Hilbert scheme of smooth space curves


<http://www.numdam.org/item?id=ASENS_1986_4_19_4_469_0>
HILBERT SCHEME OF SMOOTH SPACE CURVES

LAWRENCE EIN

Denote by $H_{d, g, n}$ the open subscheme of the Hilbert Scheme parametrizing the smooth irreducible curves of degree $d$ and genus $g$ in $\mathbb{P}^n$. The purpose of this paper is to prove that $H_{d, g, 3}$ is irreducible when $d \geq g + 3$. We also prove that every irreducible reduced curve in $\mathbb{P}^3$ with $d \geq g + 2$ is smoothable in $\mathbb{P}^3$. These results answer two questions proposed by Hartshorne and Hirschowitz ([5], 1.4). I would also like to remark that these results were asserted by Severi with an incomplete proof ([8], p. 370).

Let $\mathcal{C} \to \mathcal{M}_{g, n}$ be the universal family of smooth curves over the fine moduli space of genus $g$ curves with level $m$ structure. Suppose $\mathcal{P}ic \mathcal{C}$ is the relative Picard scheme. Set $\mathcal{W}_d = \{(\mathcal{L}, C) \in \mathcal{P}ic \mathcal{C} \mid \mathcal{L}$ is a degree $d$ line bundle on a curve $C$ and $h^0(\mathcal{L}) \geq r + 1\}$. Now suppose that $\mathcal{L}$ is a degree $d$ very ample line bundle with $h^0(\mathcal{L}) = r + 1$ and $h^1(\mathcal{L}) = \delta > 0$.

We show that if $Y$ is an irreducible component of $\mathcal{W}_d$ containing the point corresponding to $(\mathcal{L}, C)$, then $\dim Y \leq 5g - 1 - 4\delta - d$. We also show that the above inequality implies that $H_{d, g, 3}$ is irreducible when $d \geq g + 3$. More generally we prove that $H_{d, g, n}$ is irreducible when

$$d > \frac{(2n-3)g + n + 3}{n}.$$ 

I should also point out that Joe Harris has found an example where $H_{d, g, n}$ is reducible when $d \geq g + n$. Throughout the paper we shall work over the complex numbers.

I would like to thank Mark Green and Rob Lazarsfeld for many helpful discussions.

**Lemma 1.** Let $E$ be a rank $m$ locally free sheaf on a smooth irreducible curve $C$. Let $X = \mathbb{P}(E)$ and $\pi : X \to C$ be the projection map. We denote by $U$ the tautological line bundle of $\mathbb{P}(E)$. Suppose $V \subseteq H^0(U)$ is a $r + 1$-dimensional subspace. Then,

(a) The natural map $V \otimes \mathcal{O}_X \to U$ is surjective, if and only if $V \otimes \mathcal{O}_C \to \pi_* U = E$ is surjective.
(b) Assume that $|V|$ gives a birational morphism

$$f: \ X \to f(X) = Y \subseteq \mathbb{P}^r.$$ 

Set $F = \ker(V \otimes \mathcal{O}_C \to E)$.

Then there is an exact sequence,

$$0 \to (\Lambda^m E)^* \otimes \mathcal{O}_C \left( \sum_{1}^{r-m} p_j \right) \to F \to \sum_{1}^{r-m} \mathcal{O}_C(-p_j) \to 0$$

where $p_j$'s are general points on $C$.

**Proof.** — (a) Suppose that $V \otimes \mathcal{O}_X \to U$ is surjective. Let $M = \ker(V \otimes \mathcal{O}_X \to U)$. If $R = \pi^{-1}(x)$ then

$$M |_R \cong \Omega^{r-m-1}(1) \otimes (r+1-m) \mathcal{O}_{p^{r-m}}.$$ 

Hence, $R^1 \pi_* M = 0$. It follows that $V \otimes \mathcal{O}_C \to \pi_* U = E$ is surjective. Conversely, if $V \otimes \mathcal{O}_C \to E$ is surjective, then the composition $V \otimes \mathcal{O}_X \to \pi^* E \to U$ is also surjective.

(b) Set $Y = f(X)$. Choose $r-m$ general points $y_1, y_2, \ldots, y_{r-m}$ in $Y$. We may assume that $\{y_1, y_2, \ldots, y_{r-m}\}$ spans a $(r-m-1)$-plane $L$ in $\mathbb{P}^r$.

