ANNALES SCIENTIFIQUES DE L'É.N.S.

FRANCIS BONAHON

Cobordism of automorphisms of surfaces

Annales scientifiques de l'É.N.S. 4e série, tome 16, nº 2 (1983), p. 237-270

<http://www.numdam.org/item?id=ASENS_1983_4_16_2_237_0>

Ann. Sc. Éc. Norm. sup.
4e série, t. 16, 1983, p. 237 à 270.

COBORDISM OF AUTOMORPHISMS OF SURFACES

By Francis BONAHON

Topology is rich in problems whose solutions differ dramatically in high and low dimensions. One of them is the cobordism of automorphisms of manifolds : The cobordism group of orientation-preserving diffeomorphisms of closed oriented smooth n-manifolds has been computed for $n \geqq 4$ by M. Kreck ([Kr_1], [Kr_2]), and P. Melvin [Me] proved that his results extend to the case where $n=3$. When $n=2$, Kreck's invariants are still defined, but they are known ([Ca], [JJ]) to be insufficient to determine this cobordism group. We study here this last case and compute the cobordism group of diffeomorphisms of surfaces.

More precisely, in a "geometric" category CAT (= TOP, PL, DIFF, ...), let us consider CAT-automorphisms $f: F^n \to F^n$ of closed oriented n-dimensional CAT-manifolds; we do not require f to be orientation-preserving and abbreviate the notation $f: F^n \to F^n$ into (F^n, f). Two such automorphisms (F_1^n, f_1) and (F_2^n, f_2) are *cobordant* when there exists a CAT-automorphism $(M^{n+1}, \hat{f})$ of a compact oriented $(n+1)$-dimensional manifold with $\partial M^{n+1} = F_1^n \amalg (-F_2^n)$ and $\partial \hat{f} = f_1 \amalg f_2$. The cobordism classes so defined form a group $\Delta_n(\mathrm{CAT})$, where the group law is induced by disjoint sum $\amalg$. The group $\Delta_n(\mathrm{CAT})$ contains a natural subgroup $\Delta_{n+}(\mathrm{CAT})$, consisting of those cobordism classes that are represented by orientation-preserving automorphisms.

In [Kr_1] and [Kr_2], M. Kreck computes $\Delta_{n+}(\mathrm{DIFF})$, for $n \geqq 4$, in terms of ordinary cobordism groups of oriented manifolds and, when n is even, of the Witt group $W_\varepsilon(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}^\infty \oplus (\mathbb{Z}/2)^\infty \oplus (\mathbb{Z}/4)^\infty$ of isometries of free finite-dimensional $\mathbb{Z}$-modules equipped with an ε-symmetric unimodular bilinear form, where $\varepsilon = (-1)^{1/2\, n}$. P. Melvin proved [Me] that the same formulas remain valid when $n=3$, where $\Delta_{3+}(\mathrm{DIFF}) = 0$. For $n=2$, there subsists from Kreck's invariants an epimorphism $\Delta_{2+}(\mathrm{DIFF}) \to W_{-1}(\mathbb{Z}, \mathbb{Z})$, defined by considering for every automorphism (F^2, f) the induced automorphism f_* of $H_1(F)$, equipped with the intersection pairing; but A. Casson [Ca], K. Johannson and Dennis Johnson [JJ] have independently shown that this morphism is not injective.

Omitting any reference to any category since smoothing theory and the Hauptvermutung in dimensions 2 and 3 show that $\Delta_2(\mathrm{TOP}) \cong \Delta_2(\mathrm{PL}) \cong \Delta_2(\mathrm{DIFF})$, we extend here some partial results of M. Scharlemann [Sc] and prove :

Theorem :

$$\Delta_2 \cong \mathbb{Z}^\infty \oplus (\mathbb{Z}/2)^\infty, \qquad \Delta_{2+} \cong \mathbb{Z}^\infty \oplus (\mathbb{Z}/2)^\infty,$$

(*where* A^∞ *denotes the direct sum of countably many copies of* A; *the group* $\Delta_n(\mathrm{PL})$ *has clearly at most countably many elements*).

As a consequence, every element of Δ_{2+} with Kreck's invariants of order 4 has infinite order in Δ_2.

In fact, we obtain some more information on the structure of Δ_2. Let Δ_2^{P} denote the group of periodic (PL or DIFF) automorphisms of surfaces modulo cobordism by periodic automorphisms of 3-manifolds. We introduce in paragraph 6 a certain set $\mathscr{A}$ of automorphisms of surfaces, defined modulo isotopy and oriented conjugacy (the strongest equivalence relation that can reasonably be considered for the study of cobordism), for which the following holds.

THEOREM. — *The canonical map* $\Delta_2^{\mathrm{P}} \times \mathscr{A} \to \Delta_2$ *is bijective, and is an isomorphism for a suitable group structure on* $\mathscr{A}$.

The group structure of $\mathscr{A}$ is simple enough to be easily analized (§ 7) and the group Δ_2^{P} is completely determined by considerations on the fixed point set of the periodic automorphisms (§ 8). Both groups turn out to be isomorphic to $\mathbb{Z}^\infty \oplus (\mathbb{Z}/2)^\infty$, whence the computation of Δ_2 follows.

The proofs are geometric and differ completely from those in higher dimensions (in particular, there is no intervention of Kreck's invariants). The basic idea consists in modifying any null-cobordism to get a new one in a simple form (§ 5). For this purpose, we use a few tools provided by the theory of the geometric splittings of Haken 3-manifolds, namely Thurston's hyperbolization theorem, the characteristic fibered submanifold of Johannson-Jaco-Shalen (*see* § 3) and another (simpler) characteristic submanifold whose theory is developped in paragraph 2.

The results in this paper were announced in [Bo] (with a few mistakes in the algebraic computations). At the same time, A. Edmonds and J. Ewing informed us that they had obtained similar results by slightly different methods, using in particular the G-signature theorem instead of hyperbolic geometry to prove the injectivity of the map $\Delta_2^{\mathrm{P}} \to \Delta_2$ [EE].

Most of this work was carried out while I was visiting Princeton University; I would like to thank here all the members of the Department of Mathematics and especially W. P. Thurston, for their kind hospitality. I am also very indebted to L. Siebenmann for his contribution by numerous advices to the improvement of the results, the proofs and the manuscript. Lastly, I would like to thank R. Penner for carefully reading a first version of this paper.

0. Main definitions and conventions

We shall work exclusively in the category PL (=piecewise linear). Nevertheless, the proofs could easily be translated to the categories DIFF (=differentiable C^∞) or TOP (=topological); in this last case, however, periodic maps should be assumed to have a tame fixed point set.

All manifolds will be compact and orientable. This rule admits a unique exception, almost always explicitly specified when needed, for non-orientable compact surfaces occuring as bases of fibrations or quotient spaces of finite group actions on (orientable) surfaces.

An exponent often indicates the dimension of the manifolds considered. But let the reader be warned that, except for this extra information on the dimension, no difference has to be made between the notations M^n and M. However, this exponent is never omitted for traditional notations of some classical manifolds, such as the 2-sphere S^2, the 2-torus T^2, the projective plane $\mathbb{RP}^2$, the hyperbolic plane $\mathbb{H}^2$, etc.

When the opposite is not explicitly specified, every submanifold $N \subset M$ of positive codimension is assumed to be *properly embedded*, i. e. such that $N \cap \partial M = \partial N$. For a codimension 0 submanifold, it is required that its frontier $\delta N = \overline{\partial N - \partial M}$ be a codimension 1 properly embedded submanifold of M.

A 1-submanifold C^1 of a surface F^2 is *essential* when, for every base point, the homomorphisms $\pi_1(C) \to \pi_1(F)$ and $\pi_1(C, \partial C) \to \pi_1(F, \partial F)$ are injective. Equivalently, C is essential when there does not exist any disc $D^2 \subset F$ with $\delta D = \overline{\partial D - \partial F}$ a component of C (with or without boundary).

A compression disc for $F^2 \subset M^3$ is a disc $D^2 \subset M^3$ with $D \cap F = \partial D$; note that D is not properly embedded in M. Such a compression disc is *effective* when ∂D is essential in F.

A surface $F^2 \subset M^3$ is *incompressible* when:

(1) F admits no effective compression disc.

(2) No component of F is a sphere bounding a ball.

Similarly, a *∂-compression disc* for $F^2 \subset M^3$ is a disc $D^2 \subset M^3$, not properly embedded, such that $D \cap F$ is an arc contained in ∂D and $\overline{\partial D - F} = D \cap \partial M$. Again, D is *effective* when $D \cap F$ is essential in F.

A surface $F^2 \subset M^3$ is *boundary incompressible*, or *∂-incompressible*, when :

(1) F does not admit any effective ∂-compression disc.

(2) No component of F is a boundary parallel disc, i. e. there does not exist any ball $B^3 \subset M^3$ with δB a disc component of F^2.

The surface $F^2 \subset M^3$ is *essential* when it is both incompressible and ∂-incompressible.

Two closed surfaces F^2 and $G^2 \subset M^3$ are *parallel* when they are disjoint and separated by a collar $\cong F \times I$. This definition extends straightforwardly when F or G consists of components of ∂M.

The manifold M^3 is *irreducible* when it does not contain any incompressible sphere, i. e. when every sphere $\Sigma^2 \subset M^3$ bounds a ball in M^3. It is *boundary-irreducible* or *∂-irreducible* when ∂M is incompressible (extending the definition of incompressibility to boundary surfaces), i. e. when no component of M is a ball and there does not exist any disc $D^2 \subset M^3$ with ∂D essential in ∂M.

Lastlv, we often make use of the following construction: Given a codimension 1 submanifold $N^n \subset M^{n+1}$, compactify $M - N$ by adjunction of a copy of the normal S^0-bundle of N in M, with the obvious topology. The new compact manifold so constructed is said to be obtained by *splitting* M *along* N, *or by cutting* M *open along* N.

1. $\Delta_n(\mathrm{CAT})$ as a graded group

The key result of this paper will be Theorem 6.1, where we shall obtain a natural splitting of Δ_2, by geometric methods that are peculiar to the dimension considered. However, easy connectivity considerations already provide a decomposition of $\Delta_n(\mathrm{CAT})$ into a direct sum of "smaller" cobordism groups. This section is devoted to this last decomposition. The corresponding results will have no effect on the geometrical part of our study of Δ_2, and will not be used until we resume algebraic computations in paragraphs 7-8.

For notational convenience, we agree to omit any explicit reference to a category CAT (assuming a choice fixed for the whole section) and will henceforth abbreviate $\Delta_n(\mathrm{CAT})$ by Δ_n.

Consider an automorphism f of an oriented manifold F^n. To characterize the action of f on the components of F, we construct a weighted graph $\tilde{\gamma}(\mathrm{F}, f)$ in the following way: The vertices of $\tilde{\gamma}(\mathrm{F}, f)$ correspond to the components of F; an oriented edge joins the vertex associated to F_1 to the vertex associated to $f(\mathrm{F}_1)$ and this edge is weighted by the symbol $+$ or $-$ according as $f \mid \mathrm{F}_1$ preserves or reverses the orientations induced by F. Note that each component of $\tilde{\gamma}(\mathrm{F}, f)$ is homeomorphic to S^1 as a topological space and is coherently oriented by the orientations of its edges; call such a graph (homeomorphic to S^1, coherently oriented and with a weight $+$ or $-$ on each edge) a *weighted closed chain*.

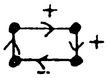
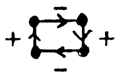
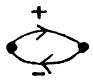
If there exists an automorphism $(\mathrm{M}^{n+1}, \hat{f})$ such that $\mathrm{F}^n \subset \partial \mathrm{M}^{n+1}$ and $f = \hat{f} \mid \mathrm{F}$, the natural map $\tilde{\gamma}(\mathrm{F}, f) \to \tilde{\gamma}(\mathrm{M}, \hat{f})$ is, above its image, a covering map respecting the orientations and the weights of the edges. It is therefore natural to identify two weighted closed chains $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ when there exists a covering map $\tilde{\gamma}_1 \to \tilde{\gamma}_2$ respecting the orientations and weights of the edges; let Γ denote the quotient of the set of weighted closed chains by the equivalence relation generated by these identifications, i. e. the equivalence relation defined by the property that $\tilde{\gamma} \sim \tilde{\gamma}'$ when they are joined by a sequence of weighted closed chains and covering maps (respecting weights and orientations) such as:

$$\tilde{\gamma} \searrow \tilde{\gamma}_1 \swarrow \tilde{\gamma}_2 \searrow \tilde{\gamma}_3 \swarrow \cdots \swarrow \tilde{\gamma}_{n-1} \searrow \tilde{\gamma}_n \swarrow \tilde{\gamma}'$$

Every automorphism (F^n, f) naturally splits into $\coprod_{\gamma \in \Gamma} (\mathrm{F}^n_\gamma, f_\gamma)$, where each component of $\tilde{\gamma}(\mathrm{F}_\gamma, f_\gamma)$ is in the class $\gamma \in \Gamma$. For every $\gamma \in \Gamma$, let $\Delta_{n\gamma}$ be the subgroup of those cobordism classes in Δ_n that are represented by automorphisms (F^n, f) where every component of $\tilde{\gamma}(\mathrm{F}, f)$ is in $\gamma \in \Gamma$. The definitions are designed so that, for the natural map:

THEOREM 1.1:

$$\Delta_n \cong \bigoplus_{\gamma \in \Gamma} \Delta_{n\gamma}. \quad \square$$

Call a weighted closed chain *primitive* when it is not a covering (respecting weights and orientations) of another chain. For instance,  is primitive, and  is not since it is a covering of 

LEMMA 1.2. — *Each class in* Γ *contains a unique primitive weighted closed chain.*

It will often be convenient to identify an element of Γ with a primitive weighted closed chain.

Proof of Lemma 1.2. — It is sufficient to prove that every weighted closed chain $\tilde{\gamma}$ is the covering of a unique primitive weighted closed chain γ. But such a γ is naturally identified with the quotient of $\tilde{\gamma}$ by its (cyclic) symmetry group. $\square$

The two simplest (primitive) weighted closed chains are  and , respectively denoted by $+$ and $-$. Note that f preserves (resp. reverses) the orientation of F precisely when $\tilde{\gamma}(\mathrm{F}, f) \in + \in \Gamma$ (resp. $\tilde{\gamma}(\mathrm{F}, f) \in - \in \mathrm{I}$), and that these notations are compatible with the definition of Δ_{n+} in the introduction.

If $\gamma \in \Gamma$ is considered as a primitive weighted closed chain, let $\nu(\gamma)$ denote its number of vertices and let its signature $\sigma(\gamma) \in \mathbb{Z}/2\mathbb{Z}$ be the number of its edges that are weighted by $-$ (mod 2).

Choose a vertex v of γ and consider an automorphism (F^n, f) such that each component of $\tilde{\gamma}(\mathrm{F}^n, f)$ is in the class $\gamma \in \Gamma$. There exists a covering $\tilde{\gamma}(\mathrm{F}, f) \to \gamma$, which induces a projection from F to the 0-skeleton of γ; let G^n be the inverse image of v by this projection and let g be $f^{\nu(\gamma)} \mid \mathrm{G}$. Then, up to (oriented) conjugacy, F splits into the disjoint union of $\nu(\gamma)$ copies of G, suitably oriented, where f sends the i-th copy to the $(i+1)$-th copy by the identity $(1 \leqq i < \nu(\gamma))$, and the last copy to the first one by g. Moreover, the orientations of the copies of G are determined by the weights of the edges of γ and g is orientation-preserving (resp. -reversing) if $\sigma(\gamma)=0$ (resp. 1). Conversely, such a (G^n, g), with g orientation-preserving or -reversing according as $\sigma(\gamma)=0$ or 1, is associated to a unique (F^n, f), up to oriented conjugacy, where each component of $\tilde{\gamma}(\mathrm{F}, f)$ is in $\gamma \in \Gamma$.

This proves :

THEOREM 1.3. — *The group* $\Delta_{n\gamma}$ *is isomorphic to* Δ_{n+} *when* $\sigma(\gamma)=0$ *and to* Δ_{n-} *when* $\sigma(\gamma)=1$. $\square$

Remark. — For a different choice of v in the above construction, the isomorphism $\Delta_{n\gamma} \cong \Delta_{n+}$ or Δ_{n-} is just changed by composition with $\mathrm{X} \mapsto \pm \mathrm{X}$. When $\sigma(\gamma)=1$, this isomorphism is even quite canonical since the orientation-reversing automorphism g realizes a conjugacy between (G, g) and $(-\mathrm{G}, g)$.

