Dusa McDuff

Local homology of groups of volume-preserving diffeomorphisms. III


<http://www.numdam.org/item?id=ASENS_1983_4_16_4_529_0>
LOCAL HOMOLOGY OF GROUPS OF VOLUME-PRESERVING Diffeomorphisms. III

By Dusa McDuff (*)

This is the last in a series of papers which study the local homology of groups of volume preserving diffeomorphisms ([10], [11]). However it may be read independently of the others, since it is self-contained apart from quoting some of their results.

Let $M$ be a compact, connected and oriented $C^\infty$-manifold without boundary, and with volume form $\omega$. Thus $\omega$ is a non-vanishing $n$-form, where $n = \dim M$, compatible with the orientation of $M$. Further, let $\text{Diff}_\omega^\bullet M$ denote the group of all $\omega$-preserving $C^\infty$-diffeomorphisms of $M$ in the compact-open $C^\infty$-topology. We will be concerned here with the "local homology" of the group $\text{Diff}_\omega^\bullet M$. As explained by Mather in [7], the local homology of a topological group $\mathcal{G}$ is the homology of the homotopy fiber $\mathbb{B} \mathcal{G}$ of the natural map $\mathbb{B} G \to \mathbb{B} \mathcal{G}$, where $G$ is the group $\mathcal{G}$ but considered with the discrete topology. This space $\mathbb{B} \mathcal{G}$ depends only on the algebraic and topological structure of the germ of $\mathcal{G}$ at the identity element $e$ (that is, of an arbitrarily small neighbourhood of $e$). In fact, it is not hard to show that if $\mathcal{G}$ is locally contractible the cohomology of $\mathbb{B} \mathcal{G}$ may be calculated from the complex of Eilenberg-MacLane cochains on this germ. Furthermore, one can define the "continuous" local cohomology of $\mathcal{G}$, which for locally contractible $\mathcal{G}$ is just the cohomology of the complex of continuous Eilenberg-MacLane cochains on the germ of $\mathcal{G}$ at $e$. When $\mathcal{G}$ is a Lie group, the van Est theorem implies that this is isomorphic to the cohomology of the Lie algebra of $\mathcal{G}$. Similarly, when $\mathcal{G} = \text{Diff}_\omega^\bullet M$, it is just the cohomology of the Lie algebra of divergence free vector fields on $M$ ([2], [5]).

Mather and Thurston showed that the local homology of the group $\text{Diff}^\bullet M$ of all diffeomorphisms of $M$ is isomorphic to the homology of the space of sections of a certain bundle over $M$ which is associated to the tangent bundle of $M$. The fiber of this bundle is made from germs of diffeomorphisms of $M$. It is suggestive, but not quite correct, to say that the fiber at $x$ is made from the set of germs of diffeomorphisms at $x$. (The trouble is that...

(*) Partially supported by NSF grant no MCS 7905795 A02.
this set has no algebraic structure.) Further, the map from $\text{B} \Diff M$ to the space of sections is essentially given by thinking of a diffeomorphism as a collection of germs, one at each point of $M$. Hence one can interpret the Mather-Thurston theorem as saying that the homology of $\Diff M$ localized at the identity may be calculated by localizing the diffeomorphisms spatially. Finally, note that because the elements of $\text{B} \Diff M$ may be thought of as holonomic or integrable sections of the fiber bundle, this theorem is very close in spirit to Gromov's work in [3] for example.

In this paper we prove the analogous result for $\Diff^\omega M$. Besides being of theoretical interest, this result is of great help in the calculation of the local homology of $\Diff^\omega M$. See [12] and in particular [6], where Hurder proves the existence of an enormous number of non-zero elements in $H_u(\text{B} \Diff^\omega M)$. Since all the classes found so far are continuous, they also live on the Lie algebra level.

Here is a precise statement of the main theorem. We state it for $\Diff^\omega(M, \text{rel} A)$, the group of $\omega$-preserving diffeomorphisms of $M$ which are the identity in some neighbourhood of $A$. Throughout we assume that the (possibly empty) subset $A$ of $M$ is closed and that $M-A$ is connected. (The latter restriction entails no loss of generality since $\Diff^\omega(M, \text{rel} A)$ decomposes as a product with one factor for each connected component of $M-A$.) The canonical $M$-bundle over $\text{B} \Diff^\omega M$ has discrete structural group and so is foliated transversely to the fibers. Its pull-back to $\text{B} \Diff^\omega M$ is isomorphic to the product $\text{B} \Diff^\omega M \times M$. Hence the space $\text{B} \Diff^\omega M \times M$ has a canonical foliation $F$ transverse to the fibers $pt \times M$. One can check that $F$ is defined by a closed $n$-form which restricts to $\omega$ on the fibers. Moreover the restriction of $F$ to $\text{B} \Diff^\omega(M, \text{rel} A) \times A$ has leaves $\text{B} \Diff^\omega(M, \text{rel} A) \times pt$ and so is trivial. (For more detail see [10] and [12].)