By the uniform position lemma [2], we may assume that

$$L \cap Y = \{y_1, y_2, \ldots, y_{r-m}\}.$$ 

Furthermore we shall assume that $f^{-1}(y_i) = q_i$ and $f$ is an isomorphism in a neighborhood of $q_i$. Set

$$Q = \{q_1, q_2, \ldots, q_{r-m}\}.$$ 

Consider the exact sequence

$$0 \to I_Q \otimes U \to U \to U|_Q \to 0,$$

where $I_Q$ is the ideal sheaf of $Q$ in $X$. Set $p_i = \pi(q_i)$ and $P = \pi(Q)$. Observe that the restriction map $V \to H^0(U|_Q)$ is surjective.

Let $W = \ker(V \to H^0(U|_Q))$. Observe that the natural map

$$\pi_* U = E \to \pi_* (U|_Q) = \sum_{i=1}^{r-m} \mathcal{O}_{p_i} = \mathcal{O}_P$$

is surjective. Set $E' = \pi_* (I_Q \otimes U)$. Observe that $E'$ is a rank $m$ locally free sheaf and $R^1 \pi_* (I_Q \otimes U) = 0.$
Consider the following diagram:

\[
\begin{array}{cccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & M' & W(\mathcal{O}_X) & I_Q\otimes U \\
\downarrow & \downarrow & \downarrow \\
0 & M & V(\mathcal{O}_X) & U \\
\downarrow & \downarrow & \downarrow \\
0 & \sum_{i=1}^r I_{x_i} & H^0(\mathcal{O}_X) & U \\
\downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0
\end{array}
\]

where

\[
M = \ker (V\otimes \mathcal{O}_X \rightarrow U) \quad \text{and} \quad M' = \ker (W(\mathcal{O}_X) \rightarrow I_Q\otimes U).
\]

Observe that \(\alpha\) is surjective because \(f^{-1}(L \cap Y) = \emptyset\).

It follows from the snake lemma \(\beta\) is also surjective.

Let \(f_i = \pi^{-1}((p_i)) \cong \mathbb{P}^{n-1}\). Consider the exact sequences,

\[
0 \rightarrow \text{Tor}_1(I_Q\otimes U, \mathcal{O}_{f_i}) \rightarrow M'\otimes \mathcal{O}_{f_i} \rightarrow W\otimes \mathcal{O}_{f_i} \rightarrow I_Q\otimes U\otimes \mathcal{O}_{f_i} \rightarrow 0,
\]

and

\[
0 \rightarrow k(q_i) \rightarrow I_Q\otimes U\otimes \mathcal{O}_{f_i} \rightarrow I_{q_i}(1) \rightarrow 0,
\]

where \(k(q_i)\) is the residue field of \(q_i\) in \(I_{q_i}(1)\). It follows from a local computation that the map

\[
H^0(W\otimes \mathcal{O}_{f_i}) \rightarrow H^0(I_Q\otimes U\otimes \mathcal{O}_{f_i})
\]

is surjective.

Also observe that \(\text{Supp} (\text{Tor}_1(I_Q\otimes U, \mathcal{O}_{f_i})) \subset q_i\).

Hence \(H^1(M'\otimes \mathcal{O}_{f_i}) = 0\). \(M'\) is torsion free and it is flat over \(C\). It follows from the theorem of base changes that \(R^1 \pi_* M' = 0\).

There is the following diagram:

\[
\begin{array}{cccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & \pi_* M' & W(\mathcal{O}_C) & E' \\
\downarrow & \downarrow & \downarrow \\
0 & F & V(\mathcal{O}_C) & E \\
\downarrow & \downarrow & \downarrow \\
0 & \sum_{i=1}^r \mathcal{O}_C(-p_i) \rightarrow H^0(\mathcal{O}_U) & \sum_{i=1}^r \mathcal{O}_{p_i} \rightarrow 0 \\
\downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0
\end{array}
\]
This showed that $F \to \sum_{i=1}^{r-m} \mathcal{O}(-p_i)$ is surjective.

