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ON A LOWER BOUND FOR THE FIRST EIGENVALUE OF THE LAPLACE OPERATOR ON A RIEMANNIAN MANIFOLD

BY ATSUSHI KASUE (*)

Introduction

Let M be a connected, compact Riemannian manifold of dimension m with smooth boundary ∂M. The Laplace operator Δ acting on functions is locally given by

$$\sum_{i,j=1}^{m} \frac{1}{\sqrt{G}} \frac{\partial}{\partial x_i}\left(\sqrt{G} g^{ij} \frac{\partial}{\partial x_j}\right),$$

where $(x_1, \ldots, x_m)$ is a local coordinate system, $g=\sum_{i,j=1}^{m} g_{ij} dx_i dx_j$ is the fundamental tensor, $G = \det(g_{ij})$ and $(g^{ij})=(g_{ij})^{-1}$. We consider the following equation:

$$(0.1) \qquad \begin{cases} \Delta u + \lambda u = 0 & \text{on } M \\ u = 0 & \text{on } \partial M. \end{cases}$$

If for some number λ there is a nontrivial solution $u(x)$ of (0. 1), we call this value of λ an eigenvalue. We write $\lambda_1(M)$ for the first eigenvalue. The purpose of the present paper is to show geometric bounds for $\lambda_1(M)$.

Let us now assume the Ricci curvature of M is bounded from below by a constant $(m-1)R$ and the trace of S_ν is bounded from above by a constant $(m-1)\Lambda$ (R, $\Lambda \in \mathbb{R}$), where S_ν denotes the second fundamental form of ∂M with respect to the unit inner normal vector field ν on ∂M (i. e., $g(S_\nu X, Y) = g(\nabla_X \nu, Y)$ for X, $Y \in T(\partial M)$). Such a manifold M is called *a Riemannian manifold of class* (R, Λ) for the sake of brevity. Recently, Li and Yau [19] have given, among other things, computable lower bounds for $\lambda_1(M)$ in terms of R, Λ and the inradius $\mathscr{I}_M$ of M (i. e., $\mathscr{I}_M := \sup\{\operatorname{dis}(x, \partial M) : x \in M\}$). Especially, their estimate is optimum in the case when $R = \Lambda = 0$ (*cf.* [*ibid.*: Theorem 11]). More precisely, they have proved that, in such a case, $\lambda_1(M)$ is greater than or equal to $\pi^2/4\mathscr{I}_M^2$; the equality is attained for a section of a flat cylinder. Their method is based on a gradient estimate of the first eigenfunction. Moreover, Gallot [8] has also showed another computable lower bound for $\lambda_1(M)$, estimating the Cheeger's isoperimetric constant in

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terms of R, Λ and $\mathscr{I}_M$(*cf.* also [9]). On the other hand, before the works mentioned above, Reilly [23] showed that if $R>0$ and $\Lambda=0$, $\lambda_1(M)$ is not less than mR and the equality holds if and only if M is isometric to a closed hemisphere of the Euclidean sphere $S^m(R)$ of constant curvature R. This result by Reilly is a generalization, to Riemannian manifolds of class (R, 0) ($R > 0$), of the well known theorem by Lichnerowicz [18] and Obata [21], which says that the first eigenvalue of the Laplace operator on a compact Riemannian manifold without boundary is greater than or equal to mR if the Ricci curvature has a positive lower bound R, and the equality holds if and only if the manifold is isometric to $S^m(R)$.

We shall now summarize our main results. In section 2, we consider the case when M is a Riemannian manifold of class (R, Λ) and show that $\lambda_1(M)$ has a lower bound depending on R, Λ and $\mathscr{I}_M$ (*cf.* Theorem 2.1). Moreover our estimate is sharp when R and Λ satisfy certain conditions which ensure us the existence of *a model space of class* (R, Λ) (*cf.* Definition 1.2). In fact, we see that the equality holds if and only if M is isometric to a model space of class (R, Λ). We note that our estimate coincides with the above one due to Li and Yau when $R=\Lambda=0$ (*cf.* Corollary 2.3) and our result contains the above theorem by Reilly as the special case: $R>0$ and $\Lambda=0$. In section 3, we consider the case when M is a domain of a complete, noncompact Riemannian manifold N and prove that if the Ricci curvature of N is bounded from below by a nonpositive constant $(m-1)R$, $\lambda_1(M)$ has a lower bound depending on R and the diameter $d(M)$ of M (*cf.* Theorem 3.1 (1)). In connection with our estimate, we must mention that, under the same assumption as above, Gallot has also given a lower estimate for $\lambda_1(M)$ in terms of R and $d(M)$ (*cf.* [8: Theorem 3.13 (i)]). It will be turn out that our estimate is sharper than his. Moreover we shall show that if the sectional curvature of N is bounded from above by a nonpositive constant K and there is *a concave function without maximum* on N, $\lambda_1(M)$ has a lower estimate depending on K and $d(M)$ (*cf.* Theorem 3.1 (2)).

The basic idea to obtain a lower bound for $\lambda_1(M)$ is a combination of an extension of a result by Barta [1] (*cf.* Lemma 1.1) and Laplacian and Hessian comparison theorems which are the refined forms of the well known Rauch's comparison theorem (*cf.* [15]).

Finally, the author would like to express sincere thanks to Professor T. Ochiai for his helpful advice and encouragement.

1. Preliminary

In this section, we shall first show a generalization of a result by Barta [1] (*cf.* Lemma 1.1), and next give the definition of a model space of class (R, Λ) (*cf.* Definition 1.2) and some notations used in Sections 2 and 3.