By Theorem 1.1 and 1.3, Δ_n is the direct sum of infinitely many copies of Δ_{n+} and Δ_{n-}. If we want to enumerate all these copies, or equivalently the elements of Γ, it is useful to know for every m the number $c_+(m)$ [resp. $c_-(m)$] of primitive weighted closed chains γ for which $\nu(\gamma)=m$ and $\sigma(\gamma)=0$ (resp. 1). Let $c(m)$ be $c_+(m)+c_-(m)$.

LEMMA 1.4. — *The numbers* $c(m)$ *and* $c_-(m)$ *are respectively determined by the induction relations :*

$$\sum_{d|m} dc(d) = 2^m,$$

$$\sum_{d|l} 2^k dc_-(2^k d) = 2^{m-1}, \text{ where } m = 2^k l \text{ with } l \text{ odd.}$$

Proof. — Just note that $mc(m)$ [resp. $mc_-(m)$] is the number of maps $\{1, 2, \ldots, m\} \to \{+, -\}$ without any period (resp. and with signature 1). □

Applying the Möbius inversion formula (*see* for instance [HW], § 16) to the expressions of 1.4, we can get explicit formulas for $c(m)$ and $c_-(m)$. Recall that the Möbius function $\mu : \mathbb{N}^* \to \{-1, 0, 1\}$ is defined by the property that $\mu(m) = 0$ when m is a multiple of a square and $\mu(p_1 p_2 \ldots p_r) = (-1)^r$ when the primes $p_1, p_2, \ldots, p_r$ are distinct $(\mu(1) = (-1)^0 = 1)$.

COROLLARY 1.5.

$$c(m) = \frac{1}{m} \sum_{d|m} \mu(d) 2^{m/d},$$

$$c_-(m) = \frac{1}{2m} \sum_{d|l} \mu(d) 2^{m/d}, \text{ where } m = 2^k l \text{ with } l \text{ odd.} \quad \square$$

PROPOSITION 1.6. — *If* (F^n, f) *represents* $X \in \Delta_n$, *the minimum number of components of* F *is* $\bigoplus_{p_\gamma(X) \neq 0} \nu(\gamma)$, *where* $p_\gamma : \Delta_n \to \Delta_{n\gamma}$ *is the projection defined by Theorem* 1.1.

Proof. — The automorphism (F, f) naturally splits into $\coprod_\gamma (F_\gamma, f_\gamma)$, where each component of $\tilde{\gamma}(F_\gamma, f_\gamma)$ is in the class $\gamma \in \Gamma$. Now, the number of components of F_γ is a multiple of $\nu(\gamma)$ and is non-zero when $p_\gamma(X) \neq 0$. The above expression consequently provides a lower bound for the number of components of F.

To check that this lower bound is actually a minimum, assume, without loss of generality, that $F_\gamma = \emptyset$ when $p_\gamma(X) = 0$. If $F_\gamma \neq \emptyset$, consider (G_γ, g_γ) associated to (F_γ, f_γ) as in the proof of 1.3 and select in G_γ a finite number of disjoint pairs of points $\{x_i', x_i''\}$ such that, after isotopy, g_γ acts on these pairs by permutation (another property will be required for these pairs later).

Let N^{n+1} be the manifold obtained from $G_\gamma^n \times I$ by glueing a 1-handle along each pair $\{x_i', x_i''\} \times \{1\}$. The automorphism $g_\gamma \times I$ of $G_\gamma \times I$ extends to an automorphism $\hat{g}_\gamma$ of N_γ^{n+1} (here we use the fact that g_γ preserves or reverses the orientation of G_γ). Identifying (G_γ, g_γ) with $(-G_\gamma \times \{0\}, \hat{g}_\gamma | G_\gamma \times \{0\})$, let G_γ' be $\partial N_\gamma - G_\gamma$ and g_γ' be $\hat{g}_\gamma | G_\gamma'$. By construction, (G_γ', g_γ') is cobordant to (G_γ, g_γ).

Let (F_γ', f_γ'), with the property that each component of $\tilde{\gamma}(F_\gamma', f_\gamma')$ is in $\gamma \in \Gamma$, be associated to (G_γ', g_γ') as in the proof of 1.3. By 1.1, (F_γ', f_γ') is cobordant to (F_γ, f_γ). Moreover, for a good choice of the pairs $\{x_i', x_i''\}$, G_γ' is connected and F_γ' consequently has exactly $\nu(\gamma)$ components.

When $p_\gamma(X)=0$, define F'_γ to be $\emptyset$. We have now constructed an automorphism $(F', f')=\coprod_\gamma (F'_\gamma, f'_\gamma)$ representing X such that the number of components of (F'_γ, f'_γ) is 0 if $p_\gamma(X)=0$ and $\nu(\gamma)$ otherwise. This ends the proof. □

2. The characteristic compression body

Let a *compression body* be any 3-manifold V^3, together with a partition $\partial V=\partial_e V \amalg \partial_i V$ of the components of its boundary into an "exterior" and "interior" part, such that no component of the interior part $\partial_i V$ is a sphere and the triad $(V; \partial_e V, \partial_i V)$ admits a handle decomposition with only handles of index 2 and 3. When F^2 is a closed surface, a *compression body for* F is, by definition, a compression body V for which $\partial_e V=F$.

Compression bodies occur naturally in the following fundamental example: In an irreducible manifold M^3, let $D^2 \subset M$ be a collection of disjoint compression discs for ∂M. If V is the union of a regular neighborhood U of $D \cup \partial M$ and of all the components of $\overline{M-U}$ that are balls, then V is a compression body for ∂M; indeed, $(V; \partial M, \partial V-\partial M)$ clearly admits the required handle decomposition, and no component of $\partial V-\partial M$ is a sphere by irreducibility of M. This is the example that justifies the terminology for the exterior and interior boundaries $\partial_e V$ and $\partial_i V$.

Note that a handlebody (i. e. a "pretzel") is just a connected compression body with empty interior boundary. As a matter of fact, the behaviour of compression bodies is very similar to that of handlebodies, in that sense that many properties of handlebodies extend naturally to compression bodies (*see* Appendix B).

THEOREM 2.1. — *Let* M^3 *be irreducible. There exists a compression body* $V^3 \subset M$ *for* ∂M, *unique up to isotopy, such that* $M-V$ *is* ∂*-irreducible* (*and irreducible*).

Remarks. — (1) From the uniqueness of V, it follows that every automorphism of M preserves V after isotopy. For this reason, in later sections, we shall call V the characteristic compression body for ∂M in M, or simply the *characteristic compression body of* M (recall that, in a group, for instance, a subgroup is characteristic when it is preserved by every automorphism of the group).

(2) Theorem 2.1 does not assert that the decomposition of $(V; \partial M, \delta V)$ into 2- and 3-handles is unique up to isotopy, or that one such decomposition is preserved by every automorphism of M. As a matter of fact, these properties notoriously fail when $M=V$ is a handlebody.

(3) The manifold $\overline{M-V}$ is obviously irreducible since M is irreducible and every component of V contains a component of ∂M.

To clarify the notion of characteristic compression body, we give some equivalent definitions before proving Theorem 2.1.

PROPOSITION 2.2. — *Let* M^3 *be irreducible and let* $V^3 \subset M$ *be a compression body for* ∂M. *The following conditions are equivalent :*

(*a*) $\overline{M-V}$ *is* ∂*-irreducible.*

(*b*) *The frontier* δV *is incompressible in* M.

(*c*) *Every surface in* M *that consists of discs can be isotoped inside* V.

(*d*) *Every compression body* $V' \subset M$ *for* ∂M *can be isotoped inside* V (*i. e.* V *is "universal"*).

Proof of $(a) \Leftrightarrow (b) \Rightarrow (c) \Leftrightarrow (d)$. — We delay the proof of $(d) \Rightarrow (b)$ until the end of that of Theorem 2.1.

The equivalence $(a) \Leftrightarrow (b)$ follows from the fact that $\delta V = \partial_i V$ is incompressible in V, which is easy to check [for every base point, the map $\pi_1(\partial_i V) \to \pi_1(V)$ is injective].

To prove $(b) \Rightarrow (c)$, suppose δV incompressible and let D be a surface in M whose components are discs. After isotopy, we can assume that the intersection of D and δV is transverse, and that the number of components of $D \cap \delta V$ cannot be reduced by any such isotopy. If $D \cap \delta V = \emptyset$, then $D \subset V$ (since $\partial D \subset \partial M \subset V$) and the conclusion sought holds. Otherwise, there exists an innermost disc $D' \subset D$ such that $D' \cap \delta V = \partial D'$. The curve $\partial D'$ bounds a disc D'' in the incompressible surface δV, and the sphere $D' \cup D''$ bounds a ball B^3 in the irreducible manifold M. But we should then be able to define, by "crushing" B, an isotopy of D that decreases the number of components of $D \cap \delta V$, which would contradict our hypothesis. The case $D \cap \delta V \neq \emptyset$ cannot therefore occur, and this ends the proof of $(b) \Rightarrow (c)$.

Any surface D in M whose components are discs is contained in a compression body $V' \subset M$ for ∂M (*see* the fundamental example at the beginning of this section). It follows that $(d) \Rightarrow (c)$.

To show that $(c) \Rightarrow (d)$, consider a compression body $V' \subset M$ for ∂M and assume V satisfies (*c*). There exists disjoint balls B_i inside V' and a disjoint union D of discs in V' such that $\partial D \subset \partial M$ and $\overline{V' - (\bigcup_i B_i)}$ is a regular neighborhood of $D \cup \partial M$: For some decomposition of $(V'; \partial M, \delta V')$ into handles of index 2 and 3, let the B_i's be the 3-handles and D consist of the cores of the 2-handles (extended to ∂M). By (*c*), D can be isotoped inside V and, after isotopy, $\overline{V' - (\bigcup_i B_i)}$ can therefore be assumed to be contained inside V. In particular, each sphere ∂B_i is now in V. From the irreducibility of V (which we immediately prove in Lemma 2.3 below) follows that the B_i's lie in V. Consequentely $V' \subset V$. □

LEMMA 2.3. — *Let* V *be a compression body. Then* V *is irreducible and every closed connected incompressible surface* F *in* V *is parallel to a component of* $\partial_i V$.

Proof of Lemma 2.3. — There exists a surface D in V, with $\partial D \subset \partial M$, which consists of discs and splits V into a manifold $\tilde{V}$ isomorphic to the disjoint union of $\partial_i V \times I$ and of some balls (the components of D are the cores of the 2-handles for a decomposition of V into handles of index 2 and 3). By definition of compression bodies, no component of $\partial_i V$ is a sphere and $\tilde{V}$ is therefore irreducible (consider its universal covering). The proposition (1.8) of [Wa_1] then implies that V is irreducible.

If F is a closed connected incompressible surface in V, it can be, as in the proof of $(a) \Rightarrow (c)$ in Proposition 2.2, isotoped so that $F \cap D = \emptyset$ (since V is irreducible). Let F still denote its image in $\tilde{V}$. By the classification of incompressible surfaces in $\partial_i V \times I$ ([Wa$_2$], Proposition 3.1), F is parallel to a component of $\partial_i V$ in $\tilde{V}$, and therefore in V. □

Proof of Theorem 2.1. – To establish the existence of a compression body $V \subset M$ for ∂M with $\overline{M-V}$ ∂-irreducible, begin with any compression body $V_0 \subset M$ for ∂M (for instance, the union of a regular neighborhood of ∂M and of the components of M that are balls). If $\overline{M-V_0}$ is not ∂-irreducible, there exists a disc D properly embedded in $\overline{M-V_0}$ such that ∂D does not bound any disc in δV_0. Let then V_1' be a regular neighborhood of $V_0 \cup D$ in M, and let V_1 be the union of V_1' and of all the components of $\overline{M-V_1'}$ that are balls. The triad $(V_1; \partial M, \delta V_1)$ has clearly a handle decomposition with only handles of index 2 and 3, and no component of δV_1 is a sphere (recall that M is irreducible); V_1 is therefore a compression body for ∂M. By the same token, we can define a sequence $V_0 \subset V_1 \subset V_2 \subset \ldots$ of compression bodies for ∂M which stops only when we reach a compression body V_n for ∂M with $\overline{M-V_n}$ ∂-irreducible (and irreducible).

Remark that δV_{i+1} is "simpler" than δV_i in some sense. To make this precise, we use a well-known *complexity* of a closed orientable surface F, namely the ∞-tuple:

$$c(F) = (\ldots, c_g(F), \ldots, c_1(F), c_0(F)) \in \mathbb{N}^{\mathbb{N}},$$

where $c_g(F)$ is the number of components of genus g of F. The complexity $c(F)$ characterizes the topological type of F. Note that, when F is connected, $c(F)$ is just the genus of F for the canonical injection of $\mathbb{N}$ in $\mathbb{N}^{\mathbb{N}}$. We order the complexities by lexicographic order (from left to right). When the two surfaces F and F′ are connected, $c(F) < c(F')$ just means that F has smaller genus than F′.

Now, in the above situation, $c(\partial_i V) < c(\partial_e V)$.

Since the set of complexities (= the set of finite $\mathbb{N}$-valued sequences) is well-ordered, the sequence (V_i) must needs stop, and there exists therefore some n for which $\overline{M-V_n}$ is ∂-irreducible (and irreducible).

To prove the uniqueness, consider two compression bodies V and $V' \subset M$ for ∂M with $\overline{M-V}$ and $\overline{M-V'}$ ∂-irreducible. By condition (d) of Proposition 2.2 [we have proved that $(a) \Rightarrow (d)$], we may assume that $V' \subset \operatorname{int} V$. The surface $\delta V'$ is incompressible in M [since we have proved $(a) \Rightarrow (b)$ in Proposition 2.2], and therefore in V; by Lemma 2.3, each of its components is consequently parallel to a component of $\partial_i V = \delta V$ in V. It follows then from a connectivity argument that $\overline{V-V'} \cong \delta V \times I \cong \delta V' \times I$; the compression bodies V and V′ are then isotopic. □

Proof of Proposition 2.2 (*end*). – We only need to prove that $(d) \Rightarrow (b)$. Consider V satisfying condition (d). We know that there exists in M a characteristic compression body V′ for ∂M which, by condition (d), we can assume contained in int V. We have now two compression bodies V and $V' \subset M$ for ∂M, with $V' \subset \operatorname{int} V$ and $\delta V'$ incompressible in M (and V). Noting that this is exactly the situation we encountered in the proof of the uniqueness of

V′, the same argument as above shows that V and V′ are isotopic. In particular, δV is isotopic to the incompressible surface $\delta V'$, which ends the proof. □

3. Essential annuli and tori

We saw in the last section that there exists in an irreducible manifold M^3 a compact codimension 0 submanifold which "engulfs" by isotopy all the discs in M. A similar engulfing phenomenon occurs for essential tori and annuli, and involves the characteristic submanifold defined by K. Johannson [Jo], W. Jaco and P. Shalen [JS].

PROPOSITION 3.1. — *Let M^3 be irreducible and ∂-irreducible. Then there exists a submanifold W^3, unique up to isotopy and called the characteristic fibered submanifold, for which the following conditions hold.*

(1) *Every component W_1 of W can be equipped with, either a Seifert fibration for which $W_1 \cap \partial M$ is vertical, or an I-bundle structure over a surface (possibly non-orientable or with boundary) for which $W_1 \cap \partial M$ is the total space of the corresponding ∂I-bundle; moreover, the pair $(W_1, W_1 \cap \partial M)$ is never isomorphic to $(T^2 \times I, T^2 \times \{0\})$.*

(2) *The frontier δW is essential and none of its components is parallel to a component of ∂M.*

(3) *For every component M_0 of $M - W$, the union $W \cup M_0$ does not satisfy* (1).

(4) *Every submanifold W′ satisfying* (1) *and* (2) *can be isotoped inside* W. □

PROPOSITION 3.2. — *With the data of Proposition* 3.1, *the characteristic fibered submanifold* W *satisfies also*:

(5) *Every essential annulus or torus which is not parallel to a boundary component can be isotoped inside* W.

(6) *Each component of $\overline{M-W}$ which does not meet $\partial(\delta W)$ and is different from $T^2 \times I$ is atoroidal and anannular.*

(7) *Every automorphism of* M *preserves* W *up to isotopy.* □

Remarks. — (*a*) For property (6), recall a manifold is anannular if it contains no essential annulus and is *atoroidal* if every incompressible torus in it is parallel to a boundary component.