Now

$$\text{rank}(\pi_* M') = 1 \quad \text{and} \quad \pi_* M' \cong (\wedge^m E)^* \otimes \mathcal{O}_C(P).$$

**Remark.** — The above construction is inspired by the techniques of Gruson and *et al.*

The fine moduli space of smooth irreducible genus $g$ curves with level $m$ structure is denoted by $M_{g,m}$. Suppose that $\mathcal{C} \to M_{g,m}$ is the universal family of curves. Let $\mathcal{P}ic \mathcal{C}$ be the relative Picard scheme. Set,

$$\mathcal{W}_d = \{(\mathcal{L}, C) \in \mathcal{P}ic \mathcal{C} | \deg \mathcal{L} = d \quad \text{and} \quad h^0(\mathcal{L}) \geq r + 1\}.$$

For the rest of the paper we shall use the following notations. We shall denote by $C$, a smooth irreducible genus $g$ curve. $\mathcal{L}$ is a degree $d$ line bundle on $C$. We shall assume $h^0(\mathcal{L}) = r + 1$, $h^1(\mathcal{L}) = \delta > 0$, and $|\mathcal{L}|$ has no base points. We denote by $f$ the natural map:

$$f: \quad C \to f(C) = C' \subseteq \mathbb{P}(H^0(\mathcal{L})) = \mathbb{P}^r.$$

Suppose that $\mathcal{O}(1)$ is the tautological line bundle of $\mathbb{P}(H^0(\mathcal{L}))$. $\mathcal{P}^1(\mathcal{O}(1))$, the first principal part of $\mathcal{O}(1)$, is isomorphic to $H^0(\mathcal{L}) \otimes \mathcal{O}_{\mathbb{P}^r}$. Set $M = f^*(\Omega^1_{\mathcal{P}^1}(1))$ and $\mathcal{P}^1(\mathcal{L}) = \text{first principal part of } \mathcal{L}$. There is the following diagram:

$$0 \to M \to H^0(\mathcal{L}) \otimes \mathcal{O}_C \to \mathcal{L} \to 0$$

$$0 \to K \otimes \mathcal{L} \to \mathcal{P}^1(\mathcal{L}) \to \mathcal{L} \to 0,$$

where $K$ is the canonical sheaf of $C$. Observe that $\mathcal{P}^1(\mathcal{L}) \otimes K \otimes \mathcal{L}^{-1} \cong \mathcal{P}^1(K)$. Hence there is the following diagram:

$$0 \to M \otimes K \otimes \mathcal{L}^{-1} \to H^0(\mathcal{L}) \otimes K \otimes \mathcal{L}^{-1} \to K \to 0$$

$$(1. A) \quad 0 \to K^2 \to \mathcal{P}^1(\mathcal{L}) \otimes K \otimes \mathcal{L}^{-1} \to K \to 0.$$  

Consider the map:

$$(1. B) \quad \mu: \quad H^0(\mathcal{L}) \otimes H^0(K \otimes \mathcal{L}^{-1}) \to H^0(\mathcal{P}^1(\mathcal{L}) \otimes K \otimes \mathcal{L}^{-1}).$$

$H^0(\mathcal{P}^1(\mathcal{L}) \otimes K \otimes \mathcal{L}^{-1})$ is naturally isomorphic to the cotangent space of $\mathcal{P}ic \mathcal{C}$ at the point $(\mathcal{L}, C)$. The image of $\mu$ is the annihilator of the Zariski tangent space of $\mathcal{W}_d$, at the point $(\mathcal{L}, C)$. See [1] for more details.

**Theorem 2.** — Suppose that $\mathcal{L}$ is a very ample degree $d$ line bundle on a smooth irreducible curve $C$ such that $h^0(\mathcal{L}) = r + 1$ and $h^1(\mathcal{L}) = \delta > 0$, where $r \geq 3$. Then,

(a) $\text{rank}(\mu) \geq 3\delta - 2 + r = 4\delta + d - g - 2$.  \hspace{1cm} (1. B)
(b) If \( Y \) is an irreducible component of \( W^r \) containing the point \((\mathcal{L}, C)\), then \( \dim Y \leq 5g - 4\delta - d - 1 \).

(c) Let \( N \) be the normal sheaf of \( C \) in \( \mathbb{P}(H^0(\mathcal{L})) \). Then \( h^1(N) \leq (r - 2)(\delta - 1) \).