1.1. — Let M be a connected, compact Riemannian manifold of dimension m with smooth boundary ∂M. We write M_o for the interior of M. A result of Barta [1] tells us that for any positive C^2-function ψ on M, we have

$$\lambda_1(M) \geqq \inf_M -\frac{\Delta\psi}{\psi}$$

We shall first extend this result in the following

LEMMA 1.1. — *Suppose there is a continuous function* ψ *on* M *such that*

$$\psi>0 \quad \text{on } M_o, \tag{1.1}$$

$$\Delta\psi+\lambda\psi\leqq 0 \quad \text{as a distribution on } M_o, \tag{1.2}$$

where λ *is a constant. Then we have*

$$\lambda_1(M)\geqq\lambda.$$

Moreover if ψ *is smooth on an open dense subset of* M, $\lambda_1(M)=\lambda$ *implies that* ψ *is the first eigenfunction* (i. e., $\Delta\psi+\lambda_1(M)\psi=0$ *on* M_o *and* $\psi=0$ *on* ∂M).

Proof. — Applying the approximation theorem by Greene and Wu [11: Lemma 1.2, Lemma 3.2 and Theorem 3.2] and the arguments in [5: p. 458], we can prove the above lemma. In fact, for any $\varepsilon>0$, the approximation theorem of Greene and Wu tells us that there exists a smooth function ψ_ε on M_o satisfying

$$|\psi_\varepsilon-\psi|<\frac{\varepsilon}{2} \tag{1.3}$$

$$\Delta\psi_\varepsilon+\lambda\psi<\varepsilon \tag{1.4}$$

on M_o. Let w be any smooth function whose support is contained in M. Then by (1.1) and (1.3), w can be represented in the form:

$$w=\psi\cdot\eta$$

on M_o and also

$$w=(\psi_\varepsilon+\varepsilon)\eta_\varepsilon$$

on M. Noting that $2(\psi_\varepsilon+\varepsilon)\eta_\varepsilon\langle\nabla\eta_\varepsilon,\nabla\psi_\varepsilon\rangle+\eta_\varepsilon^2\|\nabla\psi_\varepsilon\|^2=\langle\nabla\psi_\varepsilon,\nabla\{\eta_\varepsilon^2(\psi_\varepsilon+\varepsilon)\}\rangle$ and integrating by parts,

$$\begin{aligned}\int_M\|\nabla w\|^2-\lambda w^2&=\int_M(\psi_\varepsilon+\varepsilon)^2\|\nabla\eta_\varepsilon\|^2+2(\psi_\varepsilon+\varepsilon)\eta_\varepsilon\langle\nabla\eta_\varepsilon,\nabla\psi_\varepsilon\rangle+\eta_\varepsilon^2\|\nabla\psi_\varepsilon\|^2-\lambda(\psi_\varepsilon+\varepsilon)^2\eta_\varepsilon^2\\&=\int_M(\psi_\varepsilon+\varepsilon)^2\|\nabla\eta_\varepsilon\|^2-\eta_\varepsilon^2(\psi_\varepsilon+\varepsilon)\Delta\psi_\varepsilon-\lambda(\psi_\varepsilon+\varepsilon)^2\eta_\varepsilon^2.\end{aligned}$$

Therefore we see by (1.4) that

$$\begin{aligned}\int_M\|\nabla w\|^2-\lambda w^2&\geqq\int_M(\psi_\varepsilon+\varepsilon)^2\|\nabla\eta_\varepsilon\|^2+\eta_\varepsilon^2(\psi_\varepsilon+\varepsilon)\{\lambda(\psi-\psi_\varepsilon)-\varepsilon(\lambda+1)\}\\&\geqq\int_M\eta_\varepsilon^2(\psi_\varepsilon+\varepsilon)\{\lambda(\psi-\psi_\varepsilon)-\varepsilon(\lambda+1)\}.\end{aligned} \tag{1.5}$$

Since the right-hand side of (1.5) tends to 0 as $\varepsilon \downarrow 0$, we obtain

$$\int_M \| \nabla w \|^2 \geqq \lambda \int_M w^2.$$

This shows that $\lambda_1(M) \geqq \lambda$, by the variational characterization of $\lambda_1(M)$. Now suppose ψ is smooth on an open dense subset U of M and $\lambda_1(M) = \lambda$. Then it follows from the approximation theorem by Greene and Wu again that for any $\varepsilon > 0$ and every compact set K in $U \cap M_o$, there is a smooth function $\psi_{\varepsilon, K}$ on M_o which satisfies (1.3), (1.4) and

$$\| \nabla \psi_{\varepsilon, K} - \nabla \psi \| < \varepsilon \tag{1.6}$$

on K. Set $\eta_{\varepsilon, K} := w/(\psi_{\varepsilon, K} + \varepsilon)$. Then by (1.5) we have

$$\int_M \| \nabla w \|^2 - \lambda w^2 \geqq \int_K (\psi_{\varepsilon, K} + \varepsilon \| \nabla \eta_\varepsilon \|^2 + \int_M \eta^2_{\varepsilon, K} (\psi_{\varepsilon, K} + \varepsilon) \{ \lambda (\psi - \psi_{\varepsilon, K}) - \varepsilon(\lambda + 1) \}.$$

Since ε is any positive number, we obtain by (1.4) and (1.6)

$$\int_M \| \nabla w \|^2 - \lambda w^2 \geqq \int_K \psi^2 \| \nabla \eta \|^2 \qquad (\lambda = \lambda_1(M)). \tag{1.7}$$

Suppose w is the first eigenfunction. Then $\int_M \| \nabla w \|^2 - \lambda w^2 = - \int_M (\Delta w + \lambda w) w = 0$, so that $\nabla \eta = 0$ on K by (1.7). Since K is any compact set in U, we see that $\nabla \eta = 0$ on U and hence $\nabla \eta = 0$ on M. This implies that η is a constant on M, that is, $w = \text{const.} \times \psi$ on M. This completes the proof of Lemma 1.1.

