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WRONSKIANS AND PLÜCKER FORMULAS FOR LINEAR SYSTEMS ON CURVES

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CONTENTS

In a previous article [3] we introduced a definition of Weierstrass points of complete linear systems on curves that has several desirable properties. Firstly it is a natural definition independent of the characteristic of the base field over which the curve is defined. Secondly it always gives, for all linear systems in all characteristics, a finite number of Weierstrass points. Thirdly, the definition is phrased in terms of rank conditions on maps of bundles that define the associated maps of the curve and brings out the often overlooked connection between Weierstrass points and the stationary properties of the associated maps. Fourthly, and most importantly it leads to an interpretation of Weierstrass points as the set of zeroes of a canonical section, called the wronskian of the linear system, of a line bundle. As a result of this interpretation we can define multiplisities of Weierstrass points and give a formula for the total number of Weierstrass points of the linear system, or what is the same a generalization of the Brill-Segre formula for the total number of $(r+1)$-tuple points of an r-dimensional linear system.

In the present article we are able to go further and define what we call rank $(s+1)$-wronskian points of, not necessarily complete, linear systems of dimension r for all s such that $0 \leqq s \leqq r$. The rank $(r+1)$-wronskian points are the Weierstrass points of [3].

Our definition of wronskian points has all the features, mentioned above, of the definition of Weierstrass points. In particular we can give weights to the wronskian points and find a formula for the total number of rank $(s+1)$-wronskian points for each $s = 0, 1, \ldots, r$. These formulas turn out to be generalizations, to arbitrary characteristics, of the generalized Plücker formulas given by Cayley and Veronese. The Plücker formula corresponding to $s = r$ is the Brill-Segre formula mentioned above so that we have a very satisfactory extension of the results in [3]. This extension is made possible by a new construction of wronskian sections that is both more general and technically simpler than the one in [3]. The existence of such sections, that are generalizations of generalized wronskian determinants, was suggested by the work of G. Galbura [1] on $(r+1)$'tuple points of linear systems, and their construction is one of the main contributions of this article.

After the publication of [3], B. H. Matzat pointed out that, in [4], he gave an extension of F. K. Schmidts [7] definition of Weierstrass points in arbitrary characteristic that give the same points with the same weights as those defined in [3]. His treatment is within the algebraic framework of function fields and is based upon the wronskian of a function field as used by Schmidt. For further historical comments we refer to [3].

It is interesting to note that although the Plücker formulas that we obtain are completely similar in apparence to the classical formulas, they have the rather curious feature that the term that in the classical formulas is interpreted as the total ramification of one of the associated maps, can be negative. Hence this term has in general no geometric interpretation. In order to explain this behaviour we devote a considerable part of the article to investigate the connections between the algebraic invariants that appear in the Plücker formulas and the geometric invariants attached to the associated maps of the linear system. One of the advantages of our definition of Weierstrass, or more generally wronskian points, is that it lends itself extremely well to such investigations. It turns out that very few of the properties that hold in the classical case have natural generalizations to the general case. We give a complete set of examples showing that those properties that we are not able to generalize do indeed not have such generalizations. On the other hand we show that, when the characteristic of the ground field is zero or strictly greater than the degree of the linear system, then all the expected relations hold and we get generalizations of the classical formulas.

In our theory the associated planes of the curve, alluded to above, play a central role. These planes we prove coincide with the traditional osculating planes outside of the wronskian points and in characteristic zero they coincide everywhere. We give, however, examples showing that they are not always equal.

The search for precise global relations between multiple points and osculating planes was actualized by W. F. Pohl's work [6] on the higher order geometry of manifolds. He shows that several concepts of geometry can be formulated in terms of rank conditions on osculating bundles naturally connected with the situation at hand and that a great number of problems in geometry can be treated within the theory of first order singularities of maps of vector bundles. Our definition of wronskian points give a precise formulation of the global connections between multiple points of linear systems on a curve and the geometry of the osculating planes of the curve, within the framework of Pohl's geometry.

In accordance with the above program we describe the wronskian points in terms of rank conditions on maps of bundles that define the associated planes of the curve. Rather unexpectedly we need in general, for fixed s, several rank conditions to describe the rank $(s+1)$-wronskian points and we give examples showing that not even the Weierstrass points can be characterized as the first order singularities of a single map of bundles. It is the more surprising that it is still possible to obtain formulas of Plücker type for such points.

§ 1. Associated bundles and Weierstrass modules

In this section we shall introduce the notation used in the sequel and recall the basic facts about the bundles of principal parts on curves. We also define the two main objects of study of the article, the *associated bundles* and the *Weierstrass modules* and give the relations between these objects and the cohomology of divisors on curves. These relations will be needed to interprete Weierstrass points and multiple points in terms of properties of the Weierstrass modules. The presentation is similar to the treatment given in [3], § 1. We refer to [3] for further comments and historical remarks.

Let C be a non-singular curve of genus g and D a divisor of degree n. Moreover, let V be a linear system in $H^0(C, D)$ of (projective) dimension $r \geqq 0$.

Denote by I the ideal defining the diagonal in $C \times C$ and by $C(m)$ the subscheme of $C \times C$ defined by I^{m+1}. We have

$$(\Omega_C^1)^{\otimes m} = I^m/I^{m+1}.$$

Denote by p and q the projections of $C \times C$ onto the first and second factor. The exact sequence

$$0 \to I^{m+1} \to \mathcal{O}_{C\times C} \to \mathcal{O}_{C(m)} \to 0$$

tensored by $q^*\mathcal{O}(D)$ gives, after passing to cohomology, a long exact sequence

$$\begin{aligned}(1)\qquad & 0 \to p_*(I^{m+1} \otimes q^*\mathcal{O}(D)) \to p_*q^*\mathcal{O}(D) \to p_*(q^*\mathcal{O}(D) \mid C(m)) \\ & \to R^1p_*(I^{m+1} \otimes q^*\mathcal{O}(D)) \to R^1p_*q^*\mathcal{O}(D) \to 0\end{aligned}$$

Here we have a zero to the right because $p \mid C(m)$ is affine.

The bundle $P^m(D)$ of m'th order principal parts of D is defined by

$$\mathbb{P}^m(D) = p_*(q^*\mathcal{O}(D) \mid C(m))$$

Via the projection p the principal parts have a natural structure as an $\mathcal{O}_C$-module and $P^0(D) = \mathcal{O}(D)$.

From the exact sequence

$$0 \to I^m/I^{m+1} \to \mathcal{O}_{C(m)} \to \mathcal{O}_{C(m-1)} \to 0$$

we obtain an exact sequence

$$0 \to (\Omega_C^1)^{\otimes m} \otimes \mathcal{O}(D) \to P^m(D) \to P^{m-1}(D) \to 0. \tag{2}$$

We see that $P^m(D)$ is a locally free $\mathcal{O}_C$-module of rank $(m+1)$.

By flat base change we have that $R^i p_* q^* \mathcal{O}(D) = H^i(C, D) \otimes \mathcal{O}_C$. Let

$$v^m;\ V_C \to P^m(D)$$

be the map induced by the map

$$v^m(D);\ H^0(C, D)_C \to P^m(D)$$

defined by the sequence (1). We denote by $B^m(D)$ and $E^m(D)$ the image and cokernel of the latter map and by B^m and E^m the image and cokernel of v^m. Since $P^m(D)$ is locally free we have that $B^m(D)$ and B^m are also locally free $\mathcal{O}_C$-modules.

From the surjection $P^m(D) \to P^{m-1}(D)$ of the sequence (2) we obtain a natural commutative diagram,

$$\begin{array}{ccccccccc} 0 & \to & B^m & \longrightarrow & P^m(D) & \longrightarrow & E^m & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & B^{m-1} & \to & P^{m-1}(D) & \to & E^{m-1} & \to & 0 \end{array} \tag{3}$$

with surjective vertical maps. Moreover, we obtain from the sequence (1) an exact sequence

$$0 \to E^m(D) \to R^1 p_*(I^{m+1} \otimes q^* \mathcal{O}(D)) \to H^1(C, D)_C \to 0 \tag{4}$$

By the principle of exchange we have for all points $x \in C$ an isomorphism

$$R^1 p_*(I^{m+1} \otimes q^* \mathcal{O}(D))(x) = H^1(C, D-(m+1)x).$$

Consequently we obtain from the sequence (4) a natural commutative diagram of $\mathcal{O}_C$-modules,

$$\begin{array}{ccccccccc} 0 & \to & E^m(D)(x) & \longrightarrow & H^1(C, D-(m+1)x) & \to & H^1(C, D)_C & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & E^{m-1}(D)(x) & \to & H^1(C, D-mx) & \longrightarrow & H^1(C, D)_C & \to & 0 \end{array} \tag{5}$$

Here the horizontal sequences are exact because $H^1(C, D)_C$ is free.