(*b*) Our characteristic fibered submanifold is, for convenience, slightly different from the characteristic submanifold in [Jo] or [JS]: To recover the latter, add to the former regular neighborhoods of the components of ∂M that are tori.

Our interest in trying to confine in some submanifold all the essential discs, tori or annuli of M^3 is motivated by the following corollary of Thurston's Hyperbolization Theorem [Th_3] and of Mostow's Rigidity Theorem ([Mo], [Pr]).

PROPOSITION 3.3 (*Thurston*). — *Let M^3 be irreducible, ∂-irreducible, atoroidal and anannular. If, furthermore, each component of* M *contains an essential surface (which is always satisfied by components with non-empty boundary), then every automorphism g of* M *is isotopic to a periodic automorphism.*

Proof of Proposition 3.3. — Let $\partial_T M$ denote the union of the boundary components of M which are tori. Under the hypotheses of the proposition, Thurston's Theorem asserts that $(M - \partial_T M)$ admits a complete hyperbolic structure with finite volume and totally geodesic boundary. Consider then the double $\tilde{M}$ obtained by glueing two copies of M along $(\partial M - \partial_T M)$. The hyperbolic structure on $(M - \partial_T M)$ defines then a complete hyperbolic structure with finite volume on $(\tilde{M} - \partial_T \tilde{M})$, for which the exchange involution τ is an isometry.

Identify M with one "half" of $\tilde{M}$. The automorphism g lifts to an automorphism $\tilde{g}$ of $\tilde{M}$ which commutes with τ and coincides with g on $M \subset \tilde{M}$. By Mostow's Theorem, $\tilde{g}$ is homotopic to a (unique) automorphism $\tilde{g}'$ that is isometric on $(\tilde{M} - \partial_T \tilde{M})$. The automorphism $\tilde{g}'$ is periodic (the group of isometries of a complete finite volume hyperbolic manifold is finite) and commutes with τ (by uniqueness in Mostow's Theorem). It defines therefore a periodic automorphism g' of M. Moreover, $\tilde{g}$ and $\tilde{g}'$ induce the same outer automorphism on $\pi_1(M) \subset \pi_1(\tilde{M})$ and, by ([Wa_2], § 7), g and g' are therefore isotopic. □

4. Two lemmas on periodic maps

PROPOSITION 4.1. — *Let* V *be a compression body and* g *be an automorphism of* V *which is periodic on* $\partial_e V$. *Then* g *can be deformed to a periodic automorphism by an isotopy fixing* $\partial_e V$.

Proof. — Recall that the complexity of a closed orientable surface F is the ∞-tuple:

$$c(F) = (\ldots, c_g(F), \ldots, c_2(F), c_1(F), c_0(F)) \in \mathbb{N}^{\mathbb{N}},$$

where $c_g(F)$ is the number of components of genus g of F, and that the complexities are ordered by lexicographic order. It is easy to check that $c(\partial_i V) \leqq c(\partial_e V)$. We will prove Proposition 4.1 by induction on $c(V) = c(\partial_e V) - c(\partial_i V)$, which measures the "difference" between $\partial_e V$ and $\partial_i V$. Note that $c(V)$ belongs to the subset of the elements of $\mathbb{Z}^{\mathbb{N}}$ that are $\geqq (\ldots, 0, 0)$ for lexicographic order (from left to right) and that the induction is possible since this set, albeit much larger that $\mathbb{N}^{\mathbb{N}}$, is nevertheless well-ordered for lexicographic order.

If $c(\partial_e V) = c(\partial_i V)$, then V is isomorphic to $\partial_i V \times I$, where $\partial_i V$ corresponds to $\partial_i V \times \{0\}$ by this isomorphism. By ([Wa_2], Lemma 3.5), g can be deformed to $(g \mid \partial_i V \times \{1\}) \times Id_I$ by an isotopy fixing $\partial_e V \cong \partial_i V \times \{1\}$, whence the property follows.

In fact, this argument also holds (by a classical result on balls) when $c(V) \in \mathbb{N}^{\{0\}} \subset \mathbb{Z}^{\mathbb{N}}$. In this case, indeed, V is isomorphic to the disjoint union of $\partial_i V \times I$ and of $c_0(\partial_e V)$ balls.

Assume now Proposition 4.1 proved for every compression body V' such that $c(V') < c(V)$. The crucial step in the induction is the following.

LEMMA 4.2. — *Under the hypotheses of Proposition* 4.1 *and if* $c(V) \notin \mathbb{N}^{\{0\}}$, *i. e. if* V *requires at least one* 2*-handle in a handle decomposition of* $(V;\ \partial_e V,\ \partial_i V)$, *then there exists a simple closed connected curve* C *in* $\partial_e V$ *which bounds a disc in* V *but not in* $\partial_e V$ *and such that, for each* n, *either* $g^n(C) = C$ *or* $g^n(C) \cap C = \emptyset$.

We assume Lemma 4.2 for the while and go on with the proof of Proposition 4.1. Let D be a system of disjoint discs in V whose boundary is $\bigcup_n g^n(\mathrm{C})$. By the usual intersection reduction methods, D is unique up to isotopy fixing $\partial_e \mathrm{V}$. The automorphism g can therefore be deformed, by an isotopy fixing $\partial_e \mathrm{V}$, so that $g(\mathrm{D})=\mathrm{D}$ and $g \mid \mathrm{D}$ is periodic. Let then $\tilde{\mathrm{V}}$ be the manifold obtained by cutting V open along D and $\tilde{g}$ be the automorphism of $\tilde{\mathrm{V}}$ induced by g. The manifold $\tilde{\mathrm{V}}$ is still a compression body (*see* Corollary B.3 in Appendix B), $c(\partial_i \tilde{\mathrm{V}})=c(\partial_i \mathrm{V})$ and $c(\partial_e \tilde{\mathrm{V}})<c(\partial_e \mathrm{V})$. Apply then the induction hypothesis to $\tilde{g}$ and glue back together the deformation of $\tilde{g}$ so obtained to conclude the proof of Proposition 4.1, granting Lemma 4.2. □

Proof of Lemma 4.2. — It is here useful to leave the PL category and equip V with a smooth structure for which g is a diffeomorphism (after isotopy fixing $\partial_e \mathrm{V}$). To do this carefully, begin with deforming g so that it is periodic on a neighborhood U of $\partial_e \mathrm{V}$, then equip U with a smooth structure for which the restriction $g \mid \mathrm{U}$ is a diffeomorphism (smooth first a neighborhood of the fixed points of the non-trivial iterates of $g \mid \mathrm{U}$, and lift afterwards an appropriate smoothing of U/g), extend to V the smooth structure on U and, lastly, compose g with a small PL isotopy fixing a neighborhood of $\partial_e \mathrm{V}$. Note that the existence of a smooth rather than PL curve C satisfying the conditions of Lemma 4.2 provides a PL curve with the same properties, by lifting to $\partial_e \mathrm{V}$ a small perturbation of $(\bigcup_n g^n(\mathrm{C}))/g$ in $(\partial_e \mathrm{V})/g$.

There exists on $\partial_e \mathrm{V}$ a Riemannian metric of locally constant curvature $+1$, 0 or -1 for which the restriction of g is an isometry (*see* for instance [Th$_2$], Proposition 13.3.6; [the idea consists in choosing a suitable singular metric on the quotient $(\partial_e \mathrm{V})/g$]). For this metric, each component $\partial_0 \mathrm{V}$ of $\partial_e \mathrm{V}$ can be isometrically identified with the quotient of S^2, $\mathbb{R}^2$ or $\mathbb{H}^2$ by some discrete group of isometries isomorphic to $\pi_1(\partial_0 \mathrm{V})$. If $\partial_0 \mathrm{V}$ is not a sphere, the number of closed geodesics in $\partial_0 \mathrm{V}$ with length smaller than K is finite for every constant $\mathrm{K}>0$ [otherwise, using a fundamental domain, one easily checks that $\pi_1(\partial_0 \mathrm{V})$ would not be discrete as a subgroup of isometries of $\mathbb{R}^2$ or $\mathbb{H}^2$]. There exists therefore a simple closed geodesic C in $\partial_e \mathrm{V}$ which is length-minimizing among all the simple curves bounding a disc in V but not in $\partial_e \mathrm{V}$ [such curves exist because $c(\mathrm{V}) \notin \mathbb{N}^{\{0\}}$]. We are going to show that C satisfies the desired condition.

Since g is an isometry on $\partial_e \mathrm{V}$, the curve $g^n(\mathrm{C})$ is, for every n, a geodesic with same length as C; in particular, either $g^n(\mathrm{C})=\mathrm{C}$ or the intersection of C and $g^n(\mathrm{C})$ is transverse. Considering n such that $g^n(\mathrm{C}) \neq \mathrm{C}$, we want to prove that C does not meet $g^n(\mathrm{C})$. By hypothesis, C bounds a disc D in V and $g^n(\mathrm{C})$ bounds $\mathrm{D}'=g^n(\mathrm{D})$. By a slight perturbation of D′ [after which perhaps $\mathrm{D}' \neq g^n(\mathrm{D})$], the intersection of D and D′ can be assumed to be transverse.

Suppose in quest of a contradiction that C meets $g^n(\mathrm{C})$. There exists then an arc k component of $\mathrm{D} \cap \mathrm{D}'$ which splits D into two half-discs D_1 and D_2 and D′ into D'_1 and D'_2. Without loss of generality, we can assume the length of $\partial \mathrm{D}_1 - k$ minimum among all the possible choices for k, D_1, D_2; in particular, this implies that D_1 meets no arc component of $\mathrm{D} \cap \mathrm{D}'$ different from k, and that the length $l(\partial \mathrm{D}_1 - k)$ is not greater than $l(\partial \mathrm{D}_2 - k)$, and therefore than $1/2\, l(\partial \mathrm{D})$. Consider then the two singular discs $\mathrm{D}_1 \cup \mathrm{D}'_1$ and

$D_1 \cup D'_2$. By the above remarks, their boundaries are simple closed curves (with two corners) and:

$$l(\partial(D_1 \cup D'_1)) + l(\partial(D_1 \cup D'_2)) = 2\,l(\partial D_1 - k) + l(\partial D') \leqq l(\partial D) + l(\partial D') \leqq 2\,l(C),$$

which implies that the length of one of them, say $D_1 \cup D'_1$, is at most $l(C)$. By considering its lifting to the universal covering of the component of $\partial_e V$ that contains it, $\partial(D_1 \cup D'_1)$ cannot bound any disc in $\partial_e V$ (otherwise, two distinct geodesics of $\mathbb{R}^2$ or $\mathbb{H}^2$ would meet in at least two points); also, Dehn's lemma implies that $\partial(D_1 \cup D'_1)$ bounds a non-singular disc in V. But, by rounding the two corners of $\partial(D_1 \cup D'_1)$, one could then construct a smooth simple closed curve, bounding a disc in V but not in $\partial_e V$, that is shorter than $\partial(D_1 \cup D'_1)$, and therefore than C. This would contradict the definition of C and shows therefore that $C \cap g^n(C) = \emptyset$ if $g^n(C) \neq C$. □

PROPOSITION 4.3. — *Let* F *be a closed connected surface, possibly non-orientable, different from* S^2 *and* $\mathbb{RP}^2$, *and let* $F \tilde{\times} I$ *denote the orientation* I*-bundle over* F. *If g is an automorphism of the manifold* $F \tilde{\times} I$ *that is periodic on the boundary, then one of the two following assertions holds (possibly both).*

(*a*) *There exists a periodic automorphism g' of* $F \tilde{\times} I$ *which coincides with g on the boundary.*

(*b*) F *is a torus or a Klein bottle and, for every n and each boundary component preserved by* g^n, *the restriction of* g^n *to this torus is a translation (perhaps the identity).*

Remarks. — (1) In (*b*), we mean by translation of a torus any automorphism that lifts to a translation of $\mathbb{R}^2$ for some identification of this torus with $\mathbb{R}^2/\mathbb{Z}^2$.

(2) In (*a*), one could moreover show that g' is isotopic to g by an isotopy fixing the boundary.

Proof. — Assume first that F is orientable ($F \tilde{\times} I = F \times I$), that its genus is at least 2 and that g does not exchange the two boundary components. Let g_0 (resp. g_1) denote the automorphism of F defined by the restriction of g to $F \times \{0\}$ (resp. $F \times \{1\}$), for the standard identifications. To prove (*a*), it is sufficient to show that g_0 is conjugated to g_1 by an automorphism isotopic to the identity. For this, equip F with a (smooth) conformal structure m_0 (resp. m_1) for which g_0 (resp. g_1) is conformal (by averaging some metric). Teichmüller theory ([Te$_1$], [Te$_2$]) asserts then that every homeomorphism f of F, considered as a mapping from the Riemann surface (F, m_0) to (F, m_1), is topologically isotopic to a unique homeomorphism φ_f with constant dilatation (which measures at each point the distortion between the two conformal structures m_0 and $\varphi_f^* m_1$). Since m_0 (resp. m_1) is preserved by g_0 (resp. g_1) and since g_0 and g_1 are homotopic (and therefore isotopic), it follows from the uniqueness of the Teichmüller mappings that:

$$\varphi_{\mathrm{Id}}\, g_0 = \varphi_{g_0} = \varphi_{g_1} = g_1\, \varphi_{\mathrm{Id}}.$$

The homeomorphism φ_{Id} realizes therefore a conjugacy from g_0 to g_1, and is isotopic to the identity. By a small perturbation of φ_{Id} (consider the quotient spaces F/g_0 and F/g_1), we can lastly find a PL automorphism with the same properties. (I am indebted to L. Siebenmann for this short proof.)

If F is a torus and still g does not exchange the boundary components, it is easy to classify, up to conjugacy by isotopies, all the periodic automorphisms of F (consider the quotient space, or more precisely the quotient "orbifold", in the sense of [Th_2], § 13.3). Since g_0 and g_1 are homotopic, it then turns out that they are conjugate by an isotopy, unless they are both homotopic to the identity, in which case (*b*) holds. This ends the proof in this case.

Consider now the case where F is not orientable. Identifying F with the section $F \tilde{\times} \{1/2\} \subset F \tilde{\times} I$, we can deform g by an isotopy fixing the boundary so that $g(F)=F$ ([Wa_2], Lemma 3.5; only the case where the base of the I-bundle is orientable is explicity stated there, but the non-orientable case is similar). Then, g defines outer automorphisms of the groups below which preserve (up to inner automorphisms) the exact sequence:

$$0 \to \pi_1(F \tilde{\times} \partial I) \to \pi_1(F \tilde{\times} I) \cong \pi_1(F) \to \mathbb{Z}/2 \to 0.$$

Since g is periodic on $F \tilde{\times} \partial I$, it follows that the outer automorphism of $\pi_1(F)$ defined by $g_{|F}$ has finite order. By Nielsen's Theorem [Ni_2], $g_{|F}$ is then isotopic to a periodic map and we can therefore isotop g so that it perserves $F \tilde{\times} [1/4, 3/4]$ and that its restriction there is periodic. To end the proof, it is then sufficient to apply the study of the orientable case to $F \tilde{\times} (I -]1/4, 3/4[) \cong (F \tilde{\times} \partial I) \times I$.

The proof is similar when g exchanges the boundary components of $F \times I$ (F is then orientable). By [Wa_2], Lemma 3.5, we can assume that $g(F)=F$ where F is identified with $F \times \{1/2\}$. As above, g can be made periodic on $F \times [1/4, 3/4]$ and we end the proof by applying the previous case to $F \times [0, 1/4]$ and $F \times [1/4, 3/4]$. □

5. Splitting of cobordisms

The principal tool for our analysis of Δ_2 is Proposition 5.1 below. This section is devoted to its proof.

PROPOSITION 5.1. — *If* (F^2, f) *is null-cobordant, it bounds an automorphism* $(M^3, \hat{f})$ *where* M *splits into three pieces* V^3, M_I^3 and M_P^3, *preserved by* $\hat{f}$, *such that*:

(1) V *is a compression body for* ∂M *and* $\overline{M - V} = M_I \amalg M_P$.

(2) M_I *is an* I-*bundle over a closed, possibly non-orientable, surface.*

(3) *The restriction of* $\hat{f}$ *to* M_P *is periodic.*

Remark. — In Proposition 5.1, there does not in general exist any handle decomposition of V that is preserved by $\hat{f}$. For instance A. Fathi and F. Laudenbach [FL] have constructed an automorphism $(F, f) = \partial(V, \hat{f})$ where f is pseudo-Anosov and V is a handlebody; if $\hat{f}$ preserved any handle decomposition of V, the automorphism f would be reducible and could not be pseudo-Anosov.