1.2. — Now we shall define a special class of Riemannian manifolds with boundary. For this purpose, let us introduce the function $h_{R,\Lambda}(t)$ on $[0, \infty)$ defined by the following classical Jacobi equation:

$$h_{R,\Lambda}'' + R h_{R,\Lambda} = 0 \qquad \text{with} \qquad h_{R,\Lambda}(0) = 1 \qquad \text{and} \qquad h_{R,\Lambda}'(0) = \Lambda. \tag{1.8}$$

Set $C_1(R, \Lambda) := \inf \{ t : h_{R,\Lambda}(t) = 0,\ t > 0 \}$ $(\leqq +\infty)$ and

$$C_2(R, \Lambda) := \inf \{ t : h_{R,\Lambda}'(t) = 0,\ t > 0 \} \ (\leqq +\infty).$$

Here we understand $C_1(R, \Lambda) = +\infty$ (resp. $C_2(R, \Lambda) = +\infty$) if $h_{R,\Lambda} > 0$ (resp. $h_{R,\Lambda}'$ does not vanish on $[0, C_1(R, \Lambda)]$). Clearly, the inner radius $\mathscr{I}_M$ of a Riemannian manifold M of class (R, Λ) is less than or equal to $C_1(R, \Lambda)$. Moreover, we remark that $C_1(R, \Lambda) < +\infty$ if and only if $R > 0$, $R = 0$ and $\Lambda < 0$, or $R < 0$ and $\Lambda < -\sqrt{-R}$, and that $0 < C_2(R, \Lambda) < +\infty$ if and only if $R > 0$ and $\Lambda > 0$, or $R < 0$ and $-\sqrt{-R} < \Lambda < 0$.

DEFINITION 1.2. — A Riemannian manifold M of class (R, Λ) is called *a model space* if one of the following conditions holds:

(I) $C_1(R, \Lambda) < +\infty$ and M is isometric to the metric (closed) ball $B(R\,; C_1(R, \Lambda))$ with radius $C_1(R, \Lambda)$ in the simply connected space form $M^m(R)$ of constant curvature R.

(II) $R=0$ and $\Lambda=0$, or $0<C_2(R,\Lambda)<+\infty$. Moreover M is isometric to the warped product $[0, 2a]\times_h\Gamma$, where $h=h_{R,\Lambda}$, a is a positive number if $R=0$ and $\Lambda=0$, and $a=C_2(R,\Lambda)$ if $0<C_2(R,\Lambda)<+\infty$. (In this case, ∂M is disconnected.)

(III) $R=0$ and $\Lambda=0$, or $0<C_2(R,\Lambda)<+\infty$. Moreover ∂M is connected, there is an involutive isometry σ of ∂M without fixed points, and M is isometric to the quotient space $[0, 2a]\times_h\partial M/G_\sigma$, where a and h are the same as in (II), and G_σ is the isometry group on $[0, 2a]\times_h\partial M$ whose elements consist of the identity and the involutive isometry $\hat{\sigma}$ defined by $\hat{\sigma}((t, x)) = (2a-t, \sigma(x))$.

1.3. — Let M be a Riemannian manifold of class (R, Λ). We write ν for the unit inner normal vector field on ∂M. For a point $x\in\partial M$, we denote by $\xi(x)$ the distance between x and the cut point of N along the normal geodesic $\exp^\perp_{\partial M} t\nu(x)$. Let $(\theta_1, \ldots, \theta_{m-1})$ be a coordinate system on an open set U of ∂M. Then $(\rho, \theta_1, \ldots, \theta_{m-1})$ is a coordinate system on $\hat{U}:=\exp^\perp_{\partial M}\{t\nu(x) : x\in U, 0\leqq t<\xi(x)\}$, where $\rho:=\operatorname{dis}(\partial M, *)$. On the coordinate neighborhood $(\hat{U}, (\rho, \theta_1, \ldots, \theta_{m-1}))$, the Laplacian Δ can be expressed in the form:

$$\Delta=\frac{\partial}{\partial\rho^2}+\frac{\partial\log\sqrt{G}}{\partial\rho}\frac{\partial}{\partial\rho}+\sum_{i=1}^m\frac{\partial}{\partial\theta_i}\left(\sqrt{G}g^{ij}\frac{\partial}{\partial\theta_j}\right),$$

where $g_{ij}:=g\left(\frac{\partial}{\partial\theta_i}, \frac{\partial}{\partial\theta_j}\right)$, $(g^{ij}):=(g_{ij})^{-1}$ and $G:=\det(g_{ij})$. This shows that

$$\Delta\rho=\frac{\partial\log\sqrt{G}}{\partial\rho}$$

on $\hat{U}$. On the other hand, *since* M *is a Riemannian manifold of class* (R, Λ),

$$\Delta\rho=\frac{\partial\log\sqrt{G}}{\partial\rho}\leqq(m-1)(\log h_{R,\Lambda})'\circ\rho \tag{1.9}$$

on $\hat{U}$ *and the equality holds at a point* $p\in\hat{U}$ *if and only if the sectional curvature of any plane tangent to* $\dot{\sigma}(t)$ *is equal to* R *and* ∂M *is umbilic at* $\sigma(0)$ (i. e., $\langle S_{\dot{\sigma}(0)}X, Y\rangle=\Lambda\langle X, Y\rangle$), where $\sigma : [0, a]\to M$ is the unique normal geodesic from N to p such that $\rho(\sigma(t))=t$ (*cf.* [13], [15 : Lemma (2.8)]). Especially when M is a model space of class (R, Λ),

$$\Delta\rho=(m-1)(\log h_{R,\Lambda})'\circ\rho$$

on $\{x\in M : \rho(x)<\mathscr{I}_M\}$. Therefore the first eigenfunction Φ of a model space M of class (R, Λ) can be written in the form:

$$\Phi=\phi\circ\rho,$$

where ϕ is a smooth function on $[0, \mathscr{I}_M]$ satisfying

$$\begin{cases}\phi''+(m-1)(\log h_{R,\Lambda})'\phi'+\lambda_1(M)\phi=0 & \text{on } [0, \mathscr{I}_M]\\ \phi(0)=\phi'(\mathscr{I}_M)=0.\end{cases} \tag{1.10}$$

