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CURVES ON GENERIC HYPERSURFACES

BY HERBERT CLEMENS

1. Introduction

Let

\[ V \subseteq \mathbb{P}^n \]

be a smooth hypersurface of degree \( m \geq 2 \) in projective \( n \)-space over an algebraically closed field \( k \). By an immersed curve on \( V \), we will mean a morphism

\[ f: C \to V \]

which is everywhere of maximal rank from a complete non-singular algebraic curve \( C \). Every such mapping has a normal bundle

\[ N_{f, V} = f^*(T_V)/T_C. \]

Our purpose in this paper is to prove:

1.1. Theorem. — Let \( V \) be a generic hypersurface of degree \( m \) in \( \mathbb{P}^n \). Then \( V \) does not admit an irreducible family of immersed curves of genus \( g \) which cover a variety of codimension \( < D \) where

\[ D = \frac{2-2g}{\deg f} + m - (n+1). \]

Notice that, for example, if \( g = 0 \), Theorem 1.1 says that there are no rational curves on generic \( V \), if \( m \geq 2n-1 \).
2. Normal bundles to curves

Let $C$ be a complete non-singular curve and
\[ \phi : E \to C \]
a vector bundle of finite rank. We will call $E$ semi-positive if all quotient bundles of $E$ have non-negative degree.

2.1. Lemma. — Let
\[ E_\xi \to C \]
be an algebraic family of vector bundles of rank $r$ over $C$. If
\[ E_0 \to C \]
is semi-positive, then $E_\xi \to C$ is also semi-positive for each generic $\xi$ which specializes to 0.

Proof. — If the lemma is false, there exists a generic point $\xi'$ and a quotient bundle
\[ E_\xi' \to Q_\xi' \]
such that
\[ 0 < s = \text{rank } Q_\xi' < r \]
and
\[ \deg Q_\xi' < 0. \]

Let $L$ be a fixed line bundle on $C$ such that $L \otimes E_\xi$ is generated by global sections for all $\xi$. So we have a bundle epimorphism
\[ C \times k^N \to L \otimes E_\xi, \]
so that $L \otimes E_\xi$ is induced by a map to a Grassmann variety
\[ \varphi_\xi : C \to \text{Gr}(N-r, N) \]
of a degree equal to
\[ \deg E_\xi + r(\deg L). \]

Also $L \otimes Q_\xi$ is induced by a map
\[ \psi_\xi' : C \to \text{Gr}(N-s, N) \]
of degree equal to
\[ (2.2) \quad \deg Q_\xi' + s(\deg L). \]
Now $\psi_0$ specializes to a map

$$\psi_0: \ C \to \text{Gr}(N - s, N)$$

of degree $\leq (2, 2)$ and so gives a quotient bundle of $L \otimes E_0$ of degree $\leq (2, 2)$. Thus $E_0$ must have a quotient bundle of negative degree.

2.3. **Lemma.** — *If the global sections of $E \to C$ span the fibre of the bundle at some point $p \in C$, then $E$ is semi-positive.*

*Proof.* — The determinant bundle of any quotient bundle of $E$ has a non-trivial section.

2.4. **Lemma.** — Let

$$0 \to E_1 \to E_2 \to E_3 \to 0$$

be an exact sequence of bundles over $C$ such that $E_1$ and $E_3$ are semi-positive. Then $E_2$ is also semi-positive.

*Proof.* — Let $T$ be a sub-bundle of $E_2$ of degree greater than $\deg E_2$. Let $S$ be the minimal sub-bundle of $E_2$ containing $T$ and $E_1$. Consider the map

$$\eta: \ T \oplus E_1 \to S.$$

Then there exists a sub-bundle $K$ of $T \oplus E_1$ such that, for almost all $p \in C$, the mapping $\eta$ gives an injection

$$((T \oplus E_1)/K)_p \to S_p.$$

Since $K$ is a sub-bundle of $E_1$, $\deg L \leq \deg E_1$, so that

$$\deg((T \oplus E_1)/K) \geq \deg T.$$

Therefore $\deg S \geq \deg T$. Thus $\deg (E_2/S) < 0$ contradicting the semi-positivity of $E_3$.