To prove Proposition 5.1, we need some preliminary results.

LEMMA 5.2. — *If* (F^2, f) *is null-cobordant, it bounds an automorphism* $(M^3, \hat{f})$ *with* M *irreducible.*

Proof. — *See* Appendix A. □

Say that (F_1^2, f_1) *compresses to* (F_2^2, f_2) if $(F_1, f_1) \amalg (-F_2, f_2)$ bounds an automorphism $(V^3, \hat{f})$ such that V is a compression body for F_1 (i. e. $\partial_e V = F_1$ and $\partial_i V = -F_2$). Note that if (F_1, f_1) compresses to (F_2, f_2) and (F_2, f_2) compresses to (F_3, f_3), then (F_1, f_1) compresses to (F_3, f_3).

LEMMA 5.3. — *If* (F_1^2, f_1) *is null-cobordant, it compresses to an automorphism* (F_2^2, f_2) *bounding some* $(M_2^3, \hat{f}_2)$ *where* M_2 *is irreducible and* ∂*-irreducible.*

Proof. — By Lemma 5.2, (F_1, f_1) bounds $(M_1^3, \hat{f}_1)$ with M_1 irreducible. Consider then the characteristic compression body V of M_1 (*cf.* § 2), which we can assume preserved by $\hat{f}_1$ (Theorem 2.1) and take $(F_2, f_2) = (-\delta V, \hat{f}_1 \mid \delta V)$. □

Recall that an automorphism (F^2, f) is *reducible* when there exists an essential 1-submanifold C^1 such that $f(C) = C$ up to isotopy.

LEMMA 5.4. — *Any automorphism* (F_1^2, f_1) *compresses to an automorphism* (F_2^2, f_2) *with* f_2 *irreducible.*

Proof. — If f_1 is reducible, it preserves after isotopy an essential submanifold C^1 of F^1. Let (F_2, f_2) be constructed from (F_1, f_1) by performing a 2-surgery along each component of C and deleting the spherical components of the surface so obtained. Then (F_1, f_1) clearly compresses to (F_2, f_2). If f_2 is not irreducible, perform the same trick until an irreducible automorphism is reached (this process must stop since the compressions decrease the complexities of the surfaces). □

Proof of Proposition 5.1. — Using alternatively Lemmas 5.3 and 5.4, define a sequence of automorphisms (F_i^2, f_i) where $(F_1, f_1) = (F, f)$ and:

(1) for every i, (F_i, f_i) compresses to (F_{i+1}, f_{i+1});

(2) for every $k \geqq 1$, f_{2k+1} is irreducible and (F_{2k}, f_{2k}) bounds some (M_{2k+1}, f_{2k+1}) with M_{2k+1} irreducible and ∂-irreducible.

Since these compressions reduce the complexities of the surfaces, the "compression cobordism" between (F_i, f_i) and (F_{i+1}, f_{i+1}) consists, for i sufficiently large, of an automorphism of a product compression body $V_i \cong F_i \times I$, where F_i corresponds to $F_i \times \{0\}$ and F_{i+1} to $F_i \times \{1\}$. By [Wa$_2$], Lemma 3.5, f_i and f_{i+1} are then isotopic for the above identification $F_i \cong F_{i+1}$. There exists consequently an automorphism (F', f') and, for every i sufficiently large, an oriented isomorphism $h_i : F_i \to F'$ such that f_i is isotopic to $h_i^{-1} f' h_i$ [in other words, the sequence (F_i, f_i) "stabilizes" to (F', f') up to conjugacy and isotopy]. By construction, f' is irreducible and (F', f') bounds an automorphism $(M', \hat{f}')$ where M' is irreducible and ∂-irreducible.

Since (F, f) compresses to (F', f'), it is now sufficient to show that (F', f') bounds after isotopy some $(M_I \amalg M_P, \hat{f}'')$ where M_I is an I-bundle and $\hat{f}''$ is periodic on M_P.

Consider the characteristic fibered submanifold W of M' and isotop $\hat{f}'$ so that $\hat{f}'(W) = W$ (Proposition 3.1; recall M' is irreducible and ∂-irreducible). The irreducibility of f' implies the following properties.

CLAIM 5.5. — *The surface* δW *is closed; equivalently,* $W \cap \partial M'$ *consists of components of* $\partial M'$. *Moreover, every component of* W *that meets* $\partial M'$ *is a component of* M' *and admits an* I*-bundle structure over a closed surface.*

Proof of 5.5. — The closed 1-submanifold $\partial(\delta W)$ of $F' = \partial M'$ is essential and preserved by f'. Since f' is irreducible, it must be empty. This proves the first statement.

By definition and since $\partial(\delta W) = \emptyset$, every component W_1 of W is either an I-bundle over a closed surface, in which case it is a component of M′, or a Seifert manifold. If W_1 is not such an I-bundle, it is a Seifert manifold with non-empty boundary and different from $S^1 \times D^2$ and $T^2 \times I$ [recall that $(W_1, W_1 \cap \partial M)$ is by definition of W never isomorphic to $(T^2 \times I, T^2 \times \{0\})$]. [$Wa_1$] implies then that $\hat{f}'$ can be isotoped in order to preserve the Seifert fibration on $\bigcup_n (\hat{f}')^n(W_1)$. Then W_1 cannot meet $\partial M'$: Otherwise, f' would preserve the essential 1-submanifold $\bigcup_n (\hat{f}')^n(C)$ of F′, where C is a fiber in $W_1 \cap \partial M$. □

Let M_I consists of all the components of M′ that are I-bundles over closed surfaces, and let M″ be $M' - M_I$. Note that, by Claim 5.5, $W \cap \partial M'' = \emptyset$ and $W \cap M''$ consists only of Seifert manifolds. We are now going to modify dramatically M″ [and $(M', \hat{f}')$] so that $\hat{f}'$ be periodic on M″.

We may of course assume that no component of M′ is closed. By Propositions 3.2(6) and 3.3, $\hat{f}'$ can be isotoped so that it is periodic on the components of $\overline{M'' - W}$ that are not isomorphic to $T^2 \times I$.

Consider a component W_1 of $W \cap M''$ that is isomorphic to $T^2 \times I$. By condition (3) of Proposition 3.1, no component of $\overline{M'' - W}$ meeting W_1 is isomorphic to $T^2 \times I$. Consequently, $\hat{f}'$ is periodic on $\bigcup_n (\hat{f}')^n(\partial W_1)$. Now, by Proposition 4.3, $\hat{f}'$ can be assumed to be periodic on $\bigcup_n (\hat{f}')^n(W_1)$, except possibly in the case where, for every *m*, $(\hat{f}')^m$ is a translation on each (torus) component of $\bigcup_n (\hat{f}')^n(\partial W_1)$ it preserves.

Apply the above process to each component of $W \cap M''$ that is isomorphic to $T^2 \times I$. Let then M''_P denote the part of M″ where we have so far been able to make $\hat{f}'$ periodic: It is the union of all the components of $\overline{M'' - W}$ that are not isomorphic to $T^2 \times I$, and of some components of $W \cap M''$ isomorphic to $T^2 \times I$.

For every component W_2 of $M'' - M''_P$ that is a component of W, $\hat{f}'$ preserves up to isotopy the Seifert fibration of $\bigcup_n (\hat{f}')^n(W_2)$ [Wa_1]. If G is a boundary component of ∂W_2 and if $(\hat{f}')^n$ respects G, then $(\hat{f}')^n$ necessarily preserves a non-trivial isotopy class of simple closed curves in G (consider the fiber). By definition of M''_P, the same property holds for the other kind of components of $\delta M''_P$. The automorphism $(\delta M''_P, \hat{f}' \mid \delta M''_P)$ consequently bounds some $(V_P, \hat{f}'_P)$ where V_P consists of solid tori and $\hat{f}'_P$ is periodic (use the classification of periodic automorphisms of T^2, or an equivariant surgery argument). Consider now the manifold M_P obtained by replacing $\overline{M'' - M''_P}$ by $(-V_P)$ in M″, and let $\hat{f}_P$ be the periodic automorphism $(\hat{f}' \mid M''_P) \cup \hat{f}'_P$ of M_P. The automorphism (F', f') then bounds $(M_I \amalg M_P, (\hat{f}' \mid M_I) \amalg \hat{f}_P)$, where M_I is an I-bundle and $\hat{f}_P$ is periodic. This ends the proof of Proposition 5.1.

Remark. — In the statement of Proposition 5.1, we did not require that M be irreducible and that V be its characteristic compression body, or, equivalently, that M_P be irreducible and ∂-irreducible (and this is in general false for M and V provided by the above proof). Using the Equivariant Sphere Theorem and Loop Theorem [MY], and some cutting and pasting argument, it is nevertheless possible to add this extra condition to the conclusions of Proposition 5.1, but this is of no use for the following.

6. The canonical decomposition of Δ_2

For our purposes, it is natural to identify two automorphisms (F_1^2, f_1) and (F_2^2, f_2) when there exists an oriented isomorphism h 8 $F_1 \to F_2$ such that $hf_1 h^{-1}$ is isotopic to f_2; we shall then say that (F_1, f_1) and (F_2, f_2) are *equivalent by conjugacy and isotopy*. Let $\mathscr{F}$ be the set of such equivalence classes of automorphisms of surfaces. We often denote simply by (F, f) the class in $\mathscr{F}$ of the automorphism (F, f).

In view of the applications to cobordism, an interesting subset of $\mathscr{F}$ is $\mathscr{F}_0$, that consists of the classes of automorphisms which cannot be written as $(F, f) \amalg (-F, f) \amalg (F', f')$, with F non-empty. There is an obvious retraction $\mathscr{F} \to \mathscr{F}_0$ defined by removing all the pairs $(F, f) \amalg (-F, f)$; it transforms the monoid law $\amalg$ on $\mathscr{F}$ into a group law $\tilde{\amalg}$ on $\mathscr{F}_0$ and factors the canonical map $\mathscr{F} \to \Delta_2$ through a group homomorphism $\mathscr{F}_0 \to \Delta_2$.

Define on $\mathscr{F}$ the following relation $\prec$, which is a slight extension of the compression cobordism of paragraph 3: $(F, f) \prec (F', f')$ when $(-F, f) \amalg (F', f')$ bounds some $(M^3, \hat{f})$, where M^3 is the disjoint union of a compression body V and of an I-bundle W over a closed (possibly non-orientable) surface, such that $F = -\partial_i V$ and $F' = \partial W \amalg \partial_e V$. Note that $\prec$ is proper by [Wa_2], Lemma 3.5.

For any $X \in \Delta_2$, let (F^X, f^X) be an automorphism representing the cobordism class X, with the property that its class in $\mathscr{F}$ is minimal for $\prec$; for instance, choose (F^X, f^X) so that the complexity of F^X is minimum among all automorphisms representing X.

The automorphism f^X is clearly irreducible. A result of J. Nielsen ([Ni_1], [Ni_2]), expressed with a terminology issued from [Th_1], asserts then that (F^X, f^X) splits after isotopy into $(F_P^X, f_P^X) \amalg (F_A^X, f_A^X)$, where f_P^X is periodic and f_A^X is pseudo-Anosov. Here, an automorphism (F^2, f) is (homotopically) *pseudo-Anosov* when, for every $n \neq 0$ and every essential 1-submanifold C of F, $f^n(C)$ is never isotopic to C; equivalently, (F^2, f) is pseudo-Anosov in this sense if and only if it is topologically isotopic to a pseudo-Anosov homeomorphism in the geometric sense of [Th_1].

Let $\mathscr{A}$ be the set of those elements of $\mathscr{F}$ which may occur as the class of such a (F_A^X, f_A^X), i.e. the elements of $\mathscr{F}$ that are represented by pseudo-Anosov automorphisms and are minimal for $\prec$. It is clearly a subgroup of $\mathscr{F}_0$ [note that if (F, f) and $(F', f') \in \mathscr{F}$ are minimal for $\prec$, both belong to $\mathscr{F}_0$ and $(F, f) \tilde{\amalg} (F', f')$ is minimal for $\prec$].

Also, recall from the introduction that Δ_2^P is the *periodic cobordism group*, which consists of periodic automorphisms of oriented surfaces modulo cobordism by periodic automorphisms of 3-manifolds.

THEOREM 6.1. — *The rule* $X \to ((F_P^X, f_P^X), (F_A^X, f_A^X))$ *induces a group isomorphism* $\Delta_2 \cong \Delta_2^P \oplus \mathscr{A}$, *whose inverse is the obvious homomorphism.*

Remark. — Theorem 6.1 asserts in particular that (F_A^X, f_A^X) is well-defined up to conjugacy and isotopy. If we had required f_A^X to be a geometric pseudo-Anosov homeomorphism (in the sense of [Th_1]), (F_A^X, f_A^X) would even be unique up to mere (topological) conjugacy (two geometric pseudo-Anosov homeomorphisms of the same surface that are isotopic are conjugate).

Proof of 6.1. — To prove that the map $\Delta_2 \to \Delta_2^P \oplus \mathscr{A}$ is well-defined, consider another automorphism $(G^X, g^X) = (G_P^X, g_P^X) \amalg (G_A^X, g_A^X)$ that represents X and is minimal for $\prec$ (where g_P^X is periodic and g_A^X is pseudo-Anosov). By definition, $(F^X, f^X) \amalg (-G^X, g^X)$ bounds some $(M^3, \hat{f})$. By Proposition 5.1, this null-cobordism can be chosen so that M splits into three pieces V, M_I and M_P, preserved by $\hat{f}$, where V is a compression body for ∂M, M_I is an I-bundle and $\hat{f} \mid M_P$ is periodic.

Since both (F^X, f^X) and (G^X, g^X) are minimal for $\prec$, the compression body V is just a regular neighborhood of ∂M in M. Let M_I' (resp. M_P') denote the union of M_I (resp. M_P) and of the adjacent components of V.

Since no pseudo-Anosov surface automorphism is homotopic to a periodic one, $\partial M_P' \subset F_P^X \cup G_P^X$ and $\hat{f}$ is consequently periodic on the boundary of the product I-bundle $\overline{M_P' - M_P}$. By Proposition 4.3, $\hat{f}$ can therefore be assumed to be periodic on $M_P' = (M_P' - M_P) \cup M_P$ [case (*b*) of Proposition 4.3 cannot occur since (F^X, f^X) and (G^X, g^X) are minimal for $\prec$]. This provides a partition $M = M_I' \amalg M_P'$ of the components of M such that M_I' is an I-bundle and $\hat{f} \mid M_P'$ is periodic.

No component of M_I' can have its boundary completely contained in F^X or G^X : Otherwise, (F^X, f^X) or (G^X, g^X) would not be minimal for $\prec$. From this fact, it follows for homotopy theoretic reasons that each component of M_I' joins, either a component of F_A^X to a component of G_A^X, or a component of F_P^X to a component of G_P^X.

Let M_I'' consist of the components of M_I' that meet F_A^X (or G_A^X), and let M_P'' be $M - M_I''$. Note that $\partial M_I'' = F_A^X \cup G_A^X$ and $\partial M_P'' = F_P^X \cup G_P^X$.

The manifold M_I'' is a product I-bundle. By [Wa_2], Lemma 3.5, $\hat{f}$ can be isotoped on M_I'' so that it preserves the projection $M_I'' \to I$. It follows that (F_A^X, f_A^X) and (G_A^X, g_A^X) are equivalent by conjugacy and isotopy, i. e. represent the same element of $\mathscr{A}$.

Applying Proposition 4.3 to the I-bundle $M_I - M_I''$, $\hat{f}$ can be modified to be periodic on $M_P'' = M_P' \cup (M_I - M_I'')$ [again, case (*b*) of Proposition 4.3 cannot occur by minimality of (F^X, f^X) and (G^X, g^X)]. Hence (F_P^X, f_P^X) and $(G_P^X)\, g_P^X)$ are periodic cobordant.

This ends the proof that the rule $X \to ((F_P^X, f_P^X)\ (F_A^X, f_A^X))$ induces a map $\varphi : \Delta_2 \to \Delta_2^P \oplus \mathscr{A}$. The map φ is clearly a group homomorphism since, if (F, f) and $(F', f') \in \mathscr{F}_0$ are minimal for $\prec$, so is $(F, f) \tilde{\amalg} (F', f')$.

Let $\psi : \Delta_2^P \oplus \mathscr{A} \to \Delta_2$ be the obvious group homomorphism. By definition of φ, $\psi\varphi = \mathrm{Id}$ and φ is therefore injective.