1.4. — For the latter purpose, let us now consider the eigenvalue problem of an ordinary differential equation which is more general than (1.10). Let $F(t)$ be a continuous function on an interval $[0, \alpha)$ $(\alpha>0)$ and β a positive constant less than α. We write $\lambda(F, \beta)$ for the first eigenvalue of the following equation:

$$(1.11)\qquad \begin{cases} \phi''+F(t)\phi'+\lambda\phi=0 & \text{on } [0, \beta] \\ \phi(0)=\phi'(\beta)=0. \end{cases}$$

Note that by the change of variable: $s=T(t)$, where $T(t):=\int_0^t \exp\left[-\int_0^u F(v)dv\right]du$, equation (1.11) can be rewritten as follows:

$$(1.12)\qquad \begin{cases} \tilde{\phi}'' + \dfrac{\lambda}{(T'[G(s)])^2}\tilde{\phi}=0 & \text{on } [0, \tilde{\beta}] \\ \tilde{\phi}(0)=\tilde{\phi}'(\tilde{\beta})=0, \end{cases}$$

where $G\circ T=T\circ G=I$ and $\tilde{\beta}=T(\beta)$. Here we shall give computable lower bounds for $\lambda(F, \beta)$.

LEMMA 1.3. — *Under the above notations, we have*

$$(1.13)\qquad \lambda(F, \beta)>\left[4 \max_{0\leq t\leq\beta}\int_t^\beta (1/T'(u))du\cdot T(t)\right]^{-1}$$

Proof. — Let λ_0 be the right-hand side of (1.13) and ψ the solution of an equation:

$$\begin{cases} \psi'' + \dfrac{\lambda_0}{(T'[G(s)])^2}\psi=0 \\ \psi(0)=0 \quad \text{and} \quad \psi'(0)=1. \end{cases}$$

Let $\phi(t)$ be the first eigenfunction of (1.11). We may assume $\phi>0$ on $(0, \beta)$ and $\phi'>0$ on $[0, \beta)$. Put $\tilde{\phi}(s):=\phi\circ G(s)$. Then by simple computations, we have

$$(1.14)\qquad \tilde{\phi}(\tilde{\beta})\psi'(\tilde{\beta})=\int_0^{\tilde{\beta}}(\tilde{\phi}\psi'-\tilde{\phi}'\psi)'=(\lambda(F, \beta)-\lambda_0)\int_0^{\tilde{\beta}}\frac{\tilde{\phi}(s)\psi(s)}{(T'[G(s)])^2}ds.$$

Moreover by the definition of λ_0, we see that $\psi'>0$ on $[0, \tilde{\beta}]$ (*cf.* the proof of Lemma 4 and its corollary in [14]), and hence it follows from (1.14) that $\lambda(F, \beta)>\lambda_0$. This completes the proof of Lemma 1.3.

LEMMA 1.4. — *Suppose* F *is a constant. Then*

$$(1.15)\qquad \lambda(F.\ \beta)\geqq\frac{F^2 \exp 2F\beta}{(\exp F\beta-1+4/\pi^2)(\exp F\beta-1)^2}$$

and the equality holds if and only if $F=0$. *In this case,* $\lambda(0, \beta)=\pi^2/4F^2$.

Proof. — This is a special case of a result by Krein [17].

2. A lower bound for the first eigenvalue of a compact Riemannian manifold of class (R, Λ)

In this section, we keep the notations of Section 1 and prove the following

THEOREM 2.1. — *Let* M *be an m-dimensional compact Riemannian manifold of class* (R, Λ). *Then*

$$\lambda_1(M) \geqq \lambda(R, \Lambda, \mathscr{I}_M), \tag{2.1}$$

where $\lambda(R, \Lambda, \mathscr{I}_M)$ *is equal to* $\lambda((m-1)(\log h_{R,\Lambda})', \mathscr{I}_M)$ *if* $\mathscr{I}_M < C_1(R, \Lambda)$ (cf. *the paragraph* 1.4), *and it is equal to the first eigenvalue of the metric ball* $B(R, C_1(R, \Lambda))$ *with radius* $C_1(R, \Lambda)$ *in the simply connected space form* $M^m(R)$ *of constant curvature* R *if* $\mathscr{I}_M = C_1(R, \Lambda)$. *Moreover the equality holds in* (2.1) *if and only if* M *is a model space of class* (R, Λ).

Proof. ——We shall first show the theorem in the case when $\mathscr{I}_M < C_1(R, \Lambda)$. Put $F_{R,\Lambda} := (m-1)(\log h_{R,\Lambda})'$. Then $F_{R,\Lambda}$ is a smooth function on $[0, \mathscr{I}_M]$, since $h_{R,\Lambda}$ is positive on $[0, C_1(R, \Lambda))$. Let ϕ be the first eigenfunction of (1.11) defined by $F_{R,\Lambda}$ and $\beta = \mathscr{I}_M$. We may assume that ϕ is positive on $(0, \mathscr{I}_M]$, so that ϕ' is also positive on $[0, \mathscr{I}_M)$. Since the distance function ρ to ∂M is smooth on $M \backslash \mathscr{C}(\partial M)$, where $\mathscr{C}(\partial M)$ denotes the cut locus of ∂M, we see by (1.9) that

$$\begin{aligned} \Delta\phi \circ \rho + \lambda(F_{R,\Lambda}, \mathscr{I}_M)\phi \circ \rho &= \phi'' \circ \rho \, \| \nabla\rho \|^2 + \phi' \circ \rho\Delta\rho + \lambda(F_{R,\Lambda}, \mathscr{I}_M)\phi \circ \rho \\ &\leqq \{ \phi'' + F_{R,\Lambda}\phi' + \lambda(F_{R,\Lambda}, \mathscr{I}_M) \} \circ \rho \\ &= 0 \end{aligned} \tag{2.2}$$

on $M \backslash \mathscr{C}(\partial M)$. We note here that *inequality* (1.9) *still holds everywhere on* M *as a distribution*, although the smoothness of ρ breaks on $\mathscr{C}(\partial M)$ in general (*cf.* [15 : Corollary (2.44)]). Therefore *inequality* (2.2) *holds again on* M *as a distribution.* Thus the first assertion of the theorem follows from Lemma 1.1.