Now consider the groupoid $\Gamma^\omega$ of germs of diffeomorphisms of $\mathbb{R}^n$ which preserve the standard volume form $dx_1 \wedge \ldots \wedge dx_n$. Give $\Gamma^\omega$ the sheaf topology. The homomorphism $\Gamma^\omega_{sl} \to \mathcal{L}(n, \mathbb{R})$, which takes the germ $g$ at $x$ to its derivative $dg_x$, induces a map on classifying spaces $v: \text{B} \Gamma^\omega_{sl} \to \text{B} \mathcal{L}(n, \mathbb{R})$. We will suppose that $v$ is a Hurewicz fibration and will call its fiber $\text{B} \Gamma^\omega_{sl}$. It follows from Haefliger's general theory [4] that the foliation $F$ is classified by a commutative diagram

$$
\begin{array}{ccc}
\text{B} \Diff^\omega M \times M & \to & \text{B} \Gamma^\omega_{sl} \\
\downarrow \text{proj} & & \downarrow v \\
M & \to & \text{B} \mathcal{L}(n, \mathbb{R})
\end{array}
$$

where $\tau$ classifies the tangent bundle to $M$. Let $E_m \to M$ be the pull-back of $v$ over $\tau$. Then $F$ induces a map

$$f: \text{B} \Diff^\omega M \to S_{\omega}(M),$$

where $S_{\omega}(M)$ is the space of continuous sections of $E_m \to M$ with the compact-open topology. By choosing $F$ carefully, one can ensure that $f$ restricts to give a map

$$f: \text{B} \Diff^\omega(M, \text{rel} A) \to S_{\omega}(M, \text{rel} A),$$
where $S^0_M(M, \text{rel } A)$ is the space of sections which equal a given base section $s_0$ on $A$. (See proof of Lemma 3.1 below and [9], Appendix.) The section space $S^0_M(M, \text{rel } A)$ need not be connected and we write $S^0_M(M, \text{rel } A)$ for the connected component which contains $s_0$ and the image of $F$.

The main theorem is

**Theorem 1.** — The map

$$f: \overline{\text{Diff}}^c_0(M, \text{rel } A) \to S^0_M(M, \text{rel } A);$$

is a homology equivalence, that is, $f$ induces an isomorphism on homology for all local coefficients coming from $S^0_M(M, \text{rel } A)$.

We will see below that, except in the case $n = 2$, $A \neq \emptyset$, $\pi_1(S^0_M(M, \text{rel } A))$ is isomorphic to $H_1(\overline{\text{Diff}}^c_0(M, \text{rel } A); \mathbb{Z}) \cong H^{n-1}(M, A; \mathbb{R})$. Theorem 1 is then equivalent to the statement

$$\tilde{f}: \overline{\text{Diff}}^c_0(M, \text{rel } A) \xrightarrow{H^*} S^0_M(M, \text{rel } A),$$

where $\overline{\text{Diff}}^c_0$ denotes the kernel of the flux homomorphism $\Phi$ as defined in §2 below, and where $\tilde{S}$ is the universal cover of $S$. (When $n = 2$ and $A \neq \emptyset$ the appropriate space on the right is a cover of $S$ with fundamental group $\mathbb{R}$.) Corresponding results for non-compact $M$ are given in [10]. For example, if $A = \emptyset$, Theorem 1 holds provided that $M$ is the interior of a compact manifold of dimension $\geq 3$ such that each of its ends has infinite $\omega$-volume. Note that we do not treat the case of a non-compact manifold of finite volume.

### 2. Sketch of proof of Theorem 1

Most of the work of proving Theorem 1 was done in [10] and [11]. Suppose for the moment that $A$ is an $n$-dimensional compact submanifold of $M$ and let $A_0$ be $A$-(open collar nbhd of $\partial A$). We showed in [10] that

$$f: \overline{\text{Diff}}^c_0(M - A_0) \xrightarrow{H^*} S^0_0(M - A_0),$$

where $\tilde{\omega}$ is an extension of $\omega | M - A$ to the non-compact manifold $M - A_0$ such that every end has infinite volume, and where "c" denotes compact support. Also, by [11], we have

$$\overline{\text{Diff}}^c_0(M, \text{rel } A) \xrightarrow{H^*} \overline{\text{Diff}}^c_0(M - A_0).$$

Since $\tilde{\omega} = \omega$ on $M - A$, it follows easily that Theorem 1 holds for this $A$. By taking direct limits, one then proves Theorem 1 for all non-empty $A$.

Before going further, let us recall some facts about the fundamental groups of $\overline{\text{Diff}}^c_0 M$ and $S^0_M(M)$. Let $\text{Diff}^c_0 M$ be the identity component of $\text{Diff}_0 M$, and $\overline{\text{Diff}}^c_0 M$ be the
universal cover of $\Diff_{w_0} M$, but considered as a discrete group. It is easy to see that 
$\bar{\Diff}_{w_0} M \simeq \bar{\Diff}_{w} M$ and that $\pi_1 \bar{\Diff}_{w} M \cong \Diff_{w_0} M$. The flux homomorphism

$$\Phi: \bar{\Diff}_{w_0} M \to H^{n-1}(M; \mathbb{R}),$$

may be defined as follows [16]. An element of $\bar{\Diff}_{w_0} M$ is a pair $(g, \{ g_t \})$, where $g \in \Diff_{w_0} M$ and $\{ g_t \}$ is a homotopy class of paths joining $g_0 = \text{id}$ to $g_1 = g$. If $z$ is a singular $(n-1)$-cycle in $M$, then $\{ g_t(z) \}$ is a singular $n$-chain whose $\omega$-volume depends only on the homotopy class $\{ g_t \}$ and is zero if $z$ is a boundary. Therefore one may define $\Phi$ by the formula

$$\Phi(g, \{ g_t \})(z) = \text{vol}_w(\{ g_t(z) \}).$$

One checks that $\Phi$ is a group homomorphism by using the fact that the $g_t$ preserve $\omega$. Note also that $\Phi$ induces a homomorphism

$$\Phi: \Diff_{w_0} M \to H^{n-1}(M; \mathbb{R})/\text{ker}\Phi(\pi_1 \Diff_{w_0} M).$$

