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ZARISKI'S CONJECTURE AND RELATED PROBLEMS

BY MADHAV V. NORI

The Lefschetz hyperplane section theorem gives an understanding of the topology of a smooth projective variety X, given the topology of a general hyperplane section Y and the monodromy in a Lefschetz pencil. When X is a surface and R is a curve in X, Zariski [Z] showed how to compute $\pi_1(X-R)$, given $\pi_1(Y-R)$ and the monodromy, where Y is a smooth hyperplane section moving in a generic pencil. Ignoring the monodromy, we may conclude from his results that $\pi_1(Y-R) \rightarrow \pi_1(X-R)$ is a surjection, for a general hyperplane section Y of X.

Now let C be any ample curve on X and let U be a neighbourhood of C in the usual topology. The linear system $|mC|$ gives a projective embedding of X, for m sufficiently large. If H_0 is the hyperplane whose intersection with X is mC, then for hyperplanes H sufficiently close to H_0, $X \cap H = Y$ is contained in U, and because $\pi_1(Y-R) \rightarrow \pi_1(X-R)$ is a surjection, so is $\pi_1(U-R) \rightarrow \pi_1(X-R)$. Thus a corollary of the Lefschetz-Zariski method (for arbitrary dimension) is:

I. $\pi_1(U-R) \rightarrow \pi_1(X-R)$ *is a surjection for any neighbourhood* U *of an ample divisor* H *on* X, *and for any Zariski-closed subset* R *of* X.

The above statement easily implies (*see* 2.5) that the kernel of $\pi_1(X-D) \rightarrow \pi_1(X)$ is a central subgroup, where X is a surface, D is a nodal curve on X and each irreducible curve C contained in D is assumed to be smooth and ample. In particular, if X is simply connected, $\pi_1(X-D)$ is abelian.

We generalise the statement I above to the following Weak Lefschetz Theorem (WLT for short).

WLT. *Let* H *be a connected compact complex-analytic subspace* (*not necessarily reduced*) *of a connected complex manifold* U, *defined by a locally principal sheaf of ideals. Assume that* $\mathcal{O}_U(H)\,|\,H$ *is ample and* $\dim U \geqq 2$.

Let $q: U \rightarrow X$ *be a holomorphic, locally invertible, map, with* X *a smooth projective variety. Let* $R \subset X$ *be an arbitrary Zariski-closed subset. Put* $h = q \circ i$, *where* $i: H \rightarrow U$ *denotes the inclusion. Finally let* G *be the image of* $\pi_1(U - q^{-1}(R)) \rightarrow \pi_1(X-R)$. *Then:*

A : G *is a subgroup of finite index.*

B : *If* $q(\mathrm{H}) \cap \mathrm{R} = \emptyset$, *then the image of* $\pi_1(H) \to \pi_1(\mathrm{X}-\mathrm{R})$ *is a subgroup of finite index.*

C : *If* $\dim \mathrm{X} = \dim \mathrm{U} = 2$, *then* $[\pi_1(\mathrm{X}-\mathrm{R}) : \mathrm{G}]$ *is bounded above by* $(\mathrm{Div}\, h)^2/\mathrm{H}^2$.

See paragraph 3.16 for the definition of Div h; it is the first Chern class of $h_*(\mathcal{O}_\mathrm{H})$.

Applying WLT to the π_1 of complements of nodal curves in surfaces, we get (*see* 3.27).

II. *Let* D *and* E *be curves on a smooth projective surface* X, *that intersect transversally. Assume that* D *is nodal. Denote the number of singular points of a curve* C *by* $r(\mathrm{C})$. *Assume that* $\mathrm{C}^2 > 2r(\mathrm{C})$ *for every irreducible curve* C *contained in* D. *Then, if* N *denotes the kernel of* $\pi_1(\mathrm{X}-\mathrm{D}\cup\mathrm{E}) \to \pi_1(\mathrm{X}-\mathrm{E})$, N *is a finitely generated abelian group and the centraliser of* N *is a subgroup of finite index.*

For any irreducible curve C in $\mathbb{P}^2$ of degree d, $\mathrm{C}^2 = d^2 > (d-1)(d-2) \geqq 2r(\mathrm{C})$. Putting $\mathrm{E} = \emptyset$ in II above, we get Zariski's Conjecture: $\pi_1(\mathbb{P}^2 - \mathrm{D})$ *is abelian for any nodal curve* D *in* $\mathbb{P}^2$. This was first proved by Fulton [F] and Deligne [D] by somewhat different techniques, for the algebraic and topological cases respectively. The proof of WLT relies heavily on deformations. When $\mathrm{X} = \mathbb{P}^2$ however, these are supplied by the automorphisms of $\mathbb{P}^2$, and this gives a short proof of Zariski's Conjecture (*see* § 4).

To prove II for tame fundamental groups (*see* § 5), by Abhyankar's lemma (*see* [F]), it has to be shown that any two curves in $\varphi^{-1}(\mathrm{D})$ intersect each other, where $\varphi : \mathrm{Y} \to \mathrm{X}$ is a finite covering, unramified outside $\mathrm{D} \cup \mathrm{E}$, with X, D and E as in II. And this is so because such curves support effective Cartier divisors of positive self-intersection (*see* 5.2). This observation and WLT combine to complete the proof of II.

The deformations required to prove WLT also show (*see* 3.18).

III. *Let* $\mathrm{A}(m)$ *be the maximum number of singular points on any irreducible nodal curve linearly equivalent to* $m\mathrm{C}$, *where* C *is an ample curve on a surface* X. *Then:*

$$\overline{\lim_{m \to \infty}}\ \mathrm{A}(m)/m^2 = \mathrm{C}^2/2.$$

Applying WLT(B) to curves on surfaces, putting $\mathrm{R} = \emptyset$, we get (*see* 3.26 and 6.3).

IV. *Let* $\overline{\mathrm{C}}$ *be the non-singular model of an irreducible nodal curve* C *on a surface* X.

A : If $\mathrm{C}^2 > 2r(\mathrm{C})$, *then* $[\pi_1(\mathrm{X})$: *Image* $\pi_1(\overline{\mathrm{C}})] \leqq \mathrm{C}^2/\mathrm{C}^2 - 2r(\mathrm{C})$.

B : *If* $\mathrm{C}^2 > \max(0, 2r(\mathrm{C})-2)$, *then the normal subgroup generated by the image of* $\pi_1(\overline{\mathrm{C}})$ *is a subgroup of finite index in* $\pi_1(\mathrm{X})$.

The normal bundle of $\overline{\mathrm{C}} \to \mathrm{X}$ is a line bundle of degree $\mathrm{C}^2 - 2r(\mathrm{C})$, for an irreducible nodal curve C with $r(\mathrm{C})$ singular points. Thus if U is a "tubular neighbourhood" of $\overline{\mathrm{C}} \to \mathrm{X}$ (*see* 1.11), then the self-intersection of $\overline{\mathrm{C}}$ in U is $\mathrm{C}^2 - 2r(\mathrm{C}) > 0$, and so IV A follows immediately from WLT.

Note that by Lefschetz, if $\mathrm{C}^2 > 0$, then $\pi_1(\mathrm{C}) \to \pi_1(\mathrm{X})$ is surjective. But $\pi_1(\mathrm{C}) = \pi_1(\overline{\mathrm{C}}) \star \mathrm{F}$ where F is a free group on $r(\mathrm{C})$ generators. Thus Lefschetz allows the image of $\pi_1(\overline{\mathrm{C}}) \to \pi_1(\mathrm{X})$ to be quite small (*see* 6.2 where $\mathrm{C}^2 = 2r(\mathrm{C}) > 0$ and the image has infinite index). However we are assuming more than does Lefschetz: $\mathrm{C}^2 > 2r(\mathrm{C})$ and not merely that $\mathrm{C}^2 > 0$, and this allows us to conclude that $\pi_1(\overline{\mathrm{C}}) \to \pi_1(\mathrm{X})$ is almost surjective.

Question. – *Let* D *be an effective divisor on a surface* X *with* $D^2>0$. *Let* N *be the normal subgroup of* $\pi_1(X)$ *generated by the images of the fundamental group of the non singular models of all the curves in* D. *Is* $[\pi_1(X): N]$ *finite?*

If the answer is yes, then any surface possessing a (possibly singular) rational curve of positive self-intersection would have finite fundamental group!

IV B. answers the above question in a very special case.

Incidentally, the conclusion of II allows $\pi_1(X-D\cup E)$ to be identified to $\pi_1(T)$ where T is a certain torus-bundle on $X-E$, by 1.6 and 1.7. This has a nice corollary (*see* 2.9):

V. *The complement of the theta-divisor in a general principally polarised abelian variety has the* "integer-valued Heisenberg group" *for its fundamental group.*

The next section contains a sketch of the Lefschetz method and its adaptation to the proof of WLT. Section 1 contains the preliminaries, especially the definitions 1.1 and 1.8 and the Lemma 1.5 which are used throughout. The next section includes many corollaries of the Lefschetz-Zariski method, some perhaps new. Section 3 is devoted to deformations and the proof of WLT. Homogeneous spaces are dealt with in paragraph 4, and tame fundamental groups in paragraph 5. The examples in paragraph 6 illustrate to what extent the corollaries of WLT are best possible. Curves with other singularities are also dealt with briefly.

Acknowledgements

The report "Connectivity and its applications in Algebraic Geometry" by Fulton and Lazarsfeld [FL] shows that some of the results of paragraphs 2 and paragraphs 6 are not new.

A stronger version of the statement I of the introduction and Proposition 2.1, which even allows X to be singular, is due to Deligne (*see* 1.1 of FL). Hansen has results analogous to 4.3. Abhyankar and Prill have versions of 2.5 and 6.5 for the algebraic and topological fundamental groups respectively (*see* § 8, FL). Note, however, that their results imply that $\pi_1(\mathbb{P}^2-C)$ is abelian for an irreducible curve C whose only singularities are a nodes and b cusps of $C^2>6b+4a$, while an application of 3.27 to the blow-up of $\mathbb{P}^2$ at the cusps of C (*see* 6.5) gives the same conclusion if $C^2>6b+2a$. Thus special statement does not appear to be deducible from the connectedness theorems of Fulton-Hansen either.

0. Sketch of the main proofs

The first step (*see* Lemma 1.5 C) is to show:

(A) If $f: X\to Y$ is a dominant morphism with X and Y both smooth and connected, satisfying some further mild restrictions, then for all $p\in U$, U some non-empty Zariski-open subset of Y, there is an exact sequence:

$$\pi_1(f^{-1}(p))\to\pi_1(X)\to\pi_1(Y)\to 1.$$

To apply this to the Lefschetz-Zariski situation, let R be a Zariski-closed subset of a smooth projective variety X, and let $\mathbb{P}^*$ be the dual projective space of hyperplanes in $\mathbb{P}$ where $X \to \mathbb{P}$ is the given projective embedding. Let:

$$Z=\{(x, H)\in(X-R)\times\mathbb{P}^* \mid x\in H\}.$$

Then $Z\to X-R$ is a fibre-bundle with projective spaces as fibres and therefore $\pi_1(Z)\to\pi_1(X-R)$ is surjective. Also $Z\to\mathbb{P}^*$ satisfies the restrictions of Lemma 1.5 C, and $\mathbb{P}^*$ being simply connected, by (A) above, $\pi_1(F)\to\pi_1(Z)$ is a surjection for a general fibre F of $Z\to\mathbb{P}^*$. Of course F is simply $Y-R$, for a general hyperplane section Y of X. Thus the composite $\pi_1(Y-R)\to\pi_1(Z)\to\pi_1(X-R)$ is surjective, and this proves statement I of the introduction.

Next we adapt this argument to the proof of WLT. Part C follows from the Hodge Index Theorem (*see* Lemma 5.1 and 3.24). Note that WLT(B) is an immediate consequence of WLT(A). Simply replace U by a neighbourhood V of H such that:

(*a*) $q(V)\cap R=\emptyset$ and,

(*b*) H is a strong deformation retract of V.

Next we observe that it suffices to prove WLT(A) for $\dim X=2$. Indeed if $\dim X>2$ and Y is a general hyper plane section of X, we have just noted that $\pi_1(Y-R)\to\pi_1(X-R)$ is surjective. Put $\overline{U}=q^{-1}(Y)$, $\overline{H}=h^{-1}(Y)$, where q, i, h are as in the statement of WLT. Then $\mathcal{O}_{\overline{U}}(\overline{H})\mid\overline{H}=\mathcal{O}_U(H)\mid\overline{H}$ and is therefore ample. By induction, $[\pi_1(Y-R)$: Image $\pi_1(\overline{U}-q^{-1}(R))]$ is finite and therefore $[\pi_1(X-R)$: Image $\pi_1(U-q^{-1}(R))]$ is itself finite.

So we only need to prove WLT(A) for surfaces X. To apply (A) above, we need deformations of the morphism $mH\to X$, where mH is a m-fold thickening of H in U.

We first need:

(B) Let V be the Douady space (*see* [D2]) of U. This is the parameter-space of all compact complex-analytic subspaces of U. The m-fold thickening mH of $H\subset U$ gives a point $p(mH)$ of V. For m sufficiently large (*see* 3.10) we have:

(*a*) V is smooth at $p(mH)$,

(*b*) for z sufficiently close to $p(mH)$, the corresponding complex-analytic subspace $A\subset U$ is a (connected) compact Riemann surface and $q\mid A: A\to q(A)=B$ is birational with B a nodal curve (*see* 3.5).

The proofs of these facts depend on the notion of b-excellence of deformations (*see* 3.1-3.10) and the excellence of the Douady space V is checked in Appendix 1. Next we check that (3.12-3.15):

(C) the deformations of the closed immersion $A_0\to U$ are infinitesmally the same as deformations of the algebraic morphism $A_0\to U\to X$, for a compact complex-analytic subspace A_0 of U.

This enables one to construct for suitable $A_0\subset U$;

(D) a morphism $A\to L\times X$ of smooth algebraic varieties such that $A\to L$ is proper and flat, $A\to X$ is dominant, and also there is a point p of L such that if A_p is the fibre over p

in $A \to L$, then $A_p \to X$ can be identified with the given $A_0 \to X$. Applying (A) to this situation, we get (*see* 3.23):

(E)

$$
\text{(E)} \qquad
\begin{tikzcd}
\pi_1(A') \arrow[r] \arrow[d] & \pi_1(X') \arrow[d] \\
\pi_1(L) \arrow[r, "\theta(L)"] & \pi_1(X')/G = \text{Image } \pi_1(U')
\end{tikzcd}
$$

Here $X' = X - R$, $U' = q^{-1}(X - R)$, and similarly A' is got from A by deleting the inverse image of R in $A \to X$.

Proof. – Applying (A) to $A' \to L$, we get:

$$\pi_1(F) \to \pi_1(A') \to \pi_1(L) \to 1,$$

is exact, where F is a general fibre of $A' \to L$. However there are plenty of such general fibres "contained in U", i. e. $F \to X$ factors through $F \to U \to X$, and therefore $\pi_1(A') \to \pi_1(X')/G$ factors *via* $\theta(L) : \pi_1(L) \to \pi_1(X')/G$.

(F) The image of $\pi_1(A') \to \pi_1(X')$ is a subgroup of finite index because $A' \to X'$ is dominant (*see* 1.5 B).

In view of (F), (G) below implies WLT(A).

(G) $\theta(L)$ vanishes on a subgroup of finite index.

Indeed if T is the inverse image of the identity coset in $\pi_1(A') \to \pi_1(X')/G$, then by (G), $[\pi_1(A') : T]$ is finite. Let S be the image of T in $\pi_1(X')$. By (F), we have $[\pi_1(X') : S]$ is finite, and $S \subset G$, showing that $[\pi_1(X') : G]$ is finite, thus proving WLT(A).

In the Lefschetz situation, L is the dual projective space $\mathbb{P}^*$, so (G) is immediate.

To prove (G), we choose another compact Riemannian surface $B_0 \subset U$ and let $C_0 = A_0 \cup B_0 \subset U$. As in (D) we construct morphisms $B \to M \times X$ and $C \to N \times X$ which have as special fibres the morphisms $B_0 \to X$ and $C_0 \to X$ respectively. We take the disjoint union T of $A \times M$ and $B \times L$. There is $T \to L \times M \times X$ and the fibres of $T \to L \times M$ are the disjoint unions of the fibres of $A \to L$ and $B \to M$, which we denote by A_l and B_m respectively.

Now $C_0 = A_0 \cup B_0$ is got from the disjoint union of A_0 and B_0 by identifying the points of intersection. We identify the A_l and B_m at some points to get a family of curves (*see* 3.22) parameterized by an etale covering Z of $L \times M$, and for a suitable $z \in Z$, the corresponding curve is simply C_0. This gives a morphism $Z \to N$ because N is a "universal" family of deformations of the morphism $C_0 \to X$ (*see* 3.14). As in (E), putting $S = \pi_1(X')/G$, we get functions $\theta(M) : \pi_1(M) \to S$ and $\theta(N) : \pi_1(N) \to S$. Now Z parametrises two families of curves, $Z x_L A$ and $Z x_M B$, and these give $\theta_1(Z)$ and $\theta_2(Z)$ from $\pi_1(Z)$ to S respectively. And essentially from the definition of these θ, we have (*see* 3.23):

$$
\begin{tikzcd}
 & S & \pi_1(L) \arrow[l, "\theta(L)"'] \\
\pi_1(N) \arrow[ur, "\theta(N)"] \arrow[dr, "\theta(N)"'] & \pi_1(Z) \arrow[l] \arrow[r] \arrow[u, "\theta_1(Z)"'] \arrow[d, "\theta_2(Z)"] & \pi_1(L \times M) \arrow[u] \arrow[d] \\
 & S & \pi_1(M) \arrow[l, "\theta(M)"]
\end{tikzcd}
$$

The left side shows that $\theta_1(Z)=\theta_2(Z)$ and then the right side shows that $\theta(L)x=\theta(M)y$ for all (x, y) in the image of $\pi_1(Z)\to\pi_1(L\times M)$. But $Z\to L\times M$ is dominant and therefore this image contains $T_1\times T_2$ where $[\pi_1(L):T_1]$ and $[\pi_1(M):T_2]$ are both finite. In particular, $\theta(L)x=\theta(M)1=1$ for all $x\in T_1$ and this completes the proof of (G) and that of WLT (A).

Remark 1. — The above proof shows that even in the Lefschetz-Zariski method, the fact that $\mathbb{P}^*$ is simply connected need not be used; instead one can add linear systems to prove the result.

Remark 2. — The existence of the Douady space simplifies the writing of the proof of WLT. As a matter of fact, it can be avoided altogether, and one can work with the formal scheme of H along U, which is a purely algebraic object.