PROPOSITION 1. — *Let* $r(D) = \dim H^0(C, D) - 1$. *Then there are integers*

$$0 = b_0(D) < b_1(D) < \ldots < b_{r(D)}(D) \leqq n < b_{r(D)+1}(D) = \infty$$

such that rank $B^j(D) = (s+1)$ *for* $b_s(D) \leqq j < b_{s+1}(D)$ *and* $s = 0, 1, \ldots, r(D)$.

Proof. — It follows from the Riemann-Roch theorem that $h^1(C, D-(m+1)x) = g-n+m$ when $m \geqq n$. Consequently, when $m \geqq n$, we see from the sequence (4) that $E^m(D)$ is locally free of rank $g-n+m-h^1(C, D) = m-r(D)$. Hence

$$\text{rank } B^m(D) = (m+1)-(m-r(D)) = r(D)+1,$$

when $m \geqq n$. Moreover, the bundle $B^0(D)$ is of rank 1. Indeed, it is a subbundle of $P^0(D)=\mathcal{O}(D)$ and is non-trivial because $r(D) \geqq r \geqq 0$. It follows that the rank of $B^m(D)$ lies between 1 and $(r(D)+1)$. Moreover, we see from diagram (3) that

$$\operatorname{rank} B^m(D) - \operatorname{rank} B^{m-1}(D) \leqq 1.$$

Hence, in the chain $B^0(D) \leftarrow B^1(D) \leftarrow \ldots \leftarrow B^n(D)$ of surjections of $\mathcal{O}_C$-modules, there are exactly $r(D)$-jumps in the ranks, each jump increasing the rank by 1. The integers $b_1(D), b_2(D), \ldots, b_r(D)$ are the indices where the jumps appear.

COROLLARY 2. — *There are integers* $0=b_0<b_1<\ldots<b_r \leqq n < b_{r+1}=\infty$ *such that* $\operatorname{rank} B^j=(s+1)$ *for* $b_s \leqq j < b_{s+1}$ *and* $s=0, 1, \ldots, r$.

The following definition of a gap sequence of a linear system will be motivated in the following sections where it is related to the traditional notion of the general gap sequence of a curve (see Prop. 4(ii) and Prop. 5(i) and (ii)).

DEFINITION. — *The integers* $b_0<b_1<\ldots<b_r$ *of Corollary 2 are called the* gap sequence *of the linear system* V. *For each integer* $s=0, 1, \ldots, r$ *we denote the* $\mathcal{O}_C$*-bundle* B^{b_s} *and the* $\mathcal{O}_C$*-module* E^{b_s} *by* A^s *and* W^s *and call them the s'th* associated bundle *and s'th* Weierstrass module *of* V.

We say that the linear system V *has a* classical gap sequence *if* $b_m=m$ *for* $m=0, 1, \ldots, r$.

Remark. — The apparence of the gap sequence $b_0, b_1, \ldots, b_r$ in our enumerative formulas is the novelty of our approach. We shall for example see that when we let $b_m=m$ for $m=0, 1, \ldots, r$ in our Plücker formulas, then we obtain the classical generalized Plücker formulas. Also we show (§ 6 Theorem 4) that if the characteristic of the ground field is zero or strictly greater than n then the gap sequence is $b_m=m$ for $m=0, 1, \ldots, r$. This is the reason for using the term *classical* about the latter gap sequence.

§ 2. Wronskian points, Weierstrass points and multiple points

In this section we first define the notion of the gap sequence of a linear system *at a point* of the curve. The rather unfamiliar looking definition is motivated at the end of the section (Proposition 5 and the Remark following the proposition) where we show that for complete linear systems our definition is equivalent to the traditional definition of gap sequences.

Secondly we define the notion of *rank* $(s+1)$*-wronskian points* and give the basic properties satisfied by gap sequences and wronskian points. A rank $(r+1)$-wronskian point we call either a *Weierstrass point* or a *strictly* $(r+1)$*-tuple point* of the linear system. The first name is merited by the fact (see the Remark following Prop. 5) that our definition of a Weierstrass point is equivalent to the one given by F. K. Schmidt [7] in the case that the linear system is the complete canonical system. The second name which we introduced in [3] is motivated by the fact (see the Remark following Prop. 4) that when the linear system has a classical gap sequence the rank $(r+1)$-wronskian points are exactly the $(r+1)$-tuple points of the linear system in the traditional sence. For further references and comments about this material we refer to [3].

DEFINITION. — *An integer* $(m+1) \geqq 1$ *is called a* gap *of the linear system* V *at a point* $x \in \mathrm{C}$ *if the canonical surjection of diagram* (3), $\mathrm{E}^m(x) \to \mathrm{E}^{m-1}(x)$ *is an isomorphism.*

PROPOSITION 3. — *At each point* $x \in \mathrm{C}$ *there are exactly* $(r+1)$ *gaps of* V. *All the gaps are in the set* $\{1, 2, \ldots, (n+1)\}$.

Proof. — It follows from Corollary 2 that the module E^m is free of rank

$$(m+1)-(r+1)=m-r, \quad \text{for} \quad m \geqq n, \quad \text{and that} \quad (\dim \mathrm{E}^{m+1}(x)-\dim \mathrm{E}^m(x)) \leqq 1.$$

Moreover, it follows that $\dim \mathrm{E}^0(x) \leqq 1$. Hence in the sequence

$$\mathrm{E}^n(x) \to \mathrm{E}^{n-1}(x) \to \ldots \to \mathrm{E}^0(x) \to 0$$

there are $n+1-(n-r)=r+1$ isomorphisms and there are no gaps for $m > n$.

DEFINITION. — *We denote the gaps of* V *at a point* $x \in \mathrm{C}$ *by*

$$1 \leqq g_1(x) < g_2(x) < \ldots < g_{r+1}(x) \leqq (n+1).$$

Let s be an integer $0 \leqq s \leqq r$. *A point x of* C *is called a* rank $(s+1)$-*wronskian point of* V *if* $g_{m+1}(x) \neq b_m+1$ *for some* $m=0, 1, \ldots, s$.

A rank $(r+1)$-*wronskian point is called a* Weierstrass point *or a* strictly $(r+1)$-tuple point *of* V.

PROPOSITION 4. — *With the above notation the following assertions hold,*

(i) *We have* $g_{m+1}(x) \geqq (b_m+1)$ *for* $m=0, 1, \ldots, r$ *and for all points* $x \in \mathrm{C}$.

(ii) *We have* $g_{m+1}(x)=(b_m+1)$ *for* $m=0, 1, \ldots, r$ *and for all but a finite number of points* $x \in \mathrm{C}$. *In other words there are only a finite number of* rank $(s+1)$-*wronskian points of* V *for all s.*

(iii) *A point* $x \in \mathrm{C}$ *is a rank* $(s+1)$-*wronskian point of* V *if and only if*

$$\dim \mathrm{W}^m(x) > b_m - m$$

for some $m=0, 1, \ldots, s$.

(iv) *A point* $x \in \mathrm{C}$ *is a* rank $(s+1)$-*wronskian point of* V *if and only if the linear space of members of* V *that vanish to the order at least* b_m+1 *at x is of dimension at least* $(r+1-m)$ *for some* $m=0, 1, \ldots, s$.

Proof. — (i) Assume that $b_{m-1} \leqq j < b_m$. Then rank $\mathrm{B}^j = m$ and we have an inequality $\dim \mathrm{E}^j(x) \geqq j+1-m$ with equality for all but a finite number of points x of C.

Let $g(j)=\sup \{ i \mid g_i(x) \leqq j+1 \}$. Then, by the definition of gaps, we have that $\dim \mathrm{E}^j(x)=j+1-g(j)$. This equality together with the above inequality show that $m \geqq g(j)$ for $b_m-1 \leqq j < b_m$ and in particular that $m \geqq g(b_m-1)$. Hence $g_{m+1}(x) > b_m$.

(ii) In the proof of part (i) we observed that when $b_{m-1} \leqq j < b_m$ we have $\dim \mathrm{E}^j(x)=j+1-m$ for nearly all points $x \in \mathrm{C}$. It follows, by the definition of gaps, that for nearly all points

$x \in \mathrm{C}$ the only value of j in the interval $b_{m-1} \leqq j < b_m$ for which $(j+1)$ is a gap is $j = b_{m-1}$. The gap values are therefore $b_0+1, b_1+1, \ldots, b_r+1$ for nearly all points $x \in \mathrm{C}$.