We shall now assume the equality holds in (2.1). Then it follows from the equality discussion of Lemma 1.1 that $\phi \circ \rho$ is smooth everywhere on M, it vanishes on ∂M and it satisfies

$$\Delta\phi \circ \rho + \lambda(F_{R,\Lambda}, \mathscr{I}_M)\phi \circ \rho = 0$$

on M. Therefore by the above arguments, we get

$$\Delta\rho = F_{R,\Lambda} \circ \rho$$

on $M \backslash \mathscr{C}(\partial M)$. This shows that for any geodesic $\sigma : [0, a] \to M$ with $\rho(\sigma(t)) = t$ $(t \in [0, a])$, the sectional curvature of every plane tangent to $\dot\sigma(t)$ is equal to R and ∂M is umbilic at $\sigma(0)$ (i. e., $\langle S_{\dot\sigma(0)}X, Y \rangle = \Lambda \langle X, Y \rangle$) (*cf.* the paragraph 1.3). Moreover combining this fact with the smoothness of $\phi \circ \rho$ and the positivity of ϕ' on $[0, \mathscr{I}_M)$, we see that

$$\mathscr{C}(\partial M) = \{ x \in M : \rho(x) = \mathscr{I}_M \}.$$

Now it is not hard to see that M is a model space of class (R, Λ), which is different from

$B(R; C_1(R, \Lambda))$. When $\mathscr{I}_M$ is equal to $C_1(R, \Lambda)$, it follows from Theorem A in [16] that M is isometric to $B(R; C_1(R, \Lambda))$. This completes the proof of Theorem 2.1.

Combining Theorem 2.1 with Lemma 1.3 or Lemma 1.4, we have the following two corollaries.

COROLLARY 2.2. — *Let* M *be as in Theorem* 2.1. *Then*

$$\lambda_1(M) > \left[4 \max_{0 \leqq t \leqq \mathscr{I}_M} \int_t^{\mathscr{I}_M} h_{R,\Lambda}^{m-1}(u)du \int_0^t 1/h_{R,\Lambda}^{m-1}(u)du \right]^{-1}$$

COROLLARY 2.3. — *Let* M *be as in Theorem* 2.1. *Suppose* $R=0$ *and* $\Lambda=0$. *Then*

$$\lambda_1(M) \geqq \frac{\pi^2}{4\mathscr{I}_M^2}.$$

Moreover the equality holds if and only if M *is a model space of class* (0, 0) (*e. g., a section of a flat cylinder*).

Remark. — In the case when $R>0$, we can obtain other computable estimates for $\lambda_1(M)$, making use of a result by Friedland and Hayman [7].

3. A lower bound for the first eigenvalue of a domain in a noncompact Riemannian manifold

In this section, we shall prove the following

THEOREM 3.1. — *Let* N *be a connected, complete and noncompact Riemannian manifold without boundary and* M *a compact domain with boundary in* N.

(1) *Suppose the Ricci curvature of* N *is bounded from below by a nonpositive constant* $(m-1)R$ ($m=\dim N$). *Then*

$$\lambda_1(M) > \begin{cases} \dfrac{\pi^2}{4d(M)^2} & (R=0), \\[2ex] \dfrac{-(m-1)^2 R \exp(2(m-1)\sqrt{-R}d(M))}{(\exp(m-1)\sqrt{-R}d(M)-1+4/\pi^2)(\exp(m-1)\sqrt{-R}d(M)-1)^2} & (R<0). \end{cases}$$

(2) *Suppose the sectional curvature of* N *is bounded from above by a nonpositive constant* K *and moreover there is a concave function* $\mu : N \to \mathbb{R}$ *without maximum. Then*

$$\lambda_1(M) > \begin{cases} \dfrac{\pi^2}{4d(M)^2} & (K=0), \\[2ex] \dfrac{-(m-1)^2 K}{4(1-\exp(-(m-1)\sqrt{-K}d(M)/2))^2} & (K<0). \end{cases}$$

Remarks. — (1) The estimate of the first assertion is « sharp » if $R=0$. In fact, let Θ be the antipodal map of a sphere S^{m-1} in Euclidean space R^m and define an involutive isometry $\tilde{\Theta}: S^{m-1}\times\mathbb{R}\to S^{m-1}\times\mathbb{R}$ by $\tilde{\Theta}(\theta, t)=(\Theta(\theta), -t)$. Put $N:=S^{m-1}\times\mathbb{R}/\{\mathrm{id.}, \tilde{\Theta}\}$ and $M_t:=S^{m-1}\times[-t, t]/\{\mathrm{id.}, \tilde{\Theta}\}$. Then $\lambda_1(M_t)=\dfrac{\pi^2}{4t^2}$ and $\lim_{t\to+\infty} d(M_t)/t=1$.

(2) Let H be a connected, simply connected and complete Riemannian manifold whose sectional curvature is bounded from above by a nonpositive constant K. Then, there are many concave functions without maximum on H. Moreover if D is a freely acting, properly discontinuous group of isometries on H and $N:=H/D$ is a parabolic manifold, N possesses a concave function without maximum, where we call $N:=H/D$ a *parabolic manifold* if there is a point $z\in H(\infty)$ that is the unique fixed point of every $\varphi(\neq 1)\in D$ (*cf.* [6 : Section 7 and Section 9] for details and examples of parabolic manifolds).

(3) Let N and M be as in the second assertion of the above theorem. Then if $K<0$, we have

$$\frac{-(m-1)^2K}{4(1-\exp(-(m-1)\sqrt{-K}d(M)/2))^2}>\frac{-(m-1)^2K}{4},$$

so that

$$\lambda_1(M)>\frac{-(m-1)^2K}{4}.$$

This inequality was proved in [20] in the case when N is simply connected and the sectional curvature is bounded from above by $K<0$.