We write $\Diff_{w_0}^s M$ for the kernel of $\Phi$, and $\Diff_{w_0}^s M$ for the same group topologized as a subspace of $\Diff_{w_0} M$. (In fact $\Diff_{w_0}^s M$ is closed in $\Diff_{w_0} M$, since, as one can easily show, $\Phi(\pi_1 \Diff_{w_0} M)$ is a discrete subgroup of $H^{n-1}(M; \mathbb{R})$.) Clearly $\pi_1 \bar{\Diff}_{w_0}^s M \cong \text{ker}\Phi$. A difficult result of Thurston [16] and Banyaga [1] states that $\text{ker}\Phi$ is perfect. It follows that

$$H_1(\bar{\Diff}_{w_0}^s M; \mathbb{Z}) = 0,$$

and that

$$H_1(\bar{\Diff}_{w_0} M; \mathbb{Z}) \cong H^{n-1}(M; \mathbb{R}).$$

Note also that the map $\bar{\Diff}_{w_0}^s M \to \bar{\Diff}_{w_0} M$, when made into a fibration, is a covering map whose fiber is the discrete abelian group $H^{n-1}(M, A; \mathbb{R})$.

Now consider $\pi_1 S_{w_0}(M, \text{rel} A)$. We showed in [10] that when $n \geq 3$, $\pi_4(\bar{\Gamma}^n_A) \cong \mathbb{R}$ and $\pi_i(\bar{\Gamma}^n_A) = 0$ for $1 \leq i < n$ and $i = n + 1$. Therefore, obstruction theory implies that

$$\pi_1 S_{w_0}(M, \text{rel} A) \cong H^{n-1}(M, A; \mathbb{R}).$$

When $n = 2$ we have $\pi_1(\bar{\Gamma}^2_A) = 0$ and $\pi_2(\bar{\Gamma}^2_A) \cong \pi_3(\bar{\Gamma}^2_A) \cong \mathbb{R}$. By using obstruction theory or by looking at the fibration obtained by restricting sections to the 1-skeleton of $(M, A)$, one can show that $\pi_1 S_{w_0}(M, \text{rel} A)$ is an extension of $H^1(M, A; \mathbb{R})$ by a quotient of $\mathbb{R}$. In fact, we showed in [10], §7 that, when $A \neq \emptyset$, $\pi_1 S_{w_0}(M, \text{rel} A)$ is a central extension of $H^1(M, A; \mathbb{R})$ by $\mathbb{R}$ and so is nilpotent. In a moment we will see that $\pi_1 S_{w_0} M \cong H^1(M; \mathbb{R})$. For now, however, let $S_{w_0}(M, \text{rel} A)$ be the covering space of $S_{w_0}(M, \text{rel} A)$ corresponding to the kernel of the map

$$\pi_1(S_{w_0}(M, \text{rel} A)) \to H^{n-1}(M, A; \mathbb{R}).$$

Thus $\pi(S')$ is zero if $n \geq 3$ and is abelian otherwise.
We return to the proof of Theorem 1. Consider the commutative diagram

\[
\begin{array}{ccc}
\overline{\mathrm{B}}\mathcal{D}iff_{w0}(M, \text{rel } x_0) & \xrightarrow{f'} & \overline{\mathrm{B}}\mathcal{D}iff_{w0} M \rightarrow \overline{\mathrm{B}}\Gamma_{st}^w \\
\downarrow & & \downarrow \beta \quad \quad (*) \\
S_{w0}(M, \text{rel } x_0) & \rightarrow & S_{w0} M \rightarrow \overline{\mathrm{B}}\Gamma_{st}^w \\
\end{array}
\]

where the map \(\varepsilon\) evaluates sections at a point \(x_0 \in M\) and where \(\beta = \varepsilon \circ f\). The argument of [10], Lemma 6.1 shows that the restrictions of \(f'\) and \(f\) to \(\overline{\mathrm{B}}\mathcal{D}iff_{w0}\) lift to \(\overline{\mathrm{B}}\mathcal{D}iff^\circ\). Therefore there is a commutative diagram

\[
\begin{array}{ccc}
\overline{\mathrm{B}}\mathcal{D}iff_{w0}(M, \text{rel } x_0) & \xrightarrow{f'} & \overline{\mathrm{B}}\mathcal{D}iff_{w0} M \rightarrow \overline{\mathrm{B}}\Gamma_{st}^w \\
\downarrow & & \downarrow \beta \quad \quad (**) \\
S'_{w0}(M, \text{rel } x_0) & \rightarrow & S'_{w0} M \rightarrow \overline{\mathrm{B}}\Gamma_{st}^w \\
\end{array}
\]

Note the following

(i) The map \(f'\) in diagram (*) is a homology equivalence because Theorem 1 holds for the pair \((M, x_0)\). This immediately implies that its lift \(f\) is also a homology equivalence.