(iii) In the proof of part (i) we observed that $\dim \mathrm{W}^m(x) = b_m + 1 - g(b_m)$. If

$$\dim \mathrm{W}^m(x) > b_m - m \quad \text{for some} \quad m = 0, 1, \ldots, s$$

then we obtain that $g(b_m) \leqq m$ and consequently that $g_{m+1} > b_m + 1$. Hence x is a rank $(s+1)$-wronskian point. Conversely, if x is a rank $(s+1)$-wronskian point of V then $g_{m+1}(x) > b_m + 1$ for some $m = 0, 1, \ldots, s$ and consequently $g(b_m) < m+1$. Hence we obtain an inequality

$$\dim \mathrm{W}^m(x) = b_m + 1 - g(b_m) > b_m - m.$$

(iv) Let R denote the local ring of C at x and M its maximal ideal. Then it follows from the definition of principal parts that $\mathrm{P}^j(x) = \mathrm{R}/\mathrm{M}^{j+1}$ and that if an element $u \in \mathrm{V}$ is represented at x by a function $f \in \mathrm{R}$, then $v^j(x)(u)$ is the class of f in $\mathrm{R}/\mathrm{M}^{j+1}$. By definition u vanishes to the order at least $(j+1)$ at x if and only if the class of f is zero. We see that the space $\mathrm{V} \cap \mathrm{H}^0(\mathrm{C}, \mathrm{D} - (j+1)x)$ of members of V of multiplicity at least $(j+1)$ is at least $(r+1-m)$-dimensional if and only if $\dim(im v^j(x)) \leqq m$, that is if and only if $\dim \mathrm{E}^j(x) \geqq j+1-m$. In particular, $\dim \mathrm{V} \cap \mathrm{H}^0(\mathrm{C}, \mathrm{D} - (b_m+1)x) \geqq r+1-m$ if and only if $\dim \mathrm{W}^m(x) \geqq b_m + 1 - m$. We see that assertion (iv) follows from assertion (iii).

COROLLARY 5. — *When* V *has a classical gap sequence, that is when* $b_m = m$ *for* $m = 0, 1, \ldots, r$, *then the following assertions hold,*

(i) *We have* $g_m(x) = m$ *for* $m = 1, 2, \ldots, r+1$ *and for all but a finite number of points of* C.

(ii) *A point* $x \in \mathrm{C}$ *is a* rank $(s+1)$-*wronskian point of* V *if and only if it is a* rank $(r+1)$-*wronskian point. That is, if and only if* $\dim \mathrm{W}^r(x) > 0$.

(iii) *A point* $x \in \mathrm{C}$ *is a* rank $(s+1)$-*wronskian point of* V *if and only if* x *is an* $(r+1)$-*tuple point of* V, *that is if and only if there is a member of* V *that vanishes to order at least* $(r+1)$ *at* x.

Proof. — Assertion (i) is a reformulation of part of assertion (ii) of the proposition. If $\dim \mathrm{W}^m(x) > 0$ for some $m = 0, 1, \ldots, s$, then from the surjections $\mathrm{W}^m(x) \to \mathrm{W}^{m-1}(x)$ resulting from diagram (3), we conclude that $\dim \mathrm{W}^r(x) > 0$. Hence assertion (ii) follows from assertion (iii) of the proposition. It is clear that if $\dim \mathrm{V} \cap \mathrm{H}^0(\mathrm{C}, \mathrm{D} - (m+1)x) \geqq r+1-m$ then $\dim \mathrm{V} \cap \mathrm{H}^0(\mathrm{C}, \mathrm{D} - (m+2)x) \geqq r-m$. Consequently assertion (iii) follows from assertion (iv) of the proposition.

Assertion (ii) of the above corollary shows that when the gap sequence is classical, then the Weierstrass points, as was suggested by M. F. Pohl [6], are the first order singularities of a map of vector bundles. In the classical case this fact is used to generalize the concept of Weierstrass points to higher dimensional varieties.

In general the situation is, however, more complicated and the Weierstrass points can only, as in assertion (iii) of the above proposition, be described as a union of first order singularities. We give in § 7 a simple example showing that $\dim \mathrm{W}^r(x) > b_r - r$ is not necessary for a point to be Weierstrass. The example indicates how to construct more involved examples and that generalization to higher dimensional varieties is more delicate than in the classical case.

PROPOSITION 6. — *Fix an integer $m \geqq 0$ and a point $x \in \mathrm{C}$. The following two assertions are equivalent.*

(i) *The canonical surjection* $\mathrm{H}^1(\mathrm{C}, \mathrm{D}-(m+1)x) \rightarrow \mathrm{H}^1(\mathrm{C}, \mathrm{D}-mx)$ *of diagram* (5) *is an isomorphism.*

(ii) *The canonical surjection* $\mathrm{E}^m(\mathrm{D})(x) \rightarrow \mathrm{E}^{m-1}(\mathrm{D})(x)$ *of diagram* (5) *is an isomorphism, that is* $(m+1)$ *is a gap of* D *at* x.

Proof. — The equivalence of (i) and (ii) is immediate by diagram (5).

Remark. — Proposition 6 shows that when the linear system V is complete, then our definition of gaps coincide with the classical definition. Moreover, Proposition 4 (ii) shows that a Weierstrass point can be characterized as a point where the gap sequence is exceptional. Hence our definition of Weierstrass points coincide with the classical definition and also with the definition given in arbitrary characteristic by F. K. Schmidt [7] in the case V is complete and D a canonical divisor and extended to the present situation by B. H. Matzat [4]. Proposition 4(i) was first proved by Matzat in his dissertation (Karlsruhe, 1972). (*See* also the remark of § 2 of [3]).

§ 3. Wronskians and Plücker formulas

This section is the central part of the article. For each integer $0 \leqq s \leqq r$ we construct, in a natural way, a section of a line bundle that vanishes exactly at the rank $(s+1)$-wronskian points. In this way we can give these points a structure of a divisor. The generalized Plücker formulas express the degree of these divisors in terms of the degree of the line bundles involved. We shall discuss in later sections (§ 5 and § 6) the geometrical significance of these results.

In the article [3] we assigned a multiplicity to the Weierstrass points and thus obtained a formula for the total weight of the Weierstrass points. The below constructions generalize and at the same time vastly simplify the results of that article.

THEOREM 7. — *Let $\mathrm{Q}_p \rightarrow \mathrm{Q}_{p-1} \rightarrow \ldots \rightarrow \mathrm{Q}_1$ be a sequence of surjective maps of vector bundles on* C *with* rank $\mathrm{Q}_i = i+1 = b+i-p+1$. *Moreover, let* A *be a subbundle of* Q_p *of* rank $m+1$ *and* A_1 *its image in* Q_1. *Finally, we denote by* Ω *and* K *the kernels of the maps* $\mathrm{Q}_p \rightarrow \mathrm{Q}_{p-1}$ *and* $\mathrm{A} \rightarrow \mathrm{A}_1$. *We assume that the image of* A *in* Q_i *has* rank m for *all* $i=1, 2, \ldots, p$.

Then the map

$$v_m : \bigwedge^{m+1} \mathrm{A} \rightarrow \bigwedge^{m} \mathrm{A}_1 \otimes \Omega$$

obtained by composing the inverse of the canonical isomorphism $\Lambda^m \mathrm{A}_1 \otimes \mathrm{K} \rightarrow \Lambda^{m+1}\mathrm{A}$ *with the inclusion* $\mathrm{K} \rightarrow \Omega$ *tensored by* $\Lambda^m \mathrm{A}$, *satisfies the following property*:

Let $x \in \mathrm{C}$ *and assume that* $\dim(\mathrm{Q}_1(x)/im \mathrm{A}_1(x)) = b-p+1-m+1$, *then* $v_m(x)=0$ *if and only if* $\dim(\mathrm{Q}_p(x)/\mathrm{A}(x)) > b-m$. *Moreover we have that* v_m *is non-zero.*

Proof. — The zeroes of the map v_m are clearly the same as the zeroes of the inclusion $K \to \Omega$. In particular it follows that v_m is non-zero. Moreover, if we denote by B the image of A in Q_{p-1} we see from the natural commutative diagram

$$\begin{array}{ccc} Q_{p-1}(x) & \to & Q_1(x) \to 0 \\ \uparrow & & \uparrow \\ B(x) & \xrightarrow{\sim} & A_1(x) \end{array}$$

that $\dim(Q_1(x)/imA_1(x)) = b-p+1-m+1$, if and only if $\dim(Q_{p-1}(x)/imB(x)) = b-m$. Consequently, the assertions of the theorem follows from an inspection of the natural commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \Omega(x) & \to & Q_p(x) & \to & Q_{p-1}(x) & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & K(x) & \to & A(x) & \longrightarrow & B(x) & \longrightarrow & 0 \end{array}$$

with exact rows.

