complete the proof, we need to find a dual pair A, $\mathrm{A}' = \{p'\}$. If A, A′ is a dual pair, A symmetric, then $\{p\}'' = \{p\}$ for any $p \in \mathrm{A}$, as we have seen, and we are done. By 3.8 it only remains to analyze the possibility $\dim \mathrm{A} = \dim \mathrm{A}' = 8$ and $\dim \mathrm{M} = 24$. Fix $p \in \mathrm{A}$, $p' \in \mathrm{A}'$ and consider the set of minimal geodesics from p to p'. By construction of π_{A}, $\pi_{\mathrm{A}'}$ we see that in this case the closure G of the normal holonomy group at p' acts on $\mathrm{S}_p{}' = \mathrm{S}^{15}$, and the orbits are exactly the fibres of π_{A}. But such an action does not exist. A simple argument in our context can be given as follows: Let $h \in \mathrm{G}$ have a fixed point $x = hx$. Then the differential h_* is the identity on the normal space ν_x of the orbit $\mathrm{G}x$ at x, and thus $h = \mathrm{id}$ on the great sphere $\exp(\nu_x)$. Now let $y \in \mathrm{S}_{p'}$ be arbitrary. The intersection of $\exp(\nu_x)$ and $\exp(\nu_y)$ contains a point z, for dimension reasons. But $z = hz$ and $y \in \exp(\nu_z)$, so by the above also $y = hy$, and we conclude that $h \equiv \mathrm{id}$. Therefore, G would act freely on $\mathrm{S}_{p'}$, which is impossible, since the fibers are 7-spheres and thus not diffeomorphic to a Lie group.

Remark 4.4. — It is clear from the above proof, that part (ii) of Theorem A in the introduction holds without any restriction, if the only Riemannian fibration from the euclidian sphere $\mathrm{S}^{15}(1)$ with fiber S^7 is the Hopf fibration, up to congruence.

5. Rigidity in the non-simply connected case

In this section we assume that M is not simply connected. Let $\tilde{M} \to M$ be the universal riemannian covering of M. Clearly, $\tilde{M}$ satisfies $K \geqq 1$ and $\pi/2 \leqq \operatorname{diam}(\tilde{M}) \leqq \pi$.

We first observe

THEOREM 5.1. — *If* M *is not simply connected and say* $A' = \{p'\}$, *then* M *is isometric to real projective space* $\mathbb{RP}^n$ *of constant curvature* 1.

Proof. — Since $A' = \{p'\}$ is totally π-convex, the fiber $\pi^{-1}(p')$ consists of two points $\tilde{p}_1$ and $\tilde{p}_2$ with $d(\tilde{p}_1, \tilde{p}_2) = \pi$. Therefore $\pi_1(M) \cong \mathbb{Z}_2$, and $\tilde{M}$ is isometric to the sphere $S^n(1)$ by Toponogov's Diameter Theorem. Hence M is isometric to $\mathbb{RP}^n$.

Now let $\tilde{A}$, $\tilde{A}'$ be the inverse image of A, A′ under the covering projection $\tilde{M} \to M$. In the case left, $\tilde{A}$ and $\tilde{A}'$ are connected, totally geodesic and totally π-convex closed submanifolds of $\tilde{M}$, with dimensions a, $a' \geqq 1$. Observe that for any $\tilde{x} \in \tilde{M}$, we have $max\{d_{\tilde{A}}(\tilde{x}), d_{\tilde{A}'}(\tilde{x})\} \leqq \pi/2$, and using the notation from Section 2,

$$(\tilde{A})' = \tilde{A}' \qquad \text{and} \qquad (\tilde{A}')' = \tilde{A}.$$

Hence, by arguments exactly like those in the proof of 2.2, it follows that $\tilde{M}$ is the disjoint union

$$\tilde{M} = {}^{\varepsilon}\tilde{A} \cup \tilde{M}_{\varepsilon} \cup {}^{\varepsilon}\tilde{A}'.$$

The closure of $\tilde{M}_{\varepsilon}$ is diffeomorphic to the product $\partial^{\varepsilon}\tilde{A} \times [0,1]$. This leads to the following reduction.