(ii) The bottom row of (**) is a fibration sequence because the bottom row of (*) is, and because \(H^{*+1}(M, x_0; \mathbb{R}) \cong H^{*+1}(M; \mathbb{R})\).

(Recall that \(F \rightarrow E \rightarrow B\) is called a fibration sequence, resp. homology fibration sequence, if there is an associated inclusion of \(F\) into the homotopy fiber of \(B\) which is a weak homotopy, resp. \(\mathbb{Z}\)-homology, equivalence. Further, a \(\mathbb{Z}\)-homology equivalence is a map which induces an isomorphism on untwisted integer homology.) We will prove in §3 below that

**Proposition 2.** — The top row of (**) is a homology fibration sequence.

A comparison of the Leray-Serre spectral sequence for the rows of (**) now shows that \(\tilde{f}\) is a \(\mathbb{Z}\)-homotopy equivalence. But we saw above that \(H_1(\overline{\mathrm{B}}\mathcal{D}iff^\circ M; \mathbb{Z}) = 0\) and \(\pi_1(S_{w0} M)\) is abelian. It follows that \(\pi_1(S'_{w0} M) = 0\). Therefore \(\tilde{f}\) and \(f\) are homology equivalences. This completes the proof of Theorem 1.

### 3. Proof of Proposition 2

Let \(\mathcal{D} = \mathcal{D}iff_{w0} M\) and \(\mathcal{D}' = \mathcal{D}iff^\circ_{w0} (M, \text{rel } x_0)\). The corresponding discrete groups are denoted \(D\) and \(D'\). We want to show that the sequence

\[
\overline{\mathrm{B}}\mathcal{D}' \rightarrow \overline{\mathrm{B}}\mathcal{D} \rightarrow \overline{\mathrm{B}}\Gamma_{st}^w,
\]

is a homology fibration sequence. As in [9], we do this by considering corresponding sequences for the discrete and topologized groups.
Let $D_m$ be the groupoid whose elements are pairs $(g, x)$, $g \in D$, $x \in M$, topologized as $D \times M$, where $D$ is discrete and $M$ has its usual topology. The partial composition law is $(h, gx). (g, x) = (hg, x)$. Then $BD_m$ is the total space of the canonical $M$-bundle over $BD$, and so $M \to BD_m \to BD$ is a fibration. Note: in [9], § 3 $BD_m$ is written $D \setminus M$.) Similarly, if $D_M$ denotes the groupoid $D_m$ topologized as $D \times M$, there is a fibration $M \to B D_M \to B D$. It follows that the homotopy fiber of $BD_m \to B D_M$ is homotopy equivalent to $B D$. Further, let $\Gamma_m$ be the groupoid of germs of $\omega$-preserving diffeomorphisms of $M$, with the sheaf topology, and let $J_m$ be the groupoid of 1-jets of elements of $\Gamma_m$, with its usual topology. Since $B \Gamma_m$ classifies the same objects as $B \Gamma^n_M$, the spaces $B \Gamma_m$ and $B \Gamma^n_M$ are weakly equivalent. (Another proof of this is given in [8], §2.) Similarly $BJ_m \simeq B \mathcal{D}(n, \mathbb{R})$. Hence we may identify the homotopy fiber of the differential $\nu : B \Gamma_m \to BJ_m$ with $B \mathcal{T}_m$.

We now construct the commutative diagram

\[
\begin{array}{c}
\begin{array}{ccccccc}
\text{BD} & \to & \text{BD} & \to & \text{BD} & \to & \text{BD} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{BD'} & \to & \text{BD} & \to & \text{BD} & \to & \text{BD} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{BD'} & \to & \text{BD} & \to & \text{BD} & \to & \text{BD} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{BD'} & \to & \text{BD} & \to & \text{BD} & \to & \text{BD} \\
\end{array}
\end{array}
\]

as follows. The middle row $BD' \to BD_m \to B \Gamma_m$ consists of the classifying spaces of the exact sequence $D' \to D_m \to \Gamma_m$ of groupoids, where $D'$ is included in $D_m$ as the subobject \{ $(g, x_0) : g = \text{id near } x_0$ \} and $D_m$ is mapped to $\Gamma_m$ by taking $(g, x)$ to the germ of $g$ at $x$. Further, $F_1$ is defined to be the homotopy fiber of $\gamma_1$ at the point $\star$ in $B \Gamma_m$ which corresponds to the identity germ $(\text{id}, x_0)$ in $\Gamma_m$. Since $D'$ maps to the base point $(\text{id}, x_0)$ of $\Gamma_m$, the image of $BD'$ in $B \Gamma_m$ contracts to $\star$. (It is not equal to $\star$ since we have to take thick realizations, see [9], Appendix.) The choice of contraction determines $\alpha_1$. The bottom row is constructed similarly. Clearly, one can make the square involving $\alpha_1$, $\alpha_2$ commute. The spaces in the top row are the homotopy fibers of the corresponding vertical maps and the maps $\alpha_0$, $\gamma_0$ are induced in the obvious way by the $\alpha_i$, $\gamma_i$. Notice that $F_0$ is the homotopy fiber of both $\gamma_0$ and $F_1 \to F_2$.