Let $V$ be a smooth hypersurface of degree $m$ in $\mathbb{P}^n$ and let

$$f: \ C \to V$$

be an immersion of degree $d$. Let $W$ be a *generically chosen* hypersurface of degree $m$ in $\mathbb{P}^{n+m}$ such that

$$\mathbb{P}^n/W = V.$$

We wish to prove the following:

2.5. **Lemma.** — *The normal bundle $N_{f, W}$ to the mapping

$$f: \ C \to V \subseteq W$$

is semi-positive.*

*Proof.* — Since we assume throughout that $m \geq 2$, we can specialize $W$ to a hypersurface $X$ of degree $m$ in $\mathbb{P}^{n+m}$ which contains $\mathbb{P}^n$ and is non-singular at points of $f(C)$. By
Lemma 2.1, it will suffice to prove the assertion of the lemma for
\[ f: \mathbb{C} \to W \]
where W is generic such that it contains the \( \mathbb{P}^n \). From the sequence of normal bundles
\[ 0 \to N_{f, \mathbb{P}^n} \to N_{f, \mathbb{P}^n, W} \to f^* N_{\mathbb{P}^n, W} \to 0 \]
and the fact that \( N_{f, \mathbb{P}^n} \) is semi-positive by Lemma 2.3, we need only find some W such that \( f^* N_{\mathbb{P}^n, W} \) is semi-positive. (Use Lemma 2.1 and Lemma 2.4 to see that this is enough.)

To this end, consider the sequence
\[ 0 \to f^* N_{\mathbb{P}^n, W} \to f^* N_{\mathbb{P}^n, \mathbb{P}^n+m} \to f^* N_{\mathbb{P}^n, \mathbb{P}^n+m} \to 0. \]

If we can find some special W for which
\[ f^* N_{\mathbb{P}^n, W} \cong \mathcal{O}(m-1), \]
the proof of Lemma 2.5 will be complete. We do this by direct computation. Suppose \( f(C) \) does not intersect the linear space of codimension 2 given by
\[ x_0 = x_1 = 0 \]
in \( \mathbb{P}^n \). Then let W be the hypersurface given by
\[ x_{n+1} x_0^{m-1} + x_{n+2} x_0^{m-2} x_1 + \ldots + x_{n+m} x_1^{m-1} = 0. \]
In this case, we rewrite the map \( \lambda \) in (2.6) as
\[ f^* \mathcal{O}(1)^{\oplus m} \to f^* \mathcal{O}(m) \]
\[ \alpha_j \to \sum_{j=1}^{m-1} \alpha_j x_0^{m-1-j} x_1. \]
It is immediate to see that the kernel of this mapping is generated by
\[ (x_1, -x_0, 0, \ldots, 0) \]
\[ (0, x_1, -x_0, 0, \ldots, 0) \]
etc.
Since \( x_0 \) and \( x_1 \) do not vanish simultaneously on \( f(C) \)
\[ f^* N_{\mathbb{P}^n, W} \cong \mathcal{O}(m-1). \]

3. Proof of the main theorem

In this final section, we will prove Theorem 1.1. We let \( V \) be a generic hypersurface of degree \( m \) in \( \mathbb{P}^n \) and we suppose that there is an irreducible algebraic family \( q \) of
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immersed curves of genus $g$ on $V$ which covers a quasi-projective variety of codimension $D$ in $V$. For $f$ generic in $F$, and

$$Y \subseteq \mathbb{P}^{s+k}$$

a smooth hypersurface with $Y \cdot \mathbb{P}^n = V$, let

$$R \subseteq H^0(N_{f, Y})$$

be any subspace. We denote, for each $p \in C$, the image of the evaluation map

$$R \to (\text{fibre of } N_{f, Y} \text{ at } p)$$

$$\rho \mapsto \rho(p)$$

by $R_p$. Then there is a unique sub-bundle

$$S \subseteq N_{f, Y}$$

such that $R \subseteq H^0(S)$ and, for almost all $p \in C$, the fibre of $S$ is exactly $R_p$. Next consider the diagram

$$R \subseteq H^0(N_{f, Y})$$

$$\downarrow \phi$$

$$H^0(N_{V, Y}) \to H^0(f^* N_{V, Y}).$$

Assume now that

(3.1) $\nu(R) = \mu(H^0(N_{V, Y})).$

Then the sections of $R$ must generate the fibres of $f^* N_{V, Y}$ at each point. So

$$T = S \cap N_{f, V}$$

is a well-defined sub-bundle of $N_{f, V}$. In fact, we claim that under the above assumptions the sequence

(3.2) $0 \to N_{f, V} \to N_{f, Y} \to f^* N_{V, Y} \to 0$

must be split. To see this, notice that the mapping

$$f^* N_{V, Y} \cong S/T \to N_{f, Y}/T$$

splits the sequence.