THEOREM 5.2. — *If* M *is not simply connected and* $\operatorname{diam}(\tilde{M}) > \pi/2$, *then* $\tilde{M}$ *is isometric to* $S^n(1)$, *and the induced orthogonal action of* $\pi_1(M)$ *on* $\mathbb{R}^{n+1}$ *has a proper invariant subspace.*

Proof. — As in the proofs of 4.1 and 4.2 we see that $\tilde{A}$ (resp. $\tilde{A}'$) is either simply connected or a closed geodesic of length 2π. In the latter case we conclude $\operatorname{diam}(\tilde{M}) = \pi$, since $\tilde{A}$ is totally π-convex, and hence by Toponogov's Diameter Theorem, $\tilde{M}$ is isometric to $S^n(1)$. Otherwise, using the above topological description of $\tilde{M}$, a simple transversality argument shows for the inclusion $i_{\tilde{A}}: \tilde{A} \to \tilde{M}$, that in homotopy

$$(i_{\tilde{A}})_*: \quad \pi_q(\tilde{A}) \to \pi_q(\tilde{M})$$

is injective for $q \leqq n - a' - 2$, and surjective for $q \leqq n - a' - 1$. An analogous conclusion holds for $(i_{\tilde{A}'})_*$. This implies that $a + a' = n - 1$, since by the Main Theorem in [GS], M is homeomorphic to S^n. Then by 3.6, both $\tilde{A}$ and $\tilde{A}'$ are of constant curvature 1, and hence isometric to $S^a(1)$ and $S^{a'}(1)$, respectively. Again by Toponogov's Diameter Theorem $\tilde{M}$ is isometric to $S^n(1)$. Moreover, in either of the above cases $\tilde{A}$, $\tilde{A}'$ is a $\pi_1(M)$-invariant pair of orthogonal (complementary) great spheres in $S^n(1)$. This completes the proof (*cf.* also 5.1).

It remains to consider the case $\operatorname{diam}(\tilde{M}) = \pi/2$. In this case $\tilde{M}$ has the integral cohomology ring of a projective space $P^m(k)$, $m \geqq 2$, by 4.3. In particular, dim M is even. By Synge's Theorem $\pi_1(\mathrm{M}) \cong \mathbb{Z}_2$, and there is a fixed point free orientation reversing isometric involution $\mathrm{I}: \tilde{\mathrm{M}} \to \tilde{\mathrm{M}}$ and $\mathrm{M} = \tilde{\mathrm{M}}/\tilde{x} \sim \mathrm{I}(\tilde{x})$. Such a map does not exist for cohomological reasons when m is even. In the remaining cases $\tilde{\mathrm{M}}$ is isometric to a standard projective space by 4.3. It is however well-known [W] that the isometry group of the quaternionic projective space $\mathrm{P}^m(4) = \mathbb{H}\mathrm{P}^m$ is connected and hence contains no orientation reversing element.

Now we are only left with a question about possible fixed point free isometric involutions on $\mathrm{P}^m(2) = \mathbb{C}\mathrm{P}^m$, for $m = 2d-1$ odd and with quotients of diameter $\pi/2$. It is not difficult to see from our constructions that any quotient of $\mathbb{C}\mathrm{P}^{2d-1}$ by a fixed point free involution actually has diameter $\pi/2$. By induction on d, any two such quotients are isometric. As far as existence is concerned, the map

$$\mathrm{I}: \quad \mathbb{C}\mathrm{P}^{2d-1} \to \mathbb{C}\mathrm{P}^{2d-1}, \; [z_1, \ldots, z_{2d}] \to [\bar{z}_{d+1}, \ldots, \bar{z}_{2d}, -\bar{z}_1, \ldots, -\bar{z}_d],$$

in homogeneous coordinates is a fixed point free isometric involution on $\mathbb{C}\mathrm{P}^{2d-1}$.

Summarizing the last conclusions we have

THEOREM 5.3. — *If* M *is not simply connected and not of constant curvature* 1, *then* M *is isometric to* $\mathbb{C}\mathrm{P}^{2d-1}/\mathrm{I}$.

This finally completes the proofs of Theorems A and B in the introduction.

REFERENCES

[BB] L. BÉRARD BERGERY, *Quelques exemples de variétés Riemanniennes où toutes les géodésiques issues d'un point sont fermées et des même longueur, suivis de quelques résultats sur leur topologie* (*Ann. Inst. Fourier*, Vol. 27, 1977, pp. 231-249).