COROLLARY 8. — *For each integer $0 \leqq s \leqq r$ there exists a canonical map*

$$w_{s+1} : \bigwedge^{s+1} A^s \to (\Omega_C^1)^{\otimes(b_0+b_1+\ldots+b_s)} \otimes \mathcal{O}(D)^{s+1}$$

such that $w_{s+1}(x)=0$ if and only if $\dim W^m(x) > b_m - m$ for some $m=0, 1, \ldots, s$. In other words $w_{s+1}(x)=0$ if and only if x is a rank $(s+1)$*-wronskian point. Moreover, we have that w_{s+1} is non-zero.*

Proof. — In section 1 we defined a sequence of surjective maps

$$P^{b_s}(D) \to P^{b_s-1}(D) \to \ldots \to P^{b_s-1}(D) \to \ldots \to P^{b_1}(D) \to \ldots \to P^{b_0}(D) = \mathcal{O}(D)$$

of the bundles of principal parts. By the definition of the numbers b_m the assumptions of Theorem 6 are fulfilled with the numbers $p = b_m - b_{m-1} + 1$ and $b = b_m$ and the bundles $Q_{m-b_m+p} = P^m(D)$ and $A = A^m$. We obtain non-zero maps

$$v_m : \bigwedge^{m+1} A^m \to \bigwedge^{m} A^{m-1} \otimes (\Omega_C^1)^{\otimes b_m} \otimes \mathcal{O}(D)$$

for $m=0, 1, \ldots, s$ where v_0 is the identity $\mathcal{O}(D) \to \mathcal{O}(D)$.

Define maps u_m for $m=0, 1, \ldots, s$ by

$$u_m = v_m \otimes (\text{identity map on } (\Omega_C^1)^{\otimes(b_{m+1}+\ldots+b_s)} \otimes \mathcal{O}(D)^{s-m}).$$

We let $w_{s+1} = u_0 u_1 \ldots u_s$. Then w_{s+1} is non-zero and at a point $x \in C$ we have that $w_{s+1}(x)=0$ if and only if $v_m(x)=0$ for some $m=0, 1, \ldots, s$. If

$$\dim W^m(x) = \dim(P^{b_m}(x)/imA^m(x)) = b_m - m \quad \text{for} \quad m=0, 1, \ldots, s$$

it follows from Theorem 7 that the maps $v_1(x), v_2(x), \ldots, v_s(x)$ are non-zero.

Conversely, if $\dim(W^m(x)) > b_m - m$ for some $m=0, 1, \ldots, s$ we let t be the first integer where inequality holds. If $t=0$ we see that $v_0(x)=0$ and if $t>0$ we obtain from Theorem 7 that $v_t=0$.

DEFINITION. — *The canonical section of the line bundle*

$$(\bigwedge^{s+1} A^s)^{-1} \otimes (\Omega_C^1)^{\otimes(b_0+b_1+\ldots+b_s)} \otimes \mathcal{O}(D)^{s+1}$$

described in Corollary 7 is called the $(s+1)$'*st* wronskian *of the linear system* V. *We denote the order to which* w_{s+1} *vanishes at a point* x *by* $d_s(x)$ *and call the divisor* $\Sigma_{x\in C} d_s(x)x$ *the* $(s+1)$'*st wronskian divisor of* V. *The degree of the* $(s+1)$'*st-wronskian we denote by* d_s.

The number d_r *is called the* total weight *of the Weierstrass points of* V *or the* total number *of strictly* $(r+1)$-*tuple points.*

THEOREM 9. (THE GENERALIZED PLÜCKER FORMULAS). — *Let* $r_s=\deg A^s$. *There is a finite number of* rank $(s+1)$-*wronskian points and the weighted number of such points is given by the following two equivalent sets of relations,*

(i) $d_s=(b_0+b_1+\ldots+b_s)(2g-2)+(s+1)n-r_s$ *for* $s=0, 1, \ldots, r$.

(ii) $r_{s+1}-2r_s+r_{s-1}=(b_{s+1}-b_s)(2g-2)-d_{s+1}+2d_s-d_{s-1}$
for $s=0, \ldots, r-1$ and $d_0=n-r_0$ where we let $r_{-1}=d_{-1}=0$.

In particular we obtain that the total weight of the Weierstrass points and the total number of strictly $(r+1)$-*tuple points is equal to*
(*The Brill-Segre formula*)

$$d_r=(b_0+b_1+\ldots b_r(2g-2)+(r+1)n$$

Proof. — By definition the number d_s is the weighted sum of the zeroes of $(s+1)$'st-wronskian w_{s+1} and by corollary 8 this map is not identically zero. Consequently d_s is the degree

$$(b_0+b_1+\ldots+b_s)(2g-2)+(s+1)n-r_s$$

of the line bundle

$$(\bigwedge^{s+1} A^s)^{-1} \otimes (\Omega_C^1)^{\otimes(b_0+b_1+\ldots+b_s)} \otimes \mathcal{O}(D)^{s+1}.$$

It is immediately verified that the two sets of equations are equivalent.

Remark. — To give the Plücker equations a more familiar form we introduce the integers $f_s=d_{s+1}-2d_s+d_{s-1}$ for $s=-1, 0, \ldots, r-1$ where we let $d_{-1}=d_{-2}=0$. Then $\Sigma_{i=-1}^s (s-i)f_i=d_s-(s+1)d_0$ and we obtain the Plücker relations in the forms

$$r_s=(b_0+b_1+\ldots+b_s)(2g-2)+(s+1)(n-d_0)-\sum_{i=-1}^{s}(s-i)f_i$$

or

$$r_{s+1}-2r_s+r_{s-1}=(b_{s+1}-b_s)(2g-2)-f_s\,.$$

From the latter equations we obtain the following formula

$$(r_{s+1}-r_s)-2(r_s-r_{s-1})+(r_{s-1}-r_{s-2})=(b_{s+1}-2b_s+b_{s-1})(2g-2)-(f_s-f_{s-1})$$

which, when the linear system V has a classical gap sequence is independent of the genus.

The above expressions are identical to the classical generalized Plücker formulas of Cayley and Veronese (see e. g. [2] and [5] for recent treatments). However, much of the interest in the classical formulas comes from the fact that the number f_s can be interpreted geometrically as the total ramification of the s'th associated map or equivalently as the sum of $h_{s+1}-h_s-1$ over all points of C where $h_0, h_1, \ldots, h_r$ are the Hermite invariants of V. Such interpretations are *a priori* not possible because the maps a_s can be everywhere ramified. However, we shall show in Section 7 that even when the associated map is not everywhere ramified the integer f_s can be negative and consequently has no natural geometric interpretation.

It is remarkable that in the Brill-Segre formula all the terms retain their geometric significance. In fact even the method of using wronskians to compute the local contributions of the $(r+1)$-tuple points to d_r is reminicent of C. Segre's [8] original method. The idea of patching the classical local wronskians used by Segre into a section of a line bundle is due to G. Galbura [1]. It follows from Proposition 11 (i) and Theorem 15 (i) that, in the classical case, our $(r+1)$-wronskian is the same as the one used by Galbura. See [3], § 3 for further details.

§ 4. Local computations

In order to study the geometric significance of the local numbers $g_m(x)$ and $d_m(x)$ introduced above, we shall need to express the maps $v^m: \mathrm{V_C} \to \mathrm{P}^m(\mathrm{D})$ in local coordinates around the point $x \in \mathrm{C}$. The primary purpose of this section is to write down such expressions and to introduce some notation needed in the sequel. Since the computations are standard we merely outline the proofs and refer to [3] § 4 for details and further references. From the expressions we obtain a very useful interpretation of the gaps of V in terms of a different set of easily computable local invariants.

Fix a point x of C and a local parameter t of C at x. Let R be the local ring of C at x and let I be the kernel of the multiplication map $\mathrm{R} \otimes \mathrm{R} \to \mathrm{R}$. Then we have that $\mathrm{P}^m(\mathrm{D})_x \cong \mathrm{R} \otimes \mathrm{R}/\mathrm{I}^{m+1}$ and that under the identification $(\Omega^1_{\mathrm{C}})^{\otimes m} = \mathrm{I}^m/\mathrm{I}^{m+1}$ the generator $(dt)^m$ maps to $(t \otimes 1 - l \otimes t)^m$. For simplicity we write dt instead of $(dt)^1$. The R-module structure on $\mathrm{P}^m(\mathrm{D})_x$ corresponding to the projection p makes $\mathrm{P}^m(\mathrm{D})_x$ into a left R-module and as such it is free with a basis $1, dt, \ldots, (dt)^m$. Denote by $d^m_{\mathrm{R}}: \mathrm{R} \to \mathrm{R} \otimes \mathrm{R}/\mathrm{I}^{m+1}$ the map induced by the other projection q, that is $d^m_{\mathrm{R}}(f)$ is the class of the element $1 \otimes f$. For each element $f \in \mathrm{R}$ we denote by $d^i f$ the coefficient of $(dt)^i$ in the expression of $d^m_{\mathrm{R}} f$ in the above basis. Considering $\mathrm{P}^m(\mathrm{D})_x$ as an R-module via the first factor and writing t instead of $t \otimes 1$ we have an equality $d_{\mathrm{R}}^{m} t = t - dt$ and by the binomial theorem we obtain the formula

$$(6) \qquad d_{\mathrm{R}}^{m} t^{h} = \sum_{i=0}^{h} \binom{h}{i} t^{h-i} (dt)^{i}.$$

In particular we have that $d^{i} t^{h} = \binom{h}{i} t^{h-i}$.