1. Preliminaries

Let M be a connected complex manifold, N a closed subvariety, and $S\subset N$ an irreducible component of codimension one in M.

Put $U=\{z\in\mathbb{P}\mid |z|<2\}$. Let $f: U\to M$ be holomorphic with:

(*a*) $f^{-1}(N)=\{0\}$;

(*b*) $f(0)=p$ is a smooth point of N lying on S,

(*c*) $f'(0)\notin TN(p)=$ the tangent-space of N at p.

Then the free-homotopy class of $f\mid S^1: S^1\to M-N$ does not depend on the choice of f and thus gives a conjugacy class of elements of $\pi_1(M-N, q)$ with any base-point q.

Definition 1. — This subset of $\pi_1(M-N, q)$ will be denoted by $\gamma(M, N, S)$, or more simply by $\gamma(S)$ when the context is clear. Base points will rarely be mentioned: $\pi_1(M-N, q)$ will be abbreviated to $\pi_1(M-N)$. We have:

Fact 1.2. — $\pi_1(M)$ is the quotient of $\pi_1(M-N)$ by the subgroup generated by the subsets $\gamma(S)$ for all S as above.

Fact 1.3. — Let $h: M'\to M$ be holomorphic with M' a complex manifold and M, N, S as above. If $h(p)=q\in S$ and q is a smooth point of N, and h is transverse-regular to N at p, let S' be the unique irreducible component of $N'=h^{-1}(N)$ such that $p\in S'$. Then $\pi_1(M'-N')\to\pi_1(M-N)$ takes $\gamma(S')$ into $\gamma(S)$.

Fact 1.4 A. — If P is a principal G-bundle on a path-connected space X and G is also path-conected, the $\pi_1(P)$ is a central extension of $\pi_1(X)$.

In fact, we will only use $G=S^1$, $(S^1)^r$, and also an oriented punctured-disc bundle as follows:

Fact 1.4 B. — With M, N, S as above, assume that S is smooth and that N is a divisor with normal crossings in a neighbourhood of S. Then for a suitable tubular neighbourhood U of S, $\gamma(S)\subset\pi_1(U-N)$ is central, and is therefore a singleton.

With M, N, S as in 1.1, let $B=\{p\in N\mid N$ is not smooth of codimension 1 at $p\}$. Then B is closed and $\pi_1(M)\cong\pi_1(M-B)$. Replacing M by $M-B$, we may assume that N is the disjoint union of closed submanifolds of codimension one. Applying Van Kampen's theorem, taking tubular neighbourhoods of the components of N one at a time, we get 1.2.

The following lemma is used frequently:

LEMMA 1.5. — *Let* X *and* Y *be smooth connected varieties over* $\mathbb{C}$ *and:* $f: X \to Y$ *an arbitrary morphism. Then:*

A : *there is a non-empty Zariski-open* $U \subset Y$ *such that* $f^{-1}(U) \to U$ *is a fibre-bundle in the usual topology.*

B : *if* f *is dominant, the image of* $\pi_1(X)$ *has finite index in* $\pi_1(Y)$.

C : *if the general fibre* F *of* f *is connected and there is a codimension two subset* S *of* Y *outside which all the fibres of* f *have at least one smooth point* (*i. e.* $f^{-1}(p)$ *is generically reduced on at least one irreducible component of* $f^{-1}(p)$, $\forall p \notin S$), *then:*

$$\pi_1(F) \to \pi_1(X) \to \pi_1(Y) \to 1$$

is exact.

Proof of A. — By Hironaka's resolution of singularities, we may assume that there is a proper $\bar{f}: \bar{X} \to Y$ with $\bar{X}$ smooth, an open immersion $i: X \to \bar{X}$ such that $f = \bar{f} \circ i$ and $D = \bar{X} - i(X)$ is a divisor with normal crossings. Let D_i be the irreucible components of D for $1 \leqq i \leqq r$ and for each $S \subset \{1, 2, \ldots, r\}$ let D(S) be the intersection of the D_i for all $i \in S$. Then each D(S) is smooth by assumption and by Sard's theorem there is a Zariski-open $U \to Y$ such that $\bar{f} \mid \bar{f}^{-1}(U) \cap D(S)$ induces a surjection on all tangent-spaces, and this is enough to give local triviality in the usual topology.

Proof of B. — Let $F = f^{-1}(p)$ for any $p \in U$. Then:

$$\pi_1(F) \to \pi_1(f^{-1}(U)) \to \pi_1(U) \to \pi_0(F),$$

is exact and $\pi_0(F)$ is finite because F is an algebraic variety. Furthermore, $\pi_1(U) \to \pi_1(Y)$ is surjective, and this proves B.

Proof of C. — In view of the above exact sequence, we have to show that $\alpha(\ker b) = \ker c$ in:

$$\begin{array}{ccc} \pi_1(f^{-1}(U)) & \xrightarrow{b} & \pi_1(X) \\ {\scriptstyle\alpha}\downarrow & & \downarrow \\ \pi_1(U) & \xrightarrow{c} & \pi_1(Y) \end{array}$$

$T = \{ q \in Y \mid \dim f^{-1}(q) > \dim F \}$ has codimension $\geqq 2$ in Y, and we put $L = S \cup \bar{T}$ with S as in the statement of the lemma.

Let R be any irreducible component of $Y - U$ of codimension one in Y and let r be a smooth point of R lying outside L. By assumption $f^{-1}(r)$ has a smooth point m and because $\dim f^{-1}(r) = \dim F$, it follows that f induces a surjection on tangent-spaces at m. Let M be the unique irreducible component of $f^{-1}(R)$ to which m belongs. By 1.3, the image of $\gamma(X, X-f^{-1}(U), M)=\gamma(M)\subset\pi_1(f^{-1}(U))$ under α is contained in $\gamma(Y, Y-U, R)=\gamma(R)\subset\pi_1(U)$. The surjectivity of α shows that $\alpha(\gamma(M))=\gamma(R)$.

The $\gamma(R)$ generate $\ker c$ by 1.2, thus showing that $\alpha(\ker b)=\ker c$.

Remark 1. — Hironaka's resolution and Sard's theorem also give the conclusion of 1.5 A without the smoothness assumptions on X and Y.

Remark 2. — In quite a few applications, the $i : X\to\overline{X}, \bar{f} : \overline{X}\to Y$ as in the proof of 1.5 A are already given, so an appeal to the resolution of singularities is unnecessary here.

Next, with M a connected complex manifold and N a closed complex-analytic subset, put $\Gamma=\pi_1(M)$, $G=\pi_1(M-N)$, $H=\ker(G\to\Gamma)$ and:

$$\pi''(M, N)=G/[H, H] \qquad \text{and} \qquad \pi'(M, N)=G/[G, H].$$

In the lemmas below we identify $\pi'(M, N)$ and $\pi''(M, N)$ with the fundamental groups of certain torus-bundles on M. Since the main results of this paper provide sufficient conditions for H to be central (resp. abelian), when these conditions are satisfied $\pi_1(M-N)$ can be identified with $\pi'(M, N)$ [resp. $\pi''(M, N)$].

Let $p : \tilde{M}\to M$ be the universal cover and put $\tilde{N}=p^{-1}(N)$. Then $H=\pi_1(\tilde{M}-\tilde{N})$ and $H/[H, H]=H_1(\tilde{M}-\tilde{N})$.

I : $0\to H/[H, H]\to G/[H, H]\to G/H\to 1$ and:

II : $\pi_2(M)=H_2(\tilde{M})\to H_2(\tilde{M}, \tilde{M}-\tilde{N})\to H_1(\tilde{M}-\tilde{N})\to 0$ combine to give the exact sequence.

III : $\pi_2(M)\to H_2(\tilde{M}, \tilde{M}-\tilde{N})\to\pi''(M, N)\to\Gamma\to 1$.

Note that II is an exact sequence of Γ-modules. By definition practically, $H_0(\Gamma, H/[H, H])=H/[G, H]$, and by Hochschild-Serre (or directly by the Thom-Gysin sequence) we see that $H_0(\Gamma, H_2(\tilde{M}, \tilde{M}-\tilde{N}))=H_2(M, M-N)$. Thus II yields.

IV : $H_0(\Gamma, \pi_2)\to H_2(M, M-N)\to H/[G, H]\to 0$ is exact, from which we get the exact sequence:

V : $H_0(\Gamma, \pi_2(M))\to H_2(M, M-N)\to\pi'(M, N)\to\Gamma\to 1$.

For each irreducible codimension one subset $S\subset N$, let $L(S)\to M$ be the holomorphic line bundle whose sheaf of sections is $\mathcal{O}_M(S)$. Put $E=\bigoplus_S L(S)$. Let $p(S) : E\to L(S)$ be the projection and let $C(S)$ be the kernel of $p(S)$ and finally let F be the complement of the union of all the $C(S)$ in E.

The above F could equally well have been described as the fibre-product of the principal $\mathbb{C}^*$-bundles on M associated to the divisors S. We have:

LEMMA 1.6. — $\pi'(M, N)\cong\pi_1(F)$.

Proof. — *Put* $T=\bigcup_S C(S)$. *Then* $F=E-T$ *and* $\pi'(E, T)$ *makes sense. By* 1.4 *however, the kernel of* $\pi_1(F)\to\pi_1(E)\cong\pi_1(M)$ *is central, and therefore* $\pi_1(F)\cong\pi'(E, T)$.

The canonical sections $e(S)$ of $L(S)$ induced by $1 \in \mathcal{O}_M(S)$ give a section $e = \oplus e(S)$ of E, and $e : M \to E$ clearly takes $M - N$ into F. This induces $e_* : \pi'(M, N) \to \pi'(E, T)$, and in fact, e induces a commutative diagram of the exact sequences in V for the pairs (M, N) and (E, T).

The homomorphism $H_2(M, M-N) \to H_2(E, F)$ is easily seen to an isomorphism because e is transverse-regular to each $C(S) \subset E$, once the bad subset B of N is deleted as in the proof of 1.2. By the 5-lemma it follows that $\pi'(M, N) \to \pi'(E, T) = \pi_1(F)$ is an isomorphism.

In the above, it has tacitly been assumed that the irreducible components of N of codimension one in M are finite. To deal with $\pi''(M, N)$ it will be assumed that the irreducible components of $\tilde{N}$ of codimension one in $\tilde{M}$ are finite in number.

Now let $\tilde{E}$ be the holomorphic vector bundle on $\tilde{M}$ whose sheaf of sections is $\bigoplus_S \mathcal{O}_{\tilde{M}}(S)$, $S \subset \tilde{N}$, codim $S = 1$. The covering transformations Γ act on $\tilde{E}$ and the quotient is a vector-bundle on M. Let $\tilde{F} \subset \tilde{E}$ be the set of vectors all of whose projections are non-zero, and let F be the quotient of $\tilde{F}$ by Γ. Then:

Lemma 1.7. — $\pi''(M, N) \cong \pi_1(F)$.

The proof is similar to that of 1.6; the exact sequence III is used instead.

What follows is a formal method of separating neighbourhoods of the different branches of subvarieties that are not analytically irreducible.

Definition 1.8. — Let $h : H \to X$ be a holomorphic map of compact complex spaces (not necessarily reduced) inducing injections on Zariski-tangent-spaces everywhere. A neighbourhood of h is a triple (U, i, q) with U a Hausdorff topological space, $q : U \to X$ a local homeomorphism, $i : H \to U$ is injective and $q \circ i = h$.

Note that:

1.9. For any neighbourhood (U, i, q) of h, U acquires the structure of a complex-space such that $q : U \to X$ is a local isomorphism of complex-spaces. Furthermore, $i : H \to U$ is a closed immersion of complex-spaces.

1.10. If (U', i', q') and (U'', i'', q'') are neighbourhoods of h, so is (U, i, q) with $U = U' x_X U''$, $i = i' \times i''$.

In particular, the formal scheme of U along H does not depend on the choice of (U, i, q).

1.11. If H and X are smooth, we have *tubular neighbourhoods:* Choose a Riemannian metric on X and let U be the set of vectors in the normal bundle of h of length $< \varepsilon$. Define $i : H \to U$ by the zero section and $q : U \to X$ by the exponential map. For sufficiently small ε, q is a local diffeomorphism.

1.12. Neighbourhood of h do exist. The hypothesis on Zariski tangent-spaces implies that $F = \{(a, b) \mid h(a) = h(b), a \neq b\}$ is *closed* in $H \times H$. Choose metrics d and δ on H and X respectively. Put $3A = \inf\{d(a, b) \mid (a, b) \in F\}$. Then there is a positive B such that if $a, b \in H$ and $\delta(h(a), h(b)) < 2B$, then $d(a, b) < A$ or $d(a, b) > 2A$.

Let $R(B) = \{(a, x) \in H \times X \mid \delta(h(a), x) < B\}$. Note that if $(a_i, x) \in R(B)$ for $i = 1, 2, 3$ and $d(a_1, a_2) < A$ and $d(a_2, a_3) < A$, then $d(a_1, a_3) < 2A$ and $\delta(h(a_1), h(a_3)) < 2B$, implying that $d(a_1, a_3) < A$.

Identifying (a_1, x) and $(a_2, x) \in \mathrm{R}(\mathrm{B})$ if $d(a_1, a_2) < \mathrm{A}$, we get a topological space $\mathrm{U}(\mathrm{A}, \mathrm{B})$. Denote the equivalence class of (a, x) by $[a, x]$ and define $i : \mathrm{H} \to \mathrm{U}(\mathrm{A}, \mathrm{B})$ and $q : \mathrm{U}(\mathrm{A}, \mathrm{B}) \to \mathrm{X}$ by $i(a) = [a, h(a)]$ and $q([a, x]) = x$.

That $(\mathrm{U}(\mathrm{A}, \mathrm{B})), i, q)$ has the required properties in straightforward.

2. General hyperplane sections

X always denotes a smooth complete variety over $\mathbb{C}$ and R is an arbitrary Zariski closed subset of X throughout this section.

PROPOSITION 2.1. — *For any morphism* $\varphi : \mathrm{X} \to \mathbb{P}^{\mathrm{N}}$;

(*a*) $\varphi^{-1}(\mathrm{L})$ *is smooth and connected;*

(*b*) $\pi_1(\varphi^{-1}(\mathrm{L}) - \mathrm{R}) \to \pi_1(\mathrm{X} - \mathrm{R})$ *is a surjection for the general linear subspace* L *of* $\mathbb{P}^{\mathrm{N}}$ *with* codim $\mathrm{L} < \dim \varphi(\mathrm{X})$.

We shall prove this later and deduce first the following consequences :

DEFINITION 2.2. — An effective divisor D on X is not composed of a pencil if for some m, the rational morphism $\varphi : \mathrm{X} \to \mathbb{P}^{\mathrm{N}}$ induced by the complete linear system $|m\mathrm{D}|$ satisfies: $\dim \varphi(\mathrm{X}) \geqq 2$.

COROLLARY 2.3. — *For any neighbourhood* U (*in the usual topology*) *of the support of an effective divisor* D *not composed of a pencil,* $\pi_1(\mathrm{U} - \mathrm{R}) \to \pi_1(\mathrm{X} - \mathrm{R})$ *is surjective.*

Proof. — Let $\varphi : \mathrm{X} \to \mathbb{P}^{\mathrm{N}}$ be the rational map induced by $|m\mathrm{D}|$ with $\dim \varphi(\mathrm{X}) \geqq 2$. Desingularising the closure of the graph of φ, we get a commutative diagram:

$$\begin{tikzcd} & \overline{\mathrm{X}} \arrow[dl, "\psi"'] \arrow[dr, "\overline{\varphi}"] & \\ \mathrm{X} \arrow[rr, "\varphi"] & & \mathbb{P}^{\mathrm{N}} \end{tikzcd}$$

with:

(*a*) ψ and $\overline{\varphi}$ are morphisms;

(b) $\overline{\mathrm{X}}$ is smooth and ψ is birational.

In addition, there is a hyperplane $\mathrm{H}_0 \subset \mathbb{P}^{\mathrm{N}}$ with $\overline{\varphi^{-1}(\mathrm{H}_0)}$ contained in the support of D, and consequently, $\overline{\varphi}^{-1}(\mathrm{H}_0) \subset \psi^{-1}(\mathrm{U})$. The set V of hyperplanes H for which $\overline{\varphi}^{-1}(\mathrm{H}) \subset \psi^{-1}(\mathrm{U})$ is a neighbourhood (in the usual topology) of H_0 in the dual projective space $\mathbb{P}^*$.

The collection of H satisfying 2.1 (*a*) and 2.1 (*b*) is a Zariski-open subset W of $\mathbb{P}^*$ and therefore its intersection with V is non-empty. Taking $\mathrm{H} \in \mathrm{V} \cap \mathrm{W}$, we get a commutative diagram:

$$\begin{tikzcd} \pi_1(\overline{\varphi}^{-1}(\mathrm{H}) - \psi^{-1}(\mathrm{R})) \arrow[r, "\alpha"] & \pi_1(\psi^{-1}(\mathrm{U} - \mathrm{R})) \arrow[r, "\beta"] \arrow[d] & \pi_1(\overline{\mathrm{X}} - \psi^{-1}(\mathrm{R})) \arrow[d, "j"] \\ & \pi_1(\mathrm{U} - \mathrm{R}) \arrow[r, "\delta"] & \pi_1(\mathrm{X} - \mathrm{R}) \end{tikzcd}$$

with $\beta \circ \alpha$ a surjection, γ a surjection by Hopf's theorem, and therefore we conclude that δ is a surjection.

A special case of 2.3 is:

COROLLARY. — 2.4. — *If* C *is an irreducible curve on a surface* X *with* $C^2>0$, *for any neighbourhood* U *of* C, $\pi_1(U-R) \to \pi_1(X-R)$ *is surjective.*

COROLLARY 2.4 B. — *With* C *and* X *as in* 2.4, *if* C *intersects* R *transversally* (*in particular, if* $R=\emptyset$), *then* $\pi_1(C-C\cap R) \to \pi_1(X-R)$ *is surjective.*

Proof. — $\exists$ suitable neighbourhoods U of C such that $C-C\cap R \to U-R$ is a homotopy equivalence.

COROLLARY 2.5. — *If* D *and* E *are curves intersecting transversally on a surface* X, *and:*

(*a*) D *is nodal;*

(*b*) *every irreducible* C *contained in* D *is smooth with* $C^2>0$, *then the kernel of* $\pi_1(X-D\cup E) \to \pi_1(X-E)$ *is central.*

Proof. — By taking a tubular neighbourhood U of an irreducible curve C contained in D, $\gamma(C) \subset \pi_1(U-D\cup E)$ is central by 1.4 B, and because $\pi_1(U-D\cup E) \to \pi_1(X-D\cup E)$ is surjective by 2.4 $\gamma(C)$ is central in $\pi_1(X-D\cup E)$. The kernel in question is generated by such $\gamma(C)$, by 1.2, and this finishes the proof.