In order to prove Theorem 3.1, we shall use a Busemann function, instead of a distance function to the boundary as in the proof of Theorem 2.1.

To begin with, let us recall the definition of a Busemann function. Let N be a complete, noncompact Riemannian manifold without boundary and $\gamma: [0, +\infty)\to N$ a geodesic ray. For any $t\geqq 0$, set $B^t_\gamma:=\mathrm{dis}(\gamma(t), *)-t$. Then

$$|B^t_\gamma(x)|=|\mathrm{dis}(\gamma(t), x)-\mathrm{dis}(\gamma(0), \gamma(t))|\leqq \mathrm{dis}(\gamma(0), x),$$

by the triangle inequality, so that $\{B^t_\gamma\}_{t\geqq 0}$ is uniformly bounded on compact subsets of N. Moreover if $s<t$,

$$B^s_\gamma(x)-B^t_\gamma(x)=\mathrm{dis}(\gamma(s), x)-\mathrm{dis}(\gamma(t), x)+t-s=\mathrm{dis}(\gamma(s), x)-\mathrm{dis}(\gamma(t), x)+\mathrm{dis}(\gamma(t), \gamma(s))\geqq 0,$$

again by the triangle inequality. Thus the family $\{B^t_\gamma\}_{t\geqq 0}$ is also nonincreasing, and hence it converges to a function B_γ on N, uniformly on compact subsets. This function B_γ is called *the Busemann function associated with a geodesic ray* γ. We first note the following

FACT 3.2 (*cf.* [24 : Lemma 3.2]). — *Letting* $U_a:=\{x\in N : B_\gamma(x)>a\}$ *for* $a\in\mathbb{R}$, *we have*

$$B_\gamma=a+\mathrm{dis}(\partial U_a, *)$$

on U_a.

Moreover we have the following

LEMMA 3.3. — *Suppose the Ricci curvature of* N *is bounded from below by* $(m-1)R$

($R \leqq 0$, $m =$ dim N). *Let* ϕ *be a nondecreasing* C^2*-function defined on an interval* J. *Then we have*

$$\Delta(\phi \circ B_\gamma) \leqq (\phi'' + (m-1)\sqrt{-R}\phi') \circ B_\gamma \tag{3.1}$$

as a distribution on $B_\gamma^{-1}(J)_o$ (:=*the interior of* $B_\gamma^{-1}(J)$).

Remark. — In the case when $R=0$ and $\phi=1$, inequality (3.1) implies that B_γ is superharmonic on N. This fact was proved by Cheeger and Gromoll [4] (*cf.* also [24 : Fundamental theorem A]).

Proof of Lemma 3.3. — As the first step, let us consider the Laplacian of a distance function to a point in a complete Riemannian manifold $\tilde{N}$ without boundary. Let p be a point of $\tilde{N}$ and ρ denote the distance function to p. Let $(\rho, \theta_1, \ldots, \theta_{m-1})$ be a polar coordinates defined on $\tilde{N} \backslash \mathscr{C}(p) \cup \{p\}$, where we write $\mathscr{C}(p)$ for the cut locus of p. Set $g_{ij} := \langle \partial/\partial\theta_i, \partial/\partial\theta_j \rangle$ and $G := \det(g_{ij})$. Then we have

$$\frac{\partial \log G}{\partial \rho} \leqq (m-1)(\log f_R)' \circ \rho \tag{3.2}$$

on $\tilde{N} \backslash \mathscr{C}(p) \cup \{p\}$ (*cf.* e. g., [13]), where f_R is the solution of the classical Jacobi equation: $f_R'' + R f_R = 0$, subject to the initial conditions $f_R(0)=0$ and $f_R'(0)=1$. Since $\Delta\rho = \partial \log G/\partial\rho$ on $\tilde{N} \backslash \mathscr{C}(p) \cup \{p\}$, we see by (3.2) that

$$\Delta\rho \leqq (m-1)(\log f_R)' \circ \rho$$

on $\tilde{N} \backslash \mathscr{C}(p) \cup \{p\}$. Therefore for a nondecreasing C^2 function ϕ on J, we have

$$\begin{aligned} \Delta(\phi \circ \rho) &= \phi'' \,\|\nabla\rho\|^2 + \phi'\Delta\rho \\ &\leqq \{\phi'' + (m-1)(\log f_R)'\phi'\} \circ \rho \end{aligned} \tag{3.3}$$

on $\rho^{-1}(J)_o \backslash \mathscr{C}(p) \cup \{p\}$. We remark here that *the equality in* (3.3) *holds at a point* $x \in \rho^{-1}(J)_o \backslash \mathscr{C}(p) \cup \{p\}$ *if and only if the sectional curvature of any plane tangent to* $\dot{\sigma}(t)$ *is equal to* R, *where* $\sigma : [0, a] \to N$ *is the unique distance minimizing geodesic from p to x*. Moreover we note that *inequality* (3.3) *still holds on* $\rho^{-1}(J)_o$ *as a distribution* (*cf.* e. g., [15 : Corollary (2.42)]).

Let us now return the proof of Lemma 3.3. Applying inequality (3.3) in the sense of a distribution to $\phi \circ B_\gamma^t$ $(t \geqq 0)$, we get

$$\Delta(\phi \circ B_\gamma^t) \leqq \phi'' \circ B^t + (m-1)(\log f_R)' \circ \rho_t \cdot B^t$$

as a distribution on $(B_\gamma^t)^{-1}(J)_o$, where $\rho_t := \mathrm{dis}(\gamma(t), *)$. Therefore letting $t \to +\infty$, we obtain inequality (3.1), since $\{B_\gamma^t\}$ converges to B_γ uniformly on compact subsets and $\lim_{t \to +\infty} (\log f_R)'(t) = \sqrt{-R}$. This completes the proof of Lemma 3.2.