**Proof.** — Consider the natural embedding,

Let \( N^* \) be the conormal sheaf of \( C \) in \( \mathbb{P} \). There is the following exact sequence:

\[
0 \to N^* \otimes \mathcal{L} \to H^0(\mathcal{L}) \otimes \mathcal{O}_C \to \mathbb{P}(\mathcal{L}) \to 0.
\]

Consider the natural map

\[
F: \mathbb{P}(\mathbb{P}(\mathcal{L})) \to T \subseteq \mathbb{P}.
\]

\( T \) is the tangent surface of \( C \), and \( F \) is a birational morphism. By Lemma 1,

\[
h^1(N) = h^0(N^* \otimes K) \leq \sum_{i=1}^{r-2} h^0(K \otimes \mathcal{L}^{-1}(-p_i)) + h^0\left(\mathcal{L}^{-3} \left(\sum_{i=1}^{r-2} p_i\right)\right) = (r - 2)(\delta - 1).
\]

But \( H^0(N^* \otimes K) = \ker \mu \). Thus

\[
\text{rank}(\mu) \geq (r + 1)\delta - (r - 2)(\delta - 1) = 3\delta - 2 + r = 4\delta + d - g - 2.
\]

Since the image of \( \mu \) is the annihilator of the Zariski tangent space of \( W^r \) at \((\mathcal{L}, C)\), it follows that,

\[
\dim Y \leq (4g - 3) - (4\delta + d - g - 2) = 5g - 4\delta - d - 1.
\]

**Corollary 3.** — Assume that \( r \geq 3 \) and

\[
f: C \to f(C) = C' \subseteq \mathbb{P}(H^0(\mathcal{L}))
\]

is a birational map. Furthermore assume either \( f \) is unramified or \( P_x(C') < g + 3d - (r - 2) \). Then,

(a) \( \text{rank}(\mu) \geq 4\delta + d - g - 2 \).

(b) If \( Y \) is an irreducible component of \( W^r \) containing the point \((\mathcal{L}, C)\), then \( \dim Y \leq 5g - 1 - 4\delta - d \).

**Proof.** — Consider the natural map

\[
\varphi: H^0(\mathcal{L}) \otimes \mathcal{O}_C \to \mathbb{P}(\mathcal{L}).
\]

Set

\[
E = \text{Im}(\varphi), \quad N^* \otimes \mathcal{L} = \ker(\varphi) \quad \text{and} \quad D = \text{cok}(\varphi).
\]

Observe that \( \text{cok} \varphi \) is equal to \( \text{cok}(df: f^*\Omega_\mathcal{L} \otimes \mathcal{L} \to \Omega^1_{\mathcal{L} \otimes \mathcal{L}}) \).

It follows that \( \text{cok} \varphi \) is isomorphic to \( \Omega^1_{\mathcal{L} \otimes \mathcal{L} \otimes \mathcal{O}_R} \), where \( R \) is the ramification divisor.

Let \( X = \mathbb{P}(E) \). Consider the natural map

\[
F: X \to F(X) = T \subseteq \mathbb{P}(H^0(\mathcal{L})).
\]

ANNALES SCIENTIFIQUES DE L'ÉCOLE NORMALE SUPÉRIEURE
T is the closure of the tangent surface of the smooth part of C'. $F: X \to T$ is birational. Now

$$P_a(C') - g = \sum_{p \in C} \text{length}(\mathcal{O}_{p, C} \otimes \mathcal{O}_f(p, C)).$$

Observe that

$$\deg R = \sum_{p \in C} \text{length}(I_{p, C} \otimes \mathcal{O}_{p, C} \cdot \mathcal{O}_f(p, C)).$$

It follows that $\deg R \leq P_a(C') - g$. By Lemma 1, we can

\text{LEMMA 1.} — We can construct the following exact sequence:

$$0 \to \mathcal{L}^{-3} \otimes \mathcal{O}_C(R) \otimes \mathcal{O}_C \left( \sum_{i=1}^{r-2} p_i \right) \to N^* \otimes K \to \sum_{i=1}^{r-2} K \otimes \mathcal{L}^{-1}(-p_i) \to 0.$$

Since $\deg(R) \leq P_a - g$, it follows from our assumption

$$h^0 \left( \mathcal{L}^{-3} \left( R + \sum_{i=1}^{r-2} p_i \right) \right) = 0.$$

Thus $\dim \ker \mu = h^0(N^* \otimes K) \leq (r-2)(\delta - 1)$. As in Theorem 2, we conclude that $\rank \mu \geq 4\delta + d - g - 2$ and $\dim Y \leq 5g - 1 - 4\delta - d$.