Remark. — 2.4 for algebraic fundamental groups is equivalent to the assertion: $\varphi^{-1}(C)$ is connected for every finite covering $\varphi: Y \to X$ unramified outside R, which follows from Zariski's connectedness theorem. It follows that 2.5 for algebraic π_1 is immediately deducible from this fact, Abhyankar [A] proved this in the special case $E=\emptyset$ and $\hat{\pi}_1(X)=0$, though the proof in the general case is no different. That the right condition is "D not composed of a pencil" is suggested by Abhyankar [A].

COROLLARY 2.6. — *With* X, D *as in* 2.5, *if* $\varphi: Y \to X$ *is a desingularisation of a finite morphism unramified outside* D, *then* $\varphi_*: \pi_1(Y) \to \pi_1(X)$ *has finite kernel and cokernel, and the kernel is central.*

COROLLARY 2.7. — *If* L *is an ample line bundle on* X *and* $s \in H^0(X, L^n)$ *is a section whose vanishing defines a smooth subscheme, and* $\varphi: Y \to X$ *is the variety got from adjoining the n-th root of s, then* $\varphi_*: \pi_1(Y) \to \pi_1(X)$ *is an isomorphism, if* $\dim X \geqq 2$.

We omit the proofs of 2.6 and 2.7 which are easy consequences of 2.5. However 2.7 admits a generalisation to : $\pi_i(Y) \cong \pi_i(X)$ for $i \leqq n-1$ when $\dim X=n$ (*see* [N2]).

The following is due to Le Trang and Saito.

COROLLARY 2.8. — *Let* D *be a divisor in* $\mathbb{P}^N$ *whose only singularities outside a codimension* 3 *subset of* $\mathbb{P}^N$ *are normal crossings. Then* $\pi_1(\mathbb{P}^N-D)$ *is abelian.*

Proof. — For a general linear subspace $L \subset \mathbb{P}^N$ with $\dim L=2$;

(*a*) $\pi_1(L-D) \to \pi_1(\mathbb{P}^N-D)$ is surjective by 2.1;

(*b*) $L\cap D$ is a nodal curve in L,

so the result follows from Zariski's conjecture.

COROLLARY 2.9. — *For a principally polarized abelian variety* X *such that the singular support* S *of the theta divisor* D *is of codimension* $\geqq 3$ *in* X, $\pi_1(X-D)$ *is the "integer-valued Heisenberg group"*.

Proof. — Embed X *in* $\mathbb{P}^N$ and let Y be the intersection of X and a general linear subspace L with $2+\text{codim } L=\dim X$. Then $C=D\cap Y$ is smooth in Y and its normal bundle in Y is precisely $\mathcal{O}_X(D)\,|\,C$ and is therefore ample. Thus 2.1, 2.5 and Lefshetz hyperplane section theorem together show that $\pi_1(X-D)$ is a central extension of $\pi_1(X)$. Also D is clearly irreducible, and the result follows from 1.6, because the fundamental group of the principal $\mathbb{C}^*$-bundle on X associated to the principal polarisation is the integer-valued Heisenberg group.

The above is a special case of:

COROLLARY 2.10. — *If* X *is projective and* $Z=\bigcup_i Z_i$ *with each* Z_i *irreducible of codimension* 1 *and not composed of a pencil, and outside a codimension three subset* S *of* X *each* Z_i *is smooth and* Z *is itself a divisor with normal crossings, then* $\pi_1(X-Z)$ *is a central extension of* $\pi_1(X)$, *and is in fact isomorphic to* $\pi_1(F)$, $F\to X$ *as in* 1.6.

Proof. — Embed X in projective space and let $Y=X\cap L$ with L a genral linear subspace, so that $\dim Y=2$. Then $Y\cap Z_i=C_i$ is smooth, $Y\cap Z$ is a divisor with normal crossings on Y, and $\mathcal{O}_Y(C_i)=\mathcal{O}_X(Z_i)\,|\,Y$ and therefore C_i is not composed of a pencil, i. e. $C_i^2>0$. As above, the result follows from 2.1, 2.5 and the Lefschetz theorem.

We finally prove Proposition 2.1. The methods are the conventional ones: applying Lemma 1.5C after showing that the bad sets have codimension $\geqq 2$.

Let $f: X\to Z$, $g: Z\to\mathbb{P}^N$ with $\varphi=g\circ f$ be the Stein factorisation of the given $\varphi: X\to\mathbb{P}^N$. Let $\dim Z=k\geqq 2$, and let G be the Grassmanian of all linear subspaces of $\mathbb{P}^N$ of codimension l, where $1\leqq l<k$. By Zariski's connectedness theorem, $g^{-1}(L)$ is connected for any $L\in G$, and because the fibres of f are connected, it follows that $\varphi^{-1}(L)$ is also connected.

Let U be a Zariski-open subset of Z such that:

(*a*) $f\,|\,f^{-1}(U)$ is a smooth morphism;

(*b*) $g\,|\,U$ induces an injection on Zariski tangent spaces everywhere;

(*c*) U is itself smooth, and:

(*d*) for all $x\in U$, $f^{-1}(x)\not\subset R$;

(*e*) $Z-U$ is of pure codimension one in Z.

By the remark following the lemma below, $F=\{L\in G\,|\,U\cap g^{-1}(L)=\emptyset\}$ has codimension $\geqq 2$.

Let S and S′ be the subvarieties of $X\times G$ and $U\times G$ defined by:

$$S=\{(x, L)\,|\,\varphi(x)\in L\}$$

and:

$S'=\{(x, L)\,|\,g(x)\in L$ and $g\,|\,U$ is not transverse regular to $L\subset\mathbb{P}^N$ at $x\in U\}$. Then the projections $S\to X$ and $S'\to U$ are Zariski-locally-trivial fibre bundles with irreducible fibres

A and A′ respectively where A is the Grassmanian of codimension l subspaces of $\mathbb{P}^{N-1}$ and dim A′ = dim G − $(k+1)$. It follows that S smooth and connected of codimension l in X × G and S′ is irreducible with 1 + dim S′ = dim G. In particular, S′ → G has finite fibres outside a codimension two subset H of G.

Putting $T = F \cup H$, for all $L \in G - T$, $U \cap g^{-1}(L)$ is non-empty of dimension $\geqq k - l$ and has only finitely many singular points, and by assumptions (*a*) and (*d*) on U, it follows that $\varphi^{-1}(L) \cap f^{-1}(U) - R$ has plenty of smooth points.

Summing up, $S - (R \times G) \to G$ satisfies all the conditions of 1.5 C, and because G is simply connected, we conclude that there is a Zariski-open $W \subset G$ such that for all $L \in W$, $\pi_1(\varphi^{-1}(L) - R) \to \pi_1(S - (R \times G))$ is surjective. The projection $S - (R \times G) \to X - R$ is smooth and proper with connected fibres and therefore $\pi_1(S - (R \times G)) \to \pi_1(X - R)$ is itself surjective. Thus $\pi_1(\varphi^{-1}(L) - R) \to \pi_1(X - R)$ is surjective for all $L \in W$. The proof is complete modulo:

LEMMA. — *If* M *is irreducible*, dim M $\neq (l-1)$, *and* $\psi : M \to \mathbb{P}^N$ *is a finite morphism, define* A(M) *by:*

$$A(M) = \{ L \in G \mid \dim \psi^{-1}(L) > \dim M - l \} \quad \text{if} \quad \dim M > (l-1),$$
$$A(M) = \{ L \in G \mid \psi^{-1}(L) \neq \emptyset \} \quad \text{if} \quad \dim M < (l-1).$$

Then codim A(M) $\geqq 2$. (If dim M $= l-1$, codim A(M) = 1 *and* A(M) *is the Chow point.*)

Remark. — Denoting by $M_1, M_2, \ldots, M_h$ the irreducible components of Z − U, dim $M_j = k - 1 \geqq l$, and $\psi_j = g \mid M_j$, for $1 \leqq j \leqq h$, clearly $F = \{ L \in G \mid U \cap g^{-1}(L) = \emptyset \}$ is contained in the union of the $A(M_j)$ and therefore codim F $\geqq 2$.

Proof of Lemma. — $J = \{ (x, L) \in M \times G \mid \psi(x) \in L \}$ is irreducible of codimension l in M × G. Assume the lemma is false. Let B(M) be the inverse image of A(M) in J → G. Then:

$$\dim B(M) \geqq \dim (M \times G) - l \quad \text{if} \quad \dim M > (l-1);$$
$$\dim B(M) \geqq \dim G - 1 \quad \text{if} \quad \dim M < (l-1).$$

In the second case, dim B(M) > dim J, which is impossible.

In the first case, dim B(M) $\geqq$ dim J and therefore J = B(M) because J is irreducible. But this contradicts the surjectivity of J → G. Therefore the lemma is true.

3. Deformations of morphisms

All spaces considered here are complex analytic spaces (not necessarily reduced) and all morphisms are assumed to be holomorphic.

3.1. We are given $f_0 : P_0 \to Q$ with f_0 inducing injections on all Zariski-spaces, Q smooth, P_0 compact, and $1 + \dim P_0 = Q$.

3.2. A *deformation* of f_0 with *parameter-space* (S, s_0) is:

I. A morphism $f : P \to S \times Q$ such that $p_1 \circ f$ is flat and proper;

II. A point $s_0 \in \mathrm{S}$, an isomorphism $j : \mathrm{P}_0 \to (p_1 \circ f)^{-1} s_0$ such that $p_2 \circ f \circ j = f_0$.

For $s \in \mathrm{S}$, we put $\mathrm{P}_s = (\mathrm{p}_1 \circ f)^{-1} s$ and define $f_s : \mathrm{P}_s \to \mathrm{Q}$ by $f_s = p_2 \circ f \mid \mathrm{P}_s$. Clearly the given f defines a deformation of f_s with parameter-space (S, s).

3.3. With $\mathrm{P} \to \mathrm{S}$ as above denote the k-fold fibre-product of P over S by P_S^k, and let $q_k : P_\mathrm{S}^k \to \mathrm{S}$ be the projection. Let $\mathrm{T}(k) = \{(x_1, x_2, \ldots, x_k) \in q_k^{-1}(s_0)$ with all the x_i distinct$\}$.

For any $b \geqq 0$, the given deformation is said be *b-excellent at* s_0, if :

(*a*) S is smooth at s_0;

(*b*) if $1 \leqq k \leqq b$, then P_S^k is smooth at every point of its subset $\mathrm{T}(k)$ with dimension $=$(dim S at $s_0) + k$(dim P_0).

(*c*) $\mathrm{P}_\mathrm{S}^k \to \mathrm{Q}^k$ is a smooth morphism at every point of $\mathrm{T}(k)$ for all k such that $1 \leqq k \leqq b$.

This rather technical definition is justified by 3.5 below.

In what follows, Remmert's Theorem:

"The image of a proper holomorphic mapping is a complex-analytic subvariety"

is used freely without explicit mention.

LEMMA 3.4. — *If* $f : \mathrm{P} \to \mathrm{S} \times \mathrm{Q}$ *is a b-excellent deformation of* $f_0 : \mathrm{P}_0 \to \mathrm{Q}$ *and as before* P_s *denotes the fibre of* P *over* $s \in \mathrm{S}$, $\mathrm{B}_s(k)$ *denotes the k-fold fibre-product of* P_s *over* Q *with all k co-ordinates distinct, then for all s in the complement of a proper analytic subset* F *of* S, $\mathrm{B}_s(k)$ *is smooth and proper of dimension* $=$(dim Q)$-k$, for $k \leqq \mathrm{b}$.

Moreover if the structure-sheaf of P_0 *has no non-constant global sections, then* P_s *is smooth and connected.*

Proof. — So as not to make the notation too cumbersome, the complement of an analytic subvariety of S not containing the base-point $s_0 \in \mathrm{S}$ will be denoted by S itself. In particular we shall assume that S is smooth and connected.

Because $f_0 : \mathrm{P}_0 \to \mathrm{Q}$ induces an injection on Zariski tangent-spaces, we may assume that $f : \mathrm{P} \to \mathrm{S} \times \mathrm{Q}$ has the same property. This implies that if $\mathrm{G}(k)$ is the k-fold fibre-product of P over $\mathrm{S} \times \mathrm{Q}$, the open subset $\mathrm{F}(k)$ of $\mathrm{G}(k)$ consisting of points with all distinct entries is also closed.

Now $\mathrm{G}(k)$ is the "scheme-theoretic" inverse image of the diagonal of Q in the projection $\mathrm{P}_\mathrm{S}^k \to \mathrm{Q}^k$ which is smooth for all $k \leqq b$ at all points of $\mathrm{F}(k)$ lying over $s_0 \in \mathrm{S}$, by the assumption of b-excellence. Consequently we may assume that $\mathrm{F}(k)$ is itself smooth and has dimension $=$(dim Q)$+$(dim S)$-k$, and noting that the fibres of $\mathrm{F}(k) \to \mathrm{S}$ are precisely the $\mathrm{B}_s(k)$, the result follows from Sard's theorem. The second assertion follows from semi-continuity of $\Gamma(\mathrm{P}_s, \mathcal{O})$.

Denoting $\mathrm{P}_s \to \mathrm{Q}$ by f_s, $\forall s \in \mathrm{S} - \mathrm{F}$, we have:

(*a*) by 1-excellence, P_s is smooth and $f_s : \mathrm{P}_s \to \mathrm{Q}$ induces injections on all tangent-spaces;

(*b*) 2-excellence guarantees that dim $\mathrm{P}_s = 1 +$ dim $\mathrm{B}_s(2)$, where $\mathrm{B}_s(2) = \mathrm{P}_s \times_\mathrm{Q} \mathrm{P}_s - \Delta \mathrm{P}_s$, and therefore f_s is generically injective;

(c) b-excellence for $b=1+(\dim \mathrm{Q})$ implies that $\mathrm{B}_s(b)=\emptyset$ and therefore, for any $t\in\mathrm{Q}$, the number r of points in P_s lying over Q is $\leqq \dim$ Q. Denote these points by x_i, $1\leqq i\leqq r$. The f_s-images of neighbourhoods of x_i in P_s give smooth branches $h_i=0$ in a neighbourhood of $t\in\mathrm{Q}$, and the smoothness of $\mathrm{B}_s(r)$ at $(x_1, x_2, \ldots, x_r)$ is equivalent to the transversal intersection of these branches. Thus we have:

COROLLARY 3.5. — *If* $b=1+\dim \mathrm{Q}$, *then* $f_s(\mathrm{P}_s)$ *is a divisor with normal crossings and* P_s *is its normalisation*, $\forall\, s\in\mathrm{S}-\mathrm{F}$.

We record for later use:

LEMMA 3.6. — *Let* $f_0 : \mathrm{P}_0\to\mathrm{Q}, f'_0 : \mathrm{P}'_0\to\mathrm{Q}$ *be as in* 3.1 *and let* $f:\mathrm{P}\to\mathrm{S}\times\mathrm{Q}, f' : \mathrm{P}'\to\mathrm{S}'\times\mathrm{Q}$ *be* 1-*excellent deformations of* f_0, f'_0 *respectively. Let* $f_s : \mathrm{P}_s\to\mathrm{Q}$ *and* $f'_t : \mathrm{P}'_t\to\mathrm{Q}$ *denote the general members of these deformations. Let* R *be any proper analytic subset of* Q. *Then:*

A : *For general* $s\in\mathrm{S}, f_s(\mathrm{P}_s)\not\subset\mathrm{R}$.

B : *For general* $(s, s')\in\mathrm{S}\times\mathrm{S}'$, $\mathrm{P}_s\times_{\mathrm{Q}}\mathrm{P}'_{s'}$ *is smooth of dimension* $=(\dim\mathrm{Q})-2$, *and* if $h_s : \mathrm{P}_s\times_{\mathrm{Q}}\mathrm{P}'_{s'}\to\mathrm{Q}$ *is the given morphism, then* $\dim h_s^{-1}(\mathrm{R})\leqq(\dim\mathrm{Q})-3$.

C : *If* f *and* f' *are* 2-*excellent and* $\dim\mathrm{Q}=2$, *the projections of* $\mathrm{P}_s\times_{\mathrm{Q}}\mathrm{P}'_{s'}$ *to* P_s *and* $\mathrm{P}'_{s'}$ *are injective, for general* $(s, s')\in\mathrm{S}\times\mathrm{S}'$.

The proofs are straightforward. As a sample, we do B: denote indiscriminately by D the set of points in any space lying over $(s_0, s'_0)\in\mathrm{S}_0\times\mathrm{S}'_0$. Then $f\times f' : \mathrm{P}\times\mathrm{P}'\to\mathrm{Q}\times\mathrm{Q}$ is smooth at D and therefore $h:\mathrm{P}\times_{\mathrm{Q}}\mathrm{P}'\to\mathrm{Q}$ is smooth at D with dimension $=\dim\mathrm{S}+\dim\mathrm{S}'+\dim\mathrm{Q}-2$. Therefore $h^{-1}(\mathrm{R})$ is a proper analytic subvariety. The result follows from Sard's theorem and dimension counting.

The following are also obvious:

LEMMA 3.7. — *Let* $f:\mathrm{P}\to\mathrm{S}\times\mathrm{Q}$ *be a deformation of* $f_0 : \mathrm{P}_0\to\mathrm{Q}$ *and let* $g : \mathrm{Q}\to\mathrm{R}$ *be a local isomorphism of complex manifolds. Then* f *is b-excellent if and only if* $g\circ f$ *is a b-excellent deformation of* $g\circ f_0$.

LEMMA 3.8. — *If* $f:\mathrm{P}\to\mathrm{S}\times\mathrm{Q}$ *is a b-excellent deformation of* f_0 *with base-point* $s_0\in\mathrm{S}$ *and* $\lambda : \mathrm{S}'\to\mathrm{S}$ *is a smooth morphism at* $s'_0\in\mathrm{S}'$ *with* $\lambda(s'_0)=s_0$, *then* $\mathrm{S}'\times_{\mathrm{S}}\mathrm{P}\to\mathrm{S}'\times\mathrm{Q}$ *is b-excellent at* $s'_0\in\mathrm{S}'$.

Remark. — The definition of b-excellence has been chosen to accommodate both 3.5 and 3.7; to conclude 3.5 for $f:\mathrm{P}\to\mathrm{S}\times\mathrm{Q}$ one needs the assumption 3.3 not on all of $\mathrm{T}(k)$ but on its subset:

$$\mathrm{D}(k)=\{(x_1, x_2, \ldots, x_k)\in\mathrm{P}_0^k \mid f_0(x_1)=f_0(x_2)=\ldots=f_0(x_k),$$
$$\text{and all the } x_i \text{ are distinct}\}.$$

For instance, if $f_0 : \mathrm{P}_0\to\mathrm{Q}$ is a closed immersion, 1-excellence of f guarantees that P_s is smooth and $f_s : \mathrm{P}_s\to\mathrm{Q}$ is an embedding, for all $s\in\mathrm{S}-\mathrm{F}$.