We will prove:

**Lemma 3.1.** $\gamma_0 \sim \beta$.

**Lemma 3.2.** $\alpha_2$ is a homotopy equivalence.

**Lemma 3.3.** $\alpha_1$ is a $\mathbb{Z}$-homology equivalence.

**Proof of Proposition 2.** Since $\gamma_0 \sim \beta$, it suffices to show that $\alpha_0$ is a $\mathbb{Z}$-homology equivalence. But $BD'$ and $F_2$ are simply connected. Therefore we may apply the spectral
sequence comparison theorem to the columns $\mathcal{D}' \to \mathcal{B}' \to \mathcal{B} \mathcal{D}'$ and $F_0 \to F_1 \to F_2$. The result now follows from Lemmas 3.2 and 3.3.

It remains to prove Lemmas 3.1-3.3. The proofs of 3.1 and 3.2 are straightforward. In 3.3 we replace the groupoids $D_M$ and $\Gamma_M$ by discrete categories so that we can use Quillen’s Theorem B [13]. This is applicable because of the results of [11].

It will be convenient from now on to use the language of categories, rather than groupoids, since it is more flexible and more highly developed. Recall that a groupoid $\Gamma$ may be thought of as a topological category all of whose morphisms are invertible. The space of objects of $\mathcal{C}(\Gamma)$ is the subspace of $\Gamma$ formed by the identities, and the space of morphisms of $\mathcal{C}(\Gamma)$ is $\Gamma$ itself. Groupoid homomorphisms then correspond to continuous functors. We will assume that the reader is familiar with the basic definitions of [14] and [9], § 3.

**Proof of Lemma 3.1.** — This is just a matter of spelling out definitions.

First consider $\mathcal{F}$. Let $\mathcal{G} = \text{Diff}_{\omega_0} M$ and recall the definition of $f: \mathcal{B} \mathcal{G} \to S_{\omega_0} M$ from [8], §2. It arises from a homotopy commutative classifying diagram

$$
\begin{array}{rcl}
\mathcal{B} \mathcal{G} \times M & \longrightarrow & \mathcal{B} \Gamma_M \\
\pi = \text{proj.} & \downarrow & \downarrow \nu \\
M & \mapsto & B \Gamma_M \overset{5}{\overset{H}{\longrightarrow}} H
\end{array}
$$

for the canonical foliation on $\mathcal{B} \mathcal{G} \times M$ in the following way. We identify $S_{\omega_0} M$ with the space of pairs $(\mathcal{G}, h)$, where $\mathcal{G}$ is a map $M \to B \Gamma_M$ and $h$ is a homotopy from $\tau$ to $v \circ \mathcal{G}$. Then, given $y \in B \mathcal{G}$, we define $f(y) = (F | y \times M, H | y \times M)$, where $H$ is the indicated homotopy from $\tau \circ \pi$ to $v \circ \mathcal{G}$.

Now diagram (&&) is the realization of a diagram of categories and functors

$$
\begin{array}{rcl}
\mathcal{C}(G \backslash \mathcal{G} \times M) & \overset{\tilde{f}}{\longrightarrow} & \mathcal{C}(\Gamma_M) \\
\tilde{\pi} & \downarrow & \downarrow \tilde{\nu} \\
\mathcal{C}(\{ e \} \backslash \mathcal{X} M) & \overset{\tilde{\rho}}{\longrightarrow} & \mathcal{C} \rightarrow \mathcal{C}(J_M) \overset{\tilde{\rho}}{\longrightarrow}
\end{array}
$$

Here $\mathcal{C}(G \backslash \mathcal{G} \times M)$ is made from the action $g: (h, x) \mapsto (gh, x)$ of $G$ on $\mathcal{G} \times M$ as in [9], §3. Thus its spaces of objects and morphisms are $\mathcal{G} \times M$ and $G \times \mathcal{G} \times M$ respectively. Similarly, $\mathcal{C}(\{ e \} \backslash \mathcal{X} M)$ has $M$ as space of objects and only identity morphisms. The functor $\tilde{\pi}$ is the obvious projection, $\tilde{\tau}$ is the inclusion and $\tilde{F}$ is given by

$$
\tilde{F}(g : (h, x) \mapsto (gh, x)) = \text{germ of } g \text{ at } hx.
$$

Observe that $\tilde{\tau} \circ \tilde{\pi} \neq \tilde{\nu} \circ \tilde{F}$. However there is a natural transformation $\tilde{\mathcal{H}}$ from $\tilde{\tau} \circ \tilde{\pi}$ to $\tilde{\nu} \circ \tilde{F}$. It is a continuous map from the objects $\mathcal{G} \times M$ of $\mathcal{C}(G \backslash \mathcal{G} \times M)$ to the morphisms $J_M$ of $\mathcal{C}(J_M)$ and is defined by

$$
\tilde{\mathcal{H}}(h, x) = (dh_x, x).
$$
It follows from [9], §3, Appendix that one can realise this diagram so as to get \( \& \). In particular the (thick) realization \( G \otimes \mathcal{G} \times M \) of \( G' \otimes (G \otimes \mathcal{G} \times M) \) is homeomorphic to the product \( (G \otimes \mathcal{G}) \times M \), and \( G \otimes \mathcal{G} \simeq \mathcal{B} \mathcal{G} \). Further, by [14], §1, the realization of the natural transformation \( \mathcal{H} \) is the homotopy \( \mathcal{H} \).