To prove that φ is surjective, we need to show that, for every automorphism $(F_P, f_P) \amalg (F_A, f_A)$ where f_P is periodic and the class of (F_A, f_A) in $\mathscr{F}$ is contained in $\mathscr{A}$, (F_P, f_P) is periodic cobordant to a periodic automorphism (F'_P, f'_P) such that $(F'_P, f'_P) \amalg (F_A, f_A)$ is minimal for $\prec$. By some homotopy theoretic remarks and the usual compatibility of $\tilde{\amalg}$ with minimality, this last property is equivalent to the fact that (F'_P, f'_P) is minimal for $\prec$.

The proof of Theorem 6.1 is therefore achieved by Lemma 6.2 below.

LEMMA 6.2. — *Every periodic automorphism (F^2, f) is periodic cobordant to a periodic automorphism whose class in $\mathscr{F}$ is minimal for $\prec$.*

Proof of 6.2. — If (F^2, f) is not minimal for $\prec$, there exists by definition an automorphism $(M^3, \hat{f})$ and a partition $M = V \amalg W$ of the components of M such that:

(1) V is a compression body;

(2) W is an I-bundle over a closed surface;

(3) $F = \partial_e V \cup \partial W$ and $f = \hat{f} \mid F$;

(4) $M \not\cong F \times I$.

We would like $\hat{f}$ to be periodic. Apply Proposition 4.3 to each component of W (and to the first iterate of $\hat{f}$ that preserves it). When we get in case (*b*) of Proposition 4.3, replace the considered component of W (and its images by $\hat{f}$) by one or two solid tori and add them to V. Eventually, only case (*a*) holds and $\hat{f}$ can be assumed to be periodic on W. By Proposition 4.1, $\hat{f}$ can also be isotoped so that it is periodic on V, and therefore on M.

Let now F_1 be $-(\partial M - F)$ and let f_1 be $\hat{f} \mid F_1$. The periodic automorphisms (F, f) and (F_1, f_1) are periodic cobordant and $(F_1, f_1) \prec (F, f)$. If (F_1, f_1) is minimal for $\prec$, the property is proved. Otherwise, iterate this process and define a sequence (F_i, f_i) of periodic automorphisms periodic cobordant to (F, f) such that :

$$\ldots (F_{i+1}, f_{i+1}) \prec (F_i, f_i) \prec \ldots \prec (F_1, f_1) \prec (F, f).$$

Considering the complexities of the surface F_i, this sequence must needs stop, which happens when we reach a (F_i, f_i) that is minimal for $\prec$.

This ends the proof of Lemma 6.2, and therefore of Theorem 6.1. $\square$

An important remark for the following sections is that the proof of 6.1 is natural with respect to the graded group structure of Δ_2 defined in paragraph 1. Consequently, for the obvious definitions of $\mathscr{A}_\gamma$ and $\Delta^P_{2\gamma}$:

PROPOSITION 6.3. — *For each* $\gamma \in \Gamma$, $\Delta_{2\gamma} \cong \Delta^P_{2\gamma} \oplus \mathscr{A}_\gamma$. $\square$

7. The group $\mathscr{A}$

Let $\mathscr{A}_0 \subset \mathscr{A}$ consist of the classes of automorphisms (F, f) where f acts transitively on the set of the components of F [or, equivalently, (F, f) cannot be decomposed into $(F', f') \amalg (F'', f'')$ in a non trivial way]. If $\alpha : \mathscr{F} \to \mathscr{F}$ is the involution $(F, f) \to (-F, f)$, let $\mathscr{B}$ denote the fixed point set of α in $\mathscr{A}_0$ and choose a subset $\mathscr{C}$ such that $\mathscr{A}_0 = \mathscr{B} \amalg \mathscr{C} \amalg \alpha(\mathscr{C})$. Every element (F, f) of $\mathscr{A}$ can be written as a disjoint sum of elements of $\mathscr{A}_0$, and this decomposition is unique; if $a \in \mathscr{A}_0$, let $n_a(\mathrm{F}, f) \in \mathbb{N}$ denote the number of times a appears in this decomposition. From the definition of the group $\mathscr{A}$, it is then clear that:

PROPOSITION 7.1. — *The map $\mathscr{A} \to (\mathbb{Z}/2)^{\mathscr{B}} \oplus \mathbb{Z}^{\mathscr{C}}$ defined by:*

$$(\mathrm{F}, f) \mapsto ((\overline{n_b(\mathrm{F}, f)})_{b \in \mathscr{B}}, (n_c(\mathrm{F}, f) - n_{\alpha(c)}(\mathrm{F}, f))_{c \in \mathscr{C}}),$$

is a group isomorphism. $\square$

Let $\mathscr{F}_+$ (resp. $\mathscr{F}_-$) be the subset of the elements (F, f) of $\mathscr{F}$ where f preserves (resp. reverses) the orientation of $\mathscr{F}$, and define $\mathscr{B}_+ = \mathscr{B} \cap \mathscr{F}_+$, $\mathscr{C}_+ = \mathscr{C} \cap \mathscr{F}_+$, etc.

PROPOSITION 7.2. — *The sets $\mathscr{B}_+$, $\mathscr{C}_+$, $\mathscr{B}_-$ are infinite, and $\mathscr{C}_-$ is empty.*

COROLLARY 7.3:

$$\mathscr{A}_+ \cong \mathbb{Z}^\infty \oplus (\mathbb{Z}/2)^\infty, \qquad \mathscr{A}_- \cong (\mathbb{Z}/2)^\infty,$$
$$\mathscr{A} = \bigoplus_{\gamma \in \Gamma} \mathscr{A}_\gamma \cong \mathbb{Z}^\infty \oplus (\mathbb{Z}/2)^\infty. \quad \square$$

Proof of Proposition 7.2. — The set $\mathscr{C}_-$ is empty since $(\mathrm{F}, f) = (-\mathrm{F}, f)$ in $\mathscr{F}$ whenever f is orientation-reversing. To prove that $\mathscr{B}_+$, $\mathscr{C}_+$ and $\mathscr{B}_-$ are infinite, it will be sufficient to exhibit an infinite number of elements of each among (pseudo-)Anosov automorphisms of the torus T^2. Note that such automorphisms always belong to $\mathscr{A}$ by uniqueness of meridian discs in solid tori and of the "neck" of Klein bottles. Since there are, up to conjugacy and isotopy, infinitely many orientation-reversing Anosov automorphisms of T^2, the set $\mathscr{B}_-$ is infinite.

The elements of $\mathscr{F}$ corresponding to orientation-preserving automorphisms of T^2 are in 1-1 correspondence with the conjugacy classes of $\mathrm{SL}_2(\mathbb{Z})$ [by considering $\mathrm{H}_1(\mathrm{T}^2) \cong \mathbb{Z}^2$]. It is a pleasant exercice to show that each such conjugacy class is classified by the data of a number $\varepsilon \in \mathbb{Z}/2$ and of a sequence $(a_1 a_2 \ldots a_n)$, defined up to cyclic permutation, of rational integers with the following property: Either all the a_i's are non-null and a_i and a_{i+1} have opposite signs (including a_n and $a_{n+1} = a_1$), or the sequence is (0), (± 1) or $(0\,a)$. To check this property, associate to each such data the conjugacy class of:

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}^{\varepsilon} \begin{pmatrix} 0 & -1 \\ 1 & a_1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & a_2 \end{pmatrix} \cdots \begin{pmatrix} 0 & -1 \\ 1 & a_n \end{pmatrix}$$

and use some arguments on developments in continued fractions (compare with [BS], § 12). Moreover, the conjugacy classes of $\mathrm{SL}_2(\mathbb{Z})$ represented by Anosov automorphisms are those whose classifying sequence is different from (0), (± 1) and $(0\,a)$, and the involution defined by conjugacy with an element of $\mathrm{GL}_2(\mathbb{Z})$ of determinant -1 is translated in these data by keeping $\varepsilon \in \mathbb{Z}/2$ unchanged and replacing each $a_i \in \mathbb{Z}$ by $-a_i$. With this description, it is clear that Anosov automorphisms of T^2 define infinite subsets of $\mathscr{B}_+$ and $\mathscr{C}_+$ (*see* also [Sc], § 3). $\square$

8. The periodic cobordism group Δ_2^P

This section is devoted to the computation of Δ_2^P, and thus completes the computation of Δ_2.

Consider a periodic automorphism f of a closed oriented surface F^2. To each point $x \in F$ can be attached an element $r(f, x)$ of $\mathbb{Q}/\mathbb{Z}$ in the following way: If n is the smallest positive integer for which f^n preserves both x and the orientation of F near x, then f^n is locally conjugate to a rotation of angle $2\pi r(f, x)$ around x (the orientation is determined by that of F). Let $\mathrm{Fix}_+ f$ denote the (finite) set of points where $r(f, x) \neq 0$.

Proposition 8.1. — *If f is periodic, (F^2, f) is periodic null-cobordant [i. e. the class of (F, f) in Δ_2^P is 0] if and only if $\mathrm{Fix}_+ f$ admits a partition into pairs $\{x_i, x_1'\}$ such that:*

(1) *$r(f, x_i) + r(f, x_i') = 0$ for every i.*

(2) *For every i, $f(\{x_i, x_i'\}) = \{x_j, x_j'\}$ for some j.*

(3) *f preserves the orientation of F near x_i if and only if it preserves it near x_i'.*

Remark. — Since $r(f, f(x))$ is equal to $r(f, x)$ or to $-r(f, x)$ according as f preserves of reverses the orientation of f near x, the last condition is only relevant when $r(f, x_i) = r(f, x_1') = 1/2$ [otherwise, (3) follows from (1) and (2)]. It is also void if f is orientation-preserving or -reversing, which, by paragraph 1, we could assume as well.

Proof. — If (F^2, f) bounds $(M^3, \hat{f})$, where f and $\hat{f}$ are both periodic, consider the set $\mathrm{Fix}_+ \hat{f}$ of the points x in M such that, for some n, $\hat{f}^n$ fixes x and is a nontrivial rotation near x. It is a 1-submanifold of M, preserved by $\hat{f}$, with boundary $\mathrm{Fix}_+ f$. Moreover, if x_i and x_i' are the two boundary points of a component k_i of $\mathrm{Fix}_+ \hat{f}$, $r(f, x_i) + r(f, x_i') = 0$. We then get the partition of $\mathrm{Fix}_+ f$ sought by letting k_i range over all the arc components of $\mathrm{Fix}_+ \hat{f}$.

Conversely, if such a partition exists, choose a small disc d_i (resp. d_i') around each x_i (resp. x_i'), such that all these discs are disjoint and their union is preserved by f. Let then V be the manifold obtained from $F \times I$ by glueing a 1-handle along each pair $\{d_i \times \{1\}, d_i' \times \{1\}\}$; it is a compression body with interior boundary $\partial_i V = F \times \{0\}$ and exterior boundary $\partial_e V = \partial V - \partial_i V$. The automorphism $f \times \mathrm{Id}_I$ of $F \times I$ extends to a periodic automorphism $\hat{f}$ of V. Then, (F, f) can be identified with $(\partial_i V, \hat{f} | \partial_i V)$ and $(F', f') = (-\partial_e V, \hat{f} | \partial_e V)$ is such that $\mathrm{Fix}_+ f' = \emptyset$. To complete the proof, it is now sufficent to apply Lemma 8.2 below to (F', f'). □

Lemma 8.2. — *If f is periodic and $\mathrm{Fix}_+ f = \emptyset$, then (F^2, f) bounds a periodic automorphism of a disjoint union of handlebodies.*

Proof. — We prove 8.2 by induction on the complexity of F. Assume the lemma proved for every surface of lower complexity than F and consider the quotient space F/f; it is a surface, possibly non-orientable and/or with boundary if f does not preserve the orientation of F. We may of course assume F/f connected.

Consider first the case where the surface F/f is closed and $\dim H_1(F/f; \mathbb{Q}) \geqq 2$. The projection $p: F \to F/f$ is a covering map and this cyclic regular covering is defined by some morphism $\rho: H_1(F/f; \mathbb{Z}) \to \mathbb{Z}/n$. By the condition on $H_1(F/f; \mathbb{Q})$ there exists an indivisible class x in $H_1(F/f; \mathbb{Z})$ such that $\rho(x)=0$. Since x is indivisible, it can be represented by a simple closed curve $\tilde{C}$ in F/f [MP]. The inverse image of $\tilde{C}$ in F is a closed 1-submanifold C, which is essential since its components are non-separating. Then, construct a manifold V from $F\times I$ by glueing a 2-handle along each component of $C\times\{1\}$. For a suitable construction of V, the automorphism $f\times \mathrm{Id}_I$ of $F\times I$ extends to a periodic automorphism $\hat{f}$ of V. Since C is essential, $F'=\partial V-(F\times\{0\})$ has smaller complexity than F. Moreover, because $\rho(x)=0$, each component of C preserved by some f^m is in fact fixed by f^m; it follows that $\mathrm{Fix}_+ f'=\emptyset$, where $f'=\hat{f}|F'$. By induction hypothesis, (F', f') bounds a periodic automorphism $\hat{f}'$ of a disjoint union V' of handlebodies. If F is identified with $F\times\{0\}\subset\partial V$, (F,f) then bounds $(V\cup(-V'), \hat{f}\cup\hat{f}')$ and each component of $V\cup V'$ is clearly a handlebody.

When F/f has non-empty boundary and is not a disc, there exists a properly embedded arc k in F/f that is non-separating. Let C denote the inverse image of k in F; this is a closed essential 1-submanifold, preserved by F. Let then (F', f') be constructed as above from (F, f) and C. For each component C_1 of C and each f^m preserving C_1, $f^m|C_1$ is either the identity or a reflection; it follows that $\mathrm{Fix}_+ f'=\emptyset$. Then, apply the induction hypothesis to (F', f') to get the required property for (F, f).

To start the induction, we now only need to study the cases where F/f is a disc, a sphere, a projective plane or a Klein bottle. In the first three cases, F consists of spheres and f therefore extends to a periodic automorphism of disjoint balls. For the last case, note that F consists of tori; then surger (F,f) along the inverse image C of an essential curve in F/f to get a periodic automorphism (F', f') of a disjoint union of spheres; since (F', f') bounds a periodic automorphism of disjoint balls, it follows that (F,f) bounds a periodic automorphism of disjoint solid tori. □

The group Δ_2^P naturally splits into $\bigoplus_n \Delta_2^P(n)$, where $\Delta_2^P(n)$ is the subgroup of periodic cobordism classes of automorphisms (F, f) with f periodic of period n. As in paragraph 1, $\Delta_2^P(n)$ itself splits into $\bigoplus_{\gamma\in\Gamma} \Delta_{2\gamma}^P(n)$; note that $\Delta_{2\gamma}^P(n)\neq 0$ only if $\nu(\gamma)$ divides n, and that $\Delta_{2\gamma}^P(n)\cong\Delta_{2+}^P(n/\nu(\gamma))$ or $\Delta_{2-}^P(n/\nu(\gamma))$ according as $\sigma(\gamma)=0$ or 1.

PROPOSITION 8.3:

$$\Delta_{2+}^P(n)\cong\mathbb{Z}^{[1/2(n-1)]},$$

$$\Delta_{2-}^P(4k+2)=0,$$

$$\Delta_{2-}^P(4k)\cong(\mathbb{Z}/2)^{[1/2(k-1)]}(=0 \textit{ if } k\leqq 2)$$

(*here* [ ] *means "integral part"*).