Remark. — The definition of b-excellence and all the lemmas proved are valid for algebraic deformations too and the proofs are the same.

3.9. From now on U is a Hausdorff complex manifold of dimension l. By Douady [D 2] there is the structure of a complex analytic space on the set V' of all compact complex-subspaces of U. For every such $H \subset U$, the corresponding point of V' will be denoted by $p(H)$. The compact analytic subschemes defined by a shaaf of locally principal ideals corresponds to an open subset V of V'. Let $W \subset V \times U$ be the universal family; the sheaf of ideals J defining W in $V \times U$ is locally principal.

Denoting by $p(i, 3)$ for $i = 1, 2$, the projections from $V \times V \times U$ to $V \times U$, the product of the ideals $p(1, 3)^* J$ and $p(2, 3)^* J$ defines a closed immersion $Y \to V \times V \times U$ which is proper and flat over $V \times V$ and by the universal property of V, this defines $\alpha : V \times V \to V$ which is commutative and associative.

Let I be the locally principal sheaf of ideals defining $i : H \to U$. The closed immersion defined by I^m for $m \geqq 1$ will be denoted by $i_m : mH \to U$. Appendix 1 contains a proof of:

PROPOSITION 3.10. — *With the above notation, if* $(I/I^2)^*$ *is ample on* H, *then for all b, there is a* $m_0(b)$ *such that* $m > m_0(b) \Rightarrow W \to V \times U$ *is a b-excellent deformation of* $i_m : mH \to U$, *with base-point* $p(mH)$.

Thus, by 3.4, for m sufficiently large, mH deforms into compact submanifolds of U. To know that these submanifolds are connected, we would need.

LEMMA 3.11. — *Assume that* H *is connected and that* $i : H \to U$ *satisfies the requirements of* 3.10. *Then* $\Gamma(mH, \mathcal{O}_{mH}) = \mathbb{C}$ *for all sufficiently large m.*

Proof. — Choose a sequence of effective divisors $H = D_0 > D_1 > \ldots > D_t = 0$ so that $D_i - D_{i+1} = B_i$ is irreducible. Then $F_i = \mathcal{O}_U(-D_{i+1})/\mathcal{O}_U(-D_i)$ is an invertible sheaf on B_i. Thus there is a filtration on $\mathcal{O}_U/I$, where $I = \mathcal{O}_U(-H)$, whose associated graded equals $\bigoplus_i F_i$, and by tensoring with I^m, there is a filtration on I^m/I^{m+1} whose associated graded equals $\bigoplus_i F_i \otimes I^m$. But $\Gamma(B_i, F_i \otimes I^m) = 0$ for all $m > m_0$, and for all i, because $I^{-1} \mid B_i$ is ample. It follows that $\Gamma(U, I^m/I^{m+1}) = 0$ for all $m > m_0$.

Putting $\Gamma_m = \Gamma(mH, \mathcal{O}_{mH})$ we see therefore that $\Gamma_{m+1} \to \Gamma_m$ is injective for all $m > m_0$.

But $\Gamma = \varprojlim_m \Gamma_m$ embeds in $\prod_{p \in H} \widehat{\mathcal{O}}_{U,p}$, which is a ring with no non-zero nilpotents. Also Γ has no non-trivial idempotents by the connectedness of H. By the previous paragraph, Γ is a finite-dimensional vector-space over $\mathbb{C}$. Thus $\Gamma = \mathbb{C}$, and $\Gamma_m = \mathbb{C}$ for all $m > m_0$.

Question. — With H as in 3.11, is $\Gamma_m = \mathbb{C}$ for all $m \geqq 1$?

This is so if H is further assumed to be reduced.

LEMMA 3.12. — *Let* $q : U \to X$ *be a local isomorphism of complex manifolds. With* $i : H \to U$ *as in* 3.9, *put* $h = q \circ i$. *The composite* $W \to V \times U \to V \times X$ *defines a deformation of h parameterised by* $(V, p(H))$. *Conversely, given a deformation of h parameterised by* (S, s_0), *then there is a neighbourhood* G *of* s_0 *and a holomorphic* $g : (G, s_0) \to (V, p(H))$ *such that the deformation of* $h : H \to X$ *induced by g coincides with the restriction of the given deformation to* (G, s_0). *The germ of g is uniquely defined.*

Proof. – Let $f: \mathrm{P} \to \mathrm{S} \times \mathrm{X}$ be the given deformation. Then $\mathrm{P} \times_{\mathrm{X}} \mathrm{U} \to \mathrm{P}$ is a local isomorphism of complex-analytic spaces, equipped with a section on the closed subspace $(p_1 \circ f)^{-1} s_0$ of P. It is a property of local homeomorphisms that this section extends to a *continuous* section on a neighbourhood B of $(p_1 \circ f)^{-1} s_0$ in P; by the properness of f, we may assume $\mathrm{B} = (p_1 \circ f)^{-1} \mathrm{G}$ for a neighbourhood G of s_0 in S. In other words, we have:

$$\begin{tikzcd} \mathrm{H} \arrow[r, "i"] \arrow[d, "j"'] & \mathrm{U} \arrow[d, "q"] \\ \mathrm{B} \arrow[r, "p_2 \circ f \mid \mathrm{B}"'] \arrow[ur, "t"] & \mathrm{X} \end{tikzcd}$$

with t continuous and $j : \mathrm{H} \to (p_1 \circ f)^{-1} s_0$ as in 3.2.

Because q is a local isomorphism and $p_2 \circ f$ is holomorphic, so is t. And because $i : \mathrm{H} \to \mathrm{U}$ is a closed immersion whose ideal sheaf is invertible, the same holds for $\mathrm{B} \to \mathrm{G} \times \mathrm{U}$ if G is replaced by a smaller neighbourhood of s_0. The universal property of V now gives the required $g : \mathrm{G} \to \mathrm{V}$.

Given two commutative diagrams as above with t_1 and t_2 in place of t, the set B′ where they coincide is an open-closed subset of B and therefore B′ contains $(p_1 \circ f)^{-1} \mathrm{G}'$ for a smaller neighbourhood G′ of s_0. Again, t_1 and t_2 coincide on B′ as holomorphic maps, because q is a local isomorphism. This proves the uniqueness-statement.

In the above situation, if X is a projective variety, then H is a projective scheme and $h : \mathrm{H} \to \mathrm{X}$ is a morphism of schemes: in fact, for any ample L on X, $h^* \mathrm{L}$ is seen to ample on H, from GAGA and the vanishing theorems of Serre's FAC. The same argument shows that any holomorphic deformation of $h : \mathrm{H} \to \mathrm{X}$ with parameter-space = Spec A, A an Artin-local $\mathbb{C}$-algebra, is also algebraic. Using 3.12 we conclude:

COROLLARY 3.13. – *In 3.12, if* X *is a projective variety, then the set of holomorphic deformations of* $i : \mathrm{H} \to \mathrm{U}$ *is in bijective correspondence with algebraic deformations of* $h : \mathrm{H} \to \mathrm{X}$, *if the parameter-space* = Spec A, A *an Artin-local* $\mathbb{C}$*-algebra.*

In general, if $f_0 : \mathrm{P}_0 \to \mathrm{Q}$ is a *finite* morphism of schemes with P_0 complete, the set of *algebraic* deformations of $f_0 : \mathrm{P}_0 \to \mathrm{Q}$ parameterised by (S, s_0) being denoted by $\mathbb{F}(\mathrm{S}, s_0)$, $\mathbb{F}$ defines a functor from the category of schemes (over some fixed field k) with base-points to the category of sets. This functor $\mathbb{F}$ is not representable always, even locally in the Zariski topology. However the following will suffice for our needs.

PROPOSITION 3.14. – *Assume that* P_0 *and* Q *are projective and that the only global sections of the structure-sheaf of* P_0 *are scalars. Then there is a scheme* T *with base-point* t_0 *and* $\theta \in \mathbb{F}(\mathrm{T}, t_0)$ *with the following property: given* $s_0 \in \mathrm{S}_1$ *and* $i : \mathrm{S}_1 \to \mathrm{S}$ *a closed immersion and* $\psi \in \mathbb{F}(\mathrm{S}, s_0)$ *and* $g : (\mathrm{S}_1, s_0) \to (\mathrm{T}, t_0)$ *such that* $\mathbb{F}(i)\psi = \mathbb{F}(g)\theta$, *there exists an open subscheme* S_2 *of* S *containing* s_0 *and* $\bar{g} : (\mathrm{S}_2, s_0) \to (\mathrm{T}, t_0)$ *such that:*

$$\bar{g} \mid \mathrm{S}_2 \cap \mathrm{S}_1 = g \mid \mathrm{S}_2 \cap \mathrm{S}_1 \qquad \textit{and} \qquad \mathbb{F}(\bar{g})\theta = \mathbb{F}(j)\psi,$$

where $j : \mathrm{S}_2 \to \mathrm{S}$ *is the inclusion morphism.*

COROLLARY 3.15. — *With* $i: \mathrm{H} \rightarrow \mathrm{U}$, $q: \mathrm{U} \rightarrow \mathrm{X}$, $h=q.i$ *as in* 3.12 *and* 3.13, *assume that* $\Gamma(\mathrm{H}, \mathcal{O}_{\mathrm{H}})=\mathbb{C}$, *and let* θ *be an* algebraic *deformation of h parameterized by* (T, t_0) *as in* 3.14 *above. Let* $g:(\mathrm{G}, t_0) \rightarrow (\mathrm{V}, p(\mathrm{H}))$ *be the holomorphic map of* 3.12, *with* G *a neighbourhood of* $t_0 \in \mathrm{T}$ *in the usual topology. Then g is smooth at* t_0.

Proof. — This follows immediately from Grothendieck's criterion for formal smoothness in view of 3.13 and 3.14 (*see* EGA, Chapter IV).

3.14. Is proved in Appendix 2 and it follows from the construction that both T and the total-space of θ are quasi-projectives schemes.

3.16. FIRST CHERN CLASSES. — For any proper holomorphic $j: \mathrm{S} \rightarrow \mathrm{T}$ with finite fibres such that $j(\mathrm{S})$ contains no irreducible component of T and $j_* \mathcal{O}_{\mathrm{S}}$ has finite homological dimension at all points of T, following Mumford (*see* Chapter 5, § 3, [M 1]), one may define an effective Cartier divisor Div j on T. With this definition, we have:

3.16 A : If j, S and T are algebraic and T is smooth, then:

$$\operatorname{Div} j=\sum_{\mathrm{F}} e(\mathrm{S}; \mathrm{F})\, d(\mathrm{F})\,[j(\mathrm{F})],$$

where the F range through all irreducible components of S with $1+\dim \mathrm{F}=\dim \mathrm{T}$, and $e(\mathrm{S}; \mathrm{F})$ is the length of $\mathcal{O}_{\mathrm{S}}$ at the generic point of F, and $d(\mathrm{F})$ is the degree of $\mathrm{F} \rightarrow j(\mathrm{F})$.

A base-change lemma proved by Mumford also shows:

3.16 B : If $f: \mathrm{P} \rightarrow \mathrm{S} \times \mathrm{Q}$ is such that $p_1 \circ f: \mathrm{P} \rightarrow \mathrm{S}$ is proper and flat, and for each $s \in \mathrm{S}$, Div f_s is defined for the morphism $f_s: \mathrm{P}_s \rightarrow \mathrm{Q}$, then Div f is defined and its restriction to $s \times \mathrm{Q}$ is Div f_s.

LEMMA 3.17. — *Let* $q: \mathrm{U} \rightarrow \mathrm{X}$ *be as in* 3.12 *with* X *a smooth projective surface.*

A : *Let* A *be a smooth compact complex submanifold of dimension one in* U *and assume that* $q \mid \mathrm{A}: \mathrm{A} \rightarrow q(\mathrm{A})=\mathrm{B}$ *is birational, with* B *a nodal curve. Then:*

$$\mathrm{B}^2=\mathrm{A}^2+2\, r(\mathrm{B}),$$

where $r(\mathrm{B})$ *is, as usual, the number of singular points of* B.

B : *Let* $i: \mathrm{H} \rightarrow \mathrm{U}$, $q: \mathrm{U} \rightarrow \mathrm{X}$ *be as in* WLT *with* $\dim \mathrm{X}=2$ *and let* $h=q \circ i$. *Then, for m large, there are plenty of irreducible nodal curves* B_m *in* X, *algebraically equivalent to* $m(\operatorname{Div} h)$, *with:*

$$m^2(\operatorname{Div} h)^2-m^2(\mathrm{H}^2)=2\, r(\mathrm{B}_m).$$

Proof. — Note that $q^{-1}(\mathrm{B})=\mathrm{A}+\mathrm{R}$ is a divisor on U with R intersecting A transversally in precisely the $2 r(\mathrm{B})$ points of A lying over the nodes of B. Note next that $q^{-1}(\mathrm{B}).\mathrm{A}=\mathrm{B}.q(\mathrm{A})=\mathrm{B}.\mathrm{B}$ because $\mathrm{A} \rightarrow \mathrm{B}$ is birational. Therefore:

$$\mathrm{B}^2=\mathrm{A}^2+(\mathrm{A}.\mathrm{R})=\mathrm{A}^2+2\, r(\mathrm{B}).$$

Proof of 3.17 B. — With $\mathrm{W} \rightarrow \mathrm{V} \times \mathrm{U}$ as in 3.9, for $z \in \mathrm{V}$, let $\mathrm{W}_z \subset \mathrm{U}$ be the corresponding closed immersion.

By 3.7 and 3.10, the composite $W \to V \times X$ is 3-excellent at $p(mH) \in V$ for all m large. For z in a neighbourhood of $p(mH)$, put $W_z = A_m$ and $q(W_z) = B_m$. By 3.5, for general z, A_m is smooth and is in fact the normalisation of the nodal curve B_m. By 3.11, A_m is also connected. In addition, $A_m^2 = (mH)^2 = m^2 H^2$, and by 3.16 B, $\mathrm{Div}(q \circ i_m) = m \,\mathrm{Div}\, h$ is algebraically equivalent to $\mathrm{Div}\, q \mid A_m$, which by 3.16 A, is equal to B_m. The formula for $r(B_m)$ now follows from 3.17 A.

We apply the above remarks to the construction of nodal curves on surfaces.

PROPOSITION 3.18. — *Let* C *be an irreducible curve of positive self-intersection on a smooth projective surface* X. *Let* A(*m*) *be the maximum number of singular points on any irreducible nodal curve in the linear system* $|mC|$. *Then:*

$$\overline{\lim_{m \to \infty}}\ A(m)/m^2 = C^2/2.$$

Proof. — The formula for the genus of a curve shows that:

$$2(A(m)-1) \leqq m^2 C^2 + m C.k_X,$$

and therefore the upper limit is $\leqq C^2/2$.

To prove the other inequality, C can be replaced by any curve in the linear system of any multiple of C, and therefore we may assume that C is smooth and not rational.

Let $H \to C$ be any unramified d-fold covering and let h be the composite $H \to C \to X$. Let (U, i, q) be a tubular neighbourhood of $h : H \to X$ as in 1.11. Clearly $H^2 = d C^2$, so we may apply 3.17 B to get irreducible nodal curves B_m algebraically equivalent to $m(\mathrm{Div}\, h) = md C$ with:

$$2 r(B_m) = m^2 d(d-1) C^2.$$

Thus $2r(B_m)/(B_m^2) = 1 - (1/d)$ and taking larger and larger d, we get the result, provided we assume that algebraic equivalence = linear equivalence; in other words that $\mathrm{Pic}^0 X = 0$. This restriction will be removed in the examples below.

Remark 3.18. — It can be shown that $\pi_1(B_m) \to \pi_1(X)$ and $\pi_1(H) \to \pi_1(X)$ have the same (conjugate) image.

Example 3.19 A. — Let $\varphi : Y \to X$ be a d-fold etale covering, with X as in 3.18. With C also as in 3.18, consider the linear system $|m\varphi^{-1}(C)|$ for m large. Denoting by A_m its general member, $\varphi(A_m) = B_m$ is an irreducible nodal curve with normalisation A_m; and $B_m \in |md C|$. Applying 3.17 A to $\varphi : Y \to X$ in place of $q : U \to X$, we see that $2r(B_m)/(B_m^2) = 1 - (1/d)$ again

Thus, if the algebraic fundamental group of X is infinite [in particular, if $\mathrm{Alb}(X) \neq 0$], 3.18 is trivially proved.

3.19 B. — Assume further that $\varphi : Y \to X$ is Galois with Galois group G.

Then, for general $D \in |m\varphi^{-1}(C)|$, $\varphi^{-1}\varphi(D)$ is the union of the σD (which intersect transversally) for $\sigma \in G$, and by 2.5, $N = \ker \pi_1(Y - \varphi^{-1}\varphi(D)) \to \pi_1(Y))$ is central. Thus $\theta : \mathbb{Z}[G] \to N$ given by $\theta([\sigma]) = \gamma(\sigma D)$ for $\sigma \in G$ is surjective and well-defined.

Let I be the augmentation ideal of $\mathbb{Z}[G]$. Then the composite $\mathbb{Z}[G] \to N \to H_1(Y - \varphi^{-1}\varphi(D))$ is injective when restricted to $I \subset \mathbb{Z}[G]$, as is seen from the homology-exact-sequence of the pair $(Y, Y - \varphi^{-1}\varphi(D))$ noting that all the σD are linearly, and therefore homologically, equivalent.

Now N is also $\ker(\pi_1(X - \varphi(D)) \to \pi_1(X))$ and the conjugation-action of $\pi_1(X - \varphi(D)$ on N factors through an action ρ of G such that $\rho(\sigma)\gamma(\tau D) = \gamma(\sigma\tau D)$ for all $\sigma, \tau \in G$.

Thus, if $d = \deg \varphi = O(G) > 1$, N is not a central subgroup of $\pi_1(X - \varphi(D))$.

Putting $d = 2$, $C = \varphi(D)$, $C^2 = 4r(C)$ and the kernel of $\pi_1(X - C) \to \pi_1(X)$ is not central. In fact the subset $\gamma(C)$ of $\pi_1(X - C)$ consists of two distinct elements.

See also 6.5 and 6.6.

The following lemma will be used for the proof of WLT.

LEMMA 3.20. — *If $f : P \to S \times Q$ is algebraic (resp. holomorphic) and $p_1 \circ f : P \to S$ is proper and flat, then $S_1 = \{ s \in S \mid f$ is 1-excellent at $s \}$ is the complement of a Zariski-closed (resp. complex-analytic) subset of* S. *Here* Q *is assumed to be smooth, and* $\dim P_s + 1 = \dim Q$.