The open set of the Hilbert scheme corresponding to smooth irreducible degree $d$ genus $g$ curves in $\mathbb{P}^3$ is denoted by $H_{d, g, 3}$. If $X \in H_{d, g, 3}$, then $\chi(N_{X/\mathbb{P}^3}) = h^0(N_{X/\mathbb{P}^3}) - h^1(N_{X/\mathbb{P}^3}) = 4d$.

As in [7], one can show that each irreducible component of $H_{d, g, 3}$ has dimension greater or equal to $4d$.

\text{THEOREM 4.} — If $d \geq g + 3$, then $H_{d, g, 3}$ is irreducible.

\text{Proof.} — There is an irreducible open set of $H_{d, g, 3}$ corresponding to nonspecial curves ($h^1(\mathcal{O}_C(1)) = 0$) ([5], 6.2). Suppose for contradiction that $H_{d, g, 3}$ is reducible. Then there is an irreducible component $W$ of $H_{d, g, 3}$ such that the general curve $C$ in the family $W$ satisfies

$$h^0(\mathcal{O}_C(1)) = r+1 \quad \text{and} \quad h^1(\mathcal{O}_C(1)) = \delta > 0.$$

We denote by $H_{d, g, 3}^m$ the Hilbert scheme of degree $d$ genus $g$ smooth irreducible curves in $\mathbb{P}^3$ with level $m$ structure. Let $W_m$ be an irreducible component of $H_{d, g, 3}^m$ which maps onto $W$. Then $\dim W = \dim W_m$.

There is a natural map from

$$h: W_m \to \mathcal{W}_d \in \mathcal{P}ic \mathcal{C}.$$
Let $Y$ be an irreducible component of $\mathcal{W}_d$ containing $h(W_m)$. Let $x$ be a general point of $W_m$, then

$$\dim h^{-1} h(x) \leq \dim G(4, d+1+\delta-g) + \dim \text{Aut} \mathbb{P}^3$$

where $G(4, d+1+\delta-g)$ is the Grassman variety of 4 dimensional subspaces in a $d+1+\delta-g$-dimensional vector space. Then

$$\dim W = \dim W_m \leq \dim h^{-1} h(x) + \dim Y \leq 4d-1$$

by Theorem 2. This is a contradiction. Hence, $H^d_{\delta, 3}$ is irreducible.

**Remark.** — In [4], Harris has proved that $H^d_{\delta, 3}$ is irreducible while $d > 5/4g + 1$.

Suppose that $C'$ is an irreducible reduced degree $d$ curve in $\mathbb{P}^3$. Let

$$N_{C'/\mathbb{P}^3} = \text{Hom}(\mathcal{O}_{\mathbb{P}^3}(I_{C'}), \mathcal{O}_{C'}) = \text{Hom}(\mathcal{O}_{C}/\mathcal{I}_{C'}, \mathcal{O}_{C'})$$

be the normal sheaf of $C'$.

**Lemma 5.** $-\chi(N_{C'/\mathbb{P}^3}) = h^0(N_{C'/\mathbb{P}^3}) - h^1(N_{C'/\mathbb{P}^3}) = 4d$. Hence every irreducible component of the Hilbert scheme containing $C'$ has dimension greater or equal to $4d$.

**Proof.** — $C'$ is locally Cohen-Macaulay. We can construct an exact sequence:

$$0 \to E_2 \to E_1 \to I_{C'} \to 0$$

where $E_1$ and $E_2$ are locally free sheaves on $\mathbb{P}^3$.