COROLLARY 8.4:

$$\Delta_2^P(n)=\mathbb{Z}^K\oplus(\mathbb{Z}/2)^L,$$

with:

$$K=\sum_{m|n} c_+(m)\left[\frac{1}{2}\left(\frac{n}{m}-1\right)\right]$$

and:

$$\mathrm{L}=\sum_{4m\mid n} c_-(m)\left[\frac{1}{2}\left(\frac{n}{4m}-1\right)\right],$$

where $c_+(m)$ and $c_-(m)$ are determined by Corollary 1.5. $\square$

COROLLARY 8.5. — *For every $\gamma\in\Gamma$:*

$$\Delta_{2\gamma}\cong\mathbb{Z}^\infty\oplus(\mathbb{Z}/2)^\infty \quad \textit{and} \quad \Delta^{\mathrm{P}}_{2\gamma}\cong\mathbb{Z}^\infty \quad \textit{if } \sigma(\gamma)=0,$$
$$\Delta_{2\gamma}\cong(\mathbb{Z}/2)^\infty \quad \textit{and} \quad \Delta^{\mathrm{P}}_{2\gamma}\cong(\mathbb{Z}/2)^\infty \quad \textit{if } \sigma(\gamma)=1,$$
$$\Delta_{2}\cong\mathbb{Z}^\infty\oplus(\mathbb{Z}/2)^\infty \quad \textit{and} \quad \Delta^{\mathrm{P}}_{2}\cong\mathbb{Z}^\infty\oplus(\mathbb{Z}/2)^\infty. \quad \square$$

Proof of Proposition 8.3. — Consider (F^2, f), where f preserves the orientation of F and is periodic of period n. The projection $p:\mathrm{F}\to\mathrm{F}/f$ is a cyclic branched covering and (F, f) is consequently determined up to oriented conjugacy by the closed oriented surface F/f, the ramification points $\tilde{x}_i$ in F/f and the representation $\rho:\mathrm{H}_1(\mathrm{F}/f-\bigcup_i\{\tilde{x}_i\})\to\mathbb{Z}/n$ defined by the property that $\rho([p\gamma])=m$ for every path γ in $\mathrm{F}-p^{-1}(\bigcup_i\{\tilde{x}_i\})$ joining some x to $f^m(x)$. Note that the rotation numbers $r(f, x)$ can be easily recovered from ρ: For the natural embedding $\mathbb{Z}/n\subset\mathbb{Q}/\mathbb{Z}$, $r(f, x)=\rho([\partial\tilde{d}])$ where $\tilde{d}$ is a small disc around $\tilde{x}=p(x)$ in F/f, oriented by the orientation of F. Moreover, the number of points of the orbit of x is determined by $r(f, x)$ (together with the period n). Since f is orientation-preserving, $r(f, x)$ depends only on $\tilde{x}=p(x)$ and will also be denoted by $r(f, \tilde{x})$.

Let $\mathscr{F}^{\mathrm{P}}_+(n)$ denote the set of (oriented) conjugacy classes of automorphisms (F^2, f), where f preserves the orientation of F and is periodic of period n. If $\mathrm{C}(n)=(\mathbb{Z}/n)-\{0\}$, consider the map $\varphi:\mathscr{F}^{\mathrm{P}}_+(n)\to\mathbb{N}^{\mathrm{C}(n)}$ that "counts" for each $c\in\mathrm{C}(n)$ the number $\nu_c(f)$ of orbits $\tilde{x}\in\mathrm{F}/f$ with $r(f, \tilde{x})=c$. If $\tilde{d}_i$ is a small disc in F/f around each orbit $\tilde{x}_i$ with $r(f, \tilde{x}_i)\neq 0$, then $\sum_i[\partial\tilde{d}_i]=0$ in $\mathrm{H}_1(\mathrm{F}/f-\bigcup_i\{\tilde{x}_i\})$, and the image of φ is therefore contained in the subset of the elements $\nu\in\mathbb{N}^{\mathrm{C}(n)}$ that satisfy the condition:

$$(\star)\qquad \sum_{c\in\mathrm{C}(n)}\nu_c\,c=0 \quad \text{in } \mathbb{Q}/\mathbb{Z}.$$

Conversely, every $\nu\in\mathbb{N}^{\mathrm{C}(n)}$ satisfying $(\star)$ can easily be realized by some $(\mathrm{F}, f)\in\mathscr{F}^{\mathrm{P}}_+(n)$ (construct a suitable cyclic branched covering over S^2).

Choose now $\mathrm{A}\subset\mathrm{C}(n)$ such that $\mathrm{C}(n)=\mathrm{A}\amalg(-\mathrm{A})$ or $\mathrm{A}\amalg(-\mathrm{A})\amalg\{1/2\}$ according as n is odd or even. If $\mathrm{Fix}_+ f$ admits a partition into pairs $\{x_i, x'_i\}$ as in Proposition 8.1, note that x_i and x'_i belong to the same orbit if and only if $r(f, x_i)=r(f, x'_i)=1/2$, $n=4k$ and $x'_i=f^k(x_i)$ [recall $r(f, f(x))=r(f, x)$ since f is orientation-preserving]. It consequently follows from Proposition 8.1 that the map ψ, defined by:

$$\psi:\ \mathscr{F}^{\mathrm{P}}_+\to\mathbb{Z}^{\mathrm{A}} \qquad \text{if} \quad n\neq 2\,[4],$$
$$(\mathrm{F}, f)\mapsto(\nu_a(f)-\nu_{-a}(f))_{a\in\mathrm{A}}$$

or:

$$\psi : \mathscr{F}^{\mathrm{P}}_{+} \to \mathbb{Z}^{\mathrm{A}} \oplus \mathbb{Z}/2 \qquad \text{if} \quad n=2\,[4],$$
$$(\mathrm{F}, f) \mapsto ((\nu_a(f)-\nu_{-a}(f))_{a\in \mathrm{A}}, \overline{\nu_{1/2}(f)}),$$

induces a monomorphism $\Delta^{\mathrm{P}}_{2+}(n) \to \mathbb{Z}^{\mathrm{A}}$ or $\mathbb{Z}^{\mathrm{A}} \oplus (\mathbb{Z}/2)$. By the characterization of the image of φ, the image of ψ is easily seen to be isomorphic to $\mathbb{Z}^{\mathrm{card\,A}}$. Since card $\mathrm{A}=[1/2(n-1)]$ (and not $[1/2\,n]-1$ as written in [Bo] !), the first statement of Proposition 8.3 follows.

If (F^2, f) is an orientation-reversing automorphism of period $2\,n$, the map $p : \mathrm{F} \to \mathrm{F}/f$ is a cyclic branched covering above int (F/f). As in the previous case, (F, f) is determined up to oriented conjugacy by the surface F/f (possibly non-orientable and/or with boundary), the ramification points $\tilde{x}_i \in \mathrm{int}\ \mathrm{F}/f$ of the restriction of p above $\mathrm{int}(\mathrm{F}/f)$ and some representation $\rho : \mathrm{H}_1(\mathrm{F}/f - \bigcup_i \{\tilde{x}_i\}) \to \mathbb{Z}/2\,n$ [note that there is no ambiguity for the orientation of F, since f realizes a conjugacy between (F, f) and $(-\mathrm{F}, f)$]. As before, $r(f, x) = \rho([\partial \tilde{d}])$ where $\tilde{d}$ is a small disc around $p(x)$ in int (F, f) oriented by the orientation of F near x [and $r(f, x)=0$ if $p(x) \in \partial(\mathrm{F}/f)$].

Since $r(f, f^m(x)) = (-1)^m r(f, x)$ for every x and m, every orbit of order $4k+2$ in $\mathrm{Fix}_+ f$ admits a partition into pairs $\{x_i, f^{2k+1}(x_i)\}$ such that $r(f, x_i)+r(f^{2k+1}(x_i))=0$. By Proposition 8.1, the class of (F, f) in $\Delta^{\mathrm{P}}_{2-}(2n)$ can therefore be represented by an automorphism (F', f') such that the order of every orbit in $\mathrm{Fix}_+ f'$ is a multiple of 4. When n is odd, $\mathrm{Fix}_+ f'$ must then be empty and, consequently, $\Delta^{\mathrm{P}}_{2-}(4k+2)=0$ for every k.

Now, consider the case where $n=2k$. By the same argument as above, we can cancel every orbit where $r(f', x) = 1/2$ [because $(1/2)+(1/2)=0$ in $\mathbb{Q}/\mathbb{Z}$] and can therefore assume that $r(f', x)$ is never $1/2$. Since the order of each orbit is a multiple of 4, each $r(f', x)$ is contained in $\mathbb{Z}/k \subset \mathbb{Q}/\mathbb{Z}$. Let B be the set of pairs $\{\beta, -\beta\}$ in $\mathbb{Z}/k$ with $\beta \neq 0$, $1/2$. Proposition 8.1 shows that the map $\Phi : \Delta^{\mathrm{P}}_{2-}(4k) \to (\mathbb{Z}/2)^{\mathrm{B}}$, which counts modulo 2 the number of orbits representing each element of B [for $r(f,\ )$], is well-defined and is a monomorphism (no orbit in $\mathrm{Fix}_+ f'$ is "self-cancelable").

We claim that Φ is surjective; since card $\mathrm{B}=[1/2\,(k-1)]$, this will achieve the computation of $\Delta^{\mathrm{P}}_{2-}(4k)$. We just need to construct, for any integer l with $0<l<(1/2)k$, an orientation-reversing automorphism (F^2, f) of period $4k$ such that $r(f, x) = \pm l/k$ if x belongs to one specific orbit and $r(f, x) \notin (\mathbb{Z}/k - \{0, 1/2\})$ otherwise. For this purpose, choose in the sphere S^2 $(2l+1)$ distinct points $\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_{2l}$ and let $p : \mathrm{F}^2 \to \mathrm{S}^2$ be the $4k$-fold cyclic branched covering associated to the morphism $\rho : \mathrm{H}_1(\mathrm{S}^2 - \bigcup_i \{x_i\}) \to \mathbb{Q}/\mathbb{Z}$ defined by the property that, if $\tilde{d}_i$ is a small disc around $\tilde{x}_i$, $\rho([\partial\tilde{d}_0]) = l/k$ and $\rho([\partial\tilde{d}_i]) = -1/2k$ if $i \neq 0$. The surface F^2 has exactly two components, and consequently admits an orientation for which the covering translation f is orientation-reversing. Now, $\mathrm{Fix}_+ f$ consists of the orbit $p^{-1}(\tilde{x}_0)$ where $r(f,\ ) = \pm(l/k)$ and of the orbits $p^{-1}(\tilde{x}_i)$, with $i \neq 0$, where $r(f,\ ) = \pm 1/2k$. The automorphism (F, f) therefore satisfies the desired properties.

This ends the proof of Proposition 8.3. □

We end this section by an exercice which provides a sufficient condition for a periodic automorphism to be null-cobordant. In fact, this result generalizes to the property that the restriction $\Delta^{\mathrm{P}}_{2+} \to \mathrm{W}_{-1}(\mathbb{Z}, \mathbb{Z})$ of Kreck's homomorphism (*see* the introduction) is injective [EE].

PROPOSITION 8.6. — *If f is a periodic automorphism of* F^2 *such that* $\mathrm{X}.f_*(\mathrm{X})=0$ *for every* $\mathrm{X} \in \mathrm{H}_1(\mathrm{F})$, *then* (F, f) *is null-cobordant.*

Moreover, (F, f) *bounds a periodic automorphism of a disjoint union of handlebodies.*

Remark. — If f preserves (resp. reverses) the orientation of F, the condition that $\mathrm{X}.f_*(\mathrm{X})=0$ for every $\mathrm{X} \in \mathrm{H}_1(\mathrm{F})$ is equivalent to the property that $(f_*)^2=\mathrm{Id}$ (resp. $-\mathrm{Id}$).

Proof of 8.6. — The property is proven by induction on the complexity of F; the result is clear when F consists of spheres, which starts the induction.

When at least one component of F is not a sphere, we claim that there exists an essential curve C in F such that, for every m, either $\mathrm{C} \cap f^m(\mathrm{C}) = \emptyset$ or $f^m(\mathrm{C})=\mathrm{C}$. As in 8.2, the existence of such a curve achieves the proof by application of the induction hypothesis to the automorphism (F', f') obtained by (equivariantly) surgering (F, f) along $\bigcup_n f^n(\mathrm{C})$.

The proof of the existence of C is very close to that of Lemma 3.2 and we just sketch it. Equip F with a smooth structure and a Riemannian metric for which f is an isometric diffeomorphism, and let C′ be a simple closed geodesic that is length-minimizing among all essential curves in F. An argument similar to that used in Lemma 3.2 then shows that, for every m, either $\mathrm{C}' \cap f^m(\mathrm{C}') = \emptyset$ or $f^m(\mathrm{C}')=\mathrm{C}'$ [hint: Otherwise, select a shortest arc k in C′ that joins two intersection points in $\mathrm{C}' \cap f^m(\mathrm{C}')$ of opposite signs; k exists since $[\mathrm{C}'].f^m_*([\mathrm{C}'])=0$; one can then construct an essential curve that is shorter than C′ by adding a component of $\mathrm{C}'-f^m(\mathrm{C}')$ to k and rounding the corners, which provides a contradiction]. The PL curve C is then obtained from the smooth curve C′ by lifting a suitable small isotopy in F/f.

This ends the proof of Proposition 8.6. □

9. Cobordism and handlebodies

In the preceding sections, we obtained the algebraic structure of Δ_2. But, for a practical determination of the map $\mathscr{F} \to \Delta_2$, the following problem has still no general solution known.

(P) Given an automorphism of surface (for instance presented as a product of Dehn twists), decide whether it is null-cobordant on not.

As an approach to Problem (P), the following weaker problem is easier to handle.

(P′) Given an automorphism of a connected surface, decide whether it bounds an automorphism of a handlebody or not.

For instance, in [JJ], K. Johannson and D. Johnson exhibit an automorphism of a surface of genus 2 which induces the identity on the homology and does not extend to any handlebody (they show that any manifold obtained by perturbing with this automorphism a genus 2 Heegaard decomposition of S^3 is a homology sphere of Rohlin invariant 1). Moreover, they prove that the cobordism class of this automorphism is then non-trivial (*see* Corollary 9.4 below).

It may therefore be of some interest to give some relations between these "handlebody null-cobordisms" and the usual null-cobordisms. This is what we are going to do in this section.

The following easy corollary of Proposition 8.1 and Lemma 8.2 shows that Problem (P) and Problem (P′) are equivalent for an orientation-preserving periodic automorphism of a connected surface.

PROPOSITION 9.1. — *If an orientation-preserving periodic automorphism* (F^2, f) *of a connected surface is null-cobordant, it extends to a periodic automorphism of a handlebody.*

Proof. — By Proposition 8.1 and with the notation of § 8, the set $\{\tilde{x}\in F/f;\ r(f,\tilde{x})\neq 0, 1/2\}$ admits a partition into pairs $\{\tilde{x}_i, \tilde{x}'_i\}$, $i=1, \ldots, n$, with $r(f,\tilde{x}_i)+r(f,\tilde{x}'_i)=0$ for every i. Let $D=D_{1/2}\cup D_1\cup \ldots \cup D_n$ be a collection of disjoint discs in F/f such that:

(*a*) $(\mathrm{Fix}_+ f)/f\subset \mathrm{int}\, D$;

(*b*) $D_{1/2}\cap(\mathrm{Fix}_+ f)/f=\{\tilde{x}\in F/f;\ r(f,\tilde{x})=1/2\}$;

(*c*) $D_i\cap(\mathrm{Fix}_+ f)/f=\{\tilde{x}_i, \tilde{x}'_i\}$ for every $i=1, \ldots, n$.

If $p: F\to F/f$ denotes the natural projection, consider the periodic automorphism (F', f') constructed from (F, f) by performing an equivariant 2-surgery along the 1-manifold $p^{-1}(\partial D)$. It splits into $(F'_0, f'_0)\amalg(F'_{1/2}, f'_{1/2})\amalg\left(\coprod_{i=1}^{n}(F'_i, f'_i)\right)$ where, for $i=1/2, 1, \ldots, n$, F'_i consists of the components of F' that meet $p^{-1}(\mathrm{int}\, D_i)$.

Let $\rho: H_1(F/f-(\mathrm{Fix}_+ f)/f)\to \mathbb{Q}/\mathbb{Z}$ be the morphism classifying the cyclic branched covering $F\to F/f$. From the relation between ρ and the $r(f, x)'s$ (*see* § 8), it follows that $\rho(\partial D_i)=0$ for every $i=1/2, 1, \ldots, n$ [note that there is an even number of $\tilde{x}\in F/f$ with $r(f,\tilde{x})=1/2$]. Consequently, $\mathrm{Fix}_+ f'_0=\emptyset$, $(f'_{1/2})^m$ is for every m an orientation-preserving involution on each component of $F'_{1/2}$ it preserves and, for every $i=1, \ldots, n$, F'_i consists of spheres. Using Lemma 8.2 and Proposition 8.6, it follows that (F', f') bounds a periodic automorphism of disjoint handlebodies (and balls). Since (F, f) compresses to (F', f') by a periodic automorphism of a compression body, this ends the proof. □

An easy argument, by consideration of the induced automorphism of $H_1(T^2)$ (i. e. Kreck's invariant), shows that an automorphism of the torus T^2 is null-cobordant if and only if it extends to a solid torus. A similar property holds for automorphisms of the surface of genus 2:

PROPOSITION 9.2. — *If an automorphism* (F^2, f) *of a connected surface of genus* 2 *is null-cobordant, either it is reducible or it extends to an automorphism of a handlebody.*

Remark. — If f is reducible, it compresses to an automorphism of one or two tori. Using Propositions 5.1 and 8.1, it can be shown that an orientation-preserving automorphism of a disjoint union of tori that is null-cobordant extends to a disjoint union of solid tori and of copies of $T^2 \times I$ (but the situation is more complex for some orientation-reversing periodic automorphisms of two tori). Together with 9.2, this implies that a null-cobordant orientation-preserving automorphism of the surface of genus 2 extends, either to a handlebody, or to a manifold obtained by glueing a 1-handle over $T^2 \times I$ (with one attaching disc on each component of $T^2 \times \partial I$).