Clearly we have that $d^{i}(f \cdot g) = \sum_{j+k=i} d^{j} f \cdot d^{k} g$ for all $f, g \in \mathrm{R}$ so that if a is an element of the ground field we obtain a formula

$$(7) \qquad d^{i}(at^{h} + t^{h+1} g) = a \binom{h}{i} t^{h-i} + t^{h-i+1} g_{1}$$

for some element g_1 in R.

Choose a basis $v_0, v_1, \ldots, v_r$ of V. At a point x of C this basis determines, via the canonical map $v^0 : \mathrm{V_C} \to \mathcal{O}(\mathrm{D})$, linearly independent functions $f_0, f_1, \ldots, f_r$ in the local ring R. We can clearly choose the basis in such a way that the order of vanishing $h_i = \mathrm{order}_x f_i$ of f_i at x form a sequence $h_0 < h_1 < \ldots < h_r$. The integers h_i are local invariants. Indeed, h_i is uniquely determined as the number where the vector space spanned by $f_0, f_1, \ldots, f_r$ induces an $(i+1)$-dimensional subspace of $\mathrm{R}/(t^{h_i+1})$ and an i-dimensional subspace of $\mathrm{R}/(t^{h_i})$.

We call the integers h_i the *Hermite invariants* of V at x.

In the remaining part of this section we fix a basis $v_0, v_1, \ldots, v_r$ of V such that the orders h_i of the corresponding functions f_i are the Hermite invariants of V at x and we fix the basis 1, $dt, \ldots, (dt)^m$ *of* $\mathrm{P}^m(\mathrm{D})$ *at x. We shall also write* $f_i \equiv a_i t^{h_i} + \sum_{j=h_i+1}^{m} a_{i,j} t^{j} \pmod{t^{m+1}}$ *for* $i = 0, 1, \ldots, r$ *where the coefficients are in the ground field.*

The map

$$v_x^m : \mathrm{V} \otimes \mathrm{R} \to \mathrm{P}^m(\mathrm{D})_x$$

is expressed by the $(m+1) \times (r+1)$-matrix with entry $d^i f_j$ in row i and column j. Writing only the terms of lowest order in t it follows from formula (6) above that this matrix takes the form,

$$(8) \qquad \left| \begin{matrix} a_0 \binom{h_0}{0} t^{h_0} + \ldots, & a_1 \binom{h_1}{0} t^{h_1} + \ldots, & \ldots, & a_r \binom{h_r}{0} t^{h_r} + \ldots \\ a_0 \binom{h_0}{1} t^{h_0-1} + \ldots, & a_1 \binom{h_1}{1} t^{h_1-1} + \ldots, & \ldots, & a_r \binom{h_r}{1} t^{h_r-1} + \ldots \\ \vdots & \vdots & & \vdots \\ a_0 \binom{h_0}{m} t^{h_0-m} + \ldots, & a_1 \binom{h_1}{m} t^{h_1-m} + \ldots, & \ldots, & a_r \binom{h_r}{m} t^{h_r-m} + \ldots \end{matrix} \right|$$

Let i be the integer determined by the inequalities $h_i \leq m < h_{i+1}$. Then it is easily seen from the formula (6) that at $t = 0$ the matrix (8) takes the form

$$(9)\qquad \left|\begin{array}{ccccccc}
0 & 0 & & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & & \vdots \\
0 & & & & & & \\
 & & & & & & \\
a_0 & & & & & & \\
 & & & & & & \\
a_{0,h_0+1} & & & & & & \\
\vdots & 0 & & & & & \\
a_{0,h_1} & a_1 & & & & & \\
 & & & & & & \\
a_{0,h_1+1} & a_{1,h_1+1} & & & & & \\
\vdots & \vdots & & 0 & & & \\
a_{0,h_i} & a_{1,h_i} & \cdots & a_i & & & \\
 & & & & & & \\
a_{0,h_i+1} & a_{1,h_i+1} & & a_{i,h_i+1} & & & \\
\vdots & \vdots & & \vdots & & & \\
a_{0,m} & a_{1,m} & & a_{i,m} & 0 & \cdots & 0
\end{array}\right|$$

where column j starts with exactly h_{j-1} zeroes for $j=1, 2, \ldots, i$.

THEOREM 10. — *Fix a point x of* C *and let $h_0 < h_1 < \ldots < h_r$ be the Hermite invariants of* V *at x. Then the following assertions hold,*

(i) *For all integers $i=0, 1, \ldots, (r+1)$ we have that* $\dim(\operatorname{im} v^m(x)) = i+1$ *for all integers m satisfying the inequalities $h_i \leqq m < h_{i+1}$.*

(ii) *The inclusion* $\ker(P^m(x) \to P^{m-1}(x)) \subseteq \operatorname{im} v^m(x)$ *holds if and only if $m=h_i$ for some $i=0, 1, \ldots, r$.*

(iii) $g_{i+1}(x) = h_i + 1$ *for $i=0, 1, \ldots, r$.*

(iv) *For all but a finite number of points x of* C *we have that $b_i = h_i$ for $i=0, 1, \ldots, r$.*

Proof. — With the above choice of bases for V and $P^m(D)$ at x the map $v^m(x)$ takes the form (9) where i is determined by the inequalities $h_i \leqq m < h_{i+1}$. Assertions (i) and (ii) follow immediately from this expression.

An easy chase in the diagram (3) of Section 1 shows that the kernel of the map

$$P^m(x) \to P^{m-1}(x)$$

lies in the image of $B^m(x)$ if and only if the surjection $E^m(x) \to E^{m-1}(x)$ is an isomorphism, that is, if and only if $(m+1)$ is a gap of V at x. By assertion (ii) we obtain that $(m+1)$ is a gap if and only if $m=h_i$ for some $i=0, 1, \ldots, r$. In other words the gaps

$$g_1(x), g_2(x), \ldots, g_{r+1}(x)$$

are the integers $h_0+1, h_1+1, \ldots, h_r+1$. Assertion (iv) follows from assertion (iii) and Proposition 4 (ii).

PROPOSITION 11. — *Fix a point x of* C.

(i) *The multiplicity $d_s(x)$ of the $(s+1)$-wronskian at the point x is equal to the lowest order of the non-vanishing determinants of the $(s+1)\times(s+1)$-submatrices taken from the rows $b_0+1,\ b_1+1, \ldots, b_s+1$ of the matrix* (8).

(ii) *Assume that $b_s=s$. Let $\mathrm{N}=\binom{r+1}{s+1}-1$ and denote by $e_s(x)$ the ramification index at x of the* map $\mathrm{C}\to\mathbb{P}^{\mathrm{N}}$ *defined by the $(s+1)$'st exterior product $\Lambda^{s+1}\mathrm{V}\to\Lambda^{s+1}\mathrm{A}^s$ of the surjection $\mathrm{V}\to\mathrm{A}^s$ induced by the map v^s. Then $e_s(x)+1$ is equal to the difference between the next lowest and lowest order of the non-vanishing determinants of the $(s+1)\times(s+1)$-submatrices taken from the first $(s+1)$-rows of the matrix* (8).

Proof. — Let $m=b_s$. Then the matrix (8) represents the map v^m at x with respect to the above choice of bases $v_0, v_1, \ldots, v_r$ and $1, (dt), \ldots, (dt)^m$ of V and $\mathrm{P}^m(\mathrm{D})$. Choose a basis $u_0, u_1, \ldots, u_s$ of A^s such that the map $\mathrm{A}^s\to\mathrm{A}^m$ is the projection with center generated by $u_{m+1}, u_{m+2}, \ldots, u_s$ for $m=0, 1, \ldots, s-1$. Then the image of u_m in A^m maps under the inclusion $\mathrm{A}^m\to\mathrm{P}^{b_m}(\mathrm{D})$ to an element $c_m(dt)^{b_m}$ in the kernel of the map

$$\mathrm{P}^{b_m}(\mathrm{D})\to\mathrm{P}^{b_m-1}(\mathrm{D})$$

where c_m is in the ring R of C at x. Consequently the image of u_m in $\mathrm{P}^{b_s}(\mathrm{D})$ is of the form $c_m(dt)^{b_m}+\sum_{j=b_m+1}^{b_s}c_{m,j}(dt)^j$ with $c_{m,j}\in\mathrm{R}$.