This defines \( f \). The map \( \beta : \mathcal{B} \mathcal{G} \rightarrow \mathcal{B} \Gamma_{st}^n \) is the composite of \( f \) with evaluation at the point \( x_0 \). Since \( \mathcal{B} \Gamma_{st}^n \) is the homotopy fiber of \( v \) and \( \mathcal{B} \mathcal{G} \simeq G \otimes \mathcal{G} \), the map \( \beta \) is given by a pair \( (\beta', \beta'') \), where \( \beta' : G \otimes \mathcal{G} \rightarrow \mathcal{B} \Gamma_{st} \) and \( \beta'' \) is a homotopy from the constant map to \( v \circ \beta' \). Identifying \( \mathcal{C}(G \otimes \mathcal{G}) \) with the full subcategory of \( \mathcal{C}(G \otimes \mathcal{G} \times M) \) with objects \( \mathcal{G} \times x_0 \), one can easily check that \( \beta' \) and \( \beta'' \) are induced by the restrictions of \( \mathcal{F} \) and \( \mathcal{H} \). Finally note that \( \beta : \mathcal{D} \rightarrow \mathcal{B} \Gamma_{st}^n \) is just the restriction of \( \beta \) to \( \mathcal{D} \subset \mathcal{B} \mathcal{G} \).

Now consider \( \gamma_0 \). Instead of using the model \( D \setminus \mathcal{D} \) for \( \mathcal{B} \mathcal{D} \) in its definition, we identified \( \mathcal{B} \mathcal{D} \) with the homotopy fiber \( F' \) of \( t : D \setminus M \rightarrow D \setminus M \). (Recall that \( BD_M = D \setminus M \) and \( B_D = D \setminus M \).) Therefore in order to relate \( \gamma_0 \) to \( \beta \) we must first describe an explicit homotopy equivalence \( i : D \setminus \mathcal{D} \rightarrow F' \). This will be given by a pair \( (i', i'') \), where \( i' : D \setminus \mathcal{D} \rightarrow D \setminus M \) and \( i'' \) is a homotopy from the constant map to \( t \circ i' \). As before, we define \( i' \) and \( i'' \) on the level of categories by a diagram

\[
\mathcal{C}(D \setminus \mathcal{D}) \xrightarrow{j} \mathcal{C}(D \setminus M) \xrightarrow{i} \mathcal{C}(\{e\} \setminus x_0) \subset \mathcal{C}(\mathcal{D} \setminus M) \xrightarrow{\tilde{I}} \mathcal{C}(\mathcal{D} \setminus M) \]

Here \( j \) is the inclusion given on objects by the evaluation map \( h \mapsto h(x_0) \) at \( x_0 \), and \( \tilde{I} \) is the natural transformation from the constant functor to \( i \circ j \) given by \( \tilde{I}(h) = (h : x_0 \rightarrow h(x_0)) \). (Observe that \( I \) is a continuous map from the objects \( \mathcal{D} \) of the category \( \mathcal{C}(D \setminus \mathcal{D}) \) to the morphisms \( D \setminus M \) of \( \mathcal{C}(D \setminus M) \). Also \( e \) denotes the identity element of the group \( D \).)

We claim that the map \( i = (i', i'') \) induced by the pair \( (j, \tilde{I}) \) is a homotopy equivalence. One way to prove this is to recall that there are fibration sequences \( M \rightarrow D \setminus M \rightarrow BD, M \rightarrow \mathcal{D} \setminus M \rightarrow B \mathcal{D} \) and to compare the above diagram with the analogous diagram

\[
\mathcal{C}(D \setminus \mathcal{D}) \rightarrow \mathcal{C}(D \setminus \star) \xrightarrow{i} \mathcal{C}(\{e\} \setminus \star) \rightarrow \mathcal{C}(\mathcal{D} \setminus \star)^{\gamma_0}
\]

which expresses \( D \setminus \mathcal{D} \) as the homotopy fiber of \( BD \rightarrow \mathcal{B} \mathcal{D} \).