Continuing with the same assumptions, we wish to show that

$$L \otimes T$$

is semi-positive, where $L$, as above, is line bundle

$$f^* O_{\mathbb{P}^r}(1).$$
To see this, let \( p \in \mathbb{C} \) be a point such that the sections in the vector space \( \mathbb{R} \) given above generate the fibre of \( S \) at \( p \). Let

\[
t_p \in (\text{fibre of } T \text{ at } p).
\]

By Lemma 2.3, to prove the semi-positivity of \( L \otimes T \), it suffices to find a meromorphic section \( \tau \) of \( T \) such that:

(i) \( \tau(p) = t_p \),

(ii) the polar locus of \( \tau \) is either 0 or is a hyperplane section of \( f(\mathbb{C}) \).

To accomplish this, choose a section of \( \rho \in \mathbb{R} \) such that

\[
\rho(p) = t_p.
\]

If \( \rho \in H^0(N_f, \nu) \), set \( \tau = \rho \). If \( \rho \notin H^0(N_f, \nu) \), then by (3.1), \( \rho \) determines a non-trivial section of \( f^*N_{\nu, \nu} \) which is the restriction of a section \( \tilde{\rho} \) of \( N_{\nu, \nu} \). Now let

\[
N_{\nu, \nu} \to \mathcal{O}_\nu(1)
\]

be a projection such that \( \tilde{\rho} \) maps to a non-trivial section of \( \mathcal{O}_\nu(1) \).

Choose a base-point free pencil on \( f^*H^0(\mathcal{O}_\nu(1)) \) which comes from a two-dimensional subspace

\[
R_0 \subseteq \mathbb{R}
\]

such that \( \rho \in R_0 \). Let \( R_1 \) be an affine line in \( R_0 \) which passes through \( \rho \) but does not contain the origin of \( R_0 \). We define our section \( \tau \) of \( T \) by the rule

\[
\tau(q) = \rho'(q)
\]

where \( \rho' \) is the unique section in \( R_1 \) whose image in \( H^0(f^*\mathcal{O}_\nu(1)) \) vanishes at \( q \).

We are now ready to complete the proof of Theorem 1.1. Since \( V \) is generic, we can find an irreducible family \( F \) of curves of genus \( g \) in

\[
W \subseteq P^{n+m}
\]

such that:

(i) if \( f \in F \), then (image \( f \)) spans a linear space of dimension \( \leq n \);

(ii) for generically chosen \( f \in F \), the tangent space to \( F \) at \( f \) maps isomorphically to a subspace

\[
R \subseteq H^0(N_f, w)
\]

satisfying (3.1) for \( Y = W \),

(iii) \( f \in g \subseteq F \),

where \( g \) is the family of curves on \( V \) postulated at the beginning of paragraph 3.

(We simply use the deformations of \( f \) into curves on \( K \cdot W \) where \( K \) is a linear space of dimension \( n \) in \( P^{n+m} \).)
So we are in the situation considered earlier in paragraph 3. Thus we have associated to \( R \) the sub-bundles

\[
S \subseteq N_{f, w}
\]

and

\[
T = S \cap N_{f, v}
\]

giving a split sequence

\[
(3.3) \quad 0 \to N_{f, v}/T \to N_{f, w}/T \to L^\oplus \to 0
\]

Also \( L \otimes T \) is semi-positive.

By Lemma 2.5, \( N_{f, w} \) is semi-positive, and so therefore is

\[
N_{f, v}/T
\]

since it is a quotient of \( N_{f, w} \). In particular

\[
\deg N_{f, v}/T \geq 0.
\]

On the other hand there is a unique sub-bundle

\[
T_v \subseteq T
\]

such that the sections of the tangent space to \( g \) at \( f \), considered as a subspace of \( H^0(N_{f, v}) \), lie in \( T_v \) and generate almost all fibres of \( T_v \). Referring to the first part of paragraph 3,

\[
\text{rank } T_v = (n-2) - D
\]

so that

\[
\text{rank } (T/T_v) \leq D.
\]

Now by the adjunction formula

\[
\deg N_{f, v} = (n+1-m)(\deg L) - (2-2g).
\]

On the other hand

\[
\deg N_{f, v} = \deg (T/T_v) + \deg T_v + \deg (N_{f, v}/T) \geq \deg (T/T_v).
\]

Since \( L \otimes T \) is semi-positive

\[
\deg (L \otimes T/L \otimes T_v) \geq 0
\]

so

\[
\deg (T/T_v) \geq -rk (T/T_v) (\deg L).
\]
Putting everything together

\[(n + 1 - m)(\deg L) - (2 - 2g) \geq -(rk \, T/T_y) \, (\deg L).\]

Let

\[\alpha = \frac{2 - 2g}{\deg L}\]

Then

\[rk \, (T/T_y) \geq \alpha + m - (n + 1)\]

so that

\[D \geq \alpha + m - (n + 1).\]