Returning to the definition of v_m in Theorem 7 we see that the map

$$v_n:\Lambda^{m+1}\mathrm{A}^m\to\Lambda^m\mathrm{A}^{m-1}\otimes\Omega_{\mathrm{C}}^{\otimes b_m}\otimes\mathcal{O}(\mathrm{D})$$

sends $u_0\wedge u_1\wedge\ldots\wedge u_m$ to $c_m u_0\wedge u_1\wedge\ldots\wedge u_{m-1}\otimes(dt)^{b_m}$ so that

$$w_{s+1}(u_0\wedge u_1\wedge\ldots\wedge u_s)=c_0c_1\ldots c_s(dt)^{b_0}\otimes(dt)^{b_1}\otimes\ldots\otimes(dt)^{b_s}.$$

On the other hand the determinant of the $(s+1)\times(s+1)$-matrix taken from the rows $b_0+1, b_1+1, \ldots, b_s+1$ and the columns $j_0, j_1, \ldots, j_s$ of (8) is the product of the determinant of the $(s+1)\times(s+1)$-matrix taken from the rows $b_0+1,\ b_1+1, \ldots, b_s+1$ of the matrix B representing the map $\mathrm{A}^s\to\mathrm{P}^{b_s}(\mathrm{D})$ with respect to the bases $u_0, u_1, \ldots, u_s$ and $1, (dt), \ldots, (dt)^s$, with the determinant of the $(s+1)\times(s+1)$-matrix taken from the columns $j_0, j_1, \ldots, j_s$ of the matrix E representing the map $\mathrm{V}\to\mathrm{A}^s$ with respect to the bases $v_0, v_1, \ldots, v_r$ and $u_0, u_1, \ldots, u_s$. Assertion (i) of the proposition follows because the determinant obtained from the rows $b_0+1,\ b_1+1, \ldots, b_s+1$ is $c_0c_1\ldots c_s$ and E represents a surjective map.

When $b_s=s$ the determinant of the $(s+1)\times(s+1)$-matrix taken from the columns $j_0, j_1, \ldots, j_s$ of the matrix (8) with $m=s$ is equal to $c_0c_1\ldots c_m$ times the determinant of the corresponding submatrix of E. Assertion (ii) follows because $e_s(x)$ is one less than the next lowest order of the non-vanishing determinants of the $(s+1)\times(s+1)$-submatrices of E.

§ 5. Associated maps

The aim of the following section is to tie up the theory developed in the three first sections with the geometrical properties of the curve C. Unfortunately, the connections between the algebraic and geometric properties that are so familiar in characteristic zero do not, as we shall see, carry over to positive characteristic. As a consequence the contents of this section consist primarily of definitions and we shall have to wait for the discussion of the classical case in the next section to obtain the desired connections.

DEFINITION. — *The map*

$$a_s : \mathrm{C} \to \mathrm{Grass}\,(s, r)$$

from C *to the grassmannian of s-planes in* $\mathbb{P}^r$ *that is associated to the surjection* $\mathrm{V_C} \to \mathrm{A}^s$, *induced by the map* v^{b_s}, *we call the* s'th associated map *of* V. *The s-plane in* $\mathbb{P}^r$ *associated to a point x of* C *is called the* associated s-plane *to* V *at x and the degree* r_s *of the map* a_s *is called the* s-rank *of the linear system* V.

The degree of a_s is equal to the degree of a_s followed by the Plücker embedding $\mathrm{Grass}\,(m, r) \to \mathbb{P}^{\mathrm{N}}$, where $\mathrm{N} = \binom{r+1}{s+1} - 1$, and therefore equal to the degree of the line bundle $\Lambda^{s+1}\mathrm{A}^s$. Hence, the s-rank of V is equal to the integer r_s introduced in Theorem 9. On the other hand the degree of $\Lambda^{s+1}\mathrm{A}^s$ is equal to the first Chern class $c_1(\mathrm{A}^s)$ of A^s and this number can be interpreted geometrically as the number of associated s-planes to V that intersect a fixed $(r-s-1)$-plane of $\mathbb{P}^r$.

Let L be an s-plane in $\mathbb{P}^r$ and W the corresponding $(r-s)$-dimensional subspace of V. We say that L has *order* of *contact h* with V at x if $h+h_0$ is the lowest order of the functions in $\mathcal{O}_{\mathrm{C},r}$ associated to the vectors in W. If we, as in the preceding section, choose a basis $v_0, v_1, \ldots, v_r$ of V such that the orders $h_0, h_1, \ldots, h_r$ of the corresponding functions in $\mathcal{O}_{\mathrm{C},x}$ are the Hermite invariants of V at x, then we see that the s-plane associated to the linear subspace of V spanned by the vectors $v_{s+1}, v_{s+2}, \ldots, v_r$ is the unique s-plane with maximal order of contact with V at x and that this order of contact is $h_{s+1}-h_0$. The unique s-plane in $\mathbb{P}^r$ with maximal order of contact with V at x is called the *osculating s-plane* to V at x.

The most familiar situation is when the linear system V defines a map $f : \mathrm{C} \to \mathbb{P}^r$ and the divisor D is the inverse image of a hyperplane section in $\mathbb{P}^r$. Then $h_0 = 0$ and we obtain the familiar definitions of the order of contact of a linear space with the branch of $f(\mathrm{C})$ corresponding to x and the osculating space to $f(\mathrm{C})$ at this branch.

For each $s=0, 1, \ldots, r$ the integer $k_s(x)=h_s-h_{s-1}-1$, where we let $h_{-1}=-1$, is traditionally called the s'th *stationary index* of C at x. The point x is called *stationary* of *rank s* if $k_s(x)>0$ and the osculating s-plane is called *hyperosculating* if $\sum_{i=0}^{s} k_i(x)=h_s-s>0$, that is, if the point is stationary of some rank less than or equal to s. It is tempting, in the general situation, to call the integer $(h_s-b_s)-(h_{s-1}-b_{s-1})$ the s'th stationary index and define stationary points and hyperosculating planes in analogy with the traditional situation. We shall however, refrain from using this geometric terminology because examples (see Example 1, Section 7) show that the integer $h_s - b_s - (h_{s-1} - b_{s-1})$ can be negative.

PROPOSITION 12. — *Let* $h_0 = g_1(x) - 1$, $h_1 = g_2(x) - 1, \ldots, h_r = g_{r+1}(x) - 1$ *be the Hermite invariants of* V *at a point* x *of* C. *Moreover, let* $k_s(x) = (h_s - b_s) - (h_{s-1} - b_{s-1})$ *for* $s = 0, 1, \ldots, r$ *where we let* $b_{-1} = h_{-1} = -1$. *The following assertions are equivalent.*

(i) $$h^1(\mathrm{C}, \mathrm{D} - (b_s + 1)x) = h^1(\mathrm{C}, \mathrm{D}) + b_s - s$$

(ii) $$\dim v(\mathrm{D})^{b_s}(x) = s + 1$$

(iii) $$b_s = h_s$$

(iv) $$\sum_{m=0}^{s} k_m(x) = 0$$

Proof. — From diagram (3) of Section 1 we see that $\dim v^{b_s}(x) = s + 1$ if and only if $\dim \mathrm{E}(x)^{b_s} = b_s - s$. Consequently the equivalence of assertions (i) and (ii) follows from the top sequence of diagram (5) of Section 1.

The equivalence of assertions (ii) and (iii) follows from assertion (i) of Theorem 10 and Proposition 4 (ii).

The equivalence of assertions (iii) and (iv) follows from the above equality

$$h_s - b_s = \textstyle\sum_{i=0}^{s} k_i(x).$$

COROLLARY 13. — *Assume that the linear system* V *has a classical gap sequence. Then* $k_s(x) = h_s - h_{s-1} - 1 \geqq 0$ *and the following assertions are equivalent*

(i) $$h^1(\mathrm{C}, \mathrm{D}) = h^1(\mathrm{C}, \mathrm{D} - (s+1)x)$$

(ii) *The map* $v^s(x) : \mathrm{V} \to \mathrm{P}^s(\mathrm{D})(x)$ *is surjective.*

(iii) $$h_s = s$$

(iv) $$k_0(x) = k_1(x) = \ldots = k_s(x) = 0\,.$$

Remark. — W. F. Pohl ([6]) defined, when the characteristic of the ground field is zero, a point x of C to be *singular* of *order* s if the map $v^s(x)$ is not surjective. By the above corollary this is equivalent to the assertion that the point x is stationary of rank t for some integer $t \leqq s$. The latter assertion is, again by the corollary, equivalent to the assertion that $h_s > s$ and this is the condition for the osculating s-plane to be hyperosculating at x.