Proof of 9.2. – The null-cobordant automorphism (F^2, f) bounds some $(M^3, \hat{f})$ where M^3 splits into three parts V, M_I and M_P as in Proposition 5.1.

If $V = M$, then f extends to a handlebody and the property is proved.

If $V \neq M$ but V is nevertheless non-trivial (i. e. $V \not\cong \partial M \times I$), $(V; \partial M, \delta V)$ admits a handle decomposition with exactly one 2-handle. Now, Proposition B.1 in Appendix B proves that the core of the 2-handle (extended to ∂M) do not depend on the handle decomposition, up to isotopy. Consequently, it is preserved by $\hat{f}$ up to isotopy, and f is therefore reducible.

If V is trivial then, after isotopy, either $\hat{f}$ is periodic on M, or M is an I-bundle over the non-orientable connected surface G^2 of Euler characteristic -1 and $\hat{f}$ preserves the fibration (by [Wa_2], Lemma 3.5, extended to I-bundles with non-orientable base).

If f is periodic and orientation-preserving, apply Proposition 9.1.

If f is periodic and orientation-reversing, it is easy to see that f is reducible (lift a suitable curve in F/f).

If M is an I-bundle over G^2 and $\hat{f}$ is fiber-preserving, $\hat{f}$ induces some automorphism g of G. The automorphism g is reducible, since it is well-known that the curve splitting G into a punctured torus is unique up to isotopy. Consequently, f is reducible. □

COROLLARY 9.3 [JJ]. – *Let f be an automorphism of a connected surface* F *of genus* 2 *such that* $X . f_*(X) = 0$ *for every* $X \in H_1(F)$ [*or equivalently* $(f_*)^2 = \mathrm{Id}$ *or* $-\mathrm{Id}$ *according as f preserves or reverses the orientation of* F]. *Then* (F, f) *is null-cobordant if and only if f extends to an automorphism of a handlebody*,

Proof. – Proposition 9.3 shows that, if (F, f) is null-cobordant, it compresses to some (F', f') where F′ consists of tori (with possibly $F' = \emptyset$).

One readily checks that $X' . f'_*(X') = 0$ for every $X' \in H_1(F')$, whence it follows easily that f' is reducible and extends to an automorphism of solid tori. This ends the proof. □

Given a reducible surface automorphism, there is a natural way to surger it and obtain a new automorphism of a "smaller" surface. We consider now the inverse construction: For any automorphism (F^2, f), choose a collection of disjoint discs partitioned into pairs $\{D_i, D'_i\}$, $i = 1, \ldots, n$. If f is isotoped so that it sends each pair $\{D_i, D'_i\}$ onto a pair $\pm \{D_j, D'_j\}$ (possibly $i = j$), f extends to an automorphism f' of the surface F′ obtained by 1-surgeries along the pairs $\{D_i, D'_i\}$. We shall then say that (F', f') is a *stabilization of order n* of (F, f).

If the D_i's are fixed, there are still many possible stabilizations of (F, f). For instance, the isotopy classes of stabilizations of the identity constructed by use of the D_i's form a subgroup of $\pi_0(\mathrm{Aut}\, F')$, a semi-direct product of $\mathbb{Z}^n$ (the Dehn twists along the surgery handles) and of some braid group.

THEOREM 9.4. — *For a given surface* F^2, *there exists a constant* $K(F)$ *such that every null-cobordant automorphism* (F, f) *admits a stabilization of order* $\leqq K(F)$ *which bounds an automorphism of a disjoint union of handlebodies.*

For instance, one can take $K(F) = 2s(F)$, *where* $s(F)$ *is the sum of the genera of the components of* F.

Remark. — $2s(F)$ is not the optimum value for $K(F)$. For each surface, the proof of 9.4 provides a lower possible value for $K(F)$, but this value does not seem to admit a "nice" expression in a general formula. Moreover, even the value provided by the proof is not optimum.

Proof. — If (F, f) is null-cobordant, it bounds, by Proposition 5.1, an automorphism $(M^3, \hat{f})$ where M splits into three pieces V, M_I and M_P, each preserved by $\hat{f}$, such that:

(1) V is a compression body for $F = \partial M$ and $\overline{M - V} = M_I \cup M_P$.

(2) M_I is an I-bundle over a closed, possibly non-orientable, surface and $\hat{f} \mid M_I$ is fiber-preserving (apply [Wa_2], Lemma 3.5).

(3) $\hat{f} \mid M_P$ is periodic.

In each component of M_I, choose a fiber of the I-bundle and let U_I be a regular neighborhood in M_I of the union of these fibers. After isotopy, $\hat{f}$ can be assumed to preserve U_I. Note that $\overline{M_I - U_I}$ consists of handlebodies.

In M_P, consider the 1-submanifold $\mathrm{Fix}_+(\hat{f} \mid M_P)$ of the points where an iterate of $\hat{f}$ is locally a non-trivial rotation (*see* § 8). Let U_P be a regular neighborhood in M_P of the union of the arc components of $\mathrm{Fix}_+(\hat{f} \mid M_P)$, preserved by $\hat{f}$. Now $\mathrm{Fix}_+(\hat{f} \mid \partial(\overline{M_P - U_P})) = \emptyset$ and, by Lemma 8.2, $\hat{f} \mid \partial\,(\overline{M_P - U_P})$ therefore extends to a periodic automorphism of a disjoint union of handlebodies. Changing if necessary M by replacing $\overline{M_P - U_P}$ by these handlebodies, we can henceforth assume that $\overline{M_P - U_P}$ consists of handlebodies.

Let V′ consist of the components of V that are not handlebodies (i. e. that are not components of M). A result proved in Appendix B (Lemma B.4) asserts that, after an isotopy of $\hat{f} \mid V$ fixing $\partial_i V$, V′ admits a presentation as $(\partial_i V \times I) \cup \{\text{1-handles}\}$, where:

(1) $\partial_i V$ corresponds to $\partial_i V \times \{0\}$.

(2) The 1-handles are attached on $\partial_i V \times \{1\}$ and avoid the discs $((U_I \cup U_P) \cap \partial_i V) \times \{1\}$.

(3) $\hat{f}$ preserves $U_V = ((U_I \cup U_P) \cap \partial_i V) \times I$.

Now, if $U = U_I \cup U_P \cup U_V$, the automorphism $\hat{f} \mid \partial(\overline{M - U})$ is a stabilization of f and it bounds $\hat{f} \mid \overline{M - U}$. By construction, $M - U$ consists of handlebodies since it is obtained by glueing 1-handles on $(V - V') \amalg (\overline{M_I - U_I}) \amalg (\overline{M_P - U_P})$, that consists itself of handlebodies. The stabilization $\hat{f} \mid \partial(\overline{M - U})$ is therefore of the required type.

To end the proof, we just need to prove that the order of this stabilization is bounded by $2s(F)$. This order is the sum of the number of components of M_I and of 1/2 card $(\mathrm{Fix}_+ \hat{f} \mid \partial M_P)$. An easy computation on the Euler characteristic shows that:

$$\mathrm{card}(\mathrm{Fix}_+ \hat{f} \mid \partial M_P) \leqq 4\, s(\partial M_P),$$

whence the result follows [remark that $s(F) \geqq s(\partial_i V)$]. □

APPENDIX A. – Essential spheres in 3-manifolds
(following M. Scharlemann)

This appendix is devoted to the proof of Lemma 5.2, which we recall below. We follow here the (unpublished) exposition of M. Scharlemann in a talk given at Orsay in 1979.

LEMMA 5.2. – *If* $(F^2 f)$ *is null-cobordant, it bounds an automorphism* $(M^3, \hat{f})$ *with* M *irreducible.*

Proof. – Consider an arbitrary null-cobordism $(M^3, \hat{f})$ for $(F^2; f)$. Without loss of generality, we may assume M connected.

Let Σ^2 be a collection of disjoint spheres in M that realizes a decomposition of M into a connected sum of prime manifolds ([Kn], [Mi]). Let $M_0, M_1, \ldots, M_n$ be the components of the manifold obtained by splitting M along Σ, and let $\hat{M}_i$ be constructed from M_i by glueing a ball along every component of ∂M_i that is a "face" of Σ. The surface Σ and the indices can be chosen so that:

(1) Every $\hat{M}_i$ is a prime manifold (i. e. every separating sphere bounds a ball in $\hat{M}_i$).

(2) Every component of Σ is separating.

(3) $\hat{M}_0 \cong S^3$.

(4) For every $i \neq 0$, $\hat{M}_i \not\cong S^3$ and ∂M_i contains exactly one face of Σ, corresponding to the component Σ_i (it then follows from (2) that $\Sigma = \coprod_{i \neq 0} \Sigma_i$).

A classical result of Kneser (*see* [Mi]) asserts that the $\hat{M}_i$'s do not depend of Σ. This can be slightly improved by the following statement.

LEMMA A.1. – *If* Σ *and* Σ' *are two collections of spheres in* M^3 *that satisfy the above properties* (1) *to* (4), *there exists an automorphism* $\hat{g}$ *of* M^3 *fixing* ∂M *such that* $\hat{g}(\Sigma) = \Sigma'$.

Lemma A.1 achieves the proof of 5.2: Let $\hat{g}$ be provided by application of Lemma A.1 to $\Sigma' = \hat{f}(\Sigma)$. If M′ denotes the disjoint union of the $\hat{M}_i$'s that are not isomorphic to $S^1 \times S^2$, the automorphism $\hat{g}^{-1} \hat{f}$ of M preserves Σ and therefore induces an automorphism $\hat{f}'$ of M′. The manifold M′ is irreducible (recall that every prime connected 3-manifold is either irreducible or isomorphic to $S^1 \times S^2$), and $(F, f) = \partial (M', \hat{f}')$ since $\hat{g} \mid \partial M = \mathrm{Id}$. This ends the proof of Lemma 5.2, granting Lemma A.1. □

Proof of Lemma A.1. – Let $M'_0, M'_1, \ldots, M'_p$ denote the components of the manifold obtained by splitting M along Σ', where the indexing is coherent with conditions (1) to (4).

Consider first the case where $\Sigma \cap \Sigma' = \emptyset$. By condition (1), every separating sphere in M'_i, $i \neq 0$, either bounds a ball or is parallel to the component of $\partial M'_i$ that is a face of Σ'. Considering the (M_i)'s and (M'_j)'s as submanifolds of M, we may therefore assume, after an isotopy of Σ fixing ∂M, that $\Sigma \subset M'_0$. By a symmetric argument, for every $i \neq 0$, M_i contains exactly one component of Σ' and this component is parallel to Σ_i in M_i. It follows that Σ and Σ' are isotopic by an isotopy fixing ∂M.

Consider now the general case. We may assume that Σ and Σ' meet transversally and that the number of components of $\Sigma \cap \Sigma'$ cannot be reduced by any isotopy of Σ fixing ∂M. A standard argument then shows that, for every $i \neq 0$, no component of $\Sigma' \cap M_i$ is a disc (otherwise, one could reduce $\Sigma \cap \Sigma'$ by "crushing" some ball whose boundary is the union of a disc in Σ' and a disc in Σ_i).

If $\Sigma \cap \Sigma' \neq \emptyset$, at least one component of $\Sigma' \cap M_0$ is a disc D (by the above remark), with boundary in the component Σ_i of Σ. Let then $\Sigma_i^{\#}$ be the surface (two spheres) obtained from Σ_i by performing an embedded 2-surgery along D. There exists a simple arc k that joins the two components of $\Sigma_i^{\#}$, with $k \cap \Sigma' = \emptyset$ and $k \cap \Sigma = \partial k$: Indeed, the component of $\Sigma' \cap M_i$ that is adjacent to D is not a disc; construct k by a slight translation of an arc in this component that joins ∂D to a different component of $\Sigma' \cap \Sigma_i$ (*Fig.* 1). Let then Σ_i^* be the sphere obtained from $\Sigma_i^{\#}$ by an embedded 1-surgery along k, and let Σ^* be $(\Sigma - \Sigma_i) \cup \Sigma_i^*$ (*Fig.* 2).

By construction, $\Sigma^* \cap \Sigma' = (\Sigma \cap \Sigma') - \partial D$. We claim that there exists an automorphism $\hat{g}^*$ of M, fixing ∂M, such that $\hat{g}^*(\Sigma) = \Sigma^*$. Let D′ be a disc bounding ∂D in Σ_i and let k' be a simple arc contained in M_0, joining the point $\partial k - D'$ to a point in D and whose interior avoids Σ (but possibly $\Sigma' \cap \mathrm{int}\ k' \neq \emptyset$) (*Fig.* 1). To describe $\hat{g}^*$, it is convenient to consider the manifold $M^{\#}$ constructed by splitting M along the sphere $D \cup D'$ and by glueing a ball B^3 on the boundary of the manifold so obtained, along the "side" of $D \cup D'$ that meets k and k'. In $M^{\#}$, there exists an isotopy that translates B along $k \cup k'$ in the direction $k' - k$ and joins the identity to some automorphism $\hat{g}^{\#}$, where $\hat{g}^{\#}$ fixes B and the complement of a small neighborhood of $B \cup k \cup k'$. Since $\hat{g}^{\#}$ fixes $\partial(M^{\#} - \mathrm{int}\, B)$, it induces an automorphism $\hat{g}^*$ of M, for which $\hat{g}^*(\Sigma)$ is easily seen to be isotopic to Σ^* by an isotopy fixing ∂M (see *Fig.* 2); note that, unlike $\hat{g}^{\#}$, $\hat{g}^*$ is in general definitely not isotopic to the identity.

Iterating this process, we obtain an automorphism $\hat{g}$ fixing ∂M such that $\hat{g}(\Sigma) \cap \Sigma' = \emptyset$. The study of the case where $\Sigma \cap \Sigma' = \emptyset$ then shows that $\hat{g}$ can lastly be deformed by an isotopy fixing ∂M so that $\hat{g}(\Sigma) = \Sigma'$. $\square$

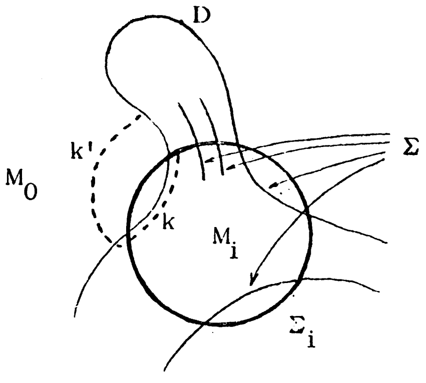


Fig. 1

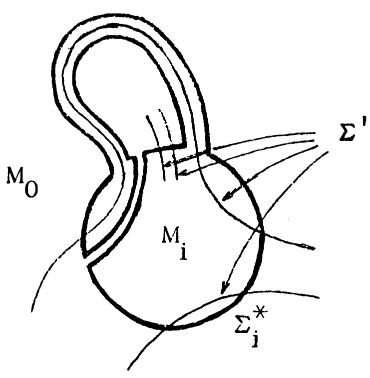


Fig. 2

APPENDIX B. – Compression bodies

The aim of this appendix is to extend to compression bodies some well-known properties of handlebodies (*see* [Z]). These results were needed in paragraph 4 and paragraph 9.

For a compression body V, consider the set $\mathscr{D}$ of all surfaces D in V with the following properties:

(1) D consists of discs with boundary in ∂_e V and splits V into a ∂-irreducible manifold.

(2) Property (1) fails if we remove one component from D.

The set $\mathscr{D}$ contains at least one element D_0 (with possibly $D_0=\emptyset$): Indeed, consider a decomposition of (V; ∂_e V, ∂_i V) into handles of index 2 and 3. The union of the cores of the 2-handles (extended to ∂_e V) satisfies (1); remove then as many components as necessary from this surface.