Proof. — Recall that $s \in S_1$ if and only if:

(*a*) S is smooth at s;

(*b*) P is smooth at each point of $P_s = (p_1 \circ f)^{-1} S$;

(*c*) $p_2 \circ f : P \to Q$ induces a surjection on tengent-spaces at each point of P_s.

From this, it follows that S_1 is Zariski-open (resp. the complement of a complex-analytic subvariety).

3.21. We now come to the proof of WLT. *The notation* R, H, U, X, i, q, h *will be as in the statement of* WLT.

It will be assumed henceforth that $\dim X = 2$.

For any $j : A \to X$, *we put* $A' = A - j^{-1}(R)$. *In particular*, $X' = X - R$, $U' = q^{-1}(X')$.

With $W \to V \times U$ as in 3.9, let:

$$V_1 = \{ z \in V \mid W \to V \times U \text{ is 1-excellent at } z \in V \}.$$

By 3.20, V_1 is open. Furthermore $p(mH) \in V_1$ and $\Gamma(mH, \mathcal{O}_{mH}) = \mathbb{C}$ for all $m > m_0$. With $\alpha : V \times V \to V$ as in 3.9, note that α is continuous (in fact holomorphic). Thus for general z_1 and z_2 in suitable neighbourhoods of $p(mH) \in V$, for some fixed $m > m_0$, we may assume:

1. z_1, z_2, and $\alpha(z_1, z_2)$ all belong to V_1, and if z_1 and z_2 represent $A_0 \subset U$ and $B_0 \subset U$ respectively, i. e. $p(A_0) = z_1$ and $p(B_0) = z_2$ with p as in 3.9, then:

2. A_0 and B_0 are smooth, irreudible and compact, by 3.5.

3. $A_0 \times_X B_0$ is finite and reduced, and its image in X does not intersect R, by 3.6 B and 3.7.

If $C_0 = A_0 \cup B_0$, then $\alpha(z_1, z_2) = p(C_0)$.

Let $A \to L \times X$, $B \to M \times X$, $C \to N \times X$ be *algebraic* deformations of the morphisms $A_0 \to X$, $B_0 \to X$, $C_0 \to X$ given by the restrictions of q, with base-points $l_0 \in L$, $m_0 \in M$ and $n_0 \in N$, satisfying the requirements of 3.14.

In view of 3.15, 3.7 and 3.8, (1) above is equivalent to:

1'. The three given algebraic deformations are 1-excellent at the base-points l_0, m_0, n_0.

Because A_0, B_0 and C_0 are connected and reduced and their images in X are not contained in R, replacing L, M and N by Zariski-neighbourhoods of l_0, m_0 and n_0, and denoting the fibres of $A \to L$, $B \to M$ and $C \to N$ by A_l, B_m and C_n, we may assume:

I : All the A_l, B_m and C_n are reduced and connected, and none of their images in X is contained in R.

II : A, B, C, L, M, N are smooth and connected. The morphisms $A \to X$, $B \to X$, $C \to X$ induce surjections on all tangent-spaces.

II is an application of Lemma 3.20: replace L by the connected component of l_0 in $\{l \in L \mid A \to L \times X \text{ is 1-excellent at } l\}$. That l_0 belongs to this subset is assured by 1'.

Because $C_0 = A_0 \cup B_0$ is got from the disjoint union of A_0 and B_0 by making certain identifications, we identify the A_l and B_m along certain points to get reduced curves which are deformations of C_0. In 3.22 below, we work this out precisely.

3.22. First note that $(A_0 . B_0) = m^2 (H^2) > 0$ and that A_0 intersects B_0 transversally in U. Indeed $A_0 \times_U B_0 \subset A_0 \times_X B_0$ which is finite and reduced by assumption (2) in 3.21. Let μ and λ be the cordinalities of $A_0 \cap B_0$ and $A_0 x_X B_0$ respectively. Then $\mu \leqq \lambda$.

Let $E(l, m) = A_l \times_X B_m$ for $(l, m) \in L \times M$. This is just the fibre of $A x_X B \to L \times M$ over $(l, m) \in L \times M$. By II, $A \to X$ and $B \to X$ are smooth morphisms, from which it follows that $A \times_X B$ is smooth with dimension $= \dim (A \times B) - \dim X = \dim L + \dim M$.

Consequently $Q = \{(l, m) \mid E(l, m) \text{ is finite and reduced}\}$ is the largest Zariski-open subset of $L \times M$ such that if E denotes its inverse image in $A \times_X B \to L \times M$, then $E \to Q$ is etale. By the properness of $A \to L$, $B \to M$, it follows that $E \to Q$ is a finite etale morphism of degree μ.

Let F be the open-closed subscheme of the μ-fold fibre-product of $E \to Q$ having all μ coordinates distinct and denote the quotient of F by the natural action of the permutation group S_μ by 0. Then $0 \to Q$ is a finite etale morphism; in fact the fibre of $0 \to Q$ over $(l, m) \in Q$ is canonically identified with the collection of all subsets of $E(l, m)$ of cardinality μ.

Denote by $p_i : F \to E$ the i-th projection and let:

$$G_i = \{(x, y) \in E \times_Q F \mid x = p_i(y)\}$$

and let G be the (disjoint) union of the G_i in $E \times_Q F$. Denote the quotient of G by S_μ by T. Then T is a closed subscheme of $E \times_Q 0$ and the projection $T \to 0$ is a finite etale morphism of degree μ. In fact with the above identification of 0, a point $(\alpha, \xi) \in E \times_Q 0$ belongs to T if and only if $\alpha \in \xi$.

For $\xi \in 0$ lying over $(l, m) \in Q$, let:

$$\xi = \{(a_1, b_1), (a_2, b_2), \ldots, (a_\mu, b_\mu)\} \subset E(l, m) = A \times_X B_m.$$

Then the set of $\xi \in 0$ for which all the a_i and the b_i are distinct points of A_l and B_m respectively form a Zariski-open subset 0_1 of 0.

Note that $E(l_0, m_0) \cong A_0 \times_X B_0$ canonically. *The subset* $A_0 \cap B_0 = A_0 x_U B_0 \subset A_0 \times_X B_0$ *of cardinality* μ *gives a point* $z \in 0_1$.

For $\xi = \{(a_1, b_1), \ldots, (a_\mu, b_\mu)\} \subset E(l, m)$, $\xi \in 0_1$, take the disjoint union of A_l and B_m and identify a_i with b_i for $1 \leqq i \leqq \mu$ to get a curve D_ξ and a morphism $D_\xi \to X$. We shall see below that the D_ξ fit together to give a flat family $D \to 0_1$.

With $T \subset E \times_Q 0$ as before, let $\overline{T} = T \cap E \times_Q 0_1$. The inclusion $E \subset A \times_X B$ gives morphisms:

$$\varphi_1 : \quad \overline{T} \to A \times_L 0_1 = D_1,$$

$$\varphi_2 : \quad \overline{T} \to B \times_M 0_1 = D_2,$$

such that $p_2 \circ \varphi_1 = p_2 \circ \varphi_2$ is the given finite etale morphism $\overline{T} \to 0_1$. It follows that both φ_1 and φ_2 are proper and induce injections on tangent-spaces; because they are also set-theoretically injective, they are closed immersions. Let T_1 and T_2 be their images and let $\psi_i : T_i \to \overline{T}$ be the inverse of φ_i for $i = 1, 2$. We want to take the disjoint union of D_1 and D_2 and identify T_1 with T_2 *via* $\varphi_2 \circ \psi_1$.

We appeal to the following lemma (*see* [N3] for a proof) where all objects are schemes of finite type over an algebraically closed field k.

LEMMA. — *Let* $i : P \to Q$ *be a closed immersion and let* $j : P \to R$ *be a finite morphism so that* $\mathcal{O}_R \to j_*(\mathcal{O}_P)$ *is injective. Assume that any finite set of points of* Q *is contained in an affine open subset.*

Then there is a commutative diagram of schemes:

$$\begin{array}{ccc} P & \xrightarrow{i} & Q \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \bar{j}} \\ R & \xrightarrow{\bar{i}} & S \end{array}$$

with $\bar{i}$ a closed immersion, $\bar{j}$ a finite morphism, $\mathcal{O}_S \to \bar{j}_*(\mathcal{O}_Q)$ a monomorphism, and $\bar{i}_*(j_* \mathcal{O}_P / \mathcal{O}_R) \to \bar{j}_*(\mathcal{O}_Q)/\mathcal{O}_S$ an isomorphism of sheaves on S.

Furthermore, given a commutative diagram:

$$\begin{array}{ccc} P & \xrightarrow{i} & Q \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j'} \\ R & \xrightarrow{i'} & S' \end{array}$$

of schemes, there is a unique morphism $\alpha : S \to S'$ such that $j' = \alpha \circ \bar{j}$, $i' = \alpha \circ \bar{i}$.

In the reference cited, this universal property is not proved, but it is in any case an immediate consequence of the first statement.

In our situation, $T_1 \coprod T_2$ is a closed subscheme of $D_1 \coprod D_2$ (which is quasi-projective) and $\psi_1 \coprod \psi_2 : T_1 \coprod T_2 \to \overline{T}$ is a finite morphism inducing the exact sequence of $\mathcal{O}_T$-modules:

$$0 \to \mathcal{O}_{\overline{T}} \to (\psi_1 \coprod \psi_2)_* \mathcal{O}_{T_1 \coprod T_2} = (\psi_1)_* \mathcal{O}_{T_1} \oplus (\psi_2)_* \mathcal{O}_{T_2} \to \mathcal{O}_{\overline{T}} \to 0.$$

Applying the lemma, we get:

$$\begin{tikzcd} T_1 \amalg T_2 \arrow[r] \arrow[d, "\psi_1 \amalg \psi_2"'] & D_1 \amalg D_2 \arrow[d, "\bar{j}=\bar{j}_1 \amalg \bar{j}_2"] \\ T \arrow[r, "\bar{i}"] & D \end{tikzcd}$$

and there are morphisms $D \to X$ and $D \to 0_1$ by the universal property. Equivalently, there is a diagram:

$$\begin{tikzcd} \overline{T} \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & D_1 \arrow[d, "\bar{j}_1"] & \\ D_2 \arrow[r, "\bar{j}_2"] & D \arrow[r] & O_1 \times X \end{tikzcd}$$

and $0 \to \mathcal{O}_D \to (\bar{j}_1)_* \mathcal{O}_{D_1} \oplus (\bar{j}_2)_* \mathcal{O}_{D_2} \to \bar{i}_* \mathcal{O}_{\overline{T}} \to 0$ is exact.

Because $D_1 \to 0_1$, $D_2 \to 0_1$ and $\overline{T} \to 0_1$ are flat morphisms, it follows that $D \to 0_1$ is flat and the above exact sequence remains exact after base-change.

For $z \in 0_1$ as before, $z = A_0 \cap B_0 \subset A_0 x_X B_0$, let D_z be the fibre of $D \to 0_1$ over z. Then:

$$\begin{tikzcd} A_0 \cap B_0 \arrow[r] \arrow[d] & A_0 \arrow[d] \\ B_0 \arrow[r] & D_z \end{tikzcd}$$

is commutative, and:

$$0 \to \mathcal{O}_{D_z} \to \mathcal{O}_{A_0} \oplus \mathcal{O}_{B_0} \to \mathcal{O}_{A_0 \cap B_0} \to 0 \text{ is exact.}$$

From this D_z is canonically identified with $A_0 \cup B_0 = C_0$ as $\mathbb{C}$-schemes.

By the assumption on $C \to N \times X$ (*see* 3.14), there is a Zariski-neighbourhood Z of z in 0_1, $\overline{D} = Z \times_{O_1} D$, and a commutative diagram:

$$\begin{tikzcd} \overline{D} \arrow[r, "g"] \arrow[d] & C \arrow[d] \\ Z \times X \arrow[r, "f \times 1_X"] & N \times X \end{tikzcd}$$

with $f(z) = n_0$.

Putting $E_1 = Z \times_{O_1} D_1$, $E_2 = Z \times_{O_1} D_2$ and using the inclusions $E_i \to \overline{D}$, we get:

I.
$$\begin{tikzcd} E_1 \arrow[r, "g_1"] \arrow[d] & C \arrow[d] \\ Z \times X \arrow[r, "f \times 1_X"] & N \times X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} E_2 \arrow[r, "g_2"] \arrow[d] & C \arrow[d] \\ Z \times X \arrow[r, "f \times 1_X"] & N \times X \end{tikzcd}$$

and by the very construction, we have:

II.
$$\begin{tikzcd} E_1 \arrow[r] \arrow[d] & A \arrow[d] \\ Z \times X \arrow[r, "u \times 1_X"] & L \times X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} E_2 \arrow[r] \arrow[d] & B \arrow[d] \\ Z \times X \arrow[r, "v \times 1_X"] & M \times X \end{tikzcd}$$

Choosing any point $s_0 \in A_0 \cap B_0$, we get a consistent system of base-points e_1, e_2, a_0, b_0, c_0 of E'_1, E'_2, A', B', C', lying over the base-points z, l_0, m_0, n_0 of Z, L, M, N.

Put $q(s_0)=x_0 \in X$.

3.23. We now apply this to π_1.

The general fibre of $A \to L$ is smooth by Sard's theorem, and is connected by 3.21 I. Also $A'_l \neq \emptyset$ for all $l \in L$ (recall that A'_l is the complement of the set of points of A_l whose images in X lie in R). Thus the hypothesis of lemma 1.5 C holds for $A' \to L$. The same is true for $B' \to M$, $C' \to N$, $E'_1 \to Z$, $E'_2 \to Z$.

LEMMA 3.23 A. — *There is a unique function* $\theta(L): \pi_1(L, l_0) \to \pi_1(X', x_0)/G$, *where* G *is the image of* $\pi_1(U', x_0)$, *such that the diagram below is commutative:*

$$\begin{tikzcd} \pi_1(A', a_0) \arrow[r] \arrow[d] & \pi_1(X', x_0) \arrow[d] \\ \pi_1(L, l_0) \arrow[r, "\theta(L)"] & \pi_1(X', x_0)/G \end{tikzcd}$$

Proof. — The uniqueness is clear because $\pi_1(A', a_0) \to \pi_1(L, l_0)$ is surjective by lemma 1.5 C.

Denote by $t: A \to L \times X$ the given morphism. Let I be any connected neighbourhood of $(p_1 \circ t)^{-1} l_0$ in A. For any $l \in L$ such that A'_l is a general fibre of $A' \to L$ in the sense of 1.5 A, and $A_l \subset I$, choose any $a \in A'_l$. By 1.5 C:

$$\ker(\pi_1(A', a) \to \pi_1(L, p_1 t(a)) \subset \mathrm{Image}(\pi_1(I', a) \to \pi_1(A', a)).$$

The truth of the above statement remains unaffected if a is replaced by any other point of $I' = I \cap A'$ because I' is path-connected. In particular, $a = a_0$ will do.

By lemma 3.12, there is a $s: I \to U$ for I small enough such that $q \circ s = p_2 \circ t \mid I$ and the restriction of s to $(p_1 \circ t)^{-1} l_0 \cong A_0$ agrees with the given immersion of A_0 in U. From this, it follows that $\pi_1(A', a_0) \to \pi_1(X', x_0)$ takes $\ker(\pi_1(A', a_0) \to \pi_1(L, l_0))$ into G, and thus defines $\theta(L): \pi_1(L, l_0) \to \pi_1(X, x_0)/G$ with the above commutative diagram.

Remark. — The very same proof works for the remaining four situations: $B' \to M$, $C' \to N$, $E'_1 \to Z$, $E'_2 \to Z$, thus defining functions $\theta(M)$, $\theta(N)$, $\theta_1(Z)$ and $\theta_2(Z)$ from $\pi_1(M, m_0)$, $\pi_1(N, n_0)$, $\pi_1(Z, z)$ and $\pi_1(Z, z)$ respectively to $\pi_1(X', x_0)/G$ with the corresponding commutative diagrams.

From the diagrams I and II of 3.22, we get:

LEMMA 3.23 B:

$$\theta(N) \circ f_* = \theta_1(Z) \quad \textit{and} \quad \theta(N) \circ f_* = \theta_2(Z);$$
$$\theta(L) \circ u_* = \theta_1(Z) \quad \textit{and} \quad \theta(M) \circ v_* = \theta_2(Z).$$

Proof. — For example, to prove $\theta(N)\circ f_* = \theta_1(Z)$, put $\theta(N)\circ f_* = \theta'_1(Z)$. The first commutative diagram of I in 3.22 and the defining property of $\theta(N)$ gives a commutative diagram:

$$\begin{array}{ccccc} \pi_1(E'_1, e_1) & \rightarrow & \pi_1(C', c_0) & \longrightarrow & \pi_1(X', x_0) \\ \downarrow & & \downarrow & & \downarrow \\ \pi_1(Z, z) & \xrightarrow{f_*} & \pi_1(N, n_0) & \xrightarrow{\theta(N)} & \pi_1(X', x_0)/G \end{array}$$

which shows that $\theta'_1(Z)$ satisfies the required commutative diagram. By the uniqueness part of 3.23 A, it follows that $\theta_1(Z) = \theta'_1(Z)$.

The same proof works for all the four cases.

As a consequence, note that $\theta_1(Z) = \theta_2(Z)$ from the first row, and $\theta(L)\,j = \theta(M)\,\delta$ for all (j, δ) in the image of $\pi_1(Z, z) \to \pi_1(L \times M, (l_0, m_0))$ from the second row of 3.23 B.

But $Z \to L \times M$ is dominant (in fact etale by very construction) and by 1.5 B the above (j, δ) form a subgroup of finite index S in $\pi_1(L, l_0) \times \pi_1(M, m_0)$. Intersecting S with the first factor, we get a subgroup T of finite index in $\pi_1(L, l_0)$ such that $\theta(L)\,j = \theta(M)\,1 =$ identity coset of $\pi_1(X, x_0)/G$ for all $j \in T$.

Let V be the inverse image of T in $\pi_1(A', a_0) \to \pi_1(L, l_0)$. Then $p = [\pi_1(A', a_0) : V]$ is finite and $\beta(V) \subset G$ where $\beta : \pi_1(A', a_0) \to \pi_1(X', x_0)$ is the given homomorphism.

Because $A' \to X$ is dominant (*see* 3.21, II), by 1.5 B again:

$$q = [\pi_1(X', x_0) : \beta\pi_1(A', a_0)]$$

is finite. It follows that:

$$[\pi_1(X', x_0) : G] \leqq p \cdot q$$

and this completes the proof of WLT (A) for surfaces.

As remarked in paragraph 0, this also proves WLT (A) and (B) in general.