We remark that Fact 3.2 and Lemma 3.3 hold for a function constructed from a divergent family of closed subsets, like a Busemann function (*cf.* [24]). More precisely, let

$\mathscr{C}=\{C_t\}_{t\in I}$ be a family of closed subsets C_t of N indexed by some interval $I=(\alpha, \beta)$. Assume $\operatorname{dis}(o, C_t)$ tends to infinity as $t \to \beta$, where $o\in M$ is a fixed point. Set $\hat{B}^t_{\mathscr{C}} := \operatorname{dis}(C_t, *) - \operatorname{dis}(0, C_t)$. Then $\hat{B}^t_{\mathscr{C}}$ is a Lipschitz continuous function with Lipschitz constant 1 and also $|\hat{B}^t_{\mathscr{C}}| \leqq \operatorname{dis}(*, o)$ by the triangle inequality. Thus $\{\hat{B}^t_{\mathscr{C}}\}$ is an equicontinuous family which is uniformly bounded on compact sets. Therefore by Ascoli's theorem, a subsequence of $\{\hat{B}^t_{\mathscr{C}}\}$, to be denoted by $\{\hat{B}^{t(j)}_{\mathscr{C}}\}$, converges to a continuous function $\hat{B}_{\mathscr{C}}$ on N uniformly on compact subsets. Then, Fact 3.2 is true for this function $\hat{B}_{\mathscr{C}}$ (*cf.* [24 : Lemma 3.2]), and also Lemma 3.3 holds, because the distance function ρ to a closed subset A satisfies inequality (3.3) as a distribution on $\rho^{-1}(I)_o \backslash A$ (*cf.* [15 : Corollary (2.24)]). Moreover we have the following

LEMMA 3.4. — *Suppose the sectional curvature of* N *is bounded from above by a nonpositive constant* K *and there is a concave function* μ *on* N *without maximum. Let* $\mathscr{C}=\{C_t\}_{t\in(-\infty, \sup \mu)}$ *be a divergent family of totally convex closed subsets*

$$C_t := \{x\in N : \mu(x) \geqq t\}$$

and $\hat{B}_{\mathscr{C}}$ *a function constructed as above by a subfamily* $\{C_{t(j)}\}$. *Then for any nonincreasing* C^2*-function* φ *on an interval* J, *we have*

$$\Delta(\varphi \circ \hat{B}_{\mathscr{C}}) \leqq \{\varphi'' + (m-1)\sqrt{-K}\varphi'\} \circ \hat{B}_{\mathscr{C}} \tag{3.4}$$

as a distribution on $\hat{B}_{\mathscr{C}}^{-1}(J)_o$.

Proof. — Since C_t is a totally convex closed subset of N for each $t\in(-\infty, \sup \mu)$, $\hat{B}^t_{\mathscr{C}}$ is a convex function on N and has continuous first derivatives on $N \backslash C_t$ (*cf.* [3 : Propositions 3.4 and 4.7] and [22 : Lemma 5]). Therefore $\hat{B}^t_{\mathscr{C}}$ is subharmonic on N (*cf.* [10, 11]) and moreover $\hat{B}^t_{\mathscr{C}}$ satisfies

$$\Delta \hat{B}^t_{\mathscr{C}} \geqq (m-1)(\log h_{K,0})' \circ \rho_t$$

as a distribution on $N \backslash C_t$, where $\rho_t := \operatorname{dis}(C_t, *)$ (*cf.* [15 : Theorem (2.49)]). Hence we have

$$\Delta(\varphi \circ \hat{B}^t_{\mathscr{C}}) \leqq \varphi'' \circ \hat{B}^t_{\mathscr{C}} + (m-1)(\log h_{K,0})' \circ \rho_t \cdot \varphi' \circ \hat{B}^t_{\mathscr{C}} \tag{3.5}$$

as a distribution on $(\hat{B}^t_{\mathscr{C}})^{-1}(J)_o \backslash C_t$. Since $\hat{B}^{t(j)}_{\mathscr{C}}$ converges to $\hat{B}_{\mathscr{C}}$ uniformly on compact sets and $\lim_{t\to\infty} (\log h_{K,0}(t))' = \sqrt{-K}$, we see by (3.5) that inequality (3.4) holds. This completes the proof of Lemma 3.4.

Proof of Theorem 3.1. — Let us first assume that the Ricci curvature of N is bounded from below by a nonpositive constant $(m-1)R$. Let B_γ be the Busemann function associated with a geodesic ray $\gamma : [0, +\infty) \to N$. Set $\overline{\delta}_\gamma(M) := \max\{B_\gamma(x) : x\in M\}$,

$$\underline{\delta}_\gamma(M) := \min\{B_\gamma(x) : x\in M\} \quad \text{and} \quad \delta_\gamma(M) := \overline{\delta}_\gamma(M) - \underline{\delta}_\gamma(M).$$

We write ϕ for the first eigenfunction of equation (1.11) defined by $F=(m-1)\sqrt{-R}$

and $\beta=\delta_\gamma(M)$. We may assume that $\phi>0$ on $(0, \delta_\gamma(M)]$. Letting $\Phi(t):=\phi(t-\delta_\gamma(M))$ for $t\in[\delta_\gamma(M), \bar{\delta}_\gamma(M)]$ and applying Lemma 3.3 to $\Phi\circ B_\gamma$, we have

$$\begin{aligned}\Delta(\Phi\circ B_\gamma)+\lambda((m-1)\sqrt{-R}, \delta_\gamma(M))\Phi\circ B_\gamma \\ &\leqq \{\Phi''+(m-1)\sqrt{-R}\Phi'+\lambda((m-1)\sqrt{-R}, \delta_\gamma(M))\Phi\}\circ B_\gamma \\ &=0\end{aligned}$$

as a distribution on M_o. Therefore it follows from Lemma 1.1 that

$$\lambda_1(M)\geqq\lambda((m-1)\sqrt{-R}, \delta_\gamma(M)).$$

Since $\delta_\gamma(M)$ is less than the diameter $d(M)$ of M by Fact 3.2, we obtain

$$\lambda_1(M)>\lambda((m-1)\sqrt{-R}, d(M)). \tag{3.6}$$

Thus the first assertion of Theorem 3.1 follows from (3.6) and Lemma 1.4.