Given an element of $\mathscr{D}$, there is a natural construction, related to classical "handle sliding" for handle decompositions, that provides many other elements of $\mathscr{D}$: Let d_1 and d_2 be two distinct components of $D\in\mathscr{D}$ and let k be a simple arc in ∂_e V that joins *a* side of d_1 to itself and whose interior meets (transversally) ∂D in exactly one point contained in ∂d_2 (see *Fig.* 3). Consider then a regular neighborhood U of $k\cup d_1$; its frontier δU consists of a disc and of an annulus A. By definition, a *sliding of* d_2 *over* d_1 *along* k is any automorphism t of V that is isotopic to a Dehn twist along A. Up to isotopy, $t(D-d_2)=D-d_2$ and $t(d_2)$ is as in Figure 3 (the two cases occur according to the direction of the Dehn twist). Note that t(D) depends only on one "half" k' of k; we shall say that t(D) *is obtained from* D *by sliding* d_2 *over* d_1 *along* k'. Note that t^{-1} is a sliding of $t(d_2)$ over d_1.

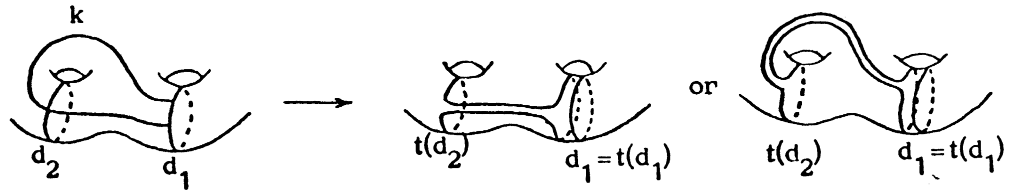


Fig. 3

Proposition B.1. – *Any two elements of* $\mathscr{D}$ *are related by a succession of slidings* (*and isotopies*).

Proof. – Consider D_0 and $D_1\in\mathscr{D}$ and isotop D_1 so that its intersection with D_0 is transverse and has minimum number of components (among all isotopies of D_1). A standard argument then shows that no component of $D_0\cap D_1$ is closed [*see* for instance the proof of $(b)\Rightarrow(c)$ in Proposition 2.2].

We now want to decrease $D_0\cap D_1$ by performing a succession of slidings on D_1. For this purpose, assume $D_0\cap D_1\neq\emptyset$ and consider the manifold $\tilde{V}_1$ constructed by cutting V open along D_1. Let $\tilde{D}_0\subset\tilde{V}_1$ be the surface obtained by splitting D_0 along $D_0\cap D_1$. Since D_0 consists of discs and no component of $D_0\cap D_1$ is closed, a component $\tilde{d}_0$ of $\tilde{D}_0$ is a disc that meets in exactly one arc the union of the faces of D_1 on $\partial\tilde{V}_1$. Let d_1^+ be the face of D_1 that meets $\tilde{d}_0$, d_1 be the corresponding component of D and d_1^- be the other face of d_1 on $\partial\tilde{V}_1$. By definition of $\mathscr{D}$, $\tilde{V}_1$ is ∂-irreducible and $\partial\tilde{d}_0$ consequently bounds a disc $\tilde{d}'_0$ in $\partial\tilde{V}_1$.

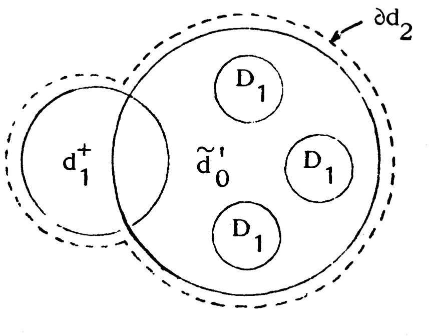


Fig. 4

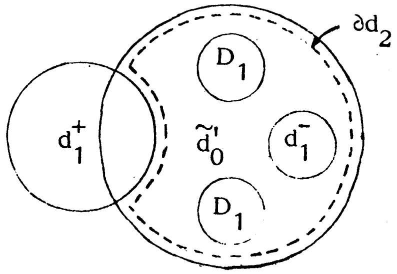


Fig. 5

If $\tilde{d}'_0$ does not contain d_1^-, let d_2 be the disc constructed by "pushing" the interior of $d_1^+ \cup \tilde{d}'_0$ inside $\tilde{V}_1$ and slightly moving its boundary so that $d_1^+ \cap \partial d_2 = \emptyset$ (*Fig.* 4). Considering d_2 as a (properly embedded) disc in V, let D_2 denote $(D_1 - d_1) \cup d_2$. One checks easily that D_2 is obtained from D_1 by a succession of slidings of d_1 over the other components of D_1 that have at least one face in $\tilde{d}'_0$.

If $\tilde{d}'_0$ contains d_1^-, let d_2 be constructed from $\overline{\tilde{d}'_0 - d_1^+}$ by pushing its interior inside $\tilde{V}_1$ and slightly moving its boundary so that $d_1^+ \cap \partial d_2 = \emptyset$ (*Fig.* 5). Again $D_2 = (D_1 - d_1) \cup d_2$ is obtained from D_1 by a succession of slidings of d_1 over the other components of D_1 with at least one face in $\tilde{d}'_0$ (but now, the slidings occur "on the d_1^--side").

In both cases, $D_0 \cap D_2$ has less components than $D_0 \cap D_1$. By iterating this process, we eventually reach a surface $D_n \in \mathscr{D}$, related to D_1 by a sequence of slidings, such that $D_n \cap D_0 = \emptyset$. Let $\tilde{V}_n$ be obtained by cutting V open along D_n. If d_0 is a component of D_0, ∂d_0 bounds a disc d'_0 in $\partial\tilde{V}_n$ ($\tilde{V}_n$ is ∂-irreducible).

There exists a component d_n of D_n with exactly one face in d'_0: Otherwise, d_0 would separate the component of V that contains it into two components, one of which is a handlebody (use the irreducibility of V), and D_0 would not satisfy the minimality condition (2) in the definition of $\mathscr{D}$. Now, the surface $D_{n+1} = (D_n - d_n) \cup d_0$ is obtained from D_n by sliding d_n over the other components of D_n with at least one face in d'_0.

By iterating this process, we eventually reach $D_p \subset D_0$. By minimality of D_0[= condition (2) in the definition of $\mathscr{D}$], it follows that, in fact, $D_p = D_0$. This ends the proof. □

COROLLARY B.2. — *If* D_0 *and* $D_1 \in \mathscr{D}$, *then* $D_1 = u_n u_{n-1} \ldots u_1(D_0)$ *where, for every* i, u_i *is a sliding of a component of* D_0 *over another one.*

Proof. — Proposition B.1 asserts that $D_1 = t_n t_{n-1} \ldots t_1(D_0)$ where, for every i, t_i is a sliding of a component of $t_{i-1} \ldots t_1(D_0)$ over another one. Note that, if $\varphi = t_{n-1} t_{n-2} \ldots t_1$, then $t'_n = \varphi^{-1} t_n \varphi$ is a sliding of a component of D_0 over another one. Since $t_n t_{n-1} \ldots t_1 = \varphi t'_n$, the result then follows by induction on n. □

COROLLARY B.3. — *In a compression body* V, *let* D *consist of discs with* $\partial D \subset \partial_e V$. *Then the manifold* $\tilde{V}$ *obtained by cutting* V *open along* D *is a compression body with interior boundary corresponding to* $\partial_i V$.

Proof. — Choose a handle decomposition of $(V; \partial_e V, \partial_i V)$ into handles of index 2 and 3 with minimum number of handles, and consider the union D_0 of the cores of the 2-handles (extended to $\partial_e V$). Then $\tilde{V}_0$, obtained by cutting V open along D_0, is isomorphic to the disjoint union of $(\partial_i V) \times I$ and of some balls. Moreover, it is easy to check that $D_0 \in \mathcal{D}$.

When $D \in \mathcal{D}$, Proposition B.1 shows that $\tilde{V} \cong \tilde{V}_0$, which proves B.3 in this case.

In the general case, D is contained in a surface D′ which consists of discs and splits V into a ∂-irreducible manifold. Let $D'' \in \mathcal{D}$ be constructed by removing from D′ as many components as necessary, and let V′ (resp. V″) denote the manifold obtained by splitting V along D′ (resp. D″). Since $\tilde{V}'' \cong \tilde{V}_0$ is irreducible and ∂-irreducible, $\tilde{V}'$ is isomorphic to the disjoint union of $\tilde{V}_0$ and of some balls, and therefore to the disjoint union of $(\partial_i V) \times I$ and of balls. It follows that $\tilde{V}$, obtained from $\tilde{V}'$ by glueing 1-handles on $\partial \tilde{V}' - \partial_i V$, is a compression body. □

In §9, we needed the following technical result.

LEMMA B.4. — *Let* V *be a compression body*, Δ *be a finite collection of disjoint discs in* $\partial_i V$ *and* g *be an automorphism of* V *preserving* $\partial_i V$ *and* Δ. *Assume that no component of* V *is a handlebody (just to lighten the notation) and choose a presentation of* V *as* $(\partial_i V \times I) \cup \{1\text{-handles}\}$, *where* $\partial_i V$ *is identified with* $\partial_i V \times \{0\}$ *and the* 1-*handles are attached on* $(\partial_i V - \Delta) \times \{1\}$.

Then, g *can be deformed, by an isotopy fixing* $\partial_i V$, *to an automorphism* g′ *such that* $g'(\Delta \times I) = \Delta \times I$.

Proof. — First of all, note that V admits such a handle decomposition: Indeed, by definition of compression bodies and since no component of V is a handlebody, the triad $(V; \partial_e V, \partial_i V)$ admits a decomposition into 2-handles (the 3-handles can easily be cancelled.) Consider then the dual decomposition.

Let D consist of the co-cores of the 1-handles of this decomposition. Clearly $D \in \mathcal{D}$.

By Corollary B.2, g can be deformed, by an isotopy fixing $\partial_i V$, to $\psi\varphi$ where $\psi(D) = D$ and the support of φ is contained in a regular neigborhood of the union of D and of a 1-subcomplex of $\partial_e V$. In particular, we may assume that the support of φ avoids $\Delta \times I \subset V$. Now, ψ can be isotoped relatively to $\partial_i V$ so that $\psi(\Delta \times I) = \Delta \times I$ (first, make it preserve the 1-handles, and then the strata $\partial_i V \times \{t\}$ by use of [Wa_2], Lemma 3.5). This ends the proof of Lemma B.4. □

REFERENCES

[Bo] F. BONAHON, *Cobordisme des difféomorphismes de surfaces* (*C. R. Acad. Sc.*, Paris, T. 290, série A, 1980, pp. 765-767).

[BS] F. BONAHON and L. C. SIEBENMANN, *New Geometric Splittings for Knots and Links, to appear.*

[Ca] A. CASSON, *Cobordism Invariants of Automorphisms of Surfaces*, Handwritten Notes, Orsay, 1979.

[EE] A. EDMONDS and J. EWING, *Remarks on the Cobordism Group of Surface Diffeomorphisms* (*Math. Ann.*, Vol. 259, 1982, pp. 497-504).

[FL] A. FATHI and F. LAUDENBACH, *Difféomorphismes pseudo-Anosov et décompositions de Heegaard* (*C. R. Acad. Sc.*, Paris, T. 291, série A, 1980, pp. 423-425).

[FLP] A. FATHI, F. LAUDENBACH and V, POENARU, *Travaux de Thurston sur les surfaces* (*Astérisque*, n° 66-67, 1979).

[HW] G. HARDY and E. WRIGHT, *An Introduction to the Theory of Numbers*, Oxford University Press, 1938.

[JS] W. JACO and P. SHALEN, *Seifert Fibered Spaces in 3-Manifolds* (*Memoirs A.M.S.*, n° 220, 1979).

[Jo] J. JOHANNSON, *Homotopy Equivalences of 3-Manifolds with Boundaries* (*Springer Lecture Notes*, n° 761, 1979).

[JJ] J. JOHANNSON and D. JOHNSON, *Non-Bounding Surface Diffeomorphisms which Act Trivially on the Homology*, preprint, 1980.

[Kn] H. KNESER, *Geschlossen Flächen in dreidimensionalen Mannigfaltigkeiten* (*Jber. Deutsch. Math. Verein.*, Vol. 38, 1929, pp. 248-260).

[Kr_1] M. KRECK, *Cobordism of Odd Dimensional Diffeomorphisms* (*Topology*, vol. 15, 1976, pp. 353-361).

[Kr_2] M. KRECK, *Bordism of Diffeomorphisms* (*Bull. Amer. Math. Soc.*, vol. 82, 1976, pp. 759-761).

[LM] S. LOPEZ DE MEDRANO, *Cobordism of Diffeomorphisms of $(k-1)$-Connected $2k$-Manifolds*, *Second Conference on Compact Transformations Groups*, 217-227, Amherst, Springer, 1972.

[MP] W. MEEKS and J. PATRUSKY, *Representing Homology Classes by Embedded Circles on a Compact Surface* (*Illinois J. Math.*, Vol. 22, 1978, pp. 262-269).

[MY] W. MEEKS and S.-T. YAU, *Topology of Three Dimensional Manifolds and the Embedding Problems in Minimal Surface Theory* (*Ann. Math.*, Vol. 112, 1980, pp. 441-484).

[Me] P. MELVIN, *Bordism of Diffeomorphisms* (*Topology*, Vol. 18, 1979, pp. 173-175).

[Mi] J. MILNOR, *A Unique Decomposition Theorem for 3-Manifolds* (*Amer. J. Math.*, 84, 1962, pp. 1-7).

[Mo] G. MOSTOW, *Quasiconformal Mappings in n-Space and the Rigidity of Hyperbolic Space Forms*, Publ. Math. Inst. des Hautes Études Scient., Paris, Vol. 34, 1968, pp. 53-104.

[Ni_1] J. NIELSEN, *Untersuchung zur Topologie der Geschlossenen Zweiseitigen*, Flachen I, II and III, Acta Mathematica, Vol. 50, 1927; Vol. 53, 1929, and Vol. 58, 1931.

[Ni_2] J. NIELSEN, *Abbildungsklassen endlicher Ordnung* (*Acta Math.*, Vol. 75, 1942, pp. 23-115).

[Pr] G. PRASAD, *Strong rigidity of $\mathbb{Q}$-rank 1 lattices* (*Invent. Math.*, Vol. 21, 1973, pp. 255-286).

[Sc] M. SCHARLEMANN, *The Subgroup of Δ_2 Generated by Automorphisms of Tori* (*Math. Ann.*, Vol. 251, 1980, pp. 263-268).

[Te_1] O. TEICHMÜLLER, *Extremale quasiconforme Abbildungen und quadratische Differentiale* (*Abh. Preuss. Akad. Wiss.*, Vol. 22, 1939, pp. 187).

[Te_2] O. TEICHMÜLLER, *Bestimmung der extremalen quasiconformem Abbildungen bei geschlossenen orientierten Riemannschen Fläschen*, Abh. Preuss. Akad. Wiss, Vol. 4, 1943, pp. 42.

[Th_1] W. THURSTON, *On the Geometry and Dynamics of Diffeomorphisms of Surfaces I*, preprint, Princeton University, 1976; for a complete account, *see* [FLP].

[Th_2] W. THURSTON, *The Geometry and Topology of 3-Manifolds* (*Mimeographed Notes*, Princeton University, 1976-1979).

[Th_3] W. THURSTON, *Hyperbolic Structures on 3-manifolds*, preprint, Princeton University, 1980.

[Wa_1] F. WALDHAUSEN, *Eine Klasse von 3-dimensionalen Mannigfaltigkeiten*, I *and* II (*Inventiones Math.*, Vol. 3, 1967, pp. 308-333 and Vol. 4, 1967, pp. 87-117).

[Wa_2] F. WALDHAUSEN, *On Irreducible Manifolds Which are Sufficently Large* (*Ann. Math.*, Vol. 87, 1968, pp. 56-88).

[Wi] H. WINKELNKEMPER, *On equators of Manifolds and the Actions of θ^n* (*Thesis*, Princeton University, 1970).

[Z] H. ZIESCHANG, *Uber einfache Kurven auf Vollbrezeln* (*Abh. Math. Semin. Univ. Hamburg*, Vol. 25, 1962, pp. 231-250).

Fr. BONAHON,
Université Paris-Sud,
Mathématiques,
Bât. 425,
91405 Orsay Cedex.

(Manuscrit reçu le 27 juin 1982.)