3.24. The Upper Bound of WLT (C).

Lemma. — *With the hypothesis of* WLT, *there is a commutative diagram:*

$$\begin{array}{ccc} & & (Y, y_0) \\ & \overset{s}{\nearrow} & \downarrow \varphi \\ (U, s_0) & \xrightarrow{q} & (X, x_0) \end{array}$$

with:

(*a*) Y *is a normal projective variety and* $\varphi : Y \to X$ *is a finite morphism unramified outside* $R \subset X$;

(*b*) *s is holomorphic and locally invertible;*

(*c*) φ *is etale at every point of* $s(U)$;

(*d*) $\pi_1(U', s_0) \to \pi_1(Y', y_0)$ *is surjective where* $Y' = \varphi^{-1}(X')$ *and* $U' = q^{-1}(X')$ *and* $X' = X - R$ *as before.*

Proof. $G = \text{Image}(\pi_1(U', s_0) \to \pi_1(X', x_0))$ is a subgroup of finite index and thus gives a connected finite-sheeted covering space $\varphi' : (Y', y_0) \to (X', x_0)$ such that $\varphi'_* \pi_1(Y', y_0) = G$, and by the lifting theorem, there is $s' : (U', s_0) \to (Y', y_0)$ such that $\varphi' \circ s'$ is the restriction of q to U'. *See* SGA 1.

It is well-known that $\varphi' : Y' \to X'$ extends to $\varphi : Y \to X$ satisfying (a) above.

Let Z be the connected component of $U \times_X Y$ containing the graph of s', and let $p : Z \to U$ denote the projection. Note that:

(a) $Z' = p^{-1}(U')$ is connected because Z is connected and normal.

(b) $p \mid Z' : Z' \to U'$ is a covering space because $\varphi' : Y' \to X'$ is a covering space.

(c) This covering space has a section, and therefore $Z' \to U'$ is an isomorphism.

(d) In addition, $p : Z \to U$ is proper and has finite fibres. Therefore $p : Z \to U$ itself an isomorphism by the analytic version of Zariski's Main Theorem.

This gives the required $s : U \to Y$ extending $s' : U' \to Y'$.

(b) and (c) in the lemma follow from the analytic irreducibility of Y, and part (d) follows from the very construction of $\varphi' : Y' \to X'$.

Proof of WLT (C). — By lemma 3.17, there are smooth compact connected curves $\overline{A} \subset U$ very close to $m H \subset U$ such that:

(a) $q(\overline{A}) = B$ is nodal,

(b) $\overline{A} \to q(\overline{A})$ is birational,

(c) $(\text{Div } h)^2 / (H^2) = (B^2) / (\overline{A}^2)$.

By the above lemma, we get a commutative diagram:

$$\begin{tikzcd} & Y \arrow[d, "\varphi"] \\ U \arrow[ur, "s"] \arrow[r, "q"'] & X \end{tikzcd}$$

and put $s(\overline{A}) = A$. By the lemma, φ is etale at every point of A and by 3.17(a), $B^2 - 2r(B) = (\overline{A}^2) > 0$. So lemma 5.1 can be applied to conclude that:

$$\deg(\varphi) \leqq B^2 / B^2 - 2r(B) = (\text{Div } h)^2 / (H^2),$$

and because $\deg(\varphi)$ is precisely the index in question, WLT (C) follows.

Some applications of WLT follow. We still assume $\dim X = 2$.

DEFINITION 3.25. — If the curve C on X is defined in a neighbourhood of $P \in C$ by $f = 0$, let $f = f_1 f_2 f_3 \ldots f_r$ be its prime factorisation in $A = \mathcal{O}_{X, P}$ and let $A(C; P) = 2(\sum_{i<j} l(A/(f_i, f_j))$ and $B(C) = C^2 - \sum_P A(C; P)$.

PROPOSITION 3.26. — *If* C *is an irreducible curve with* $B(C) > 0$ *and* $C \cap R = \emptyset$, *then the image of* $\pi_1(\overline{C}) \to \pi_1(X - R)$ *has index* $\leqq C^2 / B(C)$, *where* $\overline{C} \to C$ *is the normalisation of* C.

Remark. — For a nodal curve C, $B(C) = C^2 - 2r(C)$.

Proof. — There is a unique diagram $\overline{C} \to H \to C$ such that $\overline{C} \to H$ is set-theoretically injective and $H \to C$ induces an injection of Zariski tangent-spaces. In fact, if $P \in C$, with the notation of 3.25, there are exactly r points of H lying above P and the complete local rings of H at these points are $A/(f_i)$, $1 \leqq i \leqq r$.

Let (U, i, q) be a neighbourhood of $h: H \to X$, where h is the composite $H \to C \to X$ and consider the divisor $q^{-1}(C) = H + F$. We see that $(F.H) = C^2 - B(C)$, and also $H.q^{-1}(C) = C^2$ because $H \to C$ is birational. It follows that $H^2 = B(C) > 0$ and the proposition is proved by appealing to WLT and noting that $\overline{C} \to H$ is a homeomorphism.

PROPOSITION 3.27. — *Let* D *and* E *be curves in* X *that intersect transversally. Assume that* D *is nodal and* $C^2 > 2r(C)$ *for every irreducible curve* C *lying in* D. *Then the kernel* N *of* $\pi_1(X-(D \cup E)) \to \pi_1(X-E)$ *is abelian and its centraliser is a subgroup of finite index.*

Proof. — Fix an irreducible curve C contained in D. Let $H = \overline{C}$ the normalisation of C and let h be the composite $H \to C \to X$. For a sufficiently small tubular neighbourhood (U, i, q) of h, let $U' = q^{-1}(X')$, $X' = X - R$, $R = D \cup E$, and then $\gamma(H)$ is central in $\pi_1(U')$ by 1.4 because H is smooth and intersects the closure of $q^{-1}(R) - H$ transversally.

The image of $\gamma(H)$ in $\pi_1(U') \to \pi_1(X')$ is $\delta \in \gamma(C) \subset \pi_1(X')$ and the centraliser $C(\delta)$ of δ contains the image of $\pi_1(U')$ and is therefore a subgroup of finite index by WLT. Therefore $\gamma(C)$ is a finite set. But N is generated by the $\gamma(C)$, $C \subset D$, and is therefore finitely generated, and its centraliser is a finite intersection of subgroups of finite index, and it has finite index therefore in $\pi_1(X')$. It remains to prove that N is abelian.

Let U be a tubular neighbourhood of $H = \overline{C} \to X$ as before.

From 3.24, there is a commutative diagram:

$$\begin{tikzcd} & Y \arrow[d, "\varphi"] \\ U \arrow[ur, "\bar{s}"] \arrow[r, "q"] & X \end{tikzcd}$$

with Y a normal surface, φ a finite morphism unramified outside R (put $Y' = \varphi^{-1}(X')$), $\pi_1(U') \to \pi_1(Y')$ a surjection.

Let $Z = \{ y \in Y \mid \varphi \text{ is not etale at } y \}$. Then $Z \cap \bar{s}(H) = \emptyset$, and by Lemma 5.2, Z does not contain any irreducible component of $\varphi^{-1}(D)$. In other words φ is unramified outside E itself, and therefore N is contained in the image of $\pi_1(Y') \to \pi_1(X')$. Because $\gamma(H)$ is central in $\pi_1(U')$ and $\pi_1(U') \to \pi_1(Y')$ is onto, $\gamma(\bar{s}(H))$ is central in $\pi_1(Y')$. The image of $\gamma(\bar{s}(H))$ in $\pi_1(Y') \to \pi_1(X')$ is a member δ of $\gamma(C)$ and therefore δ commutes with N. But N, being normal, commutes with all the conjugates of δ; in other words every element of $\gamma(C)$ commutes with N. Finally N is generated by the $\gamma(C)$, $C \subset D$, and this shows that N is abelian.

4. Homogeneous spaces and Zariski's Conjecture

PROPOSITION 4.1. — *With the notation of* WLT, *if* $X = \mathbb{P}^2$, *then* $\pi_1(U - q^{-1}(R)) \to \pi_1(X - R)$ *is surjective.*

Remark. — This is stringer than WLT (C). Also there is no need to assume the ampleness of $\mathcal{O}_U(H) | H$.

Proof: Let D be the normalisation of any irreducible component of H and denote by d the composite $D \to H \to X$. Put $G = Sl(3)$.

$\theta : G \times D \to X$ given by $\theta(g, a) = g.d(a)$ makes $G \times D$ a fibre-bundle on X. This is elementary. In any case, Lemma 1.5 A gives a Zariski-open $V \subset X$ such that $\theta^{-1}(V) \to V$ is a fibre-bundle and the G-equivariance of θ shows that θ is locally trivial, in the usual topology, everywhere. Because X is simply connected, the fibres of θ are all connected. Putting $X' = X - R$, $S = \theta^{-1}(X')$, $S \to X'$ is a fibre-bundle with smooth connected fibres and therefore $\pi_1(S) \to \pi_1(X')$ is surjective.

Now the fibres of the projection $S \to G$ are of course all smooth, and they are non-empty outside a finite union of cosets of the stabiliser of the curve $d(D)$. However no subgroup of G has codimension one, and therefore 1.5 C applied to $S \to G$ allows us to conclude that $\pi_1(F) \to \pi_1(S)$ is surjective, for a general fibre F of $S \to G$, because G is simply connected.

The two surjectivities above show that for all σ in a Zariski-open $M \subset G$, $f : D \to X$ defined by $f(a) = \sigma d(a)$ induces a surjection $\pi_1(D - f^{-1}(R)) \to \pi_1(X')$. But $q : U \to X$ being a local homeomorphism, as in 3.12, for σ close to the identity, there is $g : D \to U$ such that $f = q \circ g$. Thus $\pi_1(U - q^{-1}(R)) \to \pi_1(X - R)$ is surjective.

COROLLARY (Zariski's Conjecture). — *$\pi_1(\mathbb{P}^2 - D)$ is abelian for a nodal curve D in $\mathbb{P}^2$.*

Proof. — Let H be the nonsingular model of an irreducible curve C in D and let U be a tubular neighbourhood, as in 1.11, of $H \to \mathbb{P}^2$. Put $R = D$ and apply 4.1 to conclude that $\gamma(C)$ is central in $\pi_1(\mathbb{P}^2 - D)$, and because $\pi_1(\mathbb{P}^2 - D)$ is generated by the $\gamma(C)$, the result follows.

This argument can be extended to prove the following:

PROPOSITION 4.3. — *Assume that the connected component G of the group of automorphisms of a projective variety X acts transitively. Let $D \subset X$ be a divisor such that:*

(*a*) *outside a codimension 3 subset Z of X, each singular point of D has normal crossings:*

(*b*) *no irreducible component of D is the fibre of a G-equivariant morphism $X \to Y$ with $Y = \mathbb{P}^1$ or Y = an elliptic curve.*

Then $\pi_1(X - D) \to \pi_1(X)$ has abelian kernel.

We omit the proof. In any case, by taking the intersection with a general linear subspace, it reduces to a special case of 3.27, the condition $C^2 > 2r(C)$ being a consequence of (*b*).

5. Tame fundamental groups

We work exclusively with complete normal surfaces over an algebraically closed field k.

LEMMA 5.1. — *$\varphi : Y \to X$ is a finite morphism with X smooth. On Y we have irreducible curve A such that:*

(*a*) *$A \to \varphi(A) = B$ is birational, and B is a nodal curve, with $B^2 > 2r(B)$,*

(*b*) *φ is etale at every point of A.*

Then $A^2 > 0$ and $\deg \varphi \leqq B^2 / B^2 - 2r(B)$.

Proof. — (*b*) assures us that Y is smooth at every point of A, and that the only singular points of $\varphi^{-1}(B)$ lying on A are nodes.

Consequently if $\varphi^{-1}(B)=A+R$ as divisors, A intersects R transversally in $\{P\in A \mid A$ is smooth at P and B is singular at $\varphi(P)\}$. Therefore $(A.R)=2r(B)-2r(A)$. Also $A\to B$ being birational, $B^2=B.\varphi(A)=A.\varphi^{-1}(B)=A^2+A.R$, so that $A^2-2r(A)=B^2-2r(B)$, showing finally that $0<B^2-2r(B)\leqq A^2$.

By the Hodge Index Theorem on the intersection pairing, the matrix:

$$\begin{bmatrix} (A+R)^2 & (A+R).A \\ A.(A+R) & A^2 \end{bmatrix}=\begin{bmatrix} B^2 \deg \varphi & B^2 \\ B^2 & A^2 \end{bmatrix},$$

has determinant $\leqq 0$. Thus:

$$\deg \varphi \leqq B^2/A^2 \leqq B^2/B^2-2r(B).$$

Remark. — The intersection number $(D_1.D_2)$ of Cartier divisors D_1 and D_2 on a normal surface Y is taken, by definition, to be $f^{-1}(D_1).f^{-1}(D_2)$ for any desingularisation $f:Z\to Y$.

LEMMA 5.2. — *If* $\varphi: Y\to X$ *is a finite tamely ramified morphism unramified outside* $D\cup E$ *where* D *and* E *are as in Proposition* 3.27, *then any two irreducible curves in* $\varphi^{-1}(D)$ *intersect each other.*

Proof. — Clearly we may assume that φ is Galois with Galois group G.

Let S be an irreducible curve in $\varphi^{-1}(D)$. Let:

$$G(S)=\{g\in G \mid gS=S\},\qquad Z=Y/G(S),\qquad \psi: Z\to X \quad\text{and}\quad \lambda: Y\to Z$$

the natural morphisms, and $\lambda(S)=S/G(S)=A$ and $\varphi(S)=B$. We want to apply 5.1 to $\psi: Z\to X$. By Field Theory $A\to B$ is birational and ψ is etale at the generic point of A.

For $P\in S$, $\psi: Z\to X$ is etale at $\lambda(P)$ if $G(P)=\{g\in G \mid gP=P\}$ is contained in $G(S)$, because $Y/G(P)\to X$ is etale at the image of P in $Y/G(P)$. But it is well-known from local considerations that:

(*a*) $G(P)$ is abelian, and:

(*b*) if $\varphi(P)=Q$, the inverse image of any analytically irreducible branch of $D\cup E$ at Q is analytically irreducible at P (*see* [F]), showing that $G(P)\subset G(S)$.

Applying 5.1, we see that $A^2>0$, and therefore $[S]=\lambda^{-1}(A)$ is an effective Cartier divisor on Y supported on S. If $S_1, S_2\subset\varphi^{-1}(D)$, then by the Hodge Index Theorem:

$$([S_1].[S_2])^2\geqq[S_1]^2.[S_2]^2=[G(S_1):1][G(S_2):1].(A_1^2)(A_2^2)>0,$$

where A_i is the image of S_i in $Y/G(S_i)$.

Therefore $S_1\cap S_2\neq\emptyset$. This proves the lemma.

Appealing to 5.1 also shows that:

$B^2/B^2-2r(B)\geqq\deg\psi=[G:G(S)]=$ the number of irreducible curves in $\varphi^{-1}(B)$, where S, G and G(S) are as before.

Let $I(S)=\{g\in G \mid gx=x \text{ for all } x\in S\}$. Then for any $P\in S_1\cap S_2$ with $S_i\subset\varphi^{-1}(D)$ for $i=1, 2$, $G(P)\supset I(S_1)$ and $G(P)\supset I(S_2)$, showing that $I(S_1)$ and $I(S_2)$ commute with each other. Thus the group $I(D)$ generated by all the $I(S)$, $S\subset\varphi^{-1}(D)$ is abelian. This is the ramification subgroup along D.

With S as above, $I(S)\subset\mu_n=n$-th roots of unity, for some n not divisible by the characteristic, and in fact the action of $I(S)$ on $\mathscr{I}/\mathscr{I}^2$, where $\mathscr{I}$ is the ideal-sheaf of S, is just multiplication by the corresponding root of unity. It follows that $G(S)$ is the centraliser of $I(S)$.

Therefore $Z(I(D))=Z$, the centraliser of $I(D)$, has index $[G : Z]\leqq\prod_{B\subset D}(B^2/B^2-2r(B))$.

Denoting by $\pi_t(X-R)$ the tame fundamental group of $X-R$, the above remarks show (after taking inverse limits):

PROPOSITION 5.3. — *For a smooth surface* X, *and* D *and* E *as in* 3.27, *the kernel* N *of* $\pi_t(X-D\cup E)\to\pi_t(X-D)$ *is abelian and has a finitely generated dense subgroup. Moreover its centraliser has finite index in* $\pi_t(X-D\cup E)$.

This dense subgroup has not more than $\sum_{B\subset D}[F(B)]$ generators and the index of the centraliser is $\leqq\prod_{B\subset D}[F(B)]$, where $F(B)=B^2/B^2-2r(B)$.

Denoting algebraic fundamental groups simply by π_1 we also get:

PROPOSITION 5.4. — *For any irreducible nodal curve* C *on* X *with* $C^2>2r(C)$, *the image of* $\pi_1(\overline{C})\to\pi_1(X)$ *is a subgroup of index* $\leqq C^2/C^2-2r(C)$, *where* $\overline{C}$ *is the normalisation of* C.

Proof. — Any subgroup H of finite index in $\pi_1(X)$ containing the image of $\pi_1(\overline{C})$ gives a commutative diagram:

$$\begin{tikzcd} & Y \arrow[d, "\varphi"] \\ \overline{C} \arrow[ur, "s"] \arrow[r] & X \end{tikzcd}$$

with φ a finite etale morphism. Applying Lemma 5.1, $\deg\varphi\leqq C^2/C^2-2r(C)$. But the image of $\pi_1(\overline{C})$ is the intersection of all subgroups of $\pi_1(X)$ of finite index containing it, because we are working with profinite groups. Therefore the image itself has finite index $\leqq C^2/C^2-2r(C)$.

6. Examples

We discuss to what extent the results of paragraph 2 and the WLT are best possible for nodal curves and then discuss other singularities briefly.

Let C be an irreducible nodal curve on a surface X, $\overline{C}$ its normalisation, and $h:\overline{C}\to X$ is the given morphism.

6.1. For the examples in 3.19 (A), $[\pi_1(X) : h_*\pi_1(\overline{C})]=C^2/C^2-2r(C)$ which is the upper bound imposed by WLT. Assuming Remark 3.18 however, there are C with $C^2/C^2-2r(C)$ very large and $\pi_1(X)=h_*\pi_1(\overline{C})$.

6.2. There exist C with $C^2=2r(C)>0$ and $[\pi_1(X) : h_* \pi_1(\overline{C})]$ infinite, and non-abelian kernel $(\pi_1(X-C) \to \pi_1(X))$.