Consider the following exact sequences:

$$0 \to \mathcal{H} \text{om}(I_{C'}, \mathcal{O}_{\mathbb{P}^3}) \to E_1^* \to E_2^* \to \mathcal{O}_{C'}(4) \to 0,$$

$$0 \to \mathcal{H} \text{om}(I_{C'}, \mathcal{O}_{C'}) \to E_1^*|_{C'} \to E_2^*|_{C'} \to \mathcal{O}_{C'}(4),$$

Observe that $\varphi_2 = \varphi_1 \otimes \mathcal{O}_{C'}$. Thus

$$\text{Cok} \varphi_2 = \text{Cok} \varphi_1 \otimes \mathcal{O}_{C'} = \mathcal{O}_{C'}(4).$$

Observe that

$$c_1(E_1^*) = c_1(E_2^*) \quad \text{and rank } E_1^* = 1 + \text{rank } E_2^*.$$

It follows from the Riemann-Roch theorem,

$$\chi(N_{C'/\mathbb{P}^3}) = \chi(E_1^*|_{C'}) + \chi(\mathcal{O}_{C'}(4)) - \chi(E_2^*|_{C'}) = 1 - P_d + \chi(\mathcal{O}_{C'}) + 4d = 4d.$$

$C'$ is codimension two Cohen-Macaulay. It follows that there is no local obstructions to the deformations of $C'$ ([3], 5.1). Hence the obstructions to the deformations of $C'$ in $\mathbb{P}^3$ is given by $H^1(N_{C'/\mathbb{P}^3})$. As in [7], one can show that this implies the inequality of dimension as claimed.

**Theorem 6.** — Suppose that $X$ is an irreducible reduced degree $d$ curve in $\mathbb{P}^3$. If $d \geq P_d(X) + 2$, then $X$ is smoothable.
Proof. — Let \( W \) be an irreducible component of the Hilbert scheme containing the point corresponding to \( X \). If the general member of \( W \) is smooth, then \( X \) is smoothable. Assume for contradiction that a general curve \( C' \) in \( W \) is singular. Let \( S \to W \) be the universal family of curves. Let \( p: \wtilde{S} \to S \to W \) be the normalization of \( S \). Let \( U \subseteq W \) be the open set where \( p \) is smooth. Suppose the normalization of \( C' \) is a smooth curve of genus \( g \). We can construct a variety \( U_m \) étale over \( U \) such that there is a map \( h: U_m \to \mathbb{P}^1 \). We shall divide the proof into five cases. Consider the normalization map \( \pi: C \to C' \). Set \( \pi^*\mathcal{O}_C(1)=\mathcal{O}_C(1) \).

Since \( g<\mathbb{P}_s(C') \), \( \deg\mathcal{O}_C(1) \geq g+3 \).

Case 1. — Assume that \( g=0 \).

Then \( \mathcal{O}_C(1)=\mathcal{O}_{\mathbb{P}^1}(d) \). \( C' \) is obtained by projecting the \( d \)-uple embedding of \( \mathbb{P}^1 \). The generic projection gives a smooth curve. Thus,

\[
\dim W < \dim G(4, d+1) + \dim \text{Aut} \mathbb{P}^3 - \dim \text{Aut} \mathbb{P}^1 = 4d.
\]

Case 2. — Assume that \( g=1 \).

As in Case 1, we can prove that

\[
\dim W < \dim G(4, d) + \dim \text{Aut} \mathbb{P}^3 - \dim \text{Aut} \mathbb{C} + \dim \mathcal{O}_{\mathbb{C}} = 4d.
\]

Case 3. — Assume that \( g \geq 2 \), \( \dim h(U_m)=\dim \mathcal{O}_{\mathbb{C}}=4g-3 \), and \( h^1(\mathcal{O}_C(1))=0 \).

The generic line bundle of degree \( d \geq g+3 \) is very ample. Let \( x \) be a general point of \( U_m \). Then \( \dim h^{-1}h(x)<\dim G(4, d+1-g)+\dim \text{Aut} \mathbb{P}^3 \).

Hence, \( \dim W = 4g+3 + \dim h^{-1}h(x) < 4d \).

Case 4. — Assume that \( h^1(\mathcal{O}_C(1))=0 \), \( g \geq 2 \), and \( \dim h(U_m)<4g-3 \), in this case

\[
\dim W = \dim U_m = \dim h^{-1}h(x) + \dim h(U_m) < \dim G(4, d+1-g) + \dim \text{Aut} \mathbb{P}^3 + (4g-3) \leq 4d.
\]

Case 5. — Assume that \( g \geq 2 \), and \( h^1(\mathcal{O}_C(1))=\delta>0 \).

Using Corollary 3, we can show that

\[
\dim W = \dim U_m \leq 4d-1,
\]

as in Theorem 2.

In each of the five cases, we show that \( \dim W < 4d \).

This is impossible. Thus a general curve in \( W \) is smooth.