Now we shall show the second assertion of the theorem. Let $\hat{B}_{\mathscr{C}}$ be as in Lemma 3.4. Set $\bar{\delta}_{\mathscr{C}}(M):=\max\{\hat{B}_{\mathscr{C}}(x) : x\in M\}$, $\underline{\delta}_{\mathscr{C}}(M)=\min\{\hat{B}_{\mathscr{C}}(x) : x\in M\}$ and

$$\delta_{\mathscr{C}}:=\bar{\delta}(M)-\underline{\delta}_{\mathscr{C}}(M).$$

We write ψ for the first eigenfunction of equation (1.11) defined by $F=-(m-1)\sqrt{-K}$ and $\beta=\delta_{\mathscr{C}}(M)$. We may assume that $\psi>0$ on $(0, \delta_{\mathscr{C}}(M)]$. Put $\Psi(t):=\psi(\bar{\delta}_{\mathscr{C}}(M)-t)$ for $t\in[\underline{\delta}_{\mathscr{C}}(M), \bar{\delta}_{\mathscr{C}}(M)]$. Then applying Lemma 3.4 to $\Psi\circ\hat{B}_{\mathscr{C}}$, we have

$$\begin{aligned}\Delta(\Psi\circ\hat{B}_{\mathscr{C}})+\lambda(-(m-1)\sqrt{-K}, \delta_{\mathscr{C}}(M)\Psi\circ\hat{B}_{\mathscr{C}} \\ &\leqq\{\Psi''+(m-1)\sqrt{-K}\Psi'+\lambda(-(m-1)\sqrt{-K}, \delta_{\mathscr{C}}(M))\Psi'\}\circ\hat{B}_{\mathscr{C}} \\ &=0\end{aligned}$$

as a distribution on M_o. Therefore it follows from Lemma 1.1 that

$$\lambda_1(M)\geqq\lambda(-(m-1)\sqrt{-K}, \delta_{\mathscr{C}}(M)),$$

and hence, we obtain

$$\lambda_1(M)\geqq\lambda(-(m-1)\sqrt{-K}, d(M)), \tag{3.7}$$

because $\delta_{\mathscr{C}}(M)<d(M)$. Thus the second assertion follows from (3.7), Lemma 1.4 $(K=0)$ and Lemma 1.3 $(K<0)$. This completes the proof of Theorem 3.1.

Before concluding this section, let us consider the first eigenvalue for a domain of a compact Riemannian manifold without boundary and prove the following proposition, where we shall use the same notations as in Section 1.

PROPOSITION 3.5. — *Let* N *be a connected and compact Riemannian manifold without boundary. Suppose the Ricci curvature of* N *is bounded from below by a constant* $(m-1)R$ $(m=\dim N, R\in\mathbb{R})$. *Then for any domain* M *with smooth boundary,*

$$\lambda_1(M)\geqq\lambda((m-1)F_{R,\mathscr{I}_{N\setminus M}}, d(N)-\mathscr{I}_{N\setminus M}), \tag{3.8}$$

where $F_{R,\mathscr{I}_{N\setminus M}}(t):=(\log f_R)'(t+\mathscr{I}_{N\setminus M})$ *and* f_R *is the solution of equation:* $f_R''+Rf_R=0$, *with* $f_R(0)=0$ *and* $f_R'(0)=1$. *Moreover when* $R>0$, *the equality holds in* (3.8) *if and only if* $N=S^m(R)$ *and* $M=B(R, r)$ $(0<r<\pi/\sqrt{R})$, *or* $N=P^m(R)$ *and* $M=P^m(R)\backslash B(R, r)$ $(0<r<\pi/2\sqrt{R})$, *where* $P^m(R)$ *denotes the real projective space of constant curvature* R.

Proof. — We choose a point x_o of N such that $\mathrm{dis}(x_o, M)=\mathscr{I}_{N\setminus M}$ and write ρ for the distance function to x_o. Let ϕ be the first eigenfunction of equation (1.11) defined by $F(t)=F_{R,\mathscr{I}_{N/M}}(t)$ and $\beta=d(N)-\mathscr{I}_{N\setminus M}$. We may assume that $\phi>0$ on $(0, d(N)-\mathscr{I}_{N\setminus M}]$ and $\phi'>0$ on $[0, d(N)-\mathscr{I}_{N\setminus M})$. Then $\Phi:=\phi\circ(\rho-\mathscr{I}_{N\setminus M})$ satisfies

$$\Delta\Phi+\lambda((m-1)F_{R,\mathscr{I}_{N\setminus M}}, d(N)-\mathscr{I}_{N\setminus M})\Phi\leqq 0$$

as a distribution on M_o, because of inequality (3.3) in the sense of a distribution. Therefore inequality (3.8) follows from the first assertion of Lemma 1.1. Now we assume that the equality holds in (3.8). Then the second assertion of the lemma implies that the above Φ is the first eigenfunction of M, so that $M=\{x\in N : \rho(x)\geqq r\}$ $(r=\mathscr{I}_{N\setminus M})$. Moreover by the same arguments as in the proof of Theorem 2.1, we see that $N=S^m(R)$ or $N=P^m(R)$. This completes the proof of Proposition 3.5.

Remark. — Using an isoperimetric inequality by Gromov [12], Bérard and Meyer [2] have proved that when $R>0$, $\lambda_1(M)\geqq\lambda_1(M^*)$, where M^* is the metric ball of $S^m(R)$ such that vol. (M^*)/vol. $(M)=$ Vol. $(S^m(R))$/Vol. (N), and the equality holds if and only if $N=S^m(R)$ and $M=M^*$.

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