Let $f_i : C_i \to \mathbb{P}^1$ be a double-covering ramified at $S_i \subset \mathbb{P}^1$ for $i=1,2$. Let s and t be the cardinalities of $S=S_1 \cap S_2$ and $T=S_1 \cup S_2-S$ respectively. Put $X=C_1 \times C_2$, $f=f_1 \times f_2 : X \to \mathbb{P}^1 \times \mathbb{P}^1$, and let $C=f^{-1}(\Delta \mathbb{P}^1)$. Then:

(*a*) C is irreducible if and only if $S_1 \neq S_2$;

(*b*) C is nodal and $r(C)=s$;

(*c*) $g(\overline{C})-(g(C_1)+g(C_2))=(t/2)-1$.

We shall show that $s \geqq 4$ and $S_1 \neq S_2$ imply:

(A) $[\pi_1(X) : h_* \pi_1(\overline{C})]$ is infinite.

Note that $H_1(\overline{C}, \mathbb{Q}) \to H_1(X, \mathbb{Q})$ is a surjection, and even an isomorphism if $t=2$!

(B) $\ker(\pi_1(X-C) \to \pi_1(X))$ is non-abelian if in addition, $S_1 \subset S_2$.

Case 1 : $S_1 \subset S_2$.

We see that $\overline{C} \to C_2$ is an unramified double-covering, and this makes $Y=C_1 \times \overline{C}$ a double-cover of $X=C_1 \times C_2$, and $h : \overline{C} \to X$ has then a natural lift to $\overline{h} : \overline{C} \to Y$ with $p_2 \overline{h}(x)=x$ for all $x \in \overline{C}$. It follows that $\pi_1(C_1)$ acts simply transitively on $\pi_1(Y)/\overline{h}_* \pi_1(\overline{C})$, which is contained in $\pi_1(X)/h_* \pi_1(\overline{C})$, and because $\pi_1(C_1)$ is infinite, we deduce that $[\pi_1(X) : h_* \pi_1(\overline{C})]$ is finite.

Now $(Y, Y-\overline{h}(C))$ is a fibre-bundle pair with fibre $(C_1, C_1-\{P\})$ and base-space $\overline{C}$, from which $\ker(\pi_1(Y-\overline{h}(\overline{C})) \to \pi_1(Y))$ is isomorphic to $\ker(\pi_1(C_1-\{P\}) \to \pi_1(C_1))$ which is certainly non-abelian! This combined with the facts:

(*a*) $\pi_1(Y-\theta^{-1}(C)) \to \pi_1(Y-\overline{h}(\overline{C}))$ is onto, where $\theta : Y \to X$ is the given double covering, and:

(*b*) $\ker(\pi_1(Y-\theta^{-1}(C) \to \pi_1(Y)) \cong \ker(\pi_1(X-C) \to \pi_1(X))$ proves (B).

Case 2 : $S_1 \not\subset S_2$ and $S_2 \not\subset S_1$. We sketch the proof of (A) briefly in this case. Put $G=\pi_1(\overline{C})$ and $G/N_i=\pi_1(C_i)$ for $i=1, 2$. Then $[\pi_1(X) : h_* \pi_1(\overline{C})]=[G : N_1 N_2]$, $\pi_1(Z) \cong G/N_1 N_2$ where Z is the "double-mapping cylinder": the disjoint union of $\overline{C} \times I$, C_1, C_2 with $(x, 0)$ and $(x, 1)$ *identified to* $q_1(x)$ and $q_2(x)$, for $x \in \overline{C}$ and $p_i \circ h=q_i$ for $i=1, 2$, and the $p_i : X \to C_i$ are the projections.

There is a natural $\varphi : Z \to \mathbb{P}^1$ given by $\varphi(x, t)=f_1 q_1(x)=f_2 q_2(x)$ for $x \in \overline{C}$, $t \in I$. Analysing the fibres of φ, we see that $\pi_1(Z)$ is generated by $x_1, x_2, \ldots, x_s$ subject to the relations $x_i^2=1$ for all i and $x_1 x_2 \ldots x_s=1$. It is easy to see that for $s \geqq 4$, this group is infinite; in fact for $s \geqq 5$, it is a Fuchschian subgroup Γ of $PSL_2(\mathbb{R})$ such that $h/\Gamma \cong \mathbb{P}^1$ and the elliptic fixed-points of Γ correspond to the points of $S \subset \mathbb{P}^1$.

Remark. — In the above examples, the quotient of $\pi_1(X)$ by the normal subgroup generated by $h_* \pi_1(\overline{C})$ is $(\mathbb{Z}/2\mathbb{Z})^{s-1}$.

PROPOSITION 6.3. — *If* $C^2>\max(0, 2r(C)-2)$, *then the normal subgroup generated by* $h_* \pi_1(\overline{C})$ *has finite index in* $\pi_1(X)$.

Proof. — Case 1. If $C^2 > 2r(C)$, the result follows from WLT.

Case 2. If $C^2 = 2r(C) > 0$, choose a pair of points P and Q of $\overline{C}$ such that $h(P) = h(Q)$ and $P \neq Q$. Take $\overline{C} \times \{0, 1\}$ and identify (P, 0) with (Q, 1) to get a curve F and a morphism $f: F \to X$, and let (U, i, q) be a neighbourhood of f as in 1.8. Denoting the irreducible components of F by F_1 and F_2, $F_1^2 = F_2^2 = 0$ and $(F_1 . F_2) = 1$, from which it follows that $\mathcal{O}_U(F) \mid F$ is ample. Now WLT shows that $[\pi_1(X) : f_* \pi_1(F)] \leqq 2C^2$, but $\pi_1(F) = \pi_1(F_1) * \pi_1(F_2)$ and the images of these subgroups are conjugates of $h_* \pi_1(\overline{C})$. The result follows.

Case 3. If $C^2 = (2r(C) - 1) > 0$, take $\overline{C} \times \{0, 1, 2\}$ and points P and Q as above and identify (P, i) with $(Q, i+1)$ for $i = 0, 1$, and call the resulting curve F and let $f: F \to X$ be the natural morphism. Let (U, i, q) be a neighbourhood of f, and denote the irreducible curves in F by F_i, for $i = 0, 1, 2$. Then $F_i^2 = -1$ for $i = 0, 1, 2$ and $F_0 . F_1 = F_1 . F_2 = 1$ and $F_0 . F_2 = 0$. From which $R = 2F_0 + 3F_1 + 2F_2$ is ample restricted to each F_i. Applying WLT to $R \to X$, $[\pi_1(X) : f_* \pi_1(R)]$ is finite, and as above, $f_* \pi_1(R)$ is generated by three subgroups of $\pi_1(X)$ each of which is conjugate to $h_* \pi_1(\overline{C})$, and the result follows.

The examples 6.2 raise the:

Question 6.4. — If D is an effective divisor on a surface X with $D^2 > 0$, is the normal subgroup generated by the fundamental groups of the nonsingular models of all the irreducible curves in D a subgroup of finite index in $\pi_1(X)$?

We now discuss other singularities. Let $A = \mathbb{C}[[a, b]]$, $T = \operatorname{Spec} A$, and $f \in A$ is square-free and in the square of the maximal ideal. By a sequence of blowing-up transformations we get a proper birational $\psi: S \to T$ with $\operatorname{div} \psi^*(f) = F + G$ where F is the proper transform of $f = 0$ and F meets G_{red} transversally. Let $s = G.(G + 2F)$.

For any irreducible curve C on X and for any singular point P of C, identity $\hat{\mathcal{O}}_{X, P}$ with A and let $f = 0$ be the defining equation of C here. Put $s = s(C; P)$. Let $F(C) = \sum_P s(C; P)$.

PROPOSITION 6.5. — *With the above notation, if* $C^2 > F(C)$, *then* $\pi_1(X - C) \to \pi_1(X)$ *is a central extension.*

Proof. — After a series of blowing-ups, we get $\varphi: Y \to X$ with $\varphi^{-1}(C) = C' + G$, where C' is the proper transform of C, C' meets G_{red} transversally, and $G.(G + 2C') = F(C)$. Thus $(C')^2 > 0$. Putting (Y, C', G) in place of (X, D, E) in 2.5, we see that $\gamma(C')$ is central in $\pi_1(Y - \varphi^{-1}(C))$ and therefore $\gamma(C)$ is central in $\pi_1(X - C)$.

Remark 6.6. — It is interesting to note that the $s(C; P)$ are just right! For a node, $s(C; P) = 2$ and therefore if $C^2 > 4r(C)$ for an irreducible nodal curve C, $\pi_1(X - C) \to \pi_1(X)$ is a central extension. However example 3.19 (C) shows that this is false when $C^2 = 4r(C)$.

Example 6.7. — For an ordinary cusp $(a^2 - b^3 = 0)$, $s(C; P) = 6$.

The general curve C in $\mathbb{P}^2$ of degree 6 given by $f^2 - g^3 = 0$, where f and g are homogeneous of degrees 3 and 2 respectively, is smooth outside $f = g = 0$ where its singularities are ordinary cusps. Therefore $C^2 = F(C) = 36$ in this case. This example is due to Zariski [Z]. He shows that $\pi_1(\mathbb{P}^2 - C) \cong \mathbb{Z}/(2) * \mathbb{Z}/(3)$.

In fact $\pi_1(\mathbb{P}^2 - C)$ is canonically isomorphic to $PSL_2(\mathbb{Z})$.

In any case, $G = \pi_1(\mathbb{P}^2 - C)$ has $G/[G, G] \cong \mathbb{Z}/(6)$, and it is easy to show that $[G, G]$, the fundamental group of the six-fold cyclic covering Z of $\mathbb{P}^2 - C$, is infinite ! The functions $(g^3/f^2 - g^3)^{1/3}$ and $(f^2/f^2 - g^3)^{1/2}$ on Z define a morphism $Z \to E$ where $E = \{(x, y) \mid y^2 = x^3 + 1\}$ with connected fibres. Therefore $\pi_1(Z) \to \pi_1(E)$ is a surjection, and E being the complement of a point on an elliptic curve, $\pi_1(E)$ is certainly infinite.

Example 6.8. — Let X be a smooth hypersurface in $\mathbb{P}^3$ of degree d. Let $C = H \cap X$ where H is a hyperplane in $\mathbb{P}^3$. Then $X - C$ is simply connected. This is rather striking because:

(*a*) C need not be irreducible;

(*b*) C may have arbitrary singularities;

(*c*) even if C is irreducible nodal, $2r(C) \leqq (d-1)(d-2)$ and equality can be attained, whereas $C^2 = d$;

(*d*)X is a minimal model if $d \geqq 4$.

Suppose that X contains a line L, with $L \not\subset H$, and also assume that C is irreducible. Then $P = C \cap L$ is necessarily a smooth point of C and C and L intersect transversally at P. Therefore the homomorphism $\pi_1(L - P) \to \pi_1(X - C)$ takes $\gamma(P)$ into $\gamma(C)$. But $L - P$ is the affine line and therefore $\gamma(P)$ and $\gamma(C)$ are both trivial. But $\pi_1(X)$ is the quotient of $\pi_1(X - C)$ by the subgroup generated by $\gamma(C)$, and therefore $\pi_1(X - C) \to \pi_1(X)$ is an isomorphism. Finally, X is simply connected by the Lefschetz hyperplane section theorem and therefore the proof is complete under these additional assumptions.

For the general case, first observe that:

(*a*) any hyperplane section of X is reduced;

(*b*) if X_1 and X_2 are smooth hypersurfaces in $\mathbb{P}^3$ of degree d containing C, there is a diffeomorphism $\psi : X_1 \to X_2$ such that $\psi(x) = x$ for all $x \in C$, and finally;

(*c*) if L is any line in $\mathbb{P}^3$ such that $L \not\subset H$ and $L \cap C$ is a smooth point of C, then the general hypersurface of degree d containing both C and L is smooth.

(*a*) is standard, and (*b*) and (*c*) are easy to check (though we omit to do so here).

By (*a*), for every irreducible component F of C, there is a smooth point P of C lying on F. Choose L as in (*c*) with $P = L \cap C$ and let X′ be a general hypersurface as in (*c*). Then the argument above shows that $\gamma(F) \subset \pi_1(X' - C)$ is trivial, and by (*b*), $\gamma(F)$ is also trivial in $\pi_1(X - C)$. This holds for all F and therefore $\pi_1(X - C) \to \pi_1(X)$ is an isomorphism, proving the result.

Alan Howard, *Annals of Math.*, 1966, has shown that $X - X \cap H$ is simply connected, if X is a smooth hypersurface in $\mathbb{P}^n$, $n \geqq 4$, and H is any hyperplane. This follows from 2.1 and 6.8.

APPENDIX 1

The *b*-excellence of the universal deformation.

We just prove Proposition 3.10 here. All the notation introduced in 3.9 is retained. The major step is to prove 0-excellence, i. e. the smoothness of V at $p(mH)$ for m large.

We need some algebraic preliminaries:

For every complete local Noetherian $\mathbb{C}$-algebra S with residue field $=\mathbb{C}$, fix a surjection $\beta(S) : R(S) \to S$ inducing an isomorphism of Zariski tangent-spaces, with $R(S)$ a power-series ring over $\mathbb{C}$. Let $t(S)$ be the maximal ideal of $R(S)$ and let $J(S)$ be the kernel of $\beta(S)$, and define $D^0(S)$ and $D^1(S)$ to be the dual vector-spaces of $t(S)/t(S)^2$ and $J(S)/t(S)J(S)$ respectively.

A homomorphism $f : S' \to S''$ lifts to a commutative diagram:

$$\begin{tikzcd} R(S') \arrow[r, "g"] \arrow[d, "\beta(S')"'] & R(S'') \arrow[d, "\beta(S'')"] \\ S' \arrow[r, "f"] & S'' \end{tikzcd}$$

inducing linear transformations $D^i(S'') \to D^i(S')$ for $i=0, 1$, which depend only on f and not on the choice of g; these will be denoted by $D^i(f)$ for $i=0,1$. We have:

Fact A 1 : $D^i(S) = \varinjlim_{T} D^i(T)$ for $i=0,1$ where the T range over all finite-length quotients of S.

Let $i : H \to U$ be as in 3.9 and let S be the complete local ring of V at $p(H)$. Then we have:

Fact A 2: There are $d_i : D^i(S) \to H^i(U, I^{-1}/\mathcal{O})$, where $\mathcal{O} = \mathcal{O}_U$, and d_0 is an isomorphism and d_1 is a monomorphism.

Replacing $i : H \to X$ by $i_m : m\,H \to X$ in A 2, we get: $d_i : D^i(S_m) \to H^i(U, I^{-m}/\mathcal{O})$, where S_m is the complete local ring of V at $p(m\,H)$.

Define $z : V \to V$ by $z(x) = \alpha(x, p(H))$. Because $z(p(m\,H)) = p((m+1)\,H)$, we get $f_m : S_{m+1} \to S_m$ for all $m \geqq 1$. From the definitions of the d_i, we see:

Fact A3: For all $m \geqq 1$ and $i=0, 1$, there are commutative diagrams:

$$\begin{tikzcd} D^i(S_m) \arrow[r, "d_i"] \arrow[d, "D^i(f_m)"'] & H^i(U, I^{-m}/\mathcal{O}) \arrow[d, "\delta(m,\, i)"] \\ D^i(S_{m+1}) \arrow[r, "d_i"] & H^i(U, I^{-m-1}/\mathcal{O}) \end{tikzcd}$$

where $\delta(m, i)$ is induced by the inclusion of $I^{-m}/\mathcal{O}$ in $I^{-m-1}/\mathcal{O}$.

In particular, $D^0(f_m)$ is an injection and therefore:

Corollary A4. — *$f_m : S_{m+1} \to S_m$ is a surjection for all $m \geqq 1$.*

Denote the direct limit of $V \xrightarrow{z} V \xrightarrow{z} V \to \dots$ by $\tilde{V}$ and the point $(p(H), p(2\,H), p(3\,H), \dots)$ of $\tilde{V}$ by e. Then $\tilde{V}$ is a commutative associative monoid with e as the identity and naturally we expect that "$\tilde{V}$ is smooth at e" with a suitable definition of smoothness, and this is done in A 7 below:

DEFINITION A5. — $S = \varprojlim_m S_m$. Let $J(m, k)$ be the inverse image of the k-th power of the maximal ideal of S_m in the homomorphism $S \to S_m$, and topologise S by taking the $J(m, k)$ to be a fundamental system of neighbourhoods of zero.

Now $\alpha : V \times V \to V$ induces $S_{m+n} \to S_m \hat{\otimes} S_n$, and passing to the inverse limit, we get a continuous algebra homomorphism $\psi : S \to S \hat{\otimes} S$ where $S \hat{\otimes} S$ is the inverse limit of the $S/P \otimes S/Q$ where P and Q go through all open ideals in S.

A 6 : Let B be the collection of continuous linear functionals on S. Then $S = B^* =$ the dual space of B and the open linear subspaces of S are precisely the annihilators of the finite-dimensional subspaces of B.

The above ψ and the algebra-structure on S induces the structure of a commutative associative $\mathbb{C}$-algebra with identity on B, and we also get an algebra homomorphism $\mu : B \to B \otimes B$ such that μ is co-associative, co-commutative and has a co-identity.

Now B is the union of its finitely generated subalgebras C such that $\mu(C) \subset C \otimes C$. For such a C, $M = \operatorname{Spec} C$ is a commutative affine monoid-scheme. The representations of M are unipotent because S is a local ring. Therefore M is a unipotent commutative group-scheme over $\mathbb{C}$ and so $M \cong \mathbb{G}_a^r$ (*see* Prop. 4.1, page 497, [DG]).

More canonically, let $P = \{ x \in B \mid \mu x = x \otimes 1 + 1 \otimes x \}$. Then $\operatorname{Hom}(M, \mathbb{G}_a) = P \cap C$ and $S(P \cap C) \to C$ is an isomorphism. It follows that $S(P) \to B$ is itself an isomorphism, where $S(P) =$ the symmetric algebra on P.

For any finite-dimensional $F \subset P$, $S(F) \subset S(P) = B$ and $\mu(S(F)) \subset S(F) \otimes S(F)$. Therefore $S(F)^* = R(F)$ is an algebra; in fact it is canonically the completion of $S(F^*)$ at the standard maximal ideal. The injection $S(F) \to B$ gives a surjection $S \to R(F)$ and we see.

PROPOSITION A 7. — (1) : $S \to \varprojlim_F R(F)$ *is an isomorphism of* $\mathbb{C}$-*algebras where the* F *go through all finite-dimensional subspaces of* $R(F)$, *and* (2): *the kernels of the composites* $S \to R(F) \to R(F)/t(F)^k$ *form a fundamental system of neighbourhoods of zero in* S, *where* $t(F)$ *is the maximal ideal of* $R(F)$.

COROLLARY A 8. — $\varinjlim_m D^1(S_m) = 0$.