PROPOSITION 14. — (i) *Let* L *be a linear subspace of* $\mathbb{P}^r$ *of dimension s. Then the order of contact of* L *with* C *at a point x is equal to*

$$\max\{\, m + 1 - h_0 \mid \mathbb{P}(\operatorname{im} v^m(x)) \subseteq \mathrm{L} \,\}$$

(ii) *Let* $h_0, h_1, \ldots, h_r$ *be the Hermite invariants at a point x of* C *and assume that* $m = h_s = b_s$ *for some integer s. Then the osculating and associated m-planes of* V *at x coincide.*

Proof. — Choose bases for V and $P^m(D)$ as in the previous section. With respect to this choice of bases $v^m(x)$ takes the form (9). Let W be the subspace of V corresponding to the linear space L of assertion (i). Then the inclusion $\mathbb{P}(\operatorname{im} v^m(x)) \subseteq L$ holds if and only if $W \subseteq \ker v^m(x)$. However, from the matrix (9) we see that $\ker v^m(x)$ is generated by the vectors $v_{i+1}, v_{i+2}, \ldots, v_r$ where i is determined by the inequalities $h_i \leqq m < h_{i+1}$. Consequently, if $W \leqq \ker v^m(x)$ we have that the order of contact of L with V at x is at least equal to $h_{i+1} - h_0 > m - h_0$ so the order of contact is at least equal to

$$\max \{ m+1-h_0 \mid \mathbb{P}(v^m(x)) \subseteq L \}.$$

Conversely, let $h - h_0$ be the order of contact of L with C. Then clearly $h = h_{j+1}$ for some integer j and W must be contained in the subspace of V generated by the vectors v_{j+1}, $v_{j+2}, \ldots, v_m$. Consequently $W \subseteq \ker v^m(x)$ for all integers m satisfying the inequalities $h_j \leqq m < h_{j+1}$. In particular we have that $W \subseteq \ker v^{h_{j+1}-1}(x)$ so that

$$h - h_0 = h_{j+1} - h_0 \leqq \max \{ m+1-h_0 \mid \mathbb{P}(v^m(x) \subseteq L \}$$

and the equality of assertion (i) is established.

To prove assertion (ii) it suffices to observe that if $m = h_s = b_s$ it follows from the form of the matrix (9) that the associated m-plane to V at x is defined by the equations

$$v_{s+1} = v_{s+2} = \ldots = v_r$$

and that this space is, by definition, the osculating m-plane to V at x.

Remark. — One of the strange features of the geometrical objects introduced in this section is that the osculating and associated s-planes are not always equal, not even when the linear system has a classical gap sequence (*see* Example 2 of § 7). On the other hand we shall show that they coincide when we impose appropriate conditions on the characteristic of the geound field.

§ 6. The classical situation

The most unifortunate feature of the associated maps is, as we shall indicate in the next section, that there are in general no natural relations between the geometric objects introduced so far and the invariants introduced in the first three sections. We shall, however, in the present section show that under certain restrictions on the characteristic of the ground field, the objects are related in the traditional way and that we obtain the well known formulas of enumerative geometry.

THEOREM 15. — *Assume that the characteristic of the ground field is zero or strictly greater than n. Fix a point x and let* $h_0 = g_1(x) - 1$, $h_1 = g_2(x) - 1, \ldots, h_r = g_{r+1}(x) - 1$ *be the Hermite invariants of the linear system* V *at* x.

Then the following assertions hold,

(i) *The linear system* V *has classical gap sequence, that is* $b_m = m$ *for* $m = 0, 1, \ldots, r$

(ii) *The* rank $(s+1)$-*wronskian vanishes to the order*

$$d_s(x)=\sum_{m=0}^{s}(h_m-m) \quad \textit{for} \quad s=0, 1, \ldots, r$$

at x. Equivalently the following equalities

$$d_{s+1}(x)-2d_s(x)+d_{s-1}(x)=h_{s+1}-h_s-1$$

hold for $s=-1, 0, 1, \ldots, r-1$ *where we let* $d_{-1}(x)=d_{-2}(x)=0$ *and* $h_{-1}=-1$.

(iii) *The ramification index* $e_s(x)$ *of* a_s *at* x *is equal to* $h_{s+1}-h_s-1$ *for* $s=0, 1, \ldots, r-1$.

(iv) *The Plücker formulas take the following equivalent forms*

$$r_s=(s+1)(s(g-1)+(n-d_0))-\sum_{m=0}^{s-1}(s-m)e_m$$

for $s=0, 1, \ldots, r$ *or*

$$r_{s+1}-2r_s+r_{s-1}=(2g-2)-e_s$$

for $s=0, 1, \ldots, r-1$ *and* $r_0=n-h_0=n-d_0$, *where* e_s *is the total ramification of the map* a_s. *In particular the Brill-Segre formula takes the form*

$$\sum_{m=0}^{r-1}(r-m)e_m=(r+1)(r(g-1)+n-d_0).$$

(v) *The osculating and associated s-planes at x coincide.*

Proof. — We saw at end of Section 4 that we can choose bases $v_0, v_1, \ldots, v_r$ for V and $1, dt, \ldots, (dt)^m$ for $P^m(D)$ at the point x such that the map $v^m : V_C \to P^m(D)$ takes the form (8). To prove assertion (i) it suffices to prove that for each m one of the determinants of the $(m+1)\times(m+1)$-matrices taken from the first $(m+1)$-rows of (8) is not zero. We shall prove more, namely that the matrix taken from columns $i(0)$, $i(1)$, $\ldots$, $i(m)$ has order exactly $\sum_{j=0}^{m}(h_{i(j)}-i(j))$. To this end it suffices to prove that the determinant of the matrix

$$\begin{vmatrix} \binom{h_{i(0)}}{0} & \binom{h_{i(1)}}{0} & \cdots & \binom{h_{i(m)}}{0} \\ \binom{h_{i(0)}}{1} & \binom{h_{i(1)}}{1} & \cdots & \binom{h_{i(m)}}{1} \\ \vdots & \vdots & & \vdots \\ \binom{h_{i(0)}}{m} & \binom{h_{i(1)}}{m} & \cdots & \binom{h_{i(m)}}{m} \end{vmatrix}$$

is non-zero. This is well known (see e. g. [3] § 4 for the result and references) and is easily seen by considering the determinant as an alternating polynomial of degree $\frac{m(m+1)}{2}$ in the « variables » $h_{i(0)}, h_{i(1)}, \ldots, h_{i(m)}$. As such it is divisible by $\Pi_{0\le j<k\le m}(h_{i(k)}-h_{i(j)})$

and the constant term is equal to the inverse of the number $\prod_{i=0}^{m}(i!)$. The latter number is non-zero because the characteristic is zero or greater than n and by the Riemann-Roch theorem $n \geqq r$. The number $\prod_{0 \leq j < k \leq m}(h_{i(k)} - h_{i(j)})$ is non-zero because $h_i = g_{i+1}(x) - 1 \leqq n$ by Theorem 10 (iii) and Proposition 3.

In particular we conclude that the orders of the lowest and next lowest determinants taken from the first $(m+1)$-rows are $\sum_{i=0}^{m}(h_i - i)$ and $\sum_{i=0}^{m-1}(h_i - i) + h_{m+1} - m$. Consequently assertions (ii) and (iii) follow from assertions (i) and (ii) of Proposition 11.

We see from assertions (ii) and (iii) that $e_s(x)$ is equal to the integer $f_s(x)$ introduced in the Remark following Theorem 9. The formulas of that remark are thus the same as those of assertion (iv).

To prove assertion (v) we fix a basis for A^m at x. With respect to this basis and the chosen bases for V and $P^m(D)$ the matrix (8) is expressed in the form $A \cdot B$ where B is the $(m+1) \times (r+1)$-matrix expressing the surjective map $V_C \to A^m$ and A is an $(m+1) \times (m+1)$-matrix whose determinant by Proposition 11 (i) is equal to

$$d_s(x) = \sum_{i=0}^{s} h_i - i.$$

The determinant of the $(s+1) \times (s+1)$-matrix taken from columns $i(0), i(1), \ldots, i(s)$ of B is equal to the determinant of the corresponding determinant of the matrix (8) divided by the determinant of the matrix A and has order $\sum_{j=0}^{m}(h_{i(j)} - h_j)$ by the above observations. We conclude that the determinant taken from the first $(s+1)$-columns of $A(x)$ is non-zero and that the determinants of all the other $(s+1) \times (s+1)$-submatrices of $A(x)$ are zero. Consequently, the kernel of the map $V \to A^m(x)$ is generated by the vectors

$$v_{s+1} = v_{s+2} = \ldots = v_r,$$

that is, the associated m-plane coincides with the osculating m-plane.