Finally observe that the composite \( D \setminus \mathcal{D} \xrightarrow{i} F' \xrightarrow{\gamma_0} \mathcal{B} \Gamma_{st}^n \) is given by the pair \( (\gamma_1 \circ i', \gamma_2 \circ i'') \). But \( \gamma_1 \circ i' = \beta' \) and \( \gamma_2 \circ i'' = \beta'' \) because the underlying functors and natural transformations are the same. Hence \( \beta \sim \gamma_0 \). \( \Box \)
PROOF OF LEMMA 3.2. — We must show that $B \mathcal{D}' \to B \mathcal{D}_M \to B \Gamma_M$ is a fibration sequence, where $\mathcal{D}' = \text{Diff}^\infty_0(M, \text{rel}_0)$. Let $\mathcal{D}_0 = \{ g \in \mathcal{D} : g(x_0) = x_0 \}$ and $\mathcal{D}_1 = \{ g \in \mathcal{D}_0 : dg_{x_0} = \text{id} \}$. Then $\mathcal{D}_1 \to \mathcal{D}_0 \to \mathcal{P}(n, \mathbb{R})$ is an exact sequence of groups. Since $\mathcal{D}' \simeq \mathcal{D}_1$, this implies that

$$B \mathcal{D}' \to B \mathcal{D}_0 \to B \mathcal{P}(n, \mathbb{R}),$$

is a fibration sequence. By comparing the fibrations $M \to B \mathcal{D}_0 \to B \mathcal{D}$ and $M \to B \mathcal{D}_M \to B \mathcal{D}$ one sees that the obvious inclusion $B \mathcal{D}_0 \subseteq B \mathcal{D}_M$ is a homotopy equivalence. The result now follows easily. □

PROOF OF LEMMA 3.3. — We must consider the sequence

$$BD' \to BD_M \to B \Gamma_M.$$ 

Since the groupoid homomorphism $D_M \to \Gamma_M$ is not a fibration and has no other apparent redeeming topological properties, the easiest way to understand the map $BD_M \to B \Gamma_M$ seems to be to replace the groupoids $D_M$ and $\Gamma_M$ by discrete categories, since then we may use Quillen's Theorem B.

Let $\mathcal{U} = \{ U_\alpha \}$, $\alpha \in A$, be the cover of $M$ by the interiors of all smoothly embedded closed discs. Let $\mathcal{C}(D_\alpha)$ be the discrete category with objects $\alpha \in A$ and morphisms $\alpha \to \beta$ given by all $g \in D$ such that $g U_\alpha \subseteq U_\beta$. Further, let $\mathcal{C}(E_\alpha)$ be the discrete category with the same objects as $\mathcal{C}(D_\alpha)$ and with morphisms $\alpha \to \beta$ given by the germs at $U_\alpha$ of those $g \in D$ with $g U_\alpha \subseteq U_\beta$. There are two related topological categories $\mathcal{C}(D_\alpha^\bullet)$ and $\mathcal{C}(E_\alpha^\bullet)$ whose spaces of objects consists of all pairs $(x, \alpha)$, $x \in U_\alpha$, topologized as the disjoint union $\bigsqcup_{x} U_\alpha$. Their morphisms are those morphisms $g : (x, \alpha) \to (y, \beta)$ in $\mathcal{C}(D_\alpha)$, resp. $\mathcal{C}(E_\alpha)$, which are such that $g(x) = y$ and $g U_\alpha \subseteq U_\beta$. The forgetful functors:

$$\mathcal{C}(D_\alpha^\bullet) \to \mathcal{C}(D_\alpha) \quad \text{and} \quad \mathcal{C}(E_\alpha^\bullet) \to \mathcal{C}(E_\alpha)$$

give homotopy equivalences upon realization since they induce homotopy equivalences on the spaces of objects and morphisms. There are also functors:

$$p_1 : \mathcal{C}(D_\alpha^\bullet) \to \mathcal{C}(D_M) \quad \text{and} \quad p_2 : \mathcal{C}(E_\alpha^\bullet) \to \mathcal{C}(\Gamma_M).$$

Now $p_2$ induces a homotopy equivalence by the argument of [15], § 1.

To understand $p_1$, consider the diagram

$$BD_\alpha \xrightarrow{p_1} BD_M \xrightarrow{p_2} BD$$
The homotopy fiber of $BD_{\mathbb{A}} \to BD$ is clearly $M$. We will show that the same is true for $BD_{\mathbb{A}} \to BD$. To do this, we apply

Quillen's Theorem B [13], §1. -- Let $f : \mathcal{C} \to \mathcal{C}'$ be a functor between discrete categories. If $Y \in \text{obj} \mathcal{C}$, let $Y \setminus f$ denote the category whose objects are pairs $(X, v)$, $X \in \text{obj} \mathcal{C}$, $v : Y \to fX$, and where a morphism $(X, v) \to (X', v')$ is a morphism $w : X \to X'$ in $\mathcal{C}$ such that $f(w)v = v'$. If for every morphism $Y \to Y'$ in $\mathcal{C}$ the induced functor $Y \setminus f \to Y \setminus f'$ is a homotopy equivalence (resp. Z-homology equivalence) then the sequence

$$Y \setminus f \to \mathcal{C} \to \mathcal{C'},$$

is a homotopy (resp. homology) fibration sequence.

(Following Quillen, we call a functor a homotopy equivalence, etc. if it is one upon realization.)