Proof. — Applying A1, we see that:

$$\varinjlim_m D^1(S_m) \cong \varinjlim_J D^1(S/J) \cong \varinjlim_F D^1(R(F)),$$

where the J run through all open ideals in S. But $D^1(R(F)) = 0$ because $R(F)$ is a power-series ring!

PROPOSITION A 9. — *Put* $L = (I/I^2)^*$:

I. *If* $H^1(H, L^m) = 0$ *for all large m, then* V *is smooth at* $p(mH)$ *for all large m.*

II. *If in addition,* $H^0(H, L^m)$ *has no base-points for all large m then* $W \to V \times U$ *is* 1-*excellent at* $p(mH)$ *for all large m.*

III. If L *is ample on* H, *then* $W \to V \times U$ *is b-excellent for all large m.*

Proof of I. — The cohomology sequence of:

$$0 \to I^{-m+1}/\mathcal{O} \to I^{-m}/\mathcal{O} \to i_*(L^m) \to 0,$$

where $\mathcal{O} = \mathcal{O}_U$ shows that $\delta(m-1, 1) : H^1(U, I^{-m+1}/\mathcal{O}) \to H^1(U, I^{-m}/\mathcal{O})$ is surjective for m large. Because these are finite-dimensional, $\delta(m, 1)$ is in fact an isomorphism for m large and consequently:

(IA) : $H^0(U, I^{-m}/\mathcal{O}) \to H^0(H, L^m) \to 0$ is exact for m large.

By A 2, $D^1(f_m)$ is injective for all large m and by A 8, $D^1(S_m)=0$ for all m large. This shows that S_m is a power series ring over $\mathbb{C}$, and therefore V is smooth at $p(m\mathrm{H})$. Note that $D^0(S_m) = TV(p(m\mathrm{H}))$ = the tangent-space of V at $p(m\mathrm{H})$.

Proof of II. — By the assumption and (IA) above, the invertible sheaf $I^{-m}/\mathcal{O}$ on $m\mathrm{H}$ has no base-points.

For any $x \in H$, choose a neighbourhood G of $(p(m\mathrm{H}), x)$ in $V \times U$ and a holomorphic function f on G such that $f=0$ defines $W \cap G$ as a complex-analytic subspace of G. The projection $W \to U$ is smooth at $(p(m\mathrm{H}), x) \in W$ if for some $v \in TV(p(m\mathrm{H}))$, the directional derivative $D_v f$ is non-zero at $(p(m\mathrm{H}), x)$. But $d_0 : TV(p(m\mathrm{H})) \to H^0(I^{-m}/\mathcal{O})$ is defined by $d_0(v) = D_v f/f$, and because the complete linear system of $I^{-m}/\mathcal{O}$ has no base-points, the result follows.

Remark. — Because the fibres of $W \to U$ are smooth at all points $(p(m\mathrm{H}), x)$, $x \in H$, for m large, the tangent-spaces $F_m(x)$ to these fibres are hyperplanes in $TV(p(m\mathrm{H}))$, we get a set-function $F_m : H_{red} \to \mathbb{P}(TV(p(m\mathrm{H}))^*)$ and the above actually shows that:

$$F_m = P_m \circ Q_m \qquad \text{with} \quad Q_m : H_{red} \to \mathbb{P}^{N(m)}$$

the morphism given by the complete linear system of $L^m \mid H_{red}$ and P_m is a linear inclusion of projective spaces.

Proof of III. — The b-excellence of $W \to V \times U$ at $p(m\mathrm{H})$ is equivalent, by the above remarks, to the following:

The Q_m-images of any b distinct points of H span a $(b-1)$-dimensional linear subspace, for all m large.

Let $B = \{(x_1, x_2, \ldots, x_b) \in H^b \mid i \neq j \Rightarrow x_i \neq x_j\}$. Then:

$$B_m = \{(x_1, x_2, \ldots, x_b) \in B \mid Q_m(x_i)$$

span a linear subspace of dimension $\leqq b-2\}$ is a closed subset of B. Note that:

(*a*) $B_m \cap B_n \supset B_{m+n}$ if L^m and L^n have no base-points;

(*b*) $\bigcap_m B_m = \emptyset$ by Serre's FAC, by the ampleness of L, from which it follows that $B_m = \emptyset$ for all $m > m_0$.

Actually the smoothness of U was never used. All that matters is that I is an invertible sheaf of ideals on U and $(I/I^2)^*$ is ample on H. The smoothness of V at $p(m\mathrm{H})$ and the smoothness of $\mathrm{P}_{\mathrm{S}}^k \to \mathrm{U}^k$ at all points of $\mathrm{T}(k)$ with the correct fibre-dimensions are proved without assuming that U is a manifold.

APPENDIX 2

We prove Proposition 3.14 here. The main ingredients are semi-continuity, the vanishing theorems of Serre's FAC, and Grothendieck's existence of the Hilbert-scheme. We need first some notation and well-known lemmas.

Let $p : \mathrm{X} \to \mathrm{S}$ be a proper morphism and let F be a coherent sheaf on X. We put $\mathrm{X}_{\mathrm{T}} = \mathrm{X} \times_{\mathrm{S}} \mathrm{T}$ for a morphism $\mathrm{T} \to \mathrm{S}$ and denote the projection by $p_{\mathrm{T}} : \mathrm{X}_{\mathrm{T}} \to \mathrm{T}$. The base-change of F to X_{T} is denoted by F_{T} and we put $(p_{\mathrm{T}})_* \mathrm{F}_{\mathrm{T}} = \mathrm{F}(\mathrm{T})$.

LEMMA 1. — *Given:*

$$\begin{tikzcd} \mathrm{X} \arrow[rr] \arrow[dr, "p"'] & & \mathrm{Y} \arrow[dl, "q"] \\ & \mathrm{S} & \end{tikzcd}$$

with p and q both proper, there is a unique open subscheme S' *of* S *with the property:*

A morphism $\mathrm{T} \to \mathrm{S}$ *factors via* $\mathrm{T} \to \mathrm{S}' \to \mathrm{S}$ *if and only if* $\mathrm{X}_{\mathrm{T}} \to \mathrm{Y}_{\mathrm{T}}$ *is a closed immersion.*

Proof. — Let $\mathrm{A} = \mathrm{S} - p$ (support $\Omega_{\mathrm{X}/\mathrm{Y}}$). Then $\Omega_{\mathrm{X}_{\mathrm{A}}/\mathrm{Y}_{\mathrm{A}}}$ is zero and therefore $\mathrm{B} = \mathrm{X}_{\mathrm{A}} \times_{\mathrm{Y}_{\mathrm{A}}} \mathrm{X}_{\mathrm{A}} - \Delta\, \mathrm{X}_{\mathrm{A}}$ is closed in the fibre-product. If C is the image of B in A, then it is easy to see that $\mathrm{S}' = \mathrm{A} - \mathrm{C}$ has the required property.

LEMMA 2. — *For a proper morphism* $p : \mathrm{X} \to \mathrm{S}$, $\mathcal{O}_{\mathrm{S}} \to p_* \mathcal{O}_{\mathrm{X}}$ *is an isomorphism if and only* $p_* \mathcal{O}_{\mathrm{X}}$ *is an invertible sheaf.*

Proof. — $\mathrm{A} = p_* \mathcal{O}_{\mathrm{X}}$ is a coherent sheaf of algebras on S and therefore $\mathcal{O}_{\mathrm{S}}/\mathrm{J} \to \mathrm{A}/\mathrm{JA}$ is non-zero for every maximal ideal J such that A/JA is non-zero. In particular, when A is invertible, this shows that $\mathcal{O}_{\mathrm{S}} \to \mathrm{A}$ is a surjection, and therefore an isomorphism.

LEMMA 3. — *Let* F *be a coherent sheaf on* X, $p : \mathrm{X} \to \mathrm{S}$ *is proper, and* F *is* S-*flat. The geometric points s of* S *for which* $\Gamma(\mathrm{X}_s, \mathrm{F}_s)$ *is a vector-space of rank r are the geometric points of a locally closed subscheme* $\mathrm{S}(r)$ *of* S. *Moreover a morphism* $\mathrm{T} \to \mathrm{S}$ *factors via* $\mathrm{T} \to \mathrm{S}(r) \to \mathrm{S}$ *if and only if:*

(*a*) $\mathrm{F}(\mathrm{T})$ *is locally free of rank r and:*

(*b*) *for every* $g : \mathrm{M} \to \mathrm{T}$, $g^* \mathrm{F}(\mathrm{T}) \to \mathrm{F}(\mathrm{M})$ *is an isomorphism.*

Proof. — By the semi-continuity lemma (*see* Lemma 1, page 47, [M2]) there is an affine open cover $\mathrm{U}_\alpha = \operatorname{Spec} \mathrm{A}_\alpha$ of S and a non-negative complex of free A_α-modules $\mathrm{F}_\alpha^{\bullet}$ such that for every $\mathrm{T} = \operatorname{Spec} \mathrm{B} \to \operatorname{Spec} \mathrm{A}_\alpha$:

$$\mathrm{R}^i (p_{\mathrm{T}})_* \mathrm{F}_{\mathrm{T}} \cong \mathrm{H}^i(\mathrm{B} \otimes_{\mathrm{A}_\alpha} \mathrm{F}_\alpha^{\bullet}).$$

In view of this, put $V_\alpha(t)$ = the closed subscheme of U_α given by the ideal generated by the $(t\times t)$-minors of the matrix $F_\alpha^0 \to F_\alpha^1$, and let $S_\alpha(r)$ be the locally closed subscheme of U_α given by $V_\alpha(m_\alpha - r + 1) - V_\alpha(m_\alpha - r)$ where m_α is the rank of F_α^0. Then $S_\alpha(r) \cap U_\beta = S_\beta(r) \cap U_\alpha$ and the union of the $S_\alpha(r)$ is the desired locally closed subscheme $S(r)$, as is checked from the defining property of the complex $F_\alpha^\bullet$.

LEMMA 4. — *Let $p : X \to S$ be proper and flat. With $F = \mathcal{O}_X$ in Lemma* 3, $S(1)$ *is an open subscheme of* S.

Proof. — Because $p(X)$ is an open subscheme of S by the flatness of p, we may replace S by $p(X)$ and assume that p is surjective.

Now $\mathcal{O}_S \to p_* \mathcal{O}_X$ induces an augmentation:

$$A_\alpha \xrightarrow{\varepsilon} F_\alpha^0 \to F_\alpha^1 \dots,$$

with the notation of the proof of the previous lemma. Also for any closed point s of S given by a maximal ideal J, the composite $\mathcal{O}_S/J \to p_* \mathcal{O}_X / J(p_* \mathcal{O}_X) \to (p_s)_* \mathcal{O}_{X_s}$ is injective by the surjectivity of p, and therefore $A_\alpha / JA_\alpha \to F_\alpha^0 / JF_\alpha^0$ is injective for all maximal ideals J. Thus there is $\varphi : F_\alpha^0 \to A_\alpha$ such that $\varphi . \varepsilon$ is the identity. This shows that all the $(m_\alpha \times m_\alpha)$-minors of $F_\alpha^0 \to F_\alpha^1$ are indeed zero where $m_\alpha = rk(F_\alpha^0)$ and therefore $S_\alpha(1) = U_\alpha - V_\alpha(m_\alpha - 1)$ is open in U_α (*see* proof of the previous lemma), thus completing the proof of Lemma 4.

LEMMA 5. — *Let* L *be an invertible sheaf on* X, *and $p : X \to S$ is proper and flat with no geometric fibre of p having non-constant global sections of its structure sheaf.*

Then the geometric points s of S for which $L \mid X_s$ is trivial are the geometric points of a locally closed subscheme S' of S. Moreover a morphism $T \to S$ factors via $T \to S' \to S$ if and only if $p_T^* L(T) \to L_T$ is an isomorphism. Recall that $L(T) = (p_T)_* L_T$.

Proof. — The given hypothesis and Lemma 4 together imply that $p_*(\mathcal{O}_X)$ is invertible. By Lemma 2, $\mathcal{O}_S \to p_*(\mathcal{O}_X)$ is an isomorphism. The same situation prevails even after base-change, i. e. $\mathcal{O}_T \to (p_T)_* \mathcal{O}_{X_T}$ is an isomorphism for any $T \to S$.

For $r = 1$ and $F = L$ in Lemma 3, put $A = S(1)$.

Let $T \to S$ be such that $p_T^* L(T) \to L_T$ is an isomorphism. Thus $L(T)$ is invertible because p_T is flat and surjective. Given $g : M \to T$, denote by h the morphism $X_M \to X_T$. Then $p_M^* g^* L(T) = h^* p_T^* L(T) \to h^* L_T = L_M$ is an isomorphism. Taking p_M-direct images and recalling that $\mathcal{O}_M \to (p_M)_* \mathcal{O}_{X_M}$ is an isomorphism, we get the isomorphism $g^* L(T) \to L(M)$. By the definition of A, there is a factoring $T \to A \to S$. Also the image of $X_T \to X_A$ is disjoint from G, the support of the cokernel of $p_A^* L(A) \to L_A$. Putting $S' = A - p_A(G)$, thus we have $T \to S' \to S$.

The converse is clear, because by very construction, $p_{S'}^* L(S') \to L_{S'}$ is an isomorphism, and this holds even after a base-change by $T \to S'$.

Remark. — Unlike the See-saw Theorem, page 54 of Mumford's book [M2], S' is not a closed subscheme in general.

For example, if S is the complete linear system of (1.1) curves in $\mathbb{P}^1 \times \mathbb{P}^1$ and $L = p_1^* \mathcal{O}(1) \otimes p_2^* \mathcal{O}(-1)$, then S′ is open in S: the smooth curves are in S′ and the singular ones (the pairs of lines) are not in S′.

We now come to Proposition 3.14. Let $f_0 : P_0 \to Q$ be a finite morphism of projective schemes and assume that $\Gamma(P_0, \mathcal{O})$ is the base-field. Fix once and for all an ample line bundle E on Q and a projective embedding $h_0 : P_0 \to \mathbb{P}^N$ such that:

(*a*) $h_0^* \mathcal{O}(1) \cong f_2^* E$ and:

(*b*) $H^1(P_0, f_0^* E) = 0$.

Fix further a polarisation of $Q \times \mathbb{P}^N$ and denote by Hilb the Hilbert-scheme of all closed subschemes of $Q \times \mathbb{P}^N$ having the same Hilbert polynomial as that of the closed immersion $f_0 \times h_0 : P_0 \to Q \times \mathbb{P}^N$. This gives a special k-rational point t_0 of Hilb. Letting X be the universal closed subscheme of Hilb $\times\, Q \times \mathbb{P}^N$ and applying Lemmas 4 and 5 to X → Hilb, we get a locally closed subscheme T of Hilb (containing t_0) such that the geometric points of T are precisely the closed subschemes A of $Q \times \mathbb{P}^N$ such that $\Gamma(A, \mathcal{O}_A) = \text{Const.}$ and $p_1^*(E) \otimes p_2^* \mathcal{O}(-1)$ restricted to A is trivial. By the same lemmas we know exactly when a S-valued point of Hilb is S-valued point of T.

Let $Y \to T \times Q \times \mathbb{P}^N$ be the universal closed subscheme. The projection $Y \to T \times Q$ gives a deformation of $f_0 : P_0 \to Q$ with parameter-space $= (T, t_0)$ in the sense of 3.2 and gives rise to a member θ of $\mathbb{F}(T, t_0)$. Recall that $\mathbb{F}(S, s_0)$ is the set of deformations of $f_0 : P_0 \to Q$ parametrized by (S, s_0).

LEMMA 6. — *Given* ψ *in* $\mathbb{F}(S, s_0)$,*there is an open subscheme* U *of* S *containing* s_0 *and a morphism:*

$g : (U, s_0) \to (T, t_0)$ *such that* $\mathbb{F}(\bar{g})\theta = \mathbb{F}(j)\psi$ *where:*

$j : U \to S$ *is the inclusion morphism.*

Proof. — Let $f : P \to S \times Q$ be the total-space of the deformation ψ. Because $H^1(P_0, f_0^* E) = 0$, *by semi-continuity* (*see* Theorem 3, page 53, [M2]), there is an affine neighbourhood G of s_0 in S such that $Z = (p_1 \circ f)_* (p_2 \circ f)^* E$ restricted to G is locally free and commutes with base-change. Thus if J is the maximal ideal of s_0 in S, $Z/JZ \cong \Gamma(P_0, f_0^* E)$. The embedding $h_0 : P_0 \to \mathbb{P}^N$ is determined by $\varphi_0 : k^{N+1} \to \Gamma(P_0, f_0^* E)$, and φ_0 can be extended to $\varphi : \mathcal{O}_G^{N+1} \to Z \mid G$. This in turn gives a rational map $(p_1 \circ f)^{-1} G \to G \times \mathbb{P}^N$, but the base-locus does not intersect $(p_1 \circ f)^{-1} s_0$ and therefore (replacing G by a smaller neighbourhood of s_0) we may assume that it is empty. Thus we get a morphism $(p_1 \circ f)^{-1} G \to G \times \mathbb{P}^N$. By Lemma 1 (and replacing G by...) we may assume that $(p_1 \circ f)^{-1} G \to G \times Q \times \mathbb{P}^N$is a closed immersion. By Lemma 4, we may assume that $\Gamma(A, \mathcal{O}_A) = k(s)$ for all geometric points s of S with $A = (p_1 \circ f)^{-1} s$. Also we may assume that G is connected so that the Hilbert polynomial is constant. By the definition of T, we get required morphism $(G, s_0) \to (T, t_0)$. Put U = G.

PROPOSITION 3.14. — *With the notation of the previous lemma, assume further that there is a closed subscheme* S_1 *of* S *containing* s_0, *the inclusion being denoted by* $i : S_1 \to S$, *and a morphism* $g : (S_1, s_0) \to (T, t_0)$ *such that* $\mathbb{F}(g)\theta = \mathbb{F}(i)\psi$.

Then there is an open subscheme S_2 of S containing s_0, the inclusion being denoted by $j : S_2 \to S$ and $g : (S_2, s_0) \to (T, t_0)$ such that $\bar{g} \mid S_1 \cap S_2 = g \mid S_1 \cap S_2$ and $\mathbb{F}(g)\theta = \mathbb{F}(j)\psi$.

The proof of this is a minor modification of the previous one. We may assume that S is affine and that $Z = (p_1 \circ f)_* (p_2 \circ f)^* E$ is locally free and commutes with base-change. Now $g : S_1 \to T$ gives an embedding $(p_1 \circ f)^{-1} S_1 \to S_1 \times Q \times \mathbb{P}^N$ and the projection to $\mathbb{P}^N$ is determined by $\varphi_0 : \mathcal{O}_{S_1}^{N+1} \to Z \mid S_1$. This can be extended again to $\varphi : \mathcal{O}_S^{N+1} \to Z$, and the rest of the argument goes through, word for word.

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