§ 7. Examples

The main purpose of this section is to show that the results that we have proved in the classical case and that we have not generalized do not carry over to the general case in a natural way. Most important among these results is the fundamental equality

$$e_s(x) = d_{s+1}(x) - 2d_s(x) + d_{s-1}(x)$$

which ties up the geometric properties of the associated curve with the algebraic properties of the wronskian. Example 2 below shows that the number $d_{s+1}(x) - 2d_s(x) + d_{s-1}(x)$ can be negative so that no such geometric interpretations are possible.

For computational purposes it would be desirable to have equalities of the form $e_s(x) = h_{s+1} - h_s - 1$ and that assertion (ii) of Theorem 13 should generalize to give an equality of $d_s(x)$ to, for example, the sum $\sum_{i=0}^{s}(h_i - b_i)$. However, Example 2 shows that none of these equalities hold in general.

Finally Example 2 shows that the osculting and associated planes are not equal in general.

We start with an example showing that the condition $\dim W^r(x) > b_r - r$ is not necessary for a point x to be Weierstrass. This example should be compared with the much more complicated example (1, § 6) of [3]. The purpose of that example was to show that the condition is not even necessary when the curve is embedded by the complete canonical system. Without this requirement examples are much easier to construct.

Example 1. — Let $C = \mathbb{P}^1$ and assume that the characteristic of the ground field is 2. The linear system V spanned by the sections $\{ s^4, s^3t, st^3, t^4 \}$ in $H^0(C, \mathcal{O}(4))$ separates both ordinary and infinitely near points of C and embeds C in $\mathbb{P}^3$ as a curve of degree 4. At a point $(1; a; a^3; a^4)$ the embedding can be expressed in terms of a local parameter t by the functions $g_0 = 1$, $g_1 = a + t$, $g_2 = a^3 + a^2t + at^2 + t^3$ and $g_3 = a^4 + t^4$. We choose a basis v_0, v_1, v_2, v_3 of V which determines the functions $f_0 = 1$, $f_1 = t$, $f_2 = at^2 + t^3$ and $f_3 = t^4$. It follows that $b_0 = 0$, $b_1 = 1$, $b_2 = 2$ and $b_3 = 4$ and that the Hermite invariants at the point $x = (1; 0; 0; 0)$ are $h_0 = 0$, $h_1 = 1$, $h_2 = 3$ and $h_3 = 4$. Hence x is a Weierstrass point. However with respect to the basis $1, dt, (dt)^2, (dt)^3, (dt)^4$ of $P^4(D)$ the map $v^4(x) : V \to P^4(D)(x)$ is expressed by the matrix (9),

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

so that $\dim W^3(x) = 1 = b_3 - 3$.

Note that $k_3(x) = (h_3 - b_3) - (h_2 - b_2) = -1$.

Example 2. — Let $C = \mathbb{P}^1$ and assume that the characteristic of the ground field is 3. The linear system spanned by the sections $\{ s^5, s^4t, st^4, t^5 \}$ in $H^0(C, \mathcal{O}(5))$ separates both ordinary and infinitely near points on C and embeds C into $\mathbb{P}^3$ as a curve of degree 5. At a point $(1; a; a^4; a^5)$ the embedding can be expressed in terms of a local parameter t by the functions

$$g_0 = 1, g_1 = a + t, g_2 = (a+t)^4 = a^4 + a^3t + at^3 + t^4, g_3 = (a+t)^5 = a^5 - a^4t + a^3t^2 + a^2t^3 - at^4 + t^5.$$

We choose a basis v_0, v_1, v_2, v_3 of V that determines the functions

$$f_0 = 1, \qquad f_1 = t, \qquad f_2 = a^3t^2 + a^2t^3 - at^4 + t^5, \qquad f_3 = at^3 + t^4.$$

It follows that the Hermite invariants are $h_i = i$ for $i = 0, 1, 2, 3$ when $a \neq 0$ and $h_0 = 0$, $h_1 = 1$, $h_2 = 4$, $h_3 = 5$ when $a = 0$. In particular we have that $b_i = i$ for $i = 0, 1, 2, 3$. We also see that the osculating plane at a point with $a \neq 0$ is defined by $v_3 = 0$ and has order of contact 3 with C at the point and that the osculating plane at $x = (1; 0; 0; 0)$ is given by $v_2 = 0$ and has order of contact 5 with C at the point.

With respect to the choice of basis 1, dt, $(dt)^2$, $(dt)^3$ of $P^2(D)$ the map $v^2 : V_C \to P^2(D)$ is expressed at x by the matrix (8),

$$\begin{vmatrix} 1 & t & t^5 & t^4 \\ 0 & 1 & 2t^4 & t^3 \\ 0 & 0 & t^3 & 0 \end{vmatrix} = \begin{vmatrix} 1 & t & t^5 \\ 0 & 1 & 2t^4 \\ 0 & 0 & t^3 \end{vmatrix} \quad \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & t^3 \\ 0 & 0 & 1 & 0 \end{vmatrix}$$

We see that the associated plane at x is defined by $v_3=0$. It has order of contact 4 with C at x.

The map $v^3 : V_C \to P^3(D)$ is expressed at the point $x(a)=(1; a; a^4; a^5)$ by the matrix

$$\begin{vmatrix} 1 & t & a^3t^2+a^2t^3-at^4+t^5 & at^3+t^4 \\ 0 & 1 & 2a^3t-at^3+2t^4 & t^3 \\ 0 & 0 & a^3+t^3 & 0 \\ 0 & 0 & a^2-at+2t^2 & a+t \end{vmatrix}$$

Using Proposition 11 we easily compute that we have $d_1(x(a))=d_2(x(a))=d_3(x(a))=0$ and $e_2(x(a))=2$ when $a \neq 0$ and that $d_1(x)=0$, $d_2(x)=3$, $d_3(x)=4$ and $e_2(x)=2$.

By symmetry we have that

$$d_1(y)=0, \quad d_2(y)=3, \quad d_3(y)=4 \quad \text{and} \quad e_2(y)=2, \quad \text{where} \quad y=(0;0;0;1).$$

For the numbers involved in Theorem 13 we obtain

$$\sum_{i=0}^{2}(h_i-i)=2, \qquad \sum_{i=0}^{3}(h_i-i)=4,$$

$$d_3(x)-2d_2(x)+d_1(x)=-2d_2(x)-2d_1(x)+d_0(x)=3$$

and

$$h_3-h_2-1=1, \qquad h_2-h_1-1=2$$

	$d_0(x(a))$	$d_1(x(a))$	$d_2(x(a))$	$d_3(x(a))$	$e_1(x(a))$	$e_2(x(a))$
$a \neq 0$	0	0	0	0	0	2
$a=0$	0	0	3	4	2	2
$a=\infty$	0	0	3	4	2	2

Let h_0, h_1, h_2, h_3 be the Hermite invariants at $a=0$. We have the following inequalities

$$3=d_2(x) \neq \sum_{i=0}^{2}(h_i-i)=2$$

$$3=d_2(x)-2d_1(x)+d_0(x) \neq h_2-h_1-1=2=e_1(x)$$

$$2=e_2(x) \neq h_3-h_2-1=0$$

$$-8=(2g-2)-d_2+2d_1-d_0=r_2-2r_1+r_0 \neq (2g-2)-e_1=-6$$

Incidentally the equality $e_1(x)=h_2-h_1-1=2$ holds in this case. However, modifying the above example by choosing the linear system $\{s^5, s^4t, s^2t^3, st^4, t^5\}$ one sees that one has $b_i=i$ for $i=0, 1, 2, 3, 4$ and $h_0=0, h_1=1, h_2=3, h_3=4, h_5=5$ at $a=0$ and a table

	$d_0(x(a))$	$d_1(x(a))$	$d_2(x(a))$	$d_3(x(a))$	$e_1(x(a))$	$e_2(x(a))$
$a\neq 0$	0	0	0	0	0	2
$a=0$	0	0	3	3	2	2

We obtain the inequalities

$$3=d_2(x)+\sum_{i=0}^{2}(h_i-i)=1$$

$$3=d_3(x)+\sum_{i=0}^{3}(h_i-i)=2$$

$$3=d_2(x)-2d_1(x)+d_0(x)+e_1(x)+h_2-h_1-1=1$$

$$-3=d_3(x)-2d_2(x)+d_1(x)+e_2(x)+h_3-h_2-1=0$$

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