Since in our situation $\mathcal{C}' = \mathcal{C}(D)$ has only one object $*$ and since all its morphisms are invertible, the induced functors $* \setminus f \to * \setminus f'$ have inverses. They therefore give homeomorphisms upon realization. Hence the homotopy fiber of $BD_{\mathbb{A}} \to BD$ is $\lfloor * \setminus f \rfloor$. We aim to show that $\lfloor * \setminus f \rfloor \simeq M$. Now $* \setminus f$ has objects $(\alpha, h)$, $\alpha \in \mathbb{A}$, $h \in D_\mathbb{A}$, and a morphism $(\alpha, h) \to (\beta, g)$ if and only if $gU_\alpha \supseteq U_\beta$. Consider the full subcategory $f^{-1}(\ast)$ of $* \setminus f$ with objects $(\alpha, e)$. There is a functor $\rho : * \setminus f \to f^{-1}(\ast)$ defined on objects by $\rho(\alpha, h) = (h^{-1} \alpha, e)$, where $h^{-1} \alpha \in \mathbb{A}$ satisfies $U_{h^{-1} \alpha} = h^{-1} U_\alpha$. If $i : f^{-1}(\ast) \hookrightarrow * \setminus f$ is the inclusion, then $\rho \circ i = \text{Id}$ and there is a natural transformation from $i \circ \rho$ to $\text{Id}$. Therefore $i$ and $\rho$ are adjoint functors, and so are homotopy equivalences by [14]. But $f^{-1}(\ast)$ is the full subcategory of the category of open sets and inclusions of $M$ corresponding to the cover $U$. Therefore $f^{-1}(\ast) \simeq M$ by Segal's covering lemma in [15], Prop. A.5. Hence the homotopy fiber of $BD_{\mathbb{A}} \to BD$ is $M$ as claimed. It follows that $p_1$ is an equivalence.

We now have a commutative diagram

\[
\begin{array}{ccc}
BD_{\mathbb{A}} & \xrightarrow{p_1} & BD_M \\
\downarrow \varphi & & \downarrow \psi \\
BE_{\mathbb{A}} & \xrightarrow{p_1} & BM_M
\end{array}
\]

Choose $\alpha \in \mathbb{A}$ with $x_0 \in U_\alpha$, and let $D'_\mathbb{A}$ be the group $\{ g \in D' : g = \text{id near } \overline{U}_\alpha \}$. Then $\mathcal{C}(D'_\mathbb{A})$ may be included in $\mathcal{C}(D_{\mathbb{A}})$ as the subcategory with objects $(\alpha, g)$, $g \in D'_\mathbb{A}$. Since the inclusion $BD_{D'_\mathbb{A}} \to BD'$ is a Z-homology equivalence [11], it will clearly suffice to show that:

$$BD_{D'_\mathbb{A}} \to BD_{\mathbb{A}} \to BE_{\mathbb{A}},$$

is a homology fibration sequence.
To do this we apply Quillen’s Theorem B to the functor \( q: \mathcal{C}(D_\beta) \to \mathcal{C}(E_\beta) \). For each object \( \alpha \) in \( \mathcal{C}(E_\beta) \), the category \( \alpha \setminus q \) has objects \((\gamma, h)\), where \( h \) is a germ of diffeomorphism at \( U_\gamma \) taking \( U_\gamma \) into \( U_\gamma \), and has a morphism \((\gamma, h) \to (\gamma', g h)\) for all \( g: \gamma \to \gamma' \) in \( \mathcal{C}(D_\beta) \). Let \( v \) be the morphism \( k: \beta \to \alpha \) in \( \mathcal{C}(E_\beta) \), and consider the diagram:

\[
\begin{array}{ccc}
\alpha \setminus q & \xrightarrow{v_1} & \beta \setminus q \\
\downarrow i & & \downarrow i \\
\mathcal{C}(D_\beta) & \xrightarrow{v_2} & \mathcal{C}(D_\beta),
\end{array}
\]

where the functors \( i \) are the inclusions and \( v_1 \) is induced by \( v \) in the obvious way. We define \( \rho: \alpha \setminus q \to \mathcal{C}(D_\beta) \) on morphisms by:

\[
\rho((\gamma, h) \to (\gamma', gh)) = (gh)^{-1} g h,
\]

where, for each \((\gamma, h)\), the element \( \overline{h} \in D \) is chosen to have germ \( h \) at \( U_\gamma \). The functor \( \rho: \beta \setminus q \to \mathcal{C}(D_\beta) \) is defined similarly. Finally \( v_2 \) is induced by the group homomorphism \( g \mapsto k^{-1} g k \), where \( k \in D \) is chosen to have germ \( k \) at \( U_\beta \). It is easy to check that \( i \) and \( \rho \) are adjoint, so that they are homotopy equivalences. Also, since there is a natural transformation from \( i \circ v_2 \) to \( v_1 \circ i \), the diagram is homotopy commutative. Moreover, \( v_2 \) is the composite of an isomorphism followed by the inclusion \( D_{\gamma \setminus q} \subset D_\beta \). But this inclusion is a \( Z \)-homology equivalence by [11]. Hence \( v_1 \) is also a \( Z \)-homology equivalence. Therefore Quillen’s Theorem B applies to show that \( \| \alpha \setminus q \| \to BD_\beta \to BE_\beta \) is a homology fibration sequence. Since \( BD_\gamma \cong \| \alpha \setminus q \| \), the same is true of \( BD_\alpha \to BD_\beta \to BE_\beta \).  

REFERENCES


D. McDuff,
Department of Mathematics,
State University of New York at Stony Brook,
Stony Brook,
NY 11794,
U.S.A.

(Manuscrit reçu le 11 novembre 1982.)