Lemma 7. — Assume \( f: C \to C' \subseteq \mathbb{P}(H^0(L))=\mathbb{P}^r \) is a birational map. Also assume that \( d \geq g \).

(a) Consider the multiplication map:

\[
\mu_0: H^0(L) \otimes H^0(K \otimes L^{-1}) \to H^0(K).
\]

Then \( \operatorname{rank} (\mu_0) \geq 2\delta + r - 1 = 3\delta + d - g - 1 \).
(b) \( \delta \leq \frac{2g + 1 - d}{3} \).

Proof. — Consider the exact sequence:

\[
0 \to M \to H^0(L) \otimes \mathcal{O}_C \to \mathcal{L} \to 0 \quad \text{when} \quad M = \mathcal{I} \Omega^1_{\mathbb{P}^r}(1).
\]

By Lemma 1, we can construct an exact sequence:

\[
0 \to \mathcal{L}^{-1} \otimes \mathcal{O} \left( \sum_{i=1}^{r-1} p_i \right) \to M \to \sum_{i=1}^{r-1} \mathcal{O}(-p_i) \to 0.
\]

Observe that,

\[
h^1 \left( K \otimes \mathcal{L}^{-2} \otimes \mathcal{O} \left( \sum_{i=1}^{r-1} p_i \right) \right) = h^0 \left( \mathcal{L}^2 \otimes \mathcal{O} \left( - \sum_{i=1}^{r-1} p_i \right) \right)
\]

\[
= 2d + 1 - g - (r-1) = - \chi \left( K \otimes \mathcal{L}^{-2} \otimes \mathcal{O} \left( - \sum_{i=1}^{r-1} p_i \right) \right).
\]

Thus

\[
h^0 \left( K \otimes \mathcal{L}^{-2} \otimes \mathcal{O} \left( \sum_{i=1}^{r-1} p_i \right) \right) = 0.
\]

Hence,

\[
h^0 \left( M \otimes K \otimes \mathcal{L}^{-1} \right) = \dim \ker u_0 \leq (r-1)(\delta - 1).
\]

Thus \( \mu_0 \geq 3 \delta + d - g - 1 \). Since \( g \geq \text{rank}(\mu_0) \), it follows that \( \delta \leq (2g + 1 - d)/3 \).

Theorem 8. — Let \( H_{d,g,n} \) be the open set of the Hilbert scheme of smooth irreducible degree \( d \) genus \( g \) curves in \( \mathbb{P}^n \). If \( d > (2n-3)g + n + 3)/n \), then \( H_{d,g,n} \) is irreducible.

Proof. — Let \( C \) be a smooth irreducible degree \( d \) genus \( g \) curve in \( \mathbb{P}^n \). Then \( \chi(\mathcal{N}_{C/\mathbb{P}^n}) = (n+1)d + (n-3)(1-g) \).

It follows that the dimension of each irreducible component of \( H_{d,g,n} \) is at least \( (n+1)d + (n-3)(1-g) \). Assume that \( H_{d,g,n} \) has an irreducible component \( W \) such that the general curve in the family satisfies the property \( h^0(L) = r+1 \) and \( h^1(L) = \delta > 0 \).

Then,

\[
\dim W \leq 5g - 1 - 4\delta - d + \dim G(n+1, r+1)
\]

\[
+ \dim \text{Aut} \mathbb{P}^n = 5g - 2 - 4\delta - d + (n+1)(\delta + d - g + 1),
\]

Since

\[
\delta \leq \frac{2g + 1 - d}{3} \quad \text{and} \quad d > \frac{(2n-3)g + n + 3}{n},
\]

Annales Scientifiques de l'École Normale Supérieure
it follows that $\dim W < (n+1)d + (n-3)(1-g)$ which is a contradiction.

Remark. — The above result is an improvement of a theorem of Joe Harris. In ([4], p. 72), Harris proved that $H^d_{d,g,n}$ is irreducible while $d > \frac{(2n-1)g}{n+1} + 1$.

REFERENCES


(Manuscrit reçu le 26 mars 1985,
révisé le 26 août 1985.)

L. Ein,
University of California,
Los Angeles,
Los Angeles, CA 90024,
Current Address
University of Illinois,
Chicago,
Box 4348,
Chicago, IL 60680,
U.S.A.