[Be] A. L. BESSE, *Manifolds All of Whose Geodesics Are Closed* (*Ergebnisse der Math. und ihrer Grenzgebiete*, Vol. 93, Springer-Veralg, 1978).

[B] W. BROWDER, *Higher Torsion in H-spaces* (*Trans. Amer. Math. Soc.*, Vol. 108, 1963, pp. 353-375).

[CE] J. CHEEGER and D. G. EBIN, *Comparison Theorems in Riemannian Geometry*, (North Holland Mathematical Library, Vol. 9, 1975).

[CG_1] J. CHEEGER and D. GROMOLL, *On the Lower Bound for the Injectivity Radius of* 1/4-*Pinched Manifolds* (*J. Differential Geometry*, Vol. 15, 1980, pp. 437-442).

[CG_2] J. CHEEGER and D. GROMOLL, *On the structure of Complete Manifolds of Nonnegative Curvature* (*Ann. of Math.*, Vol. 96, 1972, pp. 413-443).

[GG_1] D. GROMOLL and K. GROVE, *Rigidity of Positively Curved Manifolds with Large Diameter* (*Seminar of Differential Geometry, Ann. of Math. Studies*, Vol. 102, 1982, pp. 203-207).

[GG_2] D. GROMOLL and K. GROVE, *One-Dimensional Metric Foliations in Constant Curvature Spaces* (*Differential Geometry and Complex Analysis*, H. E. Rauch memorial volume, Springer-Verlag, 1985, pp. 165-167).

[GG_3] D. GROMOLL and K. GROVE, *The Low Dimensional Metric Foliations of Euclidian Spheres* (*J. Differential Geometry*, to appear).

[GKM] D. GROMOLL, W. KLINGENBERG and W. MEYER, *Riemannsche Geometrie im Grossen* (*Lecture Notes in Mathematics*, No. 55, Springer Verlag, second edition, 1975).

[G] M. GROMOV, *Curvature, Diameter and Betti Numbers* (*Comment. Math. Helv.*, Vol. 56, 1981, pp. 179-195).

[GS] K. GROVE and K. SHIOHAMA, *A Generalized Sphere Theorem* (*Ann. of Math.*, Vol. 106, 1977, pp. 201-211).

Differential Geometry, to appear).

[GKM] D. GROMOLL, W. KLINGENBERG and W. MEYER, *Riemannsche Geometrie im Grossen* (*Lecture Notes in Mathematics*, No. 55, Springer Verlag, second edition, 1975).

[G] M. GROMOV, *Curvature, Diameter and Betti Numbers* (*Comment. Math. Helv.*, Vol. 56, 1981, pp. 179-195).

[GS] K. GROVE and K. SHIOHAMA, *A Generalized Sphere Theorem* (*Ann. of Math.*, Vol. 106, 1977, pp. 201-211).

[KS] W. KLINGENBERG and T. SAKAI, *Injectivity Radius Estimate for 1/4-Pinched Manifolds* (*Archiv der Math.*, Vol. 39, 1980, pp. 371-376).

[P] A. POOR, *Some Results for Nonnegatively Curved Manifolds* (*J. Differential Geometry*, Vol. 9, 1974, pp. 583-600).

[Sa] T. SAKAI, *On the Diameter of Some Riemannian Manifolds* (*Archiv der Math.*, Vol. 30, 1978, pp. 427-434).

[SS] T. SAKAI and K. SHIOHAMA, *On the Structure of Positively Curved Manifolds with Certain Diameter* (*Math. Z.*, Vol. 127, 1972, pp. 75-82).

[S] K. SHIOHAMA, *Topology of Positively Curved Manifolds with a Certain Diameter* (*Minimal submanifolds and Geodesics*, Kaigai Publ., 1978, pp. 217-228).

[Wa] R. WALTER, *On the Metric Projection Onto Convex Sets in Riemannian Spaces* (*Archiv der Math.*, Vol. 25, 1974, pp. 91-98).

[W] J. A. WOLF, *Spaces of Constant Curvature*, second edition 1972, Publish or Perish.

(Manuscrit reçu le 12 mai 1986,
révisé le 7 janvier 1987).

Detlef GROMOLL,
Department of Mathematics,
S.U.N.Y.,
Stony Brook, N.Y. 11794;

Karsten GROVE,
Department of Mathematics
and
Institute for Physical Science
and Technology,
University of Maryland,
College Park